Quantum entanglement, topological order, and tensor category theory

Xiao-Gang Wen, MIT
Sept., 2014
Topological order – beyond symmetry breaking

• We used to believe that all phases and phase transitions are described by symmetry breaking

• Counter examples:
  - Quantum Hall states \( \sigma_{xy} = \frac{m}{n} \frac{e^2}{h} \)
  - Spin liquid states, Organics \( \kappa-(ET)_2X \) and herbertsmithite

• FQH states and spin-liquid states have different phases with no symmetry breaking, no crystal order, no spin order, ... so they must have a new order – **topological order** [Wen 89]
Long history of (non) topological phases

- **Symmetry breaking phases:**
  - 500(bc) Ferromagnet (exp.) ... ...

- **Topologically ordered phases:**
  - 1904 Superconductor (exp.) [Onnes 04] (Z\textsubscript{2} topo. order)
  - 1980 IQH states (exp.) [von Klitzing 80] (with no topo. exc., free fermion)
  - 1982 FQH states (exp.) [Tsui-Stormer-Gossard 82]
  - 1987 Chiral spin liquids (theo.) [Kalmeyer-Laughlin 87, Wen-Wilczek-Zee 89]
  - 1991 \textit{Z}\textsubscript{2}-spin liquids (theo.) [Read-Sachdev 91, Wen 91, Kitaev 97]
  - 1992 all Abelian FQH states (theo.) [Wen-Zee 92] (K-matrix)
  - 2000 p\textit{x}+ip\textit{y}-superconductor (theo.) [Read-Green 00] (√IQH at ν\textit{}=1)
  - 2002 hundreds symmetry enriched topological orders (theo.) [Wen 02] (PSG)
  - 2005 all 2+1D topo. orders with gapped edge (theo.) [Levin-Wen 05] (UFC)
  - 2009 ν\textit{}=\textit{5}/2 non-Abelian FQH states (exp. ?) [Willett etal 09]

- **SPT states** (no topological order and no symmetry breaking):
  - also called topological states dispite having no topological order
  - 1983 Haldane phase (theo.) [Haldane 83]
  - 1988 Haldane phase (exp. CsNiCl\textsubscript{3}) [Morra-Buyers-Armstrong-Hirakawa 88]
  - 2005 Topological insulators (theo.) [Kane-Mele 05, Bernevig-Zhang 06]
  - 2007 Topological insulators (exp.) [Molenkamp etal 07]
  - 2011 SPT states in any dim. for any symm. (theo.) [Chen-Gu-Liu-Wen 11]
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  1991 non-Abelian FQH states, (theo.) [Moore-Read 91, Wen 91] (CFT, slave)
  1991 $Z_2$-spin liquids (theo.) [Read-Sachdev 91, Wen 91, Kitaev 97]
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Topological order = patterns of long-range entanglement

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- Thinking about entanglement: [Chen-Gu-Wen 2010]
  - There are long range entangled (LRE) states
  - There are short range entangled (SRE) states

\[ |\text{LRE}\rangle \neq \text{product state} = |\text{SRE}\rangle \]

\[ \text{local unitary transformation} \]

\[ \text{LRE state} \quad \text{SRE product state} \]

\[ g_1 g_2 \]

\[ \text{topological order} \]

\[ \text{phase transition} \]

• All SRE states belong to the same trivial phase
• LRE states can belong to many different phases = different patterns of long-range entanglements = different topological orders [Wen 1989]

• How to (1) characterize and (2) classify topological orders?
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- According to Landau theory, no symm. to break → all systems belong to one trivial phase

- Thinking about entanglement: [Chen-Gu-Wen 2010]
  - There are long range entangled (LRE) states → many phases
  - There are short range entangled (SRE) states → one phase

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- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different patterns of long-range entanglements
  = different topological orders [Wen 1989]

- How to (1) characterize and (2) classify topological orders?
A complete characterization of 2+1D topo. order [Wen 1990]

To characterize topo. order = To label topo. order
To label topo. order = To find topo. inv. for topo. order

- **Topological ground state degeneracy** $D_g$ which is robust against any perturbations that can break any symmetry, but depends on the topology of space. [Wen 1989]
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- The shape of a torus is described by $\tau \in M = \frac{\text{upper-half plane}}{PSL(2, Z)}$

Ground states $\rightarrow$ a $U(D_1)$ vector bundle over the moduli space $M$.
- The local curvature is only $U(1)$: $B_{\tau_x \tau_y} = -(c_2 N^2 + c_1 N + c) \frac{1}{4 \tau_y^2}$

- **Non-Abelian geometric phase** of Dehn twist for thin-tall torus:
  $|\Psi_\alpha\rangle \rightarrow |\Psi'_\alpha\rangle = e^{ia_2 N^2 + a_1 N} T_{\alpha\beta} |\Psi_\beta\rangle$

- $90^\circ$ rotation $\hat{S}$ for square torus:
  $|\Psi_\alpha\rangle \rightarrow |\Psi'_\alpha\rangle = e^{ia_2 N^2 + a_1 N} S_{\alpha\beta} |\Psi_\beta\rangle$

- $(T, S, c)$ are topological invariant robust against any perturbations. [Wen 2012]
$(T, S, c)$ completely characterize 2+1D topological order

- We have shown how to extract topo. invariants $(T_{\alpha\beta}, S_{\alpha\beta}, c)$ from the physical ground state wave functions $\Psi(\vec{r}_1, \vec{r}_2, \cdots)$.

- The edge states of 2+1D topological order are described by “conformal field theories” [Wen, 1989], which chiral central charge is given by $c$ (the universal part of the $U(1)$ curvature).

