On Nonlinear Asymptotic Stability of the Lane-Emden Solutions for the Viscous Gaseous Star Problem

Tao Luo [a], Zhouping Xin [b], Huihui Zeng [c,d,*]

[a] *Department of Mathematics and Statistics, Georgetown University, Washington, DC, 20057, USA*
E-mail: tl48@georgetown.edu

[b] *Institute of Mathematical Sciences, The Chinese University of Hong Kong, Hong Kong*
E-mail: zpxin@ims.cuhk.edu.hk

[c] *Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China*[1]
E-mail: hhzeng@mail.tsinghua.edu.cn

[d] *Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA 02318, USA*

* Corresponding author

MSC : 35B35, 76N10
Keywords : Nonlinear asymptotic stability; Lane-Emden solutions; Vacuum free boundary problem; Global-in-time regularity uniformly up to vacuum boundary.

Abstract

This paper proves the nonlinear asymptotic stability of the Lane-Emden solutions for spherically symmetric motions of viscous gaseous stars if the adiabatic constant γ lies in the stability range $(4/3, 2)$. It is shown that for small perturbations of a Lane-Emden solution with same mass, there exists a unique global (in time) strong solution to the vacuum free boundary problem of the compressible Navier-Stokes-Poisson system with spherical symmetry for viscous stars, and the solution captures the precise physical behavior that the sound speed is $C^{1/2}$-Hölder continuous across the vacuum boundary provided that γ lies in $(4/3, 2)$. The key is to establish the global-in-time regularity uniformly up to the vacuum boundary, which ensures the large time asymptotic uniform convergence of the evolving vacuum boundary, density and velocity to those of the Lane-Emden solution with detailed convergence rates, and detailed large time behaviors of solutions near the vacuum boundary. In particular, it is shown that every spherical surface moving with the fluid converges to the sphere enclosing the same mass inside the domain of the Lane-Emden solution with a uniform convergence rate and the large time asymptotic states for the vacuum free boundary problem (1.1.2) are determined by the initial mass distribution and the total mass. To overcome the difficulty caused by the degeneracy and singular behavior near the vacuum free boundary and coordinates singularity at the symmetry center, the main ingredients of the analysis consist of combinations of some new weighted nonlinear functionals (involving both lower-order and higher-order derivatives) and space-time weighted energy estimates. The constructions of these weighted nonlinear functionals and space-time weights depend crucially on the structures of the Lane-Emden solution, the balance of pressure and gravitation, and the dissipation. Finally, the uniform boundedness of the acceleration of the vacuum boundary is also proved.

[1]Permanent address

Contents

1 Introduction **2**
1.1 Problem . . . 2
1.2 Motivations and goals . . . 4
1.3 Review of related works . . . 8

2 Lagrangian formulation and main results **9**
2.1 Lagrangian formulation . . . 9
2.2 Main theorems and remarks . . . 12

3 Proof of main results **15**
3.1 A theorem with detailed estimates . . . 15
3.2 Outline and main steps of the proofs . . . 17
3.3 Preliminaries . . . 20
3.4 Lower-order estimates . . . 25
3.5 Higher-order estimates . . . 46
3.5.1 Part I: global existence and decay of strong solutions . . . 46
3.5.2 Part II: faster decay . . . 52
3.5.3 Part III: further regularity . . . 58

4 Proof of Theorem 2.3 **59**

1 Introduction

1.1 Problem

In the fundamental hydrodynamical setting (cf. [1]), the evolving boundary of a viscous gaseous star (the interface of fluids and vacuum states) can be modeled by the following free boundary problem of the compressible Navier-Stokes-Poisson equations:

$$\begin{aligned}
&\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 && \text{in } \ \Omega(t),\\
&(\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \mathrm{div}\mathfrak{S} = -\rho\nabla_{\mathbf{x}}\Psi && \text{in } \ \Omega(t),\\
&\rho > 0 && \text{in } \ \Omega(t),\\
&\rho = 0 \ \text{ and } \ \mathfrak{S}\mathbf{n} = \mathbf{0} && \text{on } \ \Gamma(t) := \partial\Omega(t),\\
&\mathcal{V}(\Gamma(t)) = \mathbf{u}\cdot\mathbf{n}, &&\\
&(\rho,\mathbf{u}) = (\rho_0,\mathbf{u}_0) && \text{on } \ \Omega := \Omega(0).
\end{aligned} \tag{1.1.1}$$

