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Modular-type functions attached to mirror quintic Calabi–Yau varieties

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Abstract In this article we study a differential algebra of modular-type functions attached to the periods of a one-parameter family of Calabi–Yau varieties which is mirror dual to the universal family of quintic threefolds. Such an algebra is generated by seven functions satisfying functional and differential equations in parallel to the modular functional equations of classical Eisenstein series and the Ramanujan differential equation. Our result is the first example of automorphic-type functions attached to varieties whose period domain is not Hermitian symmetric. It is a reformulation and realization of a problem of Griffiths from the seventies on the existence of automorphic functions for the moduli of polarized Hodge structures.

Keywords Gauss–Manin connection · Yukawa coupling · Hodge filtration · Griffiths transversality

Mathematics Subject Classification 14N35 · 14J15 · 32G20

1 Introduction

In 1991 Candelas et al. in [2] calculated in the framework of mirror symmetry a generating function, called the Yukawa coupling, which predicts the number of rational curves of a fixed degree in a generic quintic threefold. Since then there were many efforts to relate the Yukawa coupling to classical modular or quasi-modular forms, however, there was no success. The theory of modular or quasi-modular forms is attached to elliptic curves, or more accurately, to their periods, see for instance [11,12]. In general the available automorphic form theories are attached to varieties whose Hodge structures form a Hermitian symmetric

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domain. This is not the case for the mirror quintic Calabi–Yau threefolds which are the underlying varieties of the Yukawa coupling. An attempt to formulate automorphic form theories beyond Hermitian symmetric domains was first done in the seventies by P. Griffiths in the framework of Hodge structures, see [7]. However, such a formulation has lost the generating function role of modular forms. The main aim of the present text is to reformulate and realize the construction of a modular-type function theory attached to mirror quintic threefolds. Our approach is purely geometric and it uses Grothendieck's algebraic de Rham cohomology [8] and the algebraic Gauss–Manin connection due to Katz and Oda [9], see Theorem 3. From this we derive the algebra of modular-type functions, see Theorem 1, and describe the functional equations satisfied by its elements, see Theorem 2. For preliminaries in Hodge theory and modular forms the reader is referred to Voisin's book [18] and Zagier's article [19], respectively. The present work continues and simplifies our previous article [13].

Consider the following fourth-order linear differential equation:

$$\theta^4 - z\left(\theta+\frac{1}{5}\right)\left(\theta+\frac{2}{5}\right)\left(\theta+\frac{3}{5}\right)\left(\theta+\frac{4}{5}\right) = 0, \quad \theta = z\frac{\partial}{\partial z}. \tag{1}$$

A basis of the solution space of (1) is given by:

$$\psi_i(z) = \frac{1}{i!}\frac{\partial^i}{\partial\epsilon^i}\left(5^{-5\epsilon}F(\epsilon,z)\right), \quad j = 0, 1, 2, 3,$$

where

$$F(\epsilon,z) := \sum_{n=0}^{\infty}\frac{\left(\frac{1}{5}+\epsilon\right)_n\left(\frac{2}{5}+\epsilon\right)_n\left(\frac{3}{5}+\epsilon\right)_n\left(\frac{4}{5}+\epsilon\right)_n}{(1+\epsilon)_n^4}z^{\epsilon+n}$$

and $(a)_n := a(a+1)\cdots(a+n-1)$ for $n>0$ and $(a)_0 := 1$. We use the base change

$$\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & \frac{5}{2} & -\frac{25}{12} \\ -5 & 0 & -\frac{25}{12} & 200\frac{\zeta(3)}{(2\pi i)^3} \end{pmatrix} \begin{pmatrix} \frac{1}{5^4}\psi_3 \\ \frac{2\pi i}{5^4}\psi_2 \\ \frac{(2\pi i)^2}{5^4}\psi_1 \\ \frac{(2\pi i)^3}{5^4}\psi_0 \end{pmatrix}.$$

In the new basis x_{1i} the monodromy of (1) around the singularities $z=0$ and $z=1$ is respectively given by:

$$M_0 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix}, \quad M_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that is, the analytic continuation of the 4×1 matrix $[x_{i1}]$ around the singularity $z=0$ (respectively $z=1$) is given by $M_0[x_{i1}]$, respectively $M_1[x_{i1}]$ (see for instance [5,17] and [3] for similar calculations). The functions x_{i1} can be written as periods of a holomorphic differential 3-form over topological cycles with integral coefficients (see Sect. 4).

Let

$$\tau_0 := \frac{x_{11}}{x_{21}}, \quad q := e^{2\pi i\tau_0},$$

and

$$x_{ij} := \theta^{j-1}x_{i1}, \quad i,j\in\{1,2,3,4\}.$$

Theorem 1 *Let*

$$
\begin{aligned}
t_0 &= x_{21},\\
t_1 &= 5^4 x_{21}\left((6z-1)x_{21}+5(11z-1)x_{22}+25(6z-1)x_{23}+125(z-1)x_{24}\right),\\
t_2 &= 5^4 x_{21}^2\left((2z-7)x_{21}+15(z-1)x_{22}+25(z-1)x_{23}\right),\\
t_3 &= 5^4 x_{21}^3\left((z-6)x_{21}+5(z-1)x_{22}\right),\\
t_4 &= z x_{21}^5,\\
t_5 &= 5^5(z-1)x_{21}^2\left(x_{12}x_{21}-x_{11}x_{22}\right),\\
t_6 &= 5^5(z-1)x_{21}\left(3(x_{12}x_{21}-x_{11}x_{22})+5(x_{13}x_{21}-x_{11}x_{23})\right).
\end{aligned}
$$

There are holomorphic functions h_i defined in some neighborhood of $0\in\mathbb{C}$ *such that*

$$t_i=\left(\frac{2\pi i}{5}\right)^{d_i} h_i\left(e^{2\pi i\tau_0}\right), \tag{2}$$

where

$$d_i := 3(i+1),\ i\in\{0,1,2,3,4\},\quad d_5 := 11,\quad d_6 := 8.$$

Moreover, the t_i satisfy the following ordinary differential equation:

$$
\mathrm{Ra}:\quad
\begin{cases}
\dot t_0 = \frac{1}{t_5}(6\cdot 5^4 t_0^5 + t_0t_3 - 5^4 t_4)\\
\dot t_1 = \frac{1}{t_5}(-5^8 t_0^6 + 5^5 t_0^4 t_1 + 5^8 t_0 t_4 + t_1 t_3)\\
\dot t_2 = \frac{1}{t_5}(-3\cdot 5^9 t_0^7 - 5^4 t_0^5 t_1 + 2\cdot 5^5 t_0^4 t_2 + 3\cdot 5^9 t_0^2 t_4 + 5^4 t_1 t_4 + 2t_2t_3)\\
\dot t_3 = \frac{1}{t_5}(-5^{10} t_0^8 - 5^4 t_0^5 t_2 + 3\cdot 5^5 t_0^4 t_3 + 5^{10} t_0^3 t_4 + 5^4 t_2 t_4 + 3t_3^2)\\
\dot t_4 = \frac{1}{t_5}(5^6 t_0^4 t_4 + 5 t_3 t_4)\\
\dot t_5 = \frac{1}{t_5}(-5^4 t_0^5 t_6 + 3\cdot 5^5 t_0^4 t_5 + 2t_3t_5 + 5^4 t_4 t_6)\\
\dot t_6 = \frac{1}{t_5}(3\cdot 5^5 t_0^4 t_6 - 5^5 t_0^3 t_5 - 2t_2t_5 + 3t_3t_6)
\end{cases}
\tag{3}
$$

with $\dot{*} := \frac{\partial *}{\partial \tau_0}$.

We define the weights $\deg(t_i) := d_i$. In the right hand side of Ra we have homogeneous rational functions of degree 4, 7, 10, 13, 16, 12, 9, respectively. Note that the degree of a quotient $\frac{a}{b}$ is defined to be $\deg(a)-\deg(b)$. These are the same degrees in the left hand side if we define $\deg(\dot t_i) := \deg(t_i)+1$. The ordinary differential equation Ra is a generalization of the Ramanujan differential equations satisfied by Eisenstein series, see for instance [12,19].

We write each h_i as a formal power series in q, $h_i=\sum_{n=0}^{\infty} h_{i,n}q^n$ and substitute in (3) with $\dot{*} = 5q\frac{\partial *}{\partial q}$ and we see that it determines all the coefficients $h_{i,n}$ uniquely with the initial values:

$$h_{0,0}=\frac{1}{5},\ h_{0,1}=24,\ h_{4,0}=0 \tag{4}$$

and assuming that $h_{5,0}\neq 0$. In fact the differential equation (3) seems to be the simplest way of writing the mixed recursion between $h_{i,n},\ n\geq 2$. Some of the first coefficients of the h_i are given in the table at the end of the Introduction. The differential Galois group of (1) is $\mathrm{Sp}(4,\mathbb{C})$. This together with the equality (2) implies that the functions $h_0, h_1, \ldots, h_6$ are algebraically independent over $\mathbb{C}$ (see [13], Theorem 2).