- The Dehn twist and the $90^\circ$ rotation generate the $MCG \ PSL(2, \mathbb{Z})$ of torus. $T, S$ generate a projective representation of $PSL(2, \mathbb{Z})$.

  - $\dim(T, S) \rightarrow$ number quasiparticle types
  - Eigenvalues of $T_{\alpha\beta} \rightarrow$ quasiparticle fractional statistics.
  - $S_{\alpha\beta} \rightarrow$ quasiparticle fusion $\alpha \otimes \beta = \bigoplus \gamma N_{\alpha\beta}^{\gamma}$,

$$N_{\alpha\beta}^{\gamma} = \sum_{\rho} \frac{S_{\rho\alpha} S_{\rho\beta} S_{\rho\gamma}^*}{S_{\rho1}}$$

Conjecture: $(T, S, c)$ completely characterize 2+1D topological order
Example of topological order: the dance of electrons

- Local dancing rules of a FQH liquid:
  1. Every electron dances around clock-wise
     \( \Phi_{\text{FQH}} \) only depends on \( z = x + iy \)
  2. Takes exactly two steps to go around any others
     \( \Phi_{\text{FQH}} \) ’s phase change \( 4\pi \)
     → Global dancing pattern
     \[ \Phi_{\text{FQH}}(\{z_1, \ldots, z_N\}) = \prod (z_i - z_j)^2 \]
     (for bosonic electrons)

- \( T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad S = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad c = 1 \).
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• A general theory of multi-layer Abelian FQH state:
  \[
  \prod_{l<i<j} (z_l^I - z_j^I)^{K_{ll}} \prod_{l<j; i,j} (z_l^I - z_J^I)^{K_{ll}} e^{-\frac{1}{4} \sum_{i,l} |z_l^I|^2}
  \]
  Low energy effective theory is the \( K \)-matrix
  Chern-Simons theory \( L = \frac{K_{IJ}}{4\pi} a_l \, da_j \)

• An integer number of gapless edge modes \( c = \dim(K) = \text{number of layers.} \)
Even (odd) $K$-matrix classifies all 2+1D bosonic (fermionic) Abelian topological order (but not one-to-one) [Wen-Zee 1992]

- $c =$ the difference in the positive and negative eigenvalues of $K$
- $(T, S)$ are given by

\[
T_{\vec{\alpha}\vec{\beta}} = e^{i\pi \vec{\alpha}^T K \vec{\alpha}} \delta_{\vec{\alpha}\vec{\beta}}, \quad S_{\vec{\alpha}\vec{\beta}} = \frac{e^{-i2\pi \vec{\beta}^T K \vec{\alpha}}}{\sqrt{|\det(K)|}}
\]

where $K\vec{\alpha} =$ integer vector and $K\vec{\beta} =$ integer vector. $\vec{\alpha} \sim \vec{\alpha}'$ if $\vec{\alpha}' - \vec{\alpha} =$ integer vector.
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- The $E_8$-FQH state: $\nu = 4$, no non-trivial topological quasiparticle:
  \[
  K^{E_8} = \begin{pmatrix}
  2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\
  0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\
  0 & 0 & 0 & 0 & 0 & 1 & 2 & 0
  \end{pmatrix}. \quad (T, S, c) = (1, 1, 8).\]
Example of topological order: the dance of spins

- A local dancing rules of spins:
  1. Dance while holding hands (up-spins form loop, no open ends)
  2. $\Phi_{\text{str}}(\text{loop}) = \Phi_{\text{str}}(\text{up-spin})$, $\Phi_{\text{str}}(\text{down-spin}) = \Phi_{\text{str}}(\text{closed loop})$

  $\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{loops}) = 1$

- Another local dancing rules of spins:
  1. Dance while holding hands (up-spins form loop, no open ends)
  2. $\Phi_{\text{str}}(\text{loop}) = \Phi_{\text{str}}(\text{up-spin})$, $\Phi_{\text{str}}(\text{down-spin}) = -\Phi_{\text{str}}(\text{closed loop})$

  $\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{loops}) = (-)^\# \text{ of loops}$

- Two patterns of long-range entanglement $\rightarrow$ two topo. orders: $Z_2$ topological order [Sachdev Read 91, Wen 91] and double-semion topological order. [Freedman et al 05; Levin-Wen 05]
More general patterns of long-range entanglement

Generslize the $Z_2$/double-semion dancing rule:

$$\Phi_\text{str} (\square) = \Phi_\text{str} (\square), \quad \Phi_\text{str} (\square \leftrightarrow \square) = \pm \Phi_\text{str} (\square \square)$$

**Graphic state:**

- More general wave functions are defined on graphs: within the ground state, there are $N + 1$ states on links and $N_v = N_{ij}$ states on vertices:

$$\alpha = \beta \gamma \lambda$$
More general patterns of long-range entanglement

Generlize the $\mathbb{Z}_2$/double-semion dancing rule:

$\Phi_{str}(\begin{array}{c} \alpha \beta \gamma \\ m \end{array}) = \Phi_{str}(\begin{array}{c} \alpha \beta \\ m \end{array})$, $\Phi_{str}(\begin{array}{c} \alpha \beta \gamma \\ m \end{array}) = \pm \Phi_{str}(\begin{array}{c} \alpha \beta \\ m \end{array})$

**Graphic state:**

- More general wave functions are defined on graphs: within the ground state, there are $N + 1$ states on links and $N_v = N_k^j$ states on vertices:

**More general local rule: F-move**

\[
F\text{-move: } \Phi \left( \begin{array}{c} i \\ j \\ k \\ m \\ l \end{array} \right) = \sum_{n=0}^{N} \sum_{\chi=1}^{N_{k\gamma n}} \sum_{\delta=1}^{N_{n\delta l}} F_{i\gamma j\delta; N_{k\gamma n} N_{n\delta l}} \Phi \left( \begin{array}{c} i \\ j \\ \chi \\ k \\ n \\ \delta \\ l \end{array} \right)
\]

- The matrix $F_{i\gamma j\delta; N_{k\gamma n} N_{n\delta l}} \rightarrow (F_{i\gamma j\delta})^m_{n\chi \delta}$

$= \text{local unitary transformation}$

[Levin-Wen, 2005; Chen-Gu-Wen, 2010]
Consistent conditions for $F_{ijk;m\alpha\beta}^{l;n\chi\delta}$: the pentagon identity

The two paths should lead to the same LU trans.:

$$
\sum_{t,\eta,\varphi,\kappa} F_{ijk;m\alpha\beta}^{l;n\chi\delta} F_{itl;n\varphi\chi}^{p;sk\gamma} F_{jkl;t\kappa\eta}^{p;sk\gamma} = \sum_{\epsilon} F_{mkl;n\beta\chi}^{p;q\delta\epsilon} F_{ijq;m\alpha\epsilon}^{p;s\phi\gamma}
$$

Such a set of non-linear algebraic equations is the famous pentagon identity.

- Their solution $N_{ij}^{k}, F_{ijk;m\alpha\beta}^{l;n\chi\delta} \rightarrow \text{Unitary fusion category (UFC)}$
- $\rightarrow \text{string-net states}$
\((T, S, c)\) and projective representation of \(\text{diff(torus)}\)

- For every element \(\hat{W}\) of diffeomorphism group \(\text{diff(torus)}\), we can extract a non-Abelain geometric phase \(W \in U(D_1)\) (up to an \(U(1)\) factor) from the many-body wavefunction \(\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \cdots)\).

The non-Abelain geometric phase \(\hat{W} \rightarrow W\) form a projective representation of \(\text{diff(torus)}\).
\( (T, S, c) \) and projective representation of \( \text{diff(torus)} \)

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- \( \text{MCG} = \text{diff(torus)}/\text{diff}_0(\text{torus}) \), and \( T, S \) generate the projective representation of \( \text{MCG} \).

- \( (T, S, c) \) characterize projective representation of \( \text{diff(torus)} \).
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- \((T, S, c)\) characterize projective representation of \(\text{diff}(\text{torus})\).

**Conjecture:** \([\text{Kong-Wen 2014}]\)

The set of (projective) representations of \(\text{diff}(M_{\text{space}})\) for different spaces \(M_{\text{space}}\) form a one-to-one characterization of topological order.

Non-trivial representation \(\leftrightarrow\) non-trivial topological order.

- Assume the space-time = $M \times S^1_t$ (a fiber bundle over $S^1_t$).
  Such a fiber bundle is described an element in $\hat{W} \in \text{MCG}(M)$.
  So we denote space-time = $M \times \hat{W} S^1_t$

- Volume-ind. (fixed-point) partition function

  [Kong-Wen 14]

  $Z(M \times \hat{W} S^1_t) = Z_{\text{vol-ind}}(M \times \hat{W} S^1_t)e^{-\epsilon_{\text{grnd}} V_{\text{space-time}}}$

  $Z_{\text{vol-ind}}(M \times \hat{W} S^1_t) = \text{Tr}(W)$

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  \[ Z_{\text{vol-ind}}(M \times \hat{W} S^1_t) = \text{Tr}(W) \]

• $Z_{\text{vol-ind}}(M \times S^1_t) =$ the ground state degeneracy on space $M$.

  \[ Z_{\text{vol-ind}}(S^d \times S^1_t) = 1 \text{ (stability condition)} \]

  \[ Z_{\text{vol-ind}}(S^{d-1} \times S^1 \times S^1_t) = \text{number of topological particle types.} \]

The volume-ind. partition function and the non-Abelian geometric phases are the same type of topological invariants for topologically ordered states

Xiao-Gang Wen, MIT Sept., 2014
Consider the low energy boundary effective theory of a topological order in $d+1$-dim. space-time: $Z_{\text{bndry}}(M_1^d \times M_2^d) \approx Z(M^{d+1})$

- Since the bulk is gapped, there is no interaction between the two boundaries at low energies. We expect

$$Z_{\text{bndry}}(M_1^d \times M_2^d) = Z_{\text{bndry}}(M_1^d)Z_{\text{bndry}}^*(M_2^d) \quad \text{W-twist}$$

→ the boundary effective theory is definable in the same dimension
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- If the bulk topological order is non-trivial

$$Z_{\text{bndry}}(M_1^d \times M_2^d) \neq Z_{\text{bndry}}(M_1^d)Z^*_{\text{bndry}}(M_2^d)$$

$\rightarrow$ the boundary effective theory is NOT definable in the same dimension.
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→ the boundary effective theory is NOT definable in the same dimension.

- If the bulk topological order is non-trivial, under a diffeomorphism transformation \(W\) on \(M_1^d\), the change in \(Z_{\text{bndry}}(M_1^d)\) is given by \(Z_{\text{vol-ind}}(M^{d+1})\)

→ the boundary effective theory is NOT invariant under diffeomorphism transformation, and has a gravitational anomaly.
Topo. order classify grav. anomaly in one lower dim.

- gravi. anomaly $\leftrightarrow$ the theory is not definable in the same dimension, but can appear as the boundary to a topo. orde.