Here $(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, \infty)$, ρ, $\mathbf{u}$, $\mathfrak{S}$ and Ψ denote, respectively, the space and time variable, density, velocity, stress tensor and gravitational potential; $\Omega(t) \subset \mathbb{R}^3$, $\Gamma(t)$, $\mathcal{V}(\Gamma(t))$ and $\mathbf{n}$ represent, respectively, the changing volume occupied by a fluid at time t, moving interface of fluids and vacuum states, normal velocity of $\Gamma(t)$ and exterior unit normal vector to $\Gamma(t)$. The gravitational potential is described by

$$\Psi(\mathbf{x}, t) = -G\int_{\Omega(t)} \frac{\rho(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} \ \text{ satisfying } \ \Delta\Psi = 4\pi G\rho \ \text{ in } \ \Omega(t)$$

with the gravitational constant G taken to be unity for convenience. The stress tensor is given by

$$\mathfrak{S} = pI_3 - \lambda_1\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t - \frac{2}{3}(\mathrm{div}\mathbf{u})I_3\right) - \lambda_2(\mathrm{div}\mathbf{u})I_3,$$

where I_3 is the 3×3 identical matrix, p is the pressure of the gas, $\lambda_1 > 0$ is the shear viscosity, $\lambda_2 > 0$ is the bulk viscosity, and $\nabla\mathbf{u}^t$ denotes the transpose of $\nabla\mathbf{u}$. We consider the polytropic gases for which the equation of state is given by

$$p = p(\rho) = K\rho^\gamma,$$

where $K > 0$ is a constant set to be unity for convenience, $\gamma > 1$ is the adiabatic exponent.

For a non-rotating gaseous star, it is important to consider spherically symmetric motions since the stable equilibrium configurations, which minimize the energy among all possible configurations (cf. [21]), are spherically symmetric, called Lane-Emden solutions. In this work, we are concerned with the three-dimensional spherically symmetric solutions to the free boundary problem (1.1.1) and its nonlinear asymptotic stability toward the Lane-Emden solutions. The aim is to prove the global-in-time regularity uniformly up to the vacuum boundary of solutions when $4/3 < \gamma < 2$ (the stable index) capturing an interesting behavior called the physical vacuum (cf. [2, 4, 14, 15, 23, 24, 42]) which states that the sound speed $c = \sqrt{p'(\rho)}$ is $C^{1/2}$-Hölder continuous near the vacuum boundary, as long as the initial datum is a suitably small perturbation of the Lane-Emden solution with the same total mass. Furthermore, we establish the large time asymptotic convergence of the global strong solution, in particular, the convergence of the vacuum boundary and the density, to the the Lane-Emden solutions with the detailed convergence rate as the time goes to infinity.

In the spherically symmetric setting, that is, $\Omega(t)$ is a ball with the changing radius $R(t)$,

$$\rho(\mathbf{x},t) = \rho(r,t) \ \text{ and } \ \mathbf{u}(\mathbf{x},t) = u(r,t)\mathbf{x}/r \ \text{ with } \ r = |\mathbf{x}| \in (0, R(t))\,;$$

system (1.1.1) can then be rewritten as

$$\begin{aligned}
&(r^2\rho)_t + (r^2\rho u)_r = 0 && \text{in } \ (0,\ R(t))\,,\\
&\rho(u_t + uu_r) + p_r + 4\pi\rho r^{-2}\int_0^r \rho(s,t)s^2 ds = \mu\left(\frac{(r^2u)_r}{r^2}\right)_r && \text{in } \ (0,\ R(t))\,,\\
&\rho > 0 && \text{in } \ [0,\ R(t))\,,\\
&\rho = 0 \ \text{ and } \ \frac{4}{3}\lambda_1\left(u_r - \frac{u}{r}\right) + \lambda_2\left(u_r + 2\frac{u}{r}\right) = 0 && \text{for } \ r = R(t),\\
&\dot{R}(t) = u(R(t),t) \ \text{ with } \ R(0) = R_0, \ \ u(0,t) = 0, &&\\
&(\rho, u) = (\rho_0, u_0) && \text{on } \ (0,\ R_0),
\end{aligned} \tag{1.1.2}$$