The reader who is expert in classical modular forms may ask for the functional equations of the t_i. Let $\mathbb{H}$ be the monodromy covering of $(\mathbb{C}-\{0,1\})\cup\{\infty\}$ associated with the monodromy group $\Gamma := \langle M_1, M_0\rangle$ of (1) (see §4). The set $\mathbb{H}$ is biholomorphic to the upper half plane, see [1]. This is equivalent to say that the only relation between M_0 and M_1 is $(M_0M_1)^5 = I$. We will not need to assume this statement because we do not need the coordinate system on $\mathbb{H}$ given by this biholomorphism. The monodromy group Γ acts from the left on $\mathbb{H}$ in a canonical way:

$$(A, w) \mapsto A(w) \in \mathbb{H}, \quad A \in \Gamma,\ w \in \mathbb{H}$$

and the quotient $\Gamma\backslash\mathbb{H}$ is biholomorphic to $(\mathbb{C} - \{0, 1\}) \cup \{\infty\}$. This action has one elliptic point ∞ of order 5 and two cusps 0 and 1. We can regard x_{ij} as holomorphic one-valued functions on $\mathbb{H}$. For simplicity we use the same notation for these functions: $x_{ij} : \mathbb{H} \to \mathbb{C}$. We define

$$\tau_i : \mathbb{H} \to \mathbb{C}, \quad \tau_0 := \frac{x_{11}}{x_{21}},\ \tau_1 := \frac{x_{31}}{x_{21}},\ \tau_2 := \frac{x_{41}}{x_{21}},\ \tau_3 := \frac{x_{31}x_{22} - x_{32}x_{21}}{x_{11}x_{22} - x_{12}x_{21}}$$

which are a priori meromorphic functions on $\mathbb{H}$. We will use τ_0 as a local coordinate around a point $w \in \mathbb{H}$ whenever w is not a pole of τ_0 and the derivative of τ_0 does not vanish at w. In this way we need to express the $\tau_i,\ i \in \{1, 2, 3\}$ as functions of τ_0:

$$\begin{aligned}
\tau_1 &= -\frac{25}{12} + \frac{5}{2}\tau_0(\tau_0 + 1) + \frac{\partial H}{\partial \tau_0},\\
\tau_2 &= 200\frac{\zeta(3)}{(2\pi i)^3} - \frac{5}{6}\tau_0\left(\frac{5}{2} + \tau_0^2\right) - \tau_0\frac{\partial H}{\partial \tau_0} - 2H,\\
\tau_3 &= \frac{\partial \tau_1}{\partial \tau_0},
\end{aligned}$$

where

$$H = \frac{1}{(2\pi i)^3}\sum_{n=1}^{\infty}\left(\sum_{d|n} n_d d^3\right)\frac{e^{2\pi i \tau_0 n}}{n^3} \tag{5}$$

and n_d is the virtual number of rational curves of degree d in a generic quintic threefold. The numbers n_d are also called instanton numbers or BPS degeneracies. A complete description of the image of τ_0 is not yet known. Now, the t_i are well-defined holomorphic functions on $\mathbb{H}$. The functional equations of the t_i with respect to the action of an arbitrary element of Γ are complicated mixed equalities which we have described in §4. Since Γ is generated by M_0 and M_1 it is enough to explain them for these two elements. The functional equations of the t_i with respect to the action of M_0 and written in the τ_0-coordinate are the trivial equalities $t_i(\tau_0) = t_i(\tau_0 + 1),\ i = 0, 1, \ldots, 6$.

Theorem 2 *With respect to the action of M_1, the t_i written in the τ_0-coordinate satisfy the following functional equations:*

$$\begin{aligned}
t_0(\tau_0) &= t_0\left(\frac{\tau_0}{\tau_2+1}\right)\frac{1}{\tau_2+1},\\
t_1(\tau_0) &= t_1\left(\frac{\tau_0}{\tau_2+1}\right)\frac{1}{(\tau_2+1)^2} + t_7\left(\frac{\tau_0}{\tau_2+1}\right)\frac{\tau_0\tau_3 - \tau_1}{(\tau_2+1)(\tau_0^2\tau_3 - \tau_0\tau_1 + \tau_2 + 1)}\\
&\quad + t_9\left(\frac{\tau_0}{\tau_2+1}\right)\frac{-\tau_0}{(\tau_2+1)^2} + \frac{1}{\tau_2+1},\\
t_2(\tau_0) &= t_2\left(\frac{\tau_0}{\tau_2+1}\right)\frac{1}{(\tau_2+1)^3} + t_6\left(\frac{\tau_0}{\tau_2+1}\right)\frac{\tau_0\tau_3 - \tau_1}{(\tau_2+1)^2(\tau_0^2\tau_3 - \tau_0\tau_1 + \tau_2 + 1)}
\end{aligned}$$

$$
\begin{aligned}
&\quad + t_8\left(\frac{\tau_0}{\tau_2+1}\right)\frac{-\tau_0}{(\tau_2+1)^3},\\
t_3(\tau_0) &= t_3\left(\frac{\tau_0}{\tau_2+1}\right)\frac{1}{(\tau_2+1)^4} + t_5\left(\frac{\tau_0}{\tau_2+1}\right)\frac{\tau_0\tau_3-\tau_1}{(\tau_2+1)^3(\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1)}\\
t_4(\tau_0) &= t_4\left(\frac{\tau_0}{\tau_2+1}\right)\frac{1}{(\tau_2+1)^5},\\
t_5(\tau_0) &= t_5\left(\frac{\tau_0}{\tau_2+1}\right)\frac{1}{(\tau_2+1)^2(\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1)},\\
t_6(\tau_0) &= t_6\left(\frac{\tau_0}{\tau_2+1}\right)\frac{1}{(\tau_2+1)(\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1)} + t_8\left(\frac{\tau_0}{\tau_2+1}\right)\frac{\tau_0^2}{(\tau_2+1)^3},
\end{aligned}
$$

where

$$
\begin{aligned}
t_7 &:= \frac{(5^5t_0^4+t_3)t_6-(5^5t_0^3+t_2)t_5}{5^4(t_4-t_0^5)},\\
t_8 &:= \frac{5^4(t_0^5-t_4)}{t_5},\\
t_9 &:= \frac{-5^5t_0^4-t_3}{t_5}.
\end{aligned}
$$

Since we do not know the global behavior of τ_0, the above equalities must be interpreted in the following way: for any fixed branch of $t_i(\tau_0)$ there is a path γ in the image of $\tau_0 : \mathbb{H} \to \mathbb{C}$ which connects τ_0 to $\frac{\tau_0}{\tau_2+1}$ and such that the analytic continuations of the t_i along the path γ satisfies the above equalities. For this reason it may be reasonable to use a new name for all the t_i in the right hand side of the equalities in Theorem 2. We could also state Theorem 2 without using any local coordinate system on $\mathbb{H}$: we regard t_i as holomorphic functions on $\mathbb{H}$ and, for instance, the first equality in Theorem 2 can be derived from the equalities:

$$
t_0(w) = t_0(M_1(w))\frac{1}{\tau_2(w)+1},\quad \tau_0(M_1(w)) = \frac{\tau_0(w)}{\tau_2(w)+1}.
$$

The Yukawa coupling Y turns out to be

$$
\begin{aligned}
Y &= \frac{5^8(t_4-t_0^5)^2}{t_5^3}\\
&= \left(\frac{2\pi i}{5}\right)^{-3}\left(5 + 2875\frac{q}{1-q} + 609250\cdot 2^3\frac{q^2}{1-q^2} + \cdots + n_d d^3\frac{q^d}{1-q^d} + \cdots\right)
\end{aligned}
$$

and so it satisfies the functional equation

$$
Y(\tau_0) = Y\left(\frac{\tau_0}{\tau_2+1}\right)\frac{\left(\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1\right)^3}{(\tau_2+1)^4}.
$$

The basic idea behind all the computations in Theorem 1 and Theorem 2 lies in the following geometric theorem:

Theorem 3 *Let T be the moduli of pairs $(W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4])$, where W is a mirror quintic Calabi–Yau threefold and*

$$
\begin{aligned}
&\alpha_i \in F^{4-i}/F^{5-i},\quad i \in \{1,2,3,4\}\\
&[\langle \alpha_i, \alpha_j\rangle] = \Phi.
\end{aligned}
$$

Here, $H^3_{\mathrm{dR}}(W)$ is the third algebraic de Rham cohomology of W, F^i is the ith piece of the Hodge filtration of $H^3_{\mathrm{dR}}(W)$, $\langle\cdot,\cdot\rangle$ is the intersection form in $H^3_{\mathrm{dR}}(W)$ and Φ is the constant matrix:

$$\Phi := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Then there is a unique vector field Ra *in T such the Gauss–Manin connection of the universal family of mirror quintic Calabi–Yau varieties over T composed with the vector field* Ra*, namely ∇_{Ra}, satisfies:*

$$\begin{aligned} \nabla_{\mathrm{Ra}}(\alpha_1) &= \alpha_2, \\ \nabla_{\mathrm{Ra}}(\alpha_2) &= Y\alpha_3, \\ \nabla_{\mathrm{Ra}}(\alpha_3) &= -\alpha_4, \\ \nabla_{\mathrm{Ra}}(\alpha_4) &= 0 \end{aligned}$$

for some regular function Y in T. In fact,

$$T \cong \left\{(t_0,t_1,t_2,t_3,t_4,t_5,t_6)\in\mathbb{C}^7 \mid t_5t_4(t_4-t_0^5)\neq 0\right\}, \tag{7}$$

and under this isomorphism the vector field Ra *as an ordinary differential equation is (3) and $Y=\frac{5^8(t_4-t_0)^2}{t_5^3}$ is the Yukawa coupling.*