There is an one-to-one correspondence between $d$-dimensional topological orders and $d-1$-dimensional gravitational anomalies

[Wen 2013, Kong-Wen 2014]
**Topo. order classify grav. anomaly in one lower dim.**

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**There is an one-to-one correspondence between**

- $d$-dimensional topological orders and $d-1$-dimensional gravitational anomalies

[Wen 2013, Kong-Wen 2014]

**Example 1** (gapless):

- 1+1D chiral fermion $L = i(\psi^\dagger \partial_t \psi - \psi^\dagger \partial_x \psi) \rightarrow \epsilon(k) = \nu k$ has perturbative grav. anomalous. It CANNOT appear as low energy effective theory of any well-definded local 1+1D lattice model.

- But the above chiral fermion theory CAN appear as low energy effective theory for the boundary of a 2+1D topologically ordered state – the $\nu = 1$ IQH state (which has no topological excitations).

- The same bulk $\rightarrow$ many different boundaries of the same gravitational anomaly, e.g. 3 edge modes $(\nu_1 k, -\nu_2 k, \nu_3 k)$
Example 2 (gapless):

- **1+1D chiral boson** (8 modes $c = 8$)
  \[
  L = \frac{K^E_8}{2\pi} \partial_x \phi_I \partial_t \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J.
  \]

- **Gravitational anomalous.**
  Realized as edge of
  8-layer bosonic QH state:
  \[
  \Psi^E_8 = \prod (z^I_i - z^J_j)^{K_{IJ}}.
  \]
  Filling fraction $\nu = 4$
  \[
  \text{det}(K^E_8) = 1 \rightarrow \text{no topo. exc.}
  \]

\[
K^E_8 = \begin{pmatrix}
  2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\
  0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix}
\]
Example 2 (gapless):

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  \[ L = \frac{K_{ij}^{E_8}}{2\pi} \partial_x \phi_i \partial_t \phi_j - V_{ij} \partial_x \phi_i \partial_x \phi_j. \]

- Gravitational anomalous.
  Realized as edge of 8-layer bosonic QH state:
  \[ \Psi_{E_8} = \prod (z_i^I - z_j^J)^{K_{ij}} \]
  Filling fraction $\nu = 4$
  \[ \text{det}(K_{E_8}) = 1 \rightarrow \text{no topo. exc.} \]

Example 3 (gapped):

- 2+1D theory with excitations $(1, e, m, \epsilon)$. Fusion:
  \[ e \times e = m \times m = \epsilon \times \epsilon = 1, \quad e \times m = \epsilon. \]
  Braiding: $e, m, \epsilon$ have mutual $\pi$ statistics, $e, m$ are boson $\epsilon$ is fermion.

- No gravitational anomaly. Can be realized by the toric code model.
**Example 2** (gapless):
- 1+1D chiral boson (8 modes $c = 8$)
  $$L = \frac{K_{IJ}^{E_8}}{2\pi} \partial_x \phi_I \partial_t \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J.$$
- Gravitational anomalous.
  Realized as edge of 8-layer bosonic QH state:
  $$\Psi_{E_8} = \prod (z_i^I - z_j^J)^{K_{IJ}}$$
  Filling fraction $\nu = 4$
  $$\det(K_{E_8}^{E_8}) = 1 \rightarrow \text{no topo. exc.}$$

**Example 3** (gapped):
- 2+1D theory with excitations $(1, e, m, \epsilon)$.
  Fusion:
  $$e \times e = m \times m = \epsilon \times \epsilon = 1, \ e \times m = \epsilon.$$ 
  Braiding: $e, m, \epsilon$ have mutual $\pi$ statistics, $e, m$ are boson $\epsilon$ is fermion.
- No gravitational anomaly. Can be realized by the toric code model.

**Example 4** (gapped):
- 2+1D theory with excitations $(1, \epsilon)$. $\epsilon \times \epsilon = 1$. $\epsilon$ is a fermion.
- Grav. anomalous. Cannot be realized by any 2+1D qubit model.
  But can be realized as the 2D boundary of 3+1D toric code model.
Example 2 (gapless):

- 1+1D chiral boson (8 modes \( c = 8 \))
  \[
  L = \frac{K_{E_8}^{IJ}}{2\pi} \partial_x \phi_I \partial_t \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J.
  \]
  
  - Gravitational anomalous.
  - Realized as edge of
  - 8-layer bosonic QH state:
    \[
    \Psi_{E_8} = \prod (z_I^I - z_J^J) K_{IJ}
    \]
  - Filling fraction \( \nu = 4 \)
  - \( \det(K_{E_8}^{IJ}) = 1 \rightarrow \) no topo. exc.

\[
K_{E_8}^{IJ} = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix}
\]

Example 3 (gapped):

- 2+1D theory with excitations (1, \( e, m, \epsilon \)). Fusion:
  \( e \times e = m \times m = \epsilon \times \epsilon = 1, \ e \times m = \epsilon \).
  - Braiding: \( e, m, \epsilon \) have mutual \( \pi \) statistics, \( e, m \) are boson \( \epsilon \) is fermion.

- No gravitational anomaly. Can be realized by the toric code model.

Example 4 (gapped):

- 2+1D theory with excitations (1, \( \epsilon \)). \( \epsilon \times \epsilon = 1 \). \( \epsilon \) is a fermion.

- Grav. anomalous. Cannot be realized by any 2+1D qubit model.
  - But can be realized as the 2D boundary of 3+1D toric code model.

Example 5 (gapped):

- 2+1D theory with excitations (1, \( \epsilon \)). \( \epsilon \times \epsilon = 1 \). \( \epsilon \) is a semion.