where $\mu = 4\lambda_1/3 + \lambda_2 > 0$ is the viscosity constant. $(1.1.2)_{3,4}$ state that $r = R(t)$ is the vacuum free boundary at which the normal stress $\mathfrak{S}\mathbf{n} = 0$ reduces to

$$p - \frac{4}{3}\lambda_1\left(u_r - \frac{u}{r}\right) - \lambda_2\left(u_r + 2\frac{u}{r}\right) = 0 \ \text{ for } \ r = R(t), \quad t \geq 0;$$

$(1.1.2)_5$ describes that the free boundary issues from $r = R_0$ and moves with the fluid velocity, and the center of the symmetry does not move. The initial domain is taken to be a ball $\{0 \le r \le R_0\}$, and the initial density is assumed to satisfy the following condition:

$$\rho_0(r) > 0 \ \text{ for } \ 0 \le r < R_0, \ \ \rho_0(R_0) = 0 \ \text{ and } \ -\infty < \left(\rho_0^{\gamma-1}\right)_r < 0 \ \text{ at } \ r = R_0; \quad (1.1.3)$$

so

$$\rho_0^{\gamma-1}(r) \sim R_0 - r \text{ as } r \text{ close to } R_0, \quad (1.1.4)$$

that is, the initial sound speed is $C^{1/2}$-Hölder continuous across the vacuum boundary. The unknowns here are ρ, u and $R(t)$.

The requirement (1.1.3) for the initial density near the vacuum boundary is motivated by that of the Lane-Emden solution, $\bar{\rho}$, (cf. [1, 22]) which solves

$$\partial_r(\bar{\rho}^\gamma) + 4\pi r^{-2}\bar{\rho}\int_0^r \bar{\rho}(s)s^2 ds = 0. \quad (1.1.5)$$

The solutions to (1.1.5) can be characterized by the values of γ (cf. [22]) for given finite total mass $M > 0$, if $\gamma \in (6/5, 2)$, there exists at least one compactly supported solution. For $\gamma \in (4/3, 2)$, every solution is compactly supported and unique. If $\gamma = 6/5$, the unique solution admits an explicit expression, and it has infinite support. On the other hand, for $\gamma \in (1, 6/5)$, there are no solutions with finite total mass. For $\gamma > 6/5$, let $\bar{R}$ be the radius of the stationary star giving by the Lane-Emden solution, then it holds (cf. [22, 31])

$$\bar{\rho}^{\gamma-1}(r) \sim \bar{R} - r \text{ as } r \text{ close to } \bar{R}. \quad (1.1.6)$$