The space of classical (or elliptic) modular or quasi-modular forms of a fixed degree for discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$, is finite. This simple observation is the origin of many number theoretic applications (see for instance [19]) and we may try to generalize such applications to the context of present text. Having this in mind, we have to describe the behavior of the t_i around the other cusp $z=1$. This will be done in another article. All the calculations of the present text, and in particular the calculations of the p_i and the differential equation (3), are done by Singular, see [6]. The reader who does not want to calculate everything by his own effort can obtain the corresponding Singular code and a Singular library form the Pdf file of the present text which is hyperlinked to the computer codes in my website. Many arguments of the present text work for an arbitrary Calabi–Yau differential equation in the sense of [16]. In this paper we mainly focus on the geometry of mirror quintic Calabi–Yau varieties which led us to explicit calculations. Therefore, the results for an arbitrary Calabi–Yau equation is postponed to another paper.

The present text is organized in the following way. Section 2 is dedicated to the algebro-geometric aspects of mirror quintic Calabi–Yau threefolds. In this section we describe how one can get the differential equation (3) using the Gauss–Manin connection of families of mirror quintic Calabi–Yau varieties enhanced with elements in their de Rham cohomologies. Theorem 3 is proved in this section. In Sect. 3 we describe a solution of (3). This solution is characterized by a special format of the period matrix of mirror quintic Calabi–Yau varieties. Finally in Sect. 4 we describe such a solution in terms of the periods x_{ij} and the corresponding q-expansion. In this section we first prove Theorem 2 and then Theorem 1.

Don Zagier pointed out that using the parameters t_7, t_8, t_9 the differential equation (3) must look simpler. I was able to rewrite it in the following way:

$$\begin{cases} \dot{t}_0 = t_8 - t_0 t_9 \\ \dot{t}_1 = -t_1 t_9 - 5^4 t_0 t_8 \\ \dot{t}_2 = -t_1 t_8 - 2t_2 t_9 - 3\cdot 5^5 t_0^2 t_8 \\ \dot{t}_3 = 4t_2 t_8 - 3t_3 t_9 - 5(t_7 t_8 - t_9 t_6) t_8 \\ \dot{t}_4 = -5t_4 t_9 \\ \dot{t}_5 = -t_6 t_8 - 3t_5 t_9 - t_3 \\ \dot{t}_6 = -2t_6 t_9 - t_2 - t_7 t_8 \\ \dot{t}_7 = -t_7 t_9 - t_1 \\ \dot{t}_8 = \frac{t_8^2}{t_5} t_6 - 3t_8 t_9 \\ \dot{t}_9 = \frac{t_8^2}{t_5} t_7 - t_9^2 \end{cases} \tag{8}$$

My sincere thanks go to Charles Doran, Stefan Reiter and Duco van Straten for useful conversations and their interest on the topic of the present text. In particular, I would like to thank Don Zagier whose comments on the first draft of the text motivated me to write Theorem 1 and Theorem 2 in a more elementary way and without geometric considerations. I found it useful for me and a reader who seeks for number theoretic applications similar to those for classical modular forms, see for instance [19]. I would also like to thank both mathematics institutes IMPA and MPIM for providing excellent research ambient during the preparation of the present text. After this article, there have been many developments and applications which I have collected them in a project named Gauss–Manin connection in disguise. It can be found in the author's webpage. Also in the later developments I have decided to use the name differential Calabi–Yau modular form instead of modular-type function.

	q^0	q^1	q^2	q^3	q^4	q^5	q^6
$\frac{1}{24}h_0$	$\frac{1}{120}$	1	175	117625	111784375	126958105626	160715581780591
$\frac{-1}{750}h_1$	$\frac{1}{30}$	3	930	566375	526770000	592132503858	745012928951258
$\frac{-1}{50}h_2$	$\frac{7}{10}$	107	50390	29007975	26014527500	28743493632402	35790559257796542
$\frac{-1}{5}h_3$	$\frac{6}{5}$	71	188330	100324275	86097977000	93009679497426	114266677893238146
$-h_4$	0	-1	170	41475	32183000	32678171250	38612049889554
$\frac{1}{125}h_5$	$-\frac{1}{125}$	15	938	587805	525369650	577718296190	716515428667010
$\frac{1}{25}h_6$	$-\frac{3}{5}$	187	28760	16677425	15028305250	16597280453022	20644227272244012
$\frac{1}{125}h_7$	$-\frac{1}{5}$	13	2860	1855775	1750773750	1981335668498	2502724752660128
$\frac{1}{10}h_8$	$-\frac{1}{50}$	13	6425	6744325	8719953625	12525150549888	19171976431076873
$\frac{1}{10}h_9$	$-\frac{1}{10}$	17	11185	12261425	16166719625	23478405649152	36191848368238417

2 Mirror quintic Calabi–Yau varieties

Let $W_\psi,\ \ \psi^5 \neq 1$ be the variety obtained by a resolution of singularities of the following quotient:

$$W_\psi := \left\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \mid x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \right\} / G, \tag{9}$$

where G is the group

$$G := \{(\zeta_1, \zeta_2, \ldots, \zeta_5) \mid \zeta_i^5 = 1,\ \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5 = 1\}$$

acting in a canonical way, see for instance [2]. The variety W_ψ is Calabi–Yau and it is mirror dual to the universal family of quintic varieties in $\mathbb{P}^4$. From now on we denote by W such a variety and we call it a mirror quintic Calabi–Yau threefold. This section is dedicated to algebraic-geometric aspects, such as the moduli space and the algebraic de Rham cohomology, of mirror quintic Calabi–Yau threefolds. We will use the algebraic de Rham cohomology $H^3_{\mathrm{dR}}(W)$ which is even defined for W defined over an arbitrary field of characteristic zero. The original text of Grothendieck [8] is still the main source of information on algebraic de Rham cohomology. In the present text by the moduli of the objects x we mean the set of all x modulo natural isomorphisms.

2.1 Moduli space, I

We first construct explicit affine coordinates for the moduli S of pairs (W, ω), where W is a mirror quintic Calabi–Yau threefold and ω is a holomorphic differential 3-form on W. We have

$$S \cong \mathbb{C}^2 \backslash \{(t_0^5 - t_4)t_4 = 0\},$$

where for (t_0, t_4) we associate the pair (W_{t_0,t_4}, ω_1). In the affine coordinates (x_1, x_2, x_3, x_4), that is $x_0 = 1$, W_{t_0,t_4} is given by

$$W_{t_0,t_4} := \{f(x) = 0\}/G,$$
$$f(x) := -t_4 - x_1^5 - x_2^5 - x_3^5 - x_4^5 + 5t_0x_1x_2x_3x_4,$$

and

$$\omega_1 := \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{df}.$$

The multiplicative group $G_m := \mathbb{C}^*$ acts on S by:

$$(W, \omega) \bullet k = (W, k^{-1}\omega),\ k \in G_m,\ (W, \omega) \in S.$$

In the coordinates (t_0, t_4) this corresponds to:

$$(t_0, t_4) \bullet k = (kt_0, k^5 t_4),\ (t_0, t_4) \in S,\ k \in G_m. \tag{10}$$

Two Calabi–Yau varieties in the family (9) are isomorphic if and only if they have the same ψ^5. This and (10) imply that distinct pairs (t_0, t_4) give us non-isomorphic pairs (W, ω).

2.2 Gauss–Manin connection

For a proper smooth family W/T of algebraic varieties defined over a field k of characteristic zero, we have the Gauss–Manin connection

$$\nabla : H^i_{\mathrm{dR}}(W/T) \to \Omega^1_T \otimes_{\mathcal{O}_T} H^i_{\mathrm{dR}}(W/T),$$

where $H^i_{\mathrm{dR}}(W/T)$ is the ith relative de Rham cohomology and Ω^1_T is the set of differential 1-forms on T. For simplicity we have assumed that T is affine and $H^i_{\mathrm{dR}}(W/T)$ is a $\mathcal{O}_T$-module, where $\mathcal{O}_T$ is the k-algebra of regular function on T. By definition of a connection, ∇ is k-linear and satisfies the Leibniz rule

$$\nabla(r\omega) = dr \otimes \omega + r\nabla\omega, \omega \in H^i_{\mathrm{dR}}(W/T),\ r \in \mathcal{O}_T.$$

For a vector field v in T we define

$$\nabla_v : H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}}(X)$$

to be ∇ composed with

$$v \otimes \mathrm{Id} : \Omega^1_T \otimes_{\mathcal{O}_T} H^i_{\mathrm{dR}}(X) \to \mathcal{O}_T \otimes_{\mathcal{O}_T} H^i_{\mathrm{dR}}(X) = H^i_{\mathrm{dR}}(X).$$

Sometimes it is useful to choose a basis $\omega_1, \omega_2, \ldots, \omega_h$ of the $\mathcal{O}_T$-module $H^i(X/T)$ and write the Gauss–Manin connection in this basis:

$$\nabla \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix} = A \otimes \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix}, \tag{11}$$

where A is a $h \times h$ matrix with entries in Ω^1_T. For further information on the Gauss–Manin connection see [9]. See also [14] for computational aspects of the Gauss–Manin connection.