- No grav. anomaly. Can be realized by \( \nu = 1/2 \) bosonic Laughlin state.
Classify long-range entanglement and topological order

• 1+1D: there is no topological order [Verstraete-Cirac-Latorre 05]

• 2+1D: Abelian topological order are classified by $K$-matrices
  2+1D: topological orders are classified by $(MTC, c) = (T, S, c)$?
  2+1D: topo. order with gappable edge are classified by unitary fusion categories (UFC): $\mathcal{Z}(UFC) = MTC$ [Levin-Wen 05]

\[
\Phi \left( \begin{array}{c}
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  \beta \\
  \gamma \\
  \delta \\
  \eta \\
  \iota \\
  \kappa \\
  \lambda \\
  \mu \\
  \nu \\
  \xi \\
  \tau \\
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Classify long-range entanglement and topological order

- 1+1D: there is no topological order [Verstraete-Cirac-Latorre 05]
  1+1D: anomalous topological order are classified by unitary fusion categories (UFC). [Lan-Wen 13] (anomalous topo. = gapped 2D edge)
- 2+1D: Abelian topological order are classified by $K$-matrices
  2+1D: topological orders are classified by $(MTC, c) = (T, S, c)$?
  2+1D: topo. order with gappable edge are classified by unitary fusion categories (UFC): $Z(UFC) = MTC$ [Levin-Wen 05]

$$\Phi \left( \begin{array}{c} i \\ \alpha_j \\ m \\ l \end{array} \right) = \sum F_{ijk;m\alpha\beta} \Phi \left( \begin{array}{c} j \\ \beta \\ k \end{array} \right)$$
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  \[ \Phi \left( \begin{array}{c|c|c|c|c|c|c} i & j & k \\ \hline m & \beta & \end{array} \right) = \sum F_{i j k; m \alpha \beta} \Phi \left( \begin{array}{c|c|c|c|c|c|c} i & j & k \\ \hline \alpha_m & \delta & n \chi \delta \end{array} \right) \]

- **Topological order with only trivial topological excitations**: [Kong-Wen arXiv:1405.5858; Freed arXiv:1406.7278]

  1 + 1D 2 + 1D 3 + 1D 4 + 1D 5 + 1D 6 + 1D

  Boson: 0 $\mathbb{Z}$ $E_8$ 0 $\mathbb{Z}_2$ 0 $\mathbb{Z} \oplus \mathbb{Z}$

  Fermion: $\mathbb{Z}_2$ $\mathbb{Z}$ $p + ip$ ? ? ?
Our vacuum is topo. ordered (long-range entangled) – an unification of quantum information and matter

**Long-range entangled qubits**
- **Unify:**
  - Spin-1/2
  - Fermi statistics
  - Gauge interactions
  - Chiral fermions
  - Fractional quantum number
- **New math** (algebra):
  - tensor category
  - group cohomology
- **Predict:**
  - New discrete gauge fields

**Maxwell equation**
- **Unify:**
  - Electricity
  - Magnetism
  - Light
- **New math** (geometry):
  - Fiber bundle (gauge theory)
- **Predict:**
  - $\dot{E} \rightarrow B$

Topological order and long-range entanglement provide a new conceptual lens, through which we now see a much richer world of quantum materials, and which may in time illuminate the quantum substructure of the universe itself.

Xiao-Gang Wen, MIT Sept., 2014

Quantum entanglement, topological order, and tensor category
Short-range entanglements w/ symmetry → SPT phases

For gapped systems with a symmetry $H = U_g H U^\dagger_g$, $g \in G$

- there are **LRE symmetric states** → many different phases
- there are **SRE symmetric states** → one phase (no symm. breaking)
Short-range entanglements w/ symmetry \( \rightarrow \) SPT phases

For gapped systems with a symmetry \( H = U_g H U_g^\dagger, \ g \in G \)

- there are \textbf{LRE symmetric states} \( \rightarrow \) many different phases
- there are \textbf{SRE symmetric states} \( \rightarrow \) many different phases

We may call them \textit{symmetry protected trivial} (SPT) phase

\[ g_1 \quad g_2 \]

\[ \text{SRE} \]

\[ \text{topological order} \]

\[ \text{LRE 1} \quad \text{LRE 2} \]

\[ \text{SY–SRE 1} \quad \text{SY–SRE 2} \]

\[ \text{SB–SRE 1} \quad \text{SB–SRE 2} \]

\[ \text{SB–LRE 1} \quad \text{SB–LRE 2} \]

\[ \text{SY–LRE 1} \quad \text{SY–LRE 2} \]

- SPT phases = equivalent class of \textit{symmetric} LU transformations

\[ \text{SPT 1} \quad \text{SPT 2} \]

\[ \text{phase transition} \]

\[ \text{preserve symmetry} \]

\[ \text{no symmetry} \]

\[ \text{symmetry breaking (group theory)} \]

\[ \text{topological orders} \ (???) \]

\[ \text{SPT phases} \ (???) \]
Short-range entanglements w/ symmetry $\rightarrow$ SPT phases

For gapped systems with a symmetry $H = U_g H U_g^\dagger$, $g \in G$

- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ many different phases

We may call them **symmetry protected trivial (SPT)** phase
or **symmetry protected topological (SPT)** phase

- **SPT phases** = equivalent class of *symmetric* LU transformations
- **1D Haldane phase**, [Haldane 83] 2D/3D topological insulators, [Kane-Mele 05; Bernevig-Zhang 06] [Moore-Balents 07; Fu-Kane-Mele 07] are examples of SPT phases.