1.2 Motivations and goals

The problem of nonlinear asymptotic stability of Lane-Emden solutions is of fundamental importance in both astrophysics and the theory of nonlinear PDEs. It is believed by astrophysicists that Lane-Emden solutions are stable for $4/3 < \gamma < 2$ since they minimize the total energy among all the possible configurations. The main aim of this paper is to justify rigorously the precise sense of this stability. In fact, we prove for the viscous gaseous star with $4/3 < \gamma < 2$, the Lane-Emden solution is strongly stable in the sense that it is asymptotically nonlinear stable. The first step for this purpose is to prove the global existence of strong solutions. However, due to the high degeneracy of system (1.1.2) caused by the behavior (1.1.3) near the vacuum boundary, it is a very challenging problem even for the local-in-time existence theory. Indeed, the local-in-time well-posedness of smooth solutions to vacuum free boundary problems with the behavior that the sound speed is $C^{1/2}$-Hölder continuous across vacuum boundaries was only established recently for compressible inviscid flows (cf. [3, 4, 14, 15]) (see also [29] for a local-in-time well-posedness theory in a new functional space for the three-dimensional compressible Euler-Poisson equations in spherically symmetric motions). For the vacuum free boundary problem (1.1.2) of the compressible Navier-Stokes-Poisson equations featuring the behavior (1.1.3) near the vacuum boundary, a local-in-time well-posedness theory of strong solutions was established in [12]. In order to obtain the nonlinear asymptotic stability of Lane-Emden solutions, it turns out that suitable estimates for higher order derivatives uniformly up to the vacuum boundary are necessary. Indeed, this turns out to be essential to prove the convergence of the evolving vacuum boundary and the uniform convergence of the density to those of Lane-Emden solutions, in addition to the uniform convergence of the velocity. We show the global-in-time regularity of solutions when $4/3 < \gamma < 2$ capturing the behavior (1.1.4) (or (1.1.6)) when the initial data are small perturbations of and have the same total mass as the stationary solution, $\bar\rho$, given by (1.1.5). It should be remarked that the regularity estimates near boundaries are notoriously difficult. This is particularly so for the vacuum boundary problem (1.1.2) due to the high degeneracy caused by the singular behavior of (1.1.3) near vacuum states.

Our nonlinear asymptotic stability results can be stated more precisely as follows. Suppose that the initial datum (ρ_0, u_0, R_0) is a small perturbation of the Lane-Emden solution $(\bar\rho, 0, \bar R)$ in a suitable sense (see Theorem 3.1) and has the same total mass,

$$\int_0^{R_0} r^2\rho_0(r)dr = \int_0^{\bar R} r^2\bar\rho(r)dr,$$

then there is a unique global-in-time strong solution $(\rho, u, R(t))$ $(0 \le t < +\infty)$ to (1.1.2) which is regular uniformly up to the vacuum boundary $r = R(t)$. Moreover, let $r(x,t)$ be the radius of the ball inside $B_{R(t)}(\mathbf{0})$ satisfying:

$$r_t(x,t) = u(r(x,t),t) \quad\text{and}\quad r(x,0) = r_0(x) \quad\text{for}\quad 0 \le x \le \bar R, \tag{1.2.1}$$

$$\int_0^{r_0(x)} s^2\rho_0(s)ds = \int_0^x s^2\bar\rho(s)ds \quad\text{for } 0 \le x \le \bar R. \tag{1.2.2}$$

Then

$$\lim_{t\to\infty} \|\,(r(x,t) - x,\ \rho(r(x,t),t) - \bar\rho(x),\ u(r(x,t),t))\,\|_{L^\infty_x([0,\bar R])} = 0 \tag{1.2.3}$$

with some detailed convergence rates. Notice that (1.2.1) means that the sphere $r = r(x,t)$ with the initial position $r = r_0(x)$ is moving with the fluid and (1.2.2) means that the initial mass inside the ball $B_{r_0(x)}(\mathbf{0})$ is the same as that of the Lane-Emden solution inside the ball $B_x(\mathbf{0})$ for $0 \le x \le \bar R$. It follows from the conservation of mass that the mass inside the ball $B_{r(x,t)}(\mathbf{0})$ at the instant t is the same as that of the Lane-Emden solution inside the ball $B_x(\mathbf{0})$ for $0 \le x \le \bar R$. In particular, the vacuum boundary is given by

$$R(t) = r(\bar R, t).$$

The convergence of $r(x,t)$ to x in (1.2.3) means that every spherical surface moving with the fluid converges to that inside the domain of Lane-Emden solution enclosing the same mass, in particular, the evolving vacuum boundary $R(t)$ converges to the vacuum boundary $\bar R$ as time goes to infinity. This also gives the large time asymptotic convergence of every particle moving with the fluid since the motion is radial. Moreover, the convergence (1.2.3) means that the large time asymptotic states for the free boundary problem (1.1.2) are determined completely by the initial mass distribution and total mass. Besides the above mentioned convergence (1.2.3), we also establish convergence rates of higher norms involving derivatives, and show that the vacuum boundary $R(t)$ has the regularity of $W^{2,\infty}([0,\infty))$ under a compatibility condition of the initial data with the boundary condition which implies that the acceleration of the vacuum boundary is uniformly bounded for $t \in [0, \infty)$. (Indeed, one may check from the proof that every particle moving with the fluid has the bounded acceleration for $t \in [0, \infty)$.) These results give a rather clear and complete characterization of the behavior of solutions both in large time and near the vacuum boundary.