2.3 Intersection form and Hodge filtration

For $\omega, \alpha \in H^3_{\mathrm{dR}}(W_{t_0,t_4})$ let

$$\langle \omega, \alpha \rangle := \mathrm{Tr}(\omega \cup \alpha)$$

be the intersection form. If we consider W as a complex manifold and its de Rham cohomology defined by C^∞ forms, then the intersection form is just $\langle \omega, \alpha \rangle = \frac{1}{(2\pi i)^3} \int_{W_{t_0,t_4}} \omega \wedge \alpha$. Using Poincaré duality it can be seen that it is dual to the topological intersection form in $H_3(W_{t_0,t_4}, \mathbb{Q})$, for all these see, for instance, Deligne's lectures in [4]. In $H^3_{\mathrm{dR}}(W_{t_0,t_4})$ we have the Hodge filtration

$$\{0\} = F^4 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3_{\mathrm{dR}}(W_{t_0,t_4}), \;\; \dim_{\mathbb{C}}(F^i) = 4 - i.$$

There is a relation between the Hodge filtration and the intersection form which is given by the following collection of equalities:

$$\langle F^i, F^j \rangle = 0, \quad i + j \geq 4.$$

The Griffiths transversality is a property combining the Gauss–Manin connection and the Hodge filtration. It says that the Gauss–Manin connection sends F^i to $\Omega^1_S \otimes F^{i-1}$ for $i = 1, 2, 3$. Using this we conclude that:

$$\omega_i := \frac{\partial^{i-1}}{\partial t_0^{i-1}}(\omega_1) \in F^{4-i}, \quad i = 1, 2, 3, 4. \tag{12}$$

By abuse of notation we have used $\frac{\partial}{\partial t_0}$ instead of $\nabla_{\frac{\partial}{\partial t_0}}$. The intersection form in the basis ω_i is:

$$[\langle \omega_i, \omega_j \rangle] = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{625}(t_4 - t_0^5)^{-1} \\ 0 & 0 & -\frac{1}{625}(t_4 - t_0^5)^{-1} & -\frac{1}{125} t_0^4 (t_4 - t_0^5)^{-2} \\ 0 & \frac{1}{625}(t_4 - t_0^5)^{-1} & 0 & \frac{1}{125} t_0^3 (t_4 - t_0^5)^{-2} \\ -\frac{1}{625}(t_4 - t_0^5)^{-1} & \frac{1}{125} t_0^4 (t_4 - t_0^5)^{-2} & -\frac{1}{125} t_0^3 (t_4 - t_0^5)^{-2} & 0 \end{pmatrix}.$$

For a proof see [13, p. 468].

2.4 Moduli space, II

We make the base change $\alpha = S\omega$, where S is given by

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_9 & t_8 & 0 & 0 \\ t_7 & t_6 & t_5 & 0 \\ t_1 & t_2 & t_3 & t_{10} \end{pmatrix} \tag{13}$$

and the t_i are unknown parameters, and we assume the intersection form in the α_i is given by the matrix Φ in (6):

$$\Phi = [\langle \alpha_i, \alpha_j \rangle] = S[\langle \omega_i, \omega_j \rangle]S^{\mathrm{t}}.$$

This yields to many polynomial relations between the t_i. It turns out that we can take the t_i as independent parameters and calculate all others in terms of these seven parameters:

$$\begin{aligned} t_7 t_8 - t_6 t_9 &= 3125 t_0^3 + t_2, \\ t_{10} &= -t_8 t_5, \\ t_5 t_9 &= -3125 t_0^4 - t_3, \\ t_{10} &= 625(t_4 - t_0^5). \end{aligned}$$

The expression of t_7, t_8, t_9 are given in Theorem 2. For the moduli space T introduced in Theorem 3 we get an isomorphism of sets (7), where for t in the right hand side of the isomorphism (7), we associate the pair (W_{t_0,t_4}, α). We also define

$$\tilde{t}_5 = \frac{1}{3125} t_5, \quad \tilde{t}_6 = -\frac{1}{5^6}\left(t_0^5 - t_4\right) t_6 + \frac{1}{5^{10}}\left(9375 t_0^4 + 2t_3\right) t_5$$

which correspond to the parameters t_5, t_6 in the previous article [13].

2.5 The Picard–Fuchs equation

Let us consider the one parameter family of Calabi–Yau varieties W_{t_0,t_4} with $t_0 = 1$ and $t_4 = z$ and denote by η the restriction of ω_1 to these parameters. We calculate the Picard–Fuchs equation of η with respect to the parameter z:

$$\frac{\partial^4 \eta}{\partial z^4} = \sum_{i=1}^{4} a_i(z) \frac{\partial^{i-1}\eta}{\partial z^{i-1}} \quad \text{modulo relatively exact forms.}$$

This is in fact the linear differential equation

$$I'''' = \frac{-24}{625 z^3 (z-1)} I + \frac{-24z+5}{5z^3(z-1)} I' + \frac{-72z+35}{5z^2(z-1)} I'' + \frac{-8z+6}{z(z-1)} I''', \quad ' = \frac{\partial}{\partial z} \tag{14}$$

which is calculated in [2]. This differential equation can be also written in the form (1). In [14] we have developed algorithms which calculate such differential equations. The parameter z is more convenient for our calculations than the parameter ψ and this is the reason why in this section we have used z instead of ψ. The differential equation (14) is satisfied by the periods

$$I(z) = \int_{\delta_z} \eta, \ \delta \in H_3(W_{1,z}, \mathbb{Q})$$

of the differential form η on the family $W_{1,z}$. In the basis $\frac{\partial^i \eta}{\partial z^i},\ \ i \in \{0, 1, 2, 3\}$ of $H^3_{\rm dR}(W_{1,z})$ the Gauss–Manin connection matrix has the form

$$A(z)dz := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1(z) & a_2(z) & a_3(z) & a_4(z) \end{pmatrix} dz. \tag{15}$$

2.6 Gauss–Manin connection I

We would like to calculate the Gauss–Manin connection

$$\nabla : H^3_{\rm dR}(W/S) \to \Omega^1_S \otimes_{\mathcal{O}_S} H^3_{\rm dR}(W/S)$$

of the two parameter proper family of varieties $W_{t_0,t_4},\ \ (t_0, t_4) \in S$. We calculate ∇ with respect to the basis (12) of $H^3_{\rm dR}(W/S)$. For this purpose we return back to the one parameter case. Now, consider the identity map

$$g : W_{(t_0,t_4)} \to W_{1,z},$$

which satisfies $g^*\eta = t_0\omega_1$. Under this map

$$\frac{\partial}{\partial z} = \frac{-1}{5}\frac{t_0^6}{t_4}\frac{\partial}{\partial t_0}\left(= t_0^5\frac{\partial}{\partial t_4}\right).$$

From these two equalities we obtain a matrix $\tilde{S} = \tilde{S}(t_0, t_4)$ such that

$$\left[\eta, \frac{\partial \eta}{\partial z}, \frac{\partial^2 \eta}{\partial z^2}, \frac{\partial^3 \eta_1}{\partial z^3}\right]^{\mathsf{t}} = \tilde{S}^{-1}\left[\omega_1, \omega_2, \omega_3, \omega_4\right]^{\mathsf{t}},$$

where t denotes the transpose of matrices, and the Gauss–Manin connection in the basis $\omega_i,\ \ i \in \{1, 2, 3, 4\}$ is:

$$\tilde{\rm GM} = \left(d\tilde{S} + \tilde{S}\cdot A\left(\frac{t_4}{t_0^5}\right)\cdot d\left(\frac{t_4}{t_0^5}\right)\right)\cdot \tilde{S}^{-1},$$

which is the following matrix after doing explicit calculations:

$$\begin{pmatrix} -\frac{1}{5t_4}dt_4 & dt_0 + \frac{-t_0}{5t_4}dt_4 & 0 & 0 \\ 0 & \frac{-2}{5t_4}dt_4 & dt_0 + \frac{-t_0}{5t_4}dt_4 & 0 \\ 0 & 0 & \frac{-3}{5t_4}dt_4 & dt_0 + \frac{-t_0}{5t_4}dt_4 \\ \frac{-t_0}{t_0^5-t_4}dt_0 + \frac{t_0^2}{5t_0^5t_4-5t_4^2}dt_4 & \frac{-15t_0^2}{t_0^5-t_4}dt_0 + \frac{3t_0^3}{t_0^5t_4-t_4^2}dt_4 & \frac{-25t_0^3}{t_0^5-t_4}dt_0 + \frac{5t_0^4}{t_0^5t_4-t_4^2}dt_4 & \frac{-10t_0^4}{t_0^5-t_4}dt_0 + \frac{6t_0^5+4t_4}{5t_0^5t_4-5t_4^2}dt_4 \end{pmatrix}. \tag{16}$$

Now, we calculate the Gauss–Manin connection matrix of the family W/T written in the basis $\alpha_i,\ \ i \in \{1, 2, 3, 4\}$. This is

$${\rm GM} = \left(dS + S\cdot\tilde{\rm GM}\right)\cdot S^{-1},$$

where S is the base change matrix (13). Since the matrix GM is huge and does not fit into a mathematical paper, we do not write it here.