Xiao-Gang Wen, MIT Sept., 2014
Topological states and anomalies

Topologically ordered state
- Theory with grav. anomaly

SPT state with on-site symmetry
- Theory with gauge (symm.) anomaly
- Theory with mixed gauge/grav. anomaly

SET orders
- (Tensor category w/ symmetry)

SPT orders
- (Group cohomology theory)

A classification of SPT orders

- The Haldane phase has only short-range entanglement, non-trivial only because of symmetry $\rightarrow$ notion of SPT order. [Gu-Wen 2009].
A classification of SPT orders

- The Haldane phase has only short-range entanglement, non-trivial only because of symmetry → notion of SPT order. [Gu-Wen 2009].

- **A classification SPT order** (w/ gauge anomalous boundary):
  [Chen-Gu-Liu-Wen 2011]

  **Group cohomology** $\mathcal{H}^{d+1}[G, U(1)]$ classify (one-to-one) the $d + 1$D SPT order with an on-site symmetry $G$ and whose boundary has only pure gauge anomaly.
A classification of SPT orders

- The Haldane phase has only short-range entanglement, non-trivial only because of symmetry \(\rightarrow\) notion of SPT order. [Gu-Wen 2009].

- **A classification SPT order (w/ gauge anomalous boundary)**:
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  **Group cohomology** \(H^{d+1}[G, U(1)]\) classify (one-to-one) the \(d+1\)D **SPT order** with an on-site symmetry \(G\) and whose boundary has only pure gauge anomaly.

- The boundary of non-trivial SPT order has non-trivial gauge anomaly. SPT orders described by \(H^{d+1}[G, U(1)]\) classify pure gauge anomalies in one low dim. (inc. global). [Wen 2013]
A classification of SPT orders

- The Haldane phase has only short-range entanglement, non-trivial only because of symmetry $\rightarrow$ notion of SPT order. [Gu-Wen 2009].

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- The boundary of non-trivial SPT order has non-trivial gauge anomaly. SPT orders described by $\mathcal{H}^{d+1}[G, U(1)]$ classify pure gauge anomalies in one low dim. (inc. global). [Wen 2013]

- The partition function of a SPT state $Z_{\text{vol-ind}}(M^d) = 1$ (trivial)
  - If we gauge the symmetry $Z_{\text{vol-ind}}(M^3, A) = e^{i\int_{M^3} \frac{k}{4\pi} A dA}$ (for 2+1D $U(1)$ SPT state classified by $k \in \mathcal{H}^3[U(1), U(1)] = \mathbb{Z}$).

  [Chen-Wen 2012, Lu-Vishwanath 2012, Liu-Wen 2013]

  We can probe the SPT states by gauging the symm. [Levin-Gu 12]
SPT state with mixed-gauge-grav. anomalous boundary

- Probe topological order and/or SPT order by gauging the symmetry and choosing curved space-time.
  - Gauge topological term → SPT states of group cohomology
  - Gravitational topological term → topologically ordered states
  - Mixed gauge-grav. topological term → new SPT states

- In 4D space, a $U(1)$ monople is a loop (not a point). Such a $U(1)$ monople-loop carry a gapless edge states of $k$-copy of $E_8$ bosonic IQH state.

- A new class of $U(1)$ SPT state in 3+1D labeled by $k \in \mathbb{Z}$:
  $$Z_{\text{vol-ind}}(A_i, g_{\mu\nu}) = e^{i k \int_M A \wedge \omega_3(g_{\mu\nu})}, \quad \omega_3 = \text{grav. CS term}$$

- A new class of $U(1)$ SPT state in 4+1D labeled by $k \in \mathbb{Z}_2$:
  $$Z_{\text{vol-ind}}(A_i, g_{\mu\nu}) = e^{i k \int_{M^4} A \wedge w_2(g_{\mu\nu})}, \quad w_i \text{ is the } i\text{th Stiefel-Whitney class.}$$

[Wang-Gu-Wen 14]
SPT state with mixed-gauge-grav. anomalous boundary

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- A new class of $U(1)$ SPT state in 4+1D labeled by $k \in \mathbb{Z}$:
  \[
  Z_{\text{vol-ind}}(A_i, g_{\mu\nu}) = e^{i\frac{k}{3} \int_{M^5} dA \wedge \omega_3(g_{\mu\nu})}, \quad \omega_3 = \text{grav. CS term}
  \]
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SPT state with mixed-gauge-grav. anomalous boundary

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  \[
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  - In 4D space, a \( U(1) \) monople is a loop (not a point). Such a \( U(1) \) monople-loop carry a gapless edge states of \( k \)-copy of \( E_8 \) bosonic IQH state.

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  \[
  Z_{\text{vol-ind}}(A_i, g_{\mu\nu}) = e^{i \frac{k}{2} \int_{M^4} dA \wedge w_2(g_{\mu\nu})}
  \]
  where \( w_i \) is the \( i^{th} \) Stiefel-Whitney class.

[Wang-Gu-Wen 14]

Xiao-Gang Wen, MIT Sept., 2014
Classify long-range entanglement and topological order

- **1+1D:** there is no topological order [Verstraete-Cirac-Latorre 05]

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  2+1D: topological orders are classified by \((UMTC, c) = (T, S, c)\)?