One of the crucial points of this paper is that we can obtain the decay estimate of the unweighted norm of $\|r_x(x,t) - 1\|_{L_x^2([0,\bar{R}])}$ and the uniform boundedness of $|r_x - 1|$, which are consequences of our uniform higher-order estimates and are the key to the proof of the convergence of the vacuum boundary and density. In particular, in the derivation of the decay estimate of the L_x^2-norm of $r_x(x,t) - 1$, the multipliers,

$$\int_0^x \bar{\rho}^{-\beta}(y)(r^3(y,t) - y^3)_y dy \quad \text{and} \quad \int_0^x \bar{\rho}^{-\beta}(y)(r^2(y,t)u(r(y,t),t))_y dy \quad \text{for} \quad 0 \le \beta < \gamma - 1,$$

play essential role in the construction of nonlinear functionals. It should be noted that the first multiplier is motivated by the virial equations in the study of stellar dynamics and equilibriums (cf. [20, 39]) and is used to detect the detailed balance between the pressure and self-gravitation. To the best of our knowledge, those multipliers have not been used in previous literatures.

The results obtained in the present work are among few results of global *strong* solutions to vacuum free boundary problems of compressible fluids capturing the singular behavior of (1.1.3), which is difficult and challenging due to the degeneracy caused by the physical vacuum and coordinates singularity at the center of the symmetry. We overcome this difficulty by establishing higher-order estimates involving the second-order derivatives of the velocity field, together with decay estimates of lower-order norms. This is achieved by combining some new weighted nonlinear functionals (involving both lower-order and higher-order derivatives) and space-time weighted energy estimates. The constructions of these weighted nonlinear functionals and space-time weights depend crucially on the structure of Lane-Emden solutions (in particular, the behavior (1.1.6) near the vacuum boundary), the balance between the pressure and self-gravitation, and the dissipation. In what follows, we highlight the main ideas and methods used in this article to achieve the estimates mentioned above.

The original free boundary problem (1.1.2) is reduced to an initial boundary value problem on a fixed domain $x \in [0, \bar{R}]$ by the Lagrangian particle trajectory formulation (1.2.1) and (1.2.2) with $\bar{R}$ being the radius of the Lane-Emden solution so that the domain of the Lane-Emden solution becomes the reference domain. In this formulation, essentially the basic unknown is the particle trajectory $r(x,t)$ defined in (1.2.1) and (1.2.2) (more precisely, the radius of each evolving surface inside the evolving domain, which is called the particle trajectory for simplicity here and from now on), by which the density and velocity are determined. For problem (1.1.2), this formulation is preferred because one can use it to trace each particle in the evolving domain, in particular, the evolving vacuum boundary. For higher-order estimates, due to the degeneracy of (1.1.3), the dissipation of the viscosity alone is not enough for the global-in-time estimates, and we have to make full use of the balance between the pressure and gravitation. To see this, we decompose the gradient of the pressure as two parts, the first part is to balance the gravitation and the second part is an anti-derivative of the viscosity along the particle trajectory with a degenerate weight. It should be noted that the degeneracy of this weight causes the major difficulty to obtain the higher-order estimates. Our main idea to overcome this difficulty is to introduce a quantity

$$\mathcal{G}(x,t) = \ln\left(\frac{\bar{\rho}(x)}{\rho(r(x,t),t)}\right),$$

which is the entropy relative to the Lane-Emden solution. This is a nonlinear transformation which changes the original nonlinear equation of r to an equation whose principal part is linear in $\mathcal{G}$ (see (3.2.7)). The approach has many advantages for higher-order estimates. First, the interplay among the viscosity, pressure and gravitational force can be seen easily as follows: in Lagrangian coordinates (x,t) for $x \in [0,\bar{R}]$, the viscosity term becomes $\mu\mathcal{G}_{xt}$, and the gradient of the pressure is decomposed as

$$-\left(\frac{x^2}{r^2 r_x}\right)^{\gamma} x\phi\bar{\rho} - \gamma\left(\frac{x^2}{r^2 r_x}\right)^{\gamma}\bar{\rho}^{\gamma}\mathcal{G}_x, \quad \text{where} \quad \phi(x) = x^{-3}\int_0^x 4\pi\bar{\rho}(s)s^2 ds.$$