2.7 Modular differential equation

We are in the final step of the proof of Theorem 3. We have calculated the Gauss–Manin connection GM written in the basis $\alpha_i,\ i=1,2,3,4$. It is a matter of explicit linear algebra calculations to show that there is a unique vector field Ra in T with the properties mentioned in Theorem (3), and to calculate it. In summary, the Gauss–Manin connection matrix composed with the vector field Ra and written in the basis α_i has the form:

$$\nabla_{\mathrm{Ra}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{5^8(t_4-t_0^5)^2}{t_5^3} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

It is interesting that the Yukawa coupling appears as the only non-constant term in the above matrix.

2.8 Algebraic group

There is an algebraic group which acts on the right hand side of the isomorphism (7). It corresponds to the base change in $\alpha_i,\ i\in\{1,2,3,4\}$ such that the new basis is still compatible with the Hodge filtration and we have still the intersection matrix (6):

$$G := \left\{ g=[g_{ij}]_{4\times 4}\in \mathrm{GL}(4,\mathbb{C}) \mid g_{ij}=0,\ \text{ for } j<i \text{ and } g^{\mathrm{t}}\Phi g=\Phi \right\},$$

$$\left\{ g=\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{24} \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & 0 & g_{44} \end{pmatrix},\ g_{ij}\in\mathbb{C} \begin{pmatrix} g_{11}g_{44}=1, \\ g_{22}g_{33}=1, \\ g_{12}g_{44}+g_{22}g_{34}=0, \\ g_{13}g_{44}+g_{23}g_{34}-g_{24}g_{33}=0, \end{pmatrix} \right\}.$$

G is called the Borel subgroup of $\mathrm{Sp}(4,\mathbb{C})$ respecting the Hodge flag. The action of G on the moduli T is given by:

$$(W,[\alpha_1,\alpha_2,\alpha_3,\alpha_4])\bullet g = (W,[\alpha_1,\alpha_2,\alpha_3,\alpha_4]\,g).$$

The algebraic group G is of dimension six and it has two multiplicative subgroups $G_m=(\mathbb{C}^*,\cdot)$ and four additive subgroups $G_a=(\mathbb{C},+)$ which generate it. In fact, an element $g\in G$ can be written in a unique way as the following product:

$$\begin{pmatrix} g_1^{-1} & -g_3g_1^{-1} & (-g_3g_6+g_4)g_1^{-1} & (-g_3g_4+g_5)g_1^{-1} \\ 0 & g_2^{-1} & g_6g_2^{-1} & g_4g_2^{-1} \\ 0 & 0 & g_2 & g_2g_3 \\ 0 & 0 & 0 & g_1 \end{pmatrix}$$
$$= \begin{pmatrix} g_1^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_2^{-1} & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -g_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & g_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & g_4 & 0 \\ 0 & 1 & 0 & g_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & g_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & g_6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, we have a bijection of sets $G_m \times G_m \times G_a \times G_a \times G_a \times G_a \cong G$ sending $(g_i)_{i=1,\ldots,6}$ to the above product. If we identify an element $g \in G$ with the vector $(g_i)_{i=1,\ldots,6}$ then

$$(g_1, g_2, g_3, g_4, g_5, g_6)^{-1} \\ = (g_1^{-1}, g_2^{-1}, -g_1^{-1}g_2g_3, g_1^{-1}g_2^{-1}(g_3g_6 - g_4), g_1^{-2}(-g_3^2g_6 + 2g_3g_4 - g_5), -g_2^{-2}g_6).$$

We denote by $\bullet$ the right action of G on the T space.

Proposition 1 *The action of G on the t_i (as regular functions on the affine variety T) is given by:*

$$\begin{aligned} g \bullet t_0 &= t_0g_1, \\ g \bullet t_1 &= t_1g_1^2 + t_7g_1g_2g_3 + t_9g_1g_2^{-1}g_4 - g_3g_4 + g_5, \\ g \bullet t_2 &= t_2g_1^3 + t_6g_1^2g_2g_3 + t_8g_1^2g_2^{-1}g_4, \\ g \bullet t_3 &= t_3g_1^4 + t_5g_1^3g_2g_3, \\ g \bullet t_4 &= t_4g_1^5, \\ g \bullet t_5 &= t_5g_1^3g_2, \\ g \bullet t_6 &= t_6g_1^2g_2 + t_8g_1^2g_2^{-1}g_6. \end{aligned}$$

Consequently

$$\begin{aligned} g \bullet t_7 &= t_7g_1g_2 + t_9g_1g_2^{-1}g_6 - g_3g_6 + g_4, \\ g \bullet t_8 &= t_8g_1^2g_2^{-1}, \\ g \bullet t_9 &= t_9g_1g_2^{-1} - g_3, \\ g \bullet t_{10} &= t_{10}g_1^5. \end{aligned}$$

Proof We first calculate the action of $g = (k, 1, 0, 0, 0, 0),\ k \in \mathbb{C}^*$ on t. We have an isomorphism $(W_{(t_0,t_4)}, k^{-1}\omega_1) \cong (W_{(t_0k,t_4k^5)}, \omega_1)$ given by

$$(x_1, x_2, x_3, x_4) \mapsto \left(k^{-1}x_1, k^{-1}x_2, k^{-1}x_3, k^{-1}x_4\right).$$

Under this isomorphism the vector field $k^{-1}\frac{\partial}{\partial t_0}$ is mapped to $\frac{\partial}{\partial t_0}$ and so $k^{-i}\omega_i$ is mapped to ω_i. This implies the isomorphisms

$$(W_{(t_0,t_4)}, k(t_1\omega_1 + t_2\omega_2 + t_3\omega_3 + 625(t_4 - t_0^5)\omega_4)) \\ \cong (W_{(t_0,t_4)\bullet k}, k^2t_1\omega_1 + k^3t_2\omega_2 + k^4t_3\omega_3 + 625(k^5t_4 - (kt_0)^5)\omega_4))$$

and

$$(W_{(t_0,t_4)}, S\omega) \cong (W_{(t_0,t_4)\bullet k}, S\begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k^2 & 0 & 0 \\ 0 & 0 & k^3 & 0 \\ 0 & 0 & 0 & k^4 \end{pmatrix}\omega),$$

where S is defined in (13). Therefore,

$$g \bullet t_i = t_ik^{\tilde{d}_i},\ \tilde{d}_i = i + 1,\ i \in \{0, 1, 2, 3, 4\}\ \ \tilde{d}_5 = 3,\ \ \tilde{d}_6 = 2$$

□

3 Periods

This section is dedicated to transcendental aspects of mirror quintic Calabi–Yau threefold. By this we mean the periods of meromorphic differential 3-forms over topological cycles. We first work with periods without calculating them explicitly.

3.1 Period map

We choose a symplectic basis for $H_3(W,\mathbb{Z})$, that is, a basis $\delta_i,\ i\in\{1,2,3,4\}$ such that

$$\Psi := [\langle\delta_i,\delta_j\rangle] = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is also convenient to use the basis

$$[\tilde\delta_1,\tilde\delta_2,\tilde\delta_3,\tilde\delta_4] = [\delta_1,\delta_2,\delta_3,\delta_4]\Psi^{-1} = [\delta_3,\delta_4,-\delta_1,-\delta_2].$$