  2+1D: topo. order with gappable edge are classified by unitary fusion categories (UFC): \(Z(UFC) = UMTC\) [Levin-Wen 05]

\[
\Phi \left( \alpha_{i}^{m} \beta_{l}^{j} \right) = \sum F_{i j k ; m \alpha \beta} \Phi \left( \alpha_{l}^{n} \right)
\]

- **Topological order with only trivial topological excitations:**

  [Kong-Wen 2014; Freed 2014]

\[
\begin{array}{cccccc}
1 + 1D & 2 + 1D & 3 + 1D & 4 + 1D & 5 + 1D & 6 + 1D \\
Boson: & 0 & \mathbb{Z} E_8 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z} \oplus \mathbb{Z} \\
Fermion: & \mathbb{Z}_2 & \mathbb{Z} p + ip & ? & ? & ? & ?
\end{array}
\]
Monoid and group structures of topological orders

- Let $C_d = \{ a, b, c, \cdots \}$ be a set of topologically ordered phases in $d$ dimensions. Stacking $a$-TO state and $b$-TO state → a $c$-TO state:
  
  $a \boxtimes b = c$, \hspace{1cm} a, b, c \in C_d

**Diagram:**

- $c$-TO
  - $a$-TO
  - $b$-TO

In general,

\[
Z_a \otimes \text{vol-ind}(M \ltimes \hat{W}_{S^1_t}) = Z_a \otimes \text{vol-ind}(M \ltimes \hat{W}_{S^1_t}) Z_a \otimes \text{vol-ind}(M \ltimes \hat{W}_{S^1_t})^{-1}
\]

- A topological order is invertible iff its $Z_a \otimes \text{vol-ind}(M \ltimes \hat{W}_{S^1_t}) = e^{i\theta}$.
- A topological order is invertible iff it has no topological excitations.

[Kong-Wen 14, Freed 14]
Monoid and group structures of topological orders

- Let \( C_d = \{a, b, c, \cdots\} \) be a set of topologically ordered phases in \( d \) dimensions.

  Stacking an \( a \)-TO state and a \( b \)-TO state \( \rightarrow \) a \( c \)-TO state:
  \[ a \boxtimes b = c, \quad a, b, c \in C_d \]

- \( \boxtimes \) makes \( C_d \) a monoid (a group without inverse).

Consider topological order \( a \) and its time reversal \( a^* \)

\[
Z_{\text{vol-ind}}^a(M \times \hat{W} S^1_t) = \left[Z_{\text{vol-ind}}^a(M \times \hat{W} S^1_t)\right]^*, \text{ then}
\]

\[
Z_{\text{vol-ind}}^{a \boxtimes a^*}(M \times \hat{W} S^1_t) = Z_{\text{vol-ind}}^a(M \times \hat{W} S^1_t)Z_{\text{vol-ind}}^{a^*}(M \times \hat{W} S^1_t)
\]

In general, \( Z_{\text{vol-ind}}^a(M \times \hat{W} S^1_t)Z_{\text{vol-ind}}^{a^*}(M \times \hat{W} S^1_t) \neq 1 \rightarrow a \boxtimes a^* \) is a non trivial topological order, and \( a \)-TO has no inverse.
Monoid and group structures of topological orders

- Let $C_d = \{a, b, c, \cdots\}$ be a set of topologically ordered phases in $d$ dimensions.
  
  Stacking $a$-TO state and $b$-TO state $\rightarrow$ a $c$-TO state: $a \boxtimes b = c, \ a, b, c \in C_d$

- \( \boxtimes \) make $C_d$ a monoid (a group without inverse).

Consider topological order $a$ and its time reversal $a^*$

\[
Z_{vol-ind}^a(M \times_{\widehat{W}} S^1_t) = [Z_{vol-ind}^a(M \times_{\widehat{W}} S^1_t)]^*, \ 	ext{then}
\]

\[
Z_{vol-ind}^{a \boxtimes a^*}(M \times_{\widehat{W}} S^1_t) = Z_{vol-ind}^a(M \times_{\widehat{W}} S^1_t)Z_{vol-ind}^{a^*}(M \times_{\widehat{W}} S^1_t)
\]

In general, $Z_{vol-ind}^a(M \times_{\widehat{W}} S^1_t)Z_{vol-ind}^{a^*}(M \times_{\widehat{W}} S^1_t) \neq 1 \rightarrow a \boxtimes a^*$ is a non trivial topological order, and $a$-TO has no inverse.

- A topological order is invertible iff its $Z_{vol-ind}(M \times_{\widehat{W}} S^1_t) = e^{i\theta}$

A topological order is invertible iff it has no topological excitations.

[Kong-Wen 14, Freed 14]
Classify invertible bosonic topo. order (with no topo. exc.)

In 2+1D:

- $Z_{\text{vol-ind}}(M \times_{\tilde{W}} S^1_t) = e^{\frac{i 2\pi c}{24} \int_{M \times_{\tilde{W}} S^1_t} \omega_3(g_{\mu\nu})}$ where $\omega_3$ is the gravitational Chern-Simons term: $d\omega_3 = p_1$ and $p_1$ is the first Pontryagin class.
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- The quantization of the topological term: $c = 8 \times \text{int.} \rightarrow \mathbb{Z}$-class: $\int_M \omega_3(g_{\mu\nu}) = \int_{N, \partial N = M} p_1 = \int_{N', \partial N' = M} p_1 \mod 3$, since $\int_{N_{\text{closed}}} p_1 = 0 \mod 3$. 
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  \]
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• Relation to gravitational anomaly on the boundary \( B^2 \):
  \[
  (1) \quad Z = e^{i \int_{B^2} L_{\text{eff}}^{\text{bndry}}(g_{\mu \nu})} e^{i \frac{2\pi c}{24} \int_{M^3, \partial M^3 = B^2} \omega_3(g_{\mu \nu})}
  \]
  \( e^{i \frac{2\pi c}{24} \int_{M^3, \partial M^3 = B^2} \omega_3(g_{\mu \nu})} \) is not differomorphism invariant, but
  \( e^{i \int_{B^2} L_{\text{eff}}^{\text{bndry}}(g_{\mu \nu})} e^{i \frac{2\pi c}{24} \int_{M^3, \partial M^3 = B^2} \omega_3(g_{\mu \nu})} \) is.