($\phi(x)$ is the mean density of the Lane-Emden solution inside the ball $B_x(\mathbf{0})$.) The first part of this decomposition is used to balance the gravitational force and the second part is the t-antiderivative of the viscosity multiplied by a weight which is equivalent to $\bar{\rho}^{\gamma}(x)$. This weight is degenerate on the boundary, but strictly positive in the interior. The degeneracy near the vacuum boundary is one of main obstacles in higher-order estimates, which is overcome by choosing suitable weights and multipliers, and a delicate use of the Hardy and weighted Sobolev inequalities. Indeed, in terms of $\mathcal{G}$, the principal part of (3.2.7) is

$$\mu\mathcal{G}_{xt} + \gamma\left(\frac{x^2}{r^2 r_x}\right)^{\gamma}\bar{\rho}^{\gamma}\mathcal{G}_x,$$

which is linear in $\mathcal{G}_x$ and with a degenerate damping. This structure leads to desirable estimates on $\mathcal{G}$ and their derivatives. It should be noted in this formulation, the pressure term is treated as a principal term, instead of a lower-order term, for higher-order estimates. Another advantage for this formulation is that we can get faster decay estimates by choosing a multiplier in the form of

$$x^2\bar{\rho}^{\gamma-2}\mathfrak{P}_t \quad \text{with} \quad \mathfrak{P}(x,t) = \gamma\left(\frac{x^2}{r^2 r_x}\right)^{\gamma}\bar{\rho}^{\gamma}\mathcal{G}_x + \left[\left(\frac{x^2}{r^2 r_x}\right)^{\gamma} - \left(\frac{x}{r}\right)^4\right] x\phi\bar{\rho}.$$

Here $\mathfrak{P}$ is the sum of the gradient of the pressure and gravitational force and $\mathfrak{P}_t$ represents the t-derivative in Lagrangian coordinates of $\mathfrak{P}$. This multiplier is important for getting the key new decay estimate of

$$\int_0^{\bar{R}} \bar{\rho}^{3\gamma-2} r_{xx}^2(x,t)dx \quad \text{and hence} \quad \|\left(\rho(r(x,t),t) - \bar{\rho}(x),\ u(r(x,t),t)\right)\|_{L_x^{\infty}([0,\bar{R}])}.$$

The basic strategy of this work is to use a bootstrap argument to derive the uniform boundedness of a nonlinear functional involving the solution and its first- and second-order derivatives, $\mathfrak{E}(t)$ defined in (2.2.2). To this end, the *a priori* assumption is the smallness of

$$\sup_{x\in[0,\bar{R}]} |r_x(x,t) - 1| \quad \text{and} \quad \sup_{x\in[0,\bar{R}]} |v_x(x,t)|, \quad \text{where} \quad v(x,t) = u(r(x,t),t).$$

The usual method in closing this type of *a priori* assumption is to use energy estimates and the Sobolev embedding, for example,

$$\|r_x(x,t)-1\|^2_{L^\infty_x([0,\bar{R}])} \le C\left(\|r_x(x,t)-1\|^2_{L^2_x([0,\bar{R}])} + \|r_{xx}(x,t)\|^2_{L^2_x([0,\bar{R}])}\right).$$