In this basis the intersection form is $[\langle\tilde\delta_i,\tilde\delta_j\rangle] = \Psi^{-\mathrm{t}} = \Psi$. Recall that in §2.4 a mirror quintic Calabi–Yau threefold W is equipped with a basis $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ of $H^3_{\mathrm{dR}}(W)$ compatible with the Hodge filtration and such that $[\langle\alpha_i,\alpha_j\rangle]=\Phi$. We define the period matrix to be

$$[x_{ij}] = \left[\int_{\delta_i}\alpha_j\right].$$

In this section we discard the usage of x_{ij} in the Introduction. Let $\tilde\delta_i^p\in H^3(W,\mathbb{Q})$ be the Poincaré dual of $\tilde\delta_i$, that is, it is defined by the property $\int_\delta \tilde\delta_i^p = \langle\tilde\delta_i,\delta\rangle$ for all $\delta\in H_3(W,\mathbb{Q})$. If we write α_i in terms of $\tilde\delta_i^p$ then we get:

$$[\alpha_1,\alpha_2,\alpha_3,\alpha_4] = \left[\tilde\delta_1^p,\tilde\delta_2^p,\tilde\delta_3^p,\tilde\delta_4^p\right]\left[\int_{\delta_i}\alpha_j\right],$$

that is, the coefficients of the base change matrix are the periods of the α_i over the δ_i and not the $\tilde\delta_i$. We have

$$[\langle\alpha_i,\alpha_j\rangle] = \left[\int_{\delta_i}\alpha_j\right]^{\mathrm{t}}\Psi^{-\mathrm{t}}\left[\int_{\delta_i}\alpha_j\right] \tag{18}$$

and so we get:

$$\Phi - [x_{ij}]^{\mathrm{t}}\Psi[x_{ij}] = 0. \tag{19}$$

This gives us 6 non trivial polynomial relations between periods x_{ij}:

$$\begin{aligned} x_{12}x_{31}-x_{11}x_{32}+x_{22}x_{41}-x_{21}x_{42}&=0,\\ x_{13}x_{31}-x_{11}x_{33}+x_{23}x_{41}-x_{21}x_{43}&=0,\\ x_{14}x_{31}-x_{11}x_{34}+x_{24}x_{41}-x_{21}x_{44}+1&=0,\\ x_{13}x_{32}-x_{12}x_{33}+x_{23}x_{42}-x_{22}x_{43}+1&=0,\\ x_{14}x_{32}-x_{12}x_{34}+x_{24}x_{42}-x_{22}x_{44}&=0,\\ x_{14}x_{33}-x_{13}x_{34}+x_{24}x_{43}-x_{23}x_{44}&=0. \end{aligned}$$

These equalities correspond to the entries $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$ and $(3,4)$ of (19). Taking the determinant of (19) we see that up to sign we have $\det(\mathsf{pm})=-1$. There is another effective way to calculate this determinant without the sign ambiguity. In the ideal of $\mathbb{Q}[x_{ij},\ i, j = 1, 2, 3, 4]$ generated by the polynomials $f_{12}, f_{13}, f_{14}, f_{23}, f_{2,4}, f_{34}$ in the right hand side of the above equalities, the polynomial $\det([x_{ij}])$ is reduced to -1. Let y_{ij} be indeterminate variables, $R = \mathbb{C}[y_{ij}, i, j = 1, 2, 3, 4]$ and $I := \{f \in R \mid f(\cdots, x_{ij}, \ldots) = 0\}$. The ideal I is generated by $f_{12}, f_{13}, f_{14}, f_{23}, f_{2,4}, f_{34}$, see for instance [13, Proposition 3, p. 472].

3.2 A special locus

Let

$$C^{\mathrm{t}} := [0, 1, 0, 0][\langle\tilde{\delta}_i, \tilde{\delta}_j\rangle]^{-\mathrm{t}} = [0, 0, 0, 1].$$

We are interested in the loci L of parameters $t \in T$ such that

$$\left[\int_{\delta_1} \alpha_4, \ldots, \int_{\delta_4} \alpha_4\right] = C. \tag{20}$$

Using the equality corresponding to the $(1, 4)$ entry of (18), we note that on this locus we have

$$\int_{\delta_2} \alpha_1 = 1,\ \int_{\delta_2} \alpha_i = 0,\ i \geq 2.$$

The equalities (20) define a three dimensional locus of T. We also put the following two conditions

$$\int_{\delta_1} \alpha_2 = 1, \int_{\delta_1} \alpha_3 = 0$$

in order to get a one dimensional locus. Finally, using (19) we conclude that the period matrix for points in L is of the form

$$\tau = \begin{pmatrix} \tau_0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \tau_1 & \tau_3 & 1 & 0 \\ \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix}. \tag{21}$$

The particular expressions for the $(4, 2)$ and $(4, 3)$ entries of the above matrix follow from the polynomial relations (19). The Gauss–Manin connection matrix restricted to L is:

$$\mathrm{GM}\,|_L = d\tau^{\mathrm{t}} \cdot \tau^{-\mathrm{t}} = \begin{pmatrix} 0 & d\tau_0 & -\tau_3 d\tau_0 + d\tau_1 & -\tau_1 d\tau_0 + \tau_0 d\tau_1 + d\tau_2 \\ 0 & 0 & d\tau_3 & -\tau_3 d\tau_0 + d\tau_1 \\ 0 & 0 & 0 & -d\tau_0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Griffiths transversality theorem implies that

$$-\tau_3 d\tau_0 + d\tau_1 = 0,\ \ -\tau_1 d\tau_0 + \tau_0 d\tau_1 + d\tau_2 = 0.$$

Since L is one dimensional, there are analytic relations between $\tau_0, \tau_1, \tau_2, \tau_3$. Therefore, we consider τ_0 as an independent parameter and τ_1, τ_2, τ_3 as dependent parameters on τ_0. We get

$$\tau_3 = \frac{\partial \tau_1}{\partial \tau_0},\ \frac{\partial \tau_2}{\partial \tau_0} = \tau_1 - \tau_0 \frac{\partial \tau_1}{\partial \tau_0}. \tag{22}$$

We conclude that the Gauss–Manin connection matrix restricted to L and composed with the vector field $\frac{\partial}{\partial \tau_0}$ is given by:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial \tau_3}{\partial \tau_0} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Proposition 2 *The functions $t_i(\tau_0)$ obtained by the regular functions t_i restricted to L and seen as functions in τ_0 form a solution of the ordinary differential equation* Ra.

Proof It follows from (23) and the uniqueness of the vector field Ra satisfying the equalities (17). □

3.3 The algebraic group and the special locus *L*

For any 4×4 matrix $x = [x_{ij}]$ satisfying (19) and

$$x_{11}x_{22} - x_{12}x_{21} \neq 0, \;\; x_{21} \neq 0, \tag{24}$$

there is a unique $g \in G$ such xg is of the form (21). To prove this affirmation explicitly, we take an arbitrary x and g and we write down the corresponding equations corresponding to the six entries $(2,1), (1,2), (2,2), (1,3), (2,3), (2,4)$ of xg, that is

$$xg = \begin{pmatrix} * & 1 & 0 & * \\ 1 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

For our calculations we will need the coordinates of g^{-1} in terms of x_{ij}:

$$\begin{aligned} g_1 &= x_{21}^{-1}, \\ g_2 &= \frac{-x_{21}}{x_{11}x_{22} - x_{12}x_{21}}, \\ g_3 &= \frac{-x_{22}}{x_{21}}, \\ g_4 &= \frac{-x_{12}x_{23} + x_{13}x_{22}}{x_{11}x_{22} - x_{12}x_{21}}, \\ g_5 &= \frac{x_{11}x_{22}x_{24} - x_{12}x_{21}x_{24} + x_{12}x_{22}x_{23} - x_{13}x_{22}^2}{x_{11}x_{21}x_{22} - x_{12}x_{21}^2}, \\ g_6 &= \frac{x_{11}x_{23} - x_{13}x_{21}}{x_{11}x_{22} - x_{12}x_{21}}. \end{aligned}$$

Substituting the expression of g in terms of x_{ij} in $\tau = xg$, we get:

$$\begin{pmatrix} \frac{x_{11}}{x_{21}} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{x_{31}}{x_{21}} & \frac{-x_{21}x_{32}+x_{22}x_{31}}{x_{11}x_{22}-x_{12}x_{21}} & 1 & 0 \\ \frac{x_{41}}{x_{21}} & \frac{-x_{21}x_{42}+x_{22}x_{41}}{x_{11}x_{22}-x_{12}x_{21}} & -\frac{x_{11}}{x_{21}} & 1 \end{pmatrix}.$$

Note that for the entries $(1,4)$, $(3,3)$ and $(4,3)$ of the above matrix we have used the polynomial relations (19) between periods.

4 Monodromy covering

In the previous section we described a solution of Ra locally. In this section we further study such a solution in a global context. More precisely, we describe a meromorphic map $t : \mathbb{H} \to T$ whose image is the L in the previous section, where $\mathbb{H}$ is the monodromy covering of (1).