  (2) Consider an 1+1D differomorphism \( W : B^2 \to B^2, g_{\mu \nu} \to g_{\mu \nu}^W \).
  \[
  \int_{B^2} L_{\text{eff}}^{\text{bndry}}(g_{\mu \nu}^W) - \int_{B^2} L_{\text{eff}}^{\text{bndry}}(g_{\mu \nu}) = \frac{2\pi c}{24} \int_{B^2 \times \hat{W} S^1} \omega^3(g_{\mu \nu})
  \]
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In 4+1D:

- \( Z_{\text{vol-ind}}(M \times \widetilde{W} S^1_t) = e^{i\pi \int_{M \times \widetilde{W} S^1_t} w_2 w_3} \) where \( w_i \) is the \( i^{th} \) Stiefel-Whitney class \( \rightarrow \mathbb{Z}_2 \)-class. We find \( \int_{M \times \widetilde{W} S^1_t} w_2 w_3 = 1 \) when \( M = \mathbb{C}P^2 \) and \( W : \mathbb{C}P^2 \rightarrow (\mathbb{C}P^2)^* \)

- Global grav. anomaly: for \( M = \mathbb{C}P^2 \) and \( W : \mathbb{C}P^2 \rightarrow (\mathbb{C}P^2)^* \)
  \[
  \int_M L_{\text{eff}}^{\text{bndry}}(g_{\mu\nu}^W) - \int_M L_{\text{eff}}^{\text{bndry}}(g_{\mu\nu}) = \int_{M \times \widetilde{W} S^1_t} w_2 w_3
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\int_M L_{\text{eff}}^{\text{bndry}}(g^W_{\mu\nu}) - \int_M L_{\text{eff}}^{\text{bndry}}(g_{\mu\nu}) = \int_{M \times_{\hat{W}} S^1_t} w_2 w_3
\]

In 6+1D:

• Two independent grav. Chern-Simons terms:

\[
Z_{\text{vol-ind}}(M^7) = e^{2\pi i \int_M \left[ k_1 \frac{\hat{\omega}_7 - 2\omega_7}{5} + k_2 \frac{-2\hat{\omega}_7 + 5\omega_7}{9} \right]}
\]

where \( d\omega_7 = p_2, \ d\hat{\omega}_7 = p_1 p_1 \rightarrow \mathbb{Z} \oplus \mathbb{Z}\)-class \((k_1, k_2)\). [Kong-Wen 14]

\[
\begin{array}{cccccccc}
1 + 1D & 2 + 1D & 3 + 1D & 4 + 1D & 5 + 1D & 6 + 1D \\
\text{Boson:} & 0 & \mathbb{Z} & E_8 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z} \oplus \mathbb{Z} \\
\text{Fermion:} & \mathbb{Z}_2 & \mathbb{Z} & p+ip & ? & ? & ? & ?
\end{array}
\]
Topo. orders in different dim. form a cochain complex

- Three kind of topological orders in $d$-dim.: boundary bulk
  - Exact topological order $TO_d^{\text{exct}}$ gappable definable
  - Closed topological order $TO_d^{\text{clsd}}$ gapless definable
  - Generic topological order $TO_d^{\text{gnrc}}$ may not

The gapped boundary is described by a generic topological order $TO_d$ that uniquely determines the bulk topological order $TO_{d+1}$ which is exact: $Z(\text{TO}_d) = TO_{d+1}$.

$Z(\text{Z}(\text{TO}_d)) = 1_{d+1}$ the trivial topological order.

Topological orders form a cochain complex: $\overset{\rightarrow}{Kong-Wen\ 2014}$

$Z \rightarrow \{TO_{d-1}\} \rightarrow \{TO_d\} \rightarrow \{TO_{d+1}\} \rightarrow$
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- Topological orders form a cochain complex: [Kong-Wen 2014]

$$
\begin{array}{c}
\mathbb{Z} \rightarrow \{ \text{TO}_{d-1} \} \\
\mathbb{Z} \rightarrow \{ \text{TO}_d \} \\
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\end{array}
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- Topological orders form a cochain complex: [Kong-Wen 2014]
  \[
  Z \rightarrow \{TO_{d-1}\} \rightarrow \{TO_d\} \rightarrow \{TO_{d+1}\} \rightarrow \]

- Example: 1+1D generic topological orders are classified by UFC
  2+1D closed topological orders are classified by MTC
  \[
  UFC = \{TO_{1+1}\} \xrightarrow{Z_2} MTC = \{TO_{2+1}^{\text{clsd}}\} \xrightarrow{Z_3} 1_{3+1}\]

where $Z_2$ is the Drinfeld center.
A classification of gapped quantum liquids

- **Symmetry breaking phases:** *group theory*
  No fractional statistics, no fractional quantum numbers
  Example: Ferromagnets, superfluids, *etc*
  Key: Symmetry breaking

- **Topo. ordered phases:** *n-category theory (extended TQFT)*
  Have fractional statistics, and fractional quantum numbers
  Example: FQH states, $\mathbb{Z}_2$ spin liquid states, chiral spin liquid states, *etc*
  Key: Long-range entanglement (topological order)

- **SPT ordered phases:** *group cohomology theory and beyond*
  No fractional statistics, no fractional quantum numbers
  Example: Haldane phase in 1+1D, topological insulators, *etc*
  Key: Symmetry protection

- The above three features can coexist.