However, due to the possible growth in time of the L^2-norm of r_{xx} (see (3.1.6)), the uniform bound for r_{xx} is valid only for an interval of x away from the vacuum boundary $x = \bar{R}$ (see (3.1.2), where we set $\bar{R} = 1$ for convenience). Therefore, it is difficult to obtain the smallness of $|r_x - 1|$ by the L^2-norm of its derivative, r_{xx}. Indeed, we bound $|r_x - 1|$ by a combination of the local L^2-estimate of r_{xx} in the region way from the vacuum boundary and the pointwise estimate away from the origin. In fact, away from the vacuum boundary, the L^2-estimate of r_{xx} is obtained by the weighted L^2-estimate of $\mathcal{G}_x$, while the pointwise estimate away from the origin depends crucially on decreasing property of Lane-Emden solutions. Similarly, it is hard to obtain the smallness of $|v_x|$ by the Sobolev embedding via the L^2-estimate of its derivative which may in general grow in time, so additional cares are required. Away from the vacuum, we use the weighted L^2-estimates for $\mathcal{G}_x$ and $\mathcal{G}_{xt}$ with the weight $\bar{\rho}^{\gamma-1/2}$ (see (3.3.16)), while away from the origin, it can be estimated by the L^2-estimate of $\mathcal{G}_{xt}$ with the weight x (cf. (3.3.20)) and (3.3.30)). This strategy reflects the subtlety in the study of vacuum free boundary problems for the three-dimensional spherically symmetric motions: one has to deal with the singular behavior of solutions both near the vacuum boundary and the center of the symmetry. Indeed, this subtlety was also noted in the local-in-time well-posedness theory in [12] for (1.1.2), in which a higher-order energy functional was constructed which consists of two parts, called the Eulerian energy near the origin expressed in Eulerian coordinates and the Lagrangian energy described in Lagrangian mass coordinates away from the origin. In this paper, we find a uniform way to establish higher-order estimates by using Lagrangian coordinates only through choosing suitable weights and cutoff functions which take care of both the origin and the vacuum boundary simultaneously.

1.3 Review of related works

There have been extensive works on the studies of the Euler-Poisson and the Navier-Stoke-Poisson equations with vacuum, especially in recent years. We will concentrate on those closely related to the stability of vacuum dynamics. The stability problem has been important in the theory of gaseous stars which has been studied extensively by astrophysicists (cf. [1, 41, 19]). The linear stability of Lane-Emden solutions was studied in [22]. A conditional nonlinear Lyapnov type stability theory of stationary solutions for $\gamma > 4/3$ was established in [36] using a variational approach, by assuming the existence of global solutions of the Cauchy problem for the three-dimensional compressible Euler-Poisson equations (the same type of nonlinear stability results for rotating stars were given by [27, 28]. For $\gamma \in (6/5, 4/3)$, the nonlinear dynamical instability of Lane-Emden solutions was proved by [17] and [16] in the framework of free boundary problems for Euler-Poisson systems and Navier-Stokes-Poisson equations, respectively. A nonlinear instability for $\gamma = 6/5$ was proved by [13]. For $\gamma = 4/3$, an instability was identified in [5] that a small perturbation can cause part of the mass to go off to infinity for inviscid flows.

It should be noted that the stability result in [36] is in the framework of initial value problems in the entire $\mathbb{R}^3$-space and involves only a Lyapnov functional which is essentially equivalent to a L^p-norm of difference of solutions, and the vacuum boundary cannot be traced. Another interesting work is on the vacuum free boundary problem of modified compressible Navier-Stokes-Poisson equations with spherical symmetry (cf. [10]), where the existence of a global weak solution was proved for a reduced initial boundary value problem after using the Lagrangian mass coordinates, under some constraints on the ratio of the coefficients of the shear viscosity and bulk viscosity. In contrast to the strong stability result in (1.2.3), for the global weak solutions obtained in [10], only the uniform convergence of the velocity $u(r,t)$ is proved, due to the lack of regularity near the vacuum boundary. The ideas and techniques developed in this paper can be applied to this modified compressible Navier-Stokes-Poisson equations to obtain a strong stability result as in (1.2.3). Indeed, our global-in-time regularity gives not only the decay estimates for the weighted norms