4.1 Monodromy covering

Let $\tilde{\mathbb{H}}$ be the moduli of the pairs (W, δ), where W is a mirror quintic Calabi–Yau threefold and $\delta = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ is a basis of $H_3(W, \mathbb{Z})$ such that the intersection matrix in this basis is Ψ, that is, $[\langle \delta_i, \delta_j \rangle] = \Psi$. The set $\tilde{\mathbb{H}}$ has a canonical structure of a Riemann surface, not necessarily connected. We denote by $\mathbb{H}$ the connected component of $\tilde{\mathbb{H}}$ which contains the particular pair $(W_{1,z}, \delta)$ such that the monodromies around $z = 0$ and $z = 1$ are respectively given by the matrices M_0 and M_1 in the Introduction. It is known that in the monodromy group $\Gamma := \langle M_0, M_1 \rangle$ the only relation between M_0 and M_1 is $(M_0 M_1)^5 = I$, see [1]. This is equivalent to say that $\mathbb{H}$ is biholomorphic to the upper half plane. By definition, the monodromy group Γ acts on $\mathbb{H}$ by base change in δ. The bigger group $\mathrm{Sp}(4, \mathbb{Z})$ acts also on $\tilde{\mathbb{H}}$ by base change and all connected components of $\tilde{\mathbb{H}}$ are obtained by $\mathbb{H}_\alpha := \alpha(\mathbb{H})$, $\alpha \in \mathrm{Sp}(4, \mathbb{Z})/\Gamma$:

$$\tilde{\mathbb{H}} := \cup_{\alpha \in \mathrm{Sp}(4,\mathbb{Z})/\Gamma} \tilde{\mathbb{H}}_\alpha.$$

From now on by w we denote a point (W, δ) of $\mathbb{H}$. We use the following meromorphic functions on $\mathbb{H}$:

$$\tau_i : \mathbb{H} \to \mathbb{C}, \; i \in \{0, 1, 2\}$$
$$\tau_0(w) = \frac{\int_{\delta_1} \alpha_1}{\int_{\delta_2} \alpha_1}, \; \tau_1(w) = \frac{\int_{\delta_3} \alpha_1}{\int_{\delta_2} \alpha_1}, \; \tau_2(w) = \frac{\int_{\delta_4} \alpha_1}{\int_{\delta_2} \alpha_1},$$

where α_1 is a holomorphic differential form on W. They do not depend on the choice of α_1. For simplicity, we have used the same notations τ_i as in Sect. 3.

There is a useful meromorphic function z on $\mathbb{H}$ which is obtained by identifying W with some $W_{1,z}$. It has a pole of order 5 at elliptic points which are the pairs (W, δ) with $W = W_{\psi,1}$, $\psi = 0$. In this way, we have a well-defined holomorphic function

$$\psi = z^{-\frac{1}{5}} : \mathbb{H} \to \mathbb{C}.$$

The coordinate system τ_0 is adapted for calculations around the cusp $z = 0$. Let B be the set of points $w = (W, \delta) \in \mathbb{H}$ such that either τ_0 has a pole at w or it has a critical point at w, that is, $\frac{\partial \tau_0}{\partial z}(w) = 0$. We do not know whether B is empty or not. Many functions that we are going to study are meromorphic with poles at B. The set B is characterized by the property that in its complement in $\mathbb{H}$ the inequalities (24) hold.

4.2 A particular solution

For a point $w = (W, \delta) \in \mathbb{H} \backslash B$ there is a unique basis α of $H^3_{\mathrm{dR}}(W)$ such that (W, α) is an element in the moduli space T defined in §2.4, and the period matrix $[\int_{\delta_i} \alpha_j]$ of the triple (W, δ, α) is of the form (21). This follows from the arguments in §3.3. In this way we have well-defined meromorphic maps

$$t : \mathbb{H} \to T$$

and

$$\tau : \mathbb{H} \to \mathrm{Mat}(4, \mathbb{C})$$

which are characterized by the uniqueness of the basis α and the equality:

$$\tau(w) = \left[\int_{\delta_i} \alpha_j \right].$$

If we parameterize $\mathbb{H}$ by the image of τ_0 then t is the same map as in §3.2. We conclude that the map $t : \mathbb{H} \to T$ with the coordinate system τ_0 on $\mathbb{H}$ is a solution of Ra. The functions t and τ are holomorphic outside the poles and critical points of τ_0 (this corresponds to points in which the inequalities (24) occur).

4.3 Action of the monodromy

The monodromy group $\Gamma := \langle M_0, M_1 \rangle$ acts on $\mathbb{H}$ by base change. If we choose the local coordinate system τ_0 on $\mathbb{H}$ then this action is given by:

$$A(\tau_0) = \frac{a_{11}\tau_0 + a_{12} + a_{13}\tau_1 + a_{14}\tau_2}{a_{21}\tau_0 + a_{22} + a_{23}\tau_1 + a_{24}\tau_2}, \quad A = [a_{ij}] \in \Gamma.$$

Proposition 3 *For all $A \in \Gamma$ we have*

$$t(w) = t(A(w)) \bullet g(A, w),$$

where $g(A, w) \in G$ is defined using the equality

$$A \cdot \tau(w) = \tau(A(w)) \cdot g(A, w).$$

Proof Let $w = (W, \delta) \in \mathbb{H}$ and $t(w) = (W, \alpha)$. By definition we have

$$[\int_{A(\delta)_i} \alpha_j] g(A, w)^{-1} = A\tau(w) g(A, w)^{-1} = \tau(A(w)).$$

Therefore, $t(A(w)) = (W, \alpha \cdot g(A, w)^{-1}) = t(w) \bullet g(A, w)^{-1}$. □

If we choose the coordinate system τ_0 on $\mathbb{H}$ and regard the parameters t_i and τ_i as functions in τ_0, then we have

$$t(\tau_0) = t(A(\tau_0)) \bullet g(A, \tau_0).$$

These are the functional equations of the $t_i(\tau_0)$ mentioned in the Introduction. For $A = M_0$ we have:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix} \begin{pmatrix} \tau_0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \tau_1 & \tau_3 & 1 & 0 \\ \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \tau_0 + 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \tau_1 + 5\tau + 5 & \tau_3 + 5 & 1 & 0 \\ \tau_2 - 5 - \tau_1 & -\tau_0(\tau_3 + 1) + \tau_1 & -\tau_0 - 1 & 1 \end{pmatrix}$$

which is already of the format (21). Note that

$$-(\tau_0 + 1)(\tau_3 + 5) + \tau_1 + 5\tau_0 + 5 = -\tau_0(\tau_3 + 1) + \tau_1.$$

Therefore, $M_0(\tau_0) = \tau_0 + 1$ and $g(M_0, \tau_0)$ is the identity matrix. The corresponding functional equation of t_i simply says that t_i is invariant under $\tau_0 \mapsto \tau_0 + 1$:

$$t_i(\tau_0) = t_i(\tau_0 + 1),\ i \in \{0, 1, \ldots, 6\}.$$

For $A = M_1$ we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \tau_1 & \tau_3 & 1 & 0 \\ \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix} = \begin{pmatrix} \tau_0 & 1 & 0 & 0 \\ \tau_2 + 1 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \\ \tau_1 & \tau_3 & 1 & 0 \\ \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\tau_0}{\tau_2+1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{\tau_1}{\tau_2+1} & \frac{\tau_0\tau_1\tau_3-\tau_1^2+\tau_2\tau_3+\tau_3}{\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1} & 1 & 0 \\ \frac{\tau_2}{\tau_2+1} & \frac{-\tau_0\tau_3+\tau_1}{\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1} & \frac{-\tau_0}{\tau_2+1} & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} (\tau_2+1) & (-\tau_0\tau_3+\tau_1) & (-\tau_0) & 1 \\ 0 & \frac{\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1}{\tau_2+1} & \frac{\tau_0^2}{\tau_2+1} & \frac{-\tau_0}{\tau_2+1} \\ 0 & 0 & \frac{\tau_2+1}{\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1} & \frac{\tau_0\tau_3-\tau_1}{\tau_0^2\tau_3-\tau_0\tau_1+\tau_2+1} \\ 0 & 0 & 0 & \frac{1}{\tau_2+1} \end{pmatrix},$$

where the element of the algebraic group G in the right hand side has the coordinates:

$$\begin{aligned} g_1 &= \frac{1}{\tau_2 + 1}, \\ g_2 &= \frac{\tau_2 + 1}{\tau_0^2\tau_3 - \tau_0\tau_1 + \tau_2 + 1}, \\ g_3 &= \frac{\tau_0\tau_3 - \tau_1}{\tau_2 + 1}, \\ g_4 &= \frac{-\tau_0}{\tau_0^2\tau_3 - \tau_0\tau_1 + \tau_2 + 1}, \\ g_5 &= \frac{1}{\tau_0^2\tau_3 - \tau_0\tau_1 + \tau_2 + 1}, \\ g_6 &= \frac{\tau_0^2}{\tau_0^2\tau_3 - \tau_0\tau_1 + \tau_2 + 1}. \end{aligned}$$

In this case we have

$$M_1(\tau_0) = \frac{\tau_0}{\tau_2 + 1}.$$

The corresponding functional equations of the t_i can be written immediately. These are presented in Theorem 2.

4.4 The solution in terms of periods

In this section we explicitly calculate the map t. For $w = (W, \delta) \in \mathbb{H}$ we identify W with $W_{1,z}$ and hence we obtain a unique point $\tilde{z} = (1, 0, 0, 0, z, 1, 0) \in T$. Now, we have a well-defined period map

$$\mathsf{pm} : \mathbb{H} \to \mathrm{Mat}(4, \mathbb{C}),$$
$$w = (W_{1,z}, \{\delta_{i,z},\ i = 1, 2, 3, 4\}) \mapsto \left[\int_{\delta_{i,z}} \alpha_j\right].$$

We write $\mathsf{pm}(w)g(w) = \tau(w)$, where $\tau(w)$ is of the form (21) and $g(w) \in G$. We have

$$t(w) = \tilde{z} \bullet g(w).$$

For the one dimensional locus $\tilde{z} \in T$, we have $\alpha = S\omega$ and $\omega = T\tilde{\eta}$, where

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5^5 & -5^{4(z-1)} & 0 & 0 \\ -\frac{5}{z-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 5^4(z-1) \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -5 & 0 & 0 \\ 2 & 15 & 25 & 0 \\ -6 & -55 & -150 & -125 \end{pmatrix}$$

and

$$\tilde{\eta} = [\eta, \theta\eta, \theta^2\eta, \theta^3\eta]^{\mathsf{t}},\ \ \theta = z\frac{\partial}{\partial z}.$$

Therefore, $\alpha = ST\eta$. Restricted to $\tilde{z}$-locus we have $\alpha_1 = \omega_1 = \eta$ and by our definition of the x_{ij} in the introduction

$$x_{ij} = \theta^{j-1}\int_{\delta_i} \eta, \quad i, j = 1, 2, 3, 4.$$

Therefore,

$$\mathsf{pm}(w) = [x_{ij}](ST)^{\mathsf{t}}.$$

Now, the map $w \mapsto t(w)$, where the domain $\mathbb{H}$ is equipped with the coordinate system z, is given by the expressions for t_i in Theorem 1. We conclude that if we write the t_i in terms of τ_0 then we get functions which are solutions to Ra. Note that

$$\frac{\partial}{\partial \tau_0} = 2\pi i q\frac{\partial}{\partial q} = \left(z\frac{\partial \frac{x_{11}}{x_{21}}}{\partial z}\right)^{-1} z\frac{\partial}{\partial z} = \frac{x_{21}^2}{x_{12}x_{21} - x_{11}x_{22}}\theta.$$

4.5 Calculating periods

In this section we calculate the periods x_{ij} explicitly. This will finish the proof of our main theorems announced in the Introduction.

We restrict the parameter $t \in T$ to the one dimensional loci $\tilde{z}$ given by $t_0 = 1, t_1 = t_2 = t_3 = 0, t_4 = z, t_5 = 1, t_6 = 0$. On this locus $\eta = \omega_1 = \alpha_1$. We know that the integrals $\int_\delta \eta,\ \delta \in H_3(W_{1,z}, \mathbb{Q})$ satisfy the linear differential equation (14). Four linearly independent solutions of (14) are given by $\psi_0, \psi_1, \psi_2, \psi_3$ in the Introduction, see for instance [17] and [3]. In fact, there are four topological cycles with complex coefficients $\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4 \in H_3(W_{1,z}, \mathbb{C})$ such that $\int_{\hat{\delta}_i} \eta = \frac{(2\pi i)^{i-1}}{5^4}\psi_{4-i}$. Note that the pair $(W_{1,z}, 5\eta)$ is isomorphic to the pair (W_ψ, Ω) used in [2]. We use a new basis given by

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & d & \frac{d}{2} & -b \\ -d & 0 & -b & -a \end{pmatrix} \begin{pmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \\ \hat{\delta}_3 \\ \hat{\delta}_4 \end{pmatrix},$$

where

$$a = \frac{c_3}{(2\pi i)^3}\zeta(3) = \frac{-200}{(2\pi i)^3}\zeta(3), \quad b = c_2 \cdot H/24 = \frac{25}{12}, \; d = H^3 = 5,$$

(these notations are used in [17]). The monodromies around $z = 0$ and $z = 1$ written in the basis δ_i are respectively

$$M_0 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix} \quad M_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $k = 2b + \frac{d}{6} = 5$, see [3]. In fact, $\delta_i \in H_3(W_{1,z}, \mathbb{Z})$ for $i = 1, 2, 3, 4$. This follows from the calculations in [2] and the expressions for monodromy matrices. In summary, we have

$$\begin{aligned}
x_{11} &= \int_{\delta_1} \eta = \frac{1}{5^2}\left(\frac{2\pi i}{5}\right)^2 \psi_1(\tilde{z}), \\
x_{21} &= \int_{\delta_2} \eta = \frac{1}{5}\left(\frac{2\pi i}{5}\right)^3 \psi_0, \\
x_{31} &= \int_{\delta_3} \eta = \frac{d}{125}\psi_2(\tilde{z})\frac{2\pi i}{5} + \frac{d}{50}\cdot\left(\frac{2\pi i}{5}\right)^2 \cdot \psi_1(\tilde{z}) - \frac{b}{5}\cdot\left(\frac{2\pi i}{5}\right)^3 \cdot \psi_0(\tilde{z}), \\
x_{41} &= \int_{\delta_4} \eta = \frac{-d}{5^4}\psi_3(\tilde{z}) + \frac{-b}{5^2}\cdot\left(\frac{2\pi i}{5}\right)^2 \cdot \psi_1(\tilde{z}) - \frac{a}{5}\cdot\left(\frac{2\pi i}{5}\right)^3 \cdot \psi_0(\tilde{z}),
\end{aligned}$$

where $\tilde{z} = \frac{z}{5^5}$. We have also

$$\begin{aligned}
\tau_0 &= \frac{\int_{\delta_1}\eta}{\int_{\delta_2}\eta} = \frac{1}{2\pi i}\frac{\psi_1(\tilde{z})}{\psi_0(\tilde{z})}, \\
\tau_1 &= \frac{\int_{\delta_3}\eta}{\int_{\delta_2}\eta} = d\left(\frac{1}{2}\tau_0^2 + \frac{1}{5}H'\right) + \frac{d}{2}\tau_0 - b = -b + \frac{d}{2}\tau_0(\tau_0+1) + \frac{d}{5}H', \\
\tau_2 &= \frac{\int_{\delta_4}\eta}{\int_{\delta_2}\eta} = -d\left(\frac{-1}{3}\tau_0^3 + \tau_0\left(\frac{1}{2}\tau_0^2 + \frac{1}{5}H'\right) + \frac{2}{5}H\right) - b\tau_0 - a \\
&= -a - b\tau_0 - \frac{d}{6}\tau_0^3 - \frac{d}{5}\tau_0 H' - \frac{2d}{5}H,
\end{aligned}$$

where H is defined in (5). We have used the equalities

$$\frac{\psi_2}{\psi_0} - \frac{1}{2}\left(\frac{\psi_1}{\psi_0}\right)^2 = \frac{1}{5}\left(\sum_{n=1}^{\infty}\left(\sum_{d|n} n_d d^3\right)\frac{q^n}{n^2}\right),$$

$$\frac{1}{3}\left(\frac{\psi_1}{\psi_0}\right)^3 - \frac{\psi_1}{\psi_0}\frac{\psi_2}{\psi_0} + \frac{\psi_3}{\psi_0} = \frac{2}{5}\sum_{n=1}^{\infty}\left(\sum_{d|n} n_d d^3\right)\frac{q^n}{n^3}$$

see for instance [10,15]. We can use the explicit series

$$\psi_0(\tilde{z}) = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \tilde{z}^m$$

$$\psi_1(\tilde{z}) = \ln(\tilde{z})\psi_0(\tilde{z}) + 5\tilde{\psi}_1(\tilde{z}), \quad \tilde{\psi}_1(\tilde{z}) = \sum_{m=1}^{\infty} \frac{(5m)!}{(m!)^5} \left(\sum_{k=m+1}^{5m} \frac{1}{k} \right) \tilde{z}^m$$

and calculate the q-expansion of $t_i(\tau_0)$ around the cusp $z = 0$. There is another way of doing this using the differential equation Ra. We just use the above equalities to obtain the initial values (4) in the Introduction. We write each h_i as a formal power series in q, $h_i = \sum_{n=0}^{\infty} t_{i,n} q^n$, and substitute it in (3) with $\dot{t} := 5q\frac{\partial t}{\partial q}$. Let

$$T_n = [t_{0,n}, t_{1,n}, t_{2,n}, t_{3,n}, t_{4,n}, t_{5,n}, t_{6,n}].$$

Comparing the coefficients of q^0 and q^1 in both sides of Ra we get:

$$T_0 = \left[\frac{1}{5}, -25, -35, -6, 0, -1, -15 \right],$$
$$T_1 = [24, -2250, -5350, -355, 1, 1875, 4675].$$

Comparing the coefficients of q^n, $n \geq 2$ we get a recursion of the following type:

$$(A_0 + 5nI_{7\times 7})T_n^{\mathsf{t}} = \text{A function of the entries of } T_0,\ T_1, \ldots, T_{n-1},$$

where

$$A_0 = \left[\frac{\partial(t_5 \mathrm{Ra}_i)}{\partial t_j} \right]_{i,j=0,1,\ldots,6} \text{evaluated at } t = T_0, \quad \mathrm{Ra} = \sum_{i=0}^{6} \mathrm{Ra}_i \frac{\partial}{\partial t_i}.$$

The matrix $A_0 + 5nI_{7\times 7}$, $n \geq 2$ is invertible and so we get a recursion in T_n.

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