$$\|\bar{\rho}^{\gamma/2}(r_x(x,t)-1)\|_{L^2_x([0,\bar{R}])} \quad \text{and} \quad \|xv_x\|_{L^2_x([0,\bar{R}])}$$

as in [10], but also the decay estimates of the unweighted norms of

$$\|r_x(x,t)-1\|_{L^2_x([0,\bar{R}])}, \quad \|v_x\|_{L^\infty_x([0,\bar{R}])}$$

and some uniform estimates on the second derivatives valid up to the vacuum boundary, which are crucial to the nonlinear asymptotic stability for this modified model. Furthermore, our theory holds without the restrictions on the viscosity coefficients as in [10]. This will be reported in a forthcoming paper (cf. [30]).

We conclude the introduction by noting that there are also other prior results on free boundary problems involving vacuum for compressible Navier-Stokes equations besides the ones aforementioned. For the one-dimensional motions, there are many results concerning global weak solutions to free boundary problems of the Navier-Stokes equations, one may refer to [32, 35, 26, 9, 18, 7, 43, 44, 18, 45] and references therein. As for the spherically symmetric motions, global existence and stability of weak solutions were obtained in [33, 34] to compressible Navier-Stokes equations for gases surrounding a solid ball (a hard core) without self-gravitation. However, those results are restricted to cut-off domains excluding a neighborhood of the origin. It should be noted that for a modified system of Navier-Stokes equations, a global existence of weak solutions with spherical symmetry containing the origin was established in [11] for which the density does not vanish on the boundary. For a class of free boundary problems of compressible Navier-Stokes-Poisson equations away from vacuum states, the readers may refer to [37, 38] for the local-in-time well-posedness results and [40] for linearized stability results of stationary solutions.

2 Lagrangian formulation and main results

2.1 Lagrangian formulation

First, we recall some properties of Lane-Emden solutions. For $\gamma \in (4/3, 2)$, it is known that for any given finite positive total mass, there exists a unique solution to equation (1.1.5) whose support is compact (cf. [22]). Without abusing notations, x will denote the distance from the origin for the Lane-Emden solution. Therefore, for any $M \in (0,\infty)$, there exists a unique function $\bar\rho(x)$ such that

$$\bar\rho_0 := \bar\rho(0) > 0, \quad \bar\rho(x) > 0 \ \text{ for } \ x \in \left(0,\ \bar R\right), \quad \bar\rho\left(\bar R\right) = 0, \quad M = \int_0^{\bar R} 4\pi\bar\rho(s)s^2 ds; \ (2.1.1)$$

$$-\infty < \bar\rho_x < 0 \ \text{ for } \ x \in (0,\ \bar R) \ \text{ and } \ \bar\rho(x) \le \bar\rho_0 \ \text{ for } \ x \in \left(0,\ \bar R\right);$$

$$(\bar\rho^\gamma)_x = -x\phi\bar\rho, \quad \text{where } \ \phi := x^{-3}\int_0^x 4\pi\bar\rho(s)s^2 ds \in \left[M/\bar R^3,\ 4\pi\bar\rho_0/3\right]; \qquad (2.1.2)$$

for a certain finite positive constant $\bar R$ (indeed, $\bar R$ is determined by M and γ). Note that

$$\left(\bar\rho^{\gamma-1}\right)_x = \frac{\gamma-1}{\gamma}\bar\rho^{-1}\left(\bar\rho^\gamma\right)_x = -\frac{\gamma-1}{\gamma}x\phi.$$

It then follows from (2.1.1) and (2.1.2) that $\bar\rho$ satisfies the physical vacuum condition, that is,

$$\bar\rho^{\gamma-1}(x) \sim \bar R - x \ \text{ as } \ x \quad \text{close to } \bar R.$$

More precisely, there exists a constant C depending on M and γ such that

$$C^{-1}\left(\bar R - x\right) \le \bar\rho^{\gamma-1}(x) \le C\left(\bar R - x\right), \quad x \in \left(0,\ \bar R\right). \qquad (2.1.3)$$

We adopt a particle trajectory Lagrangian formulation for (1.1.2) as follows. Let x be the reference variable and define the Lagrangian variable $r(x,t)$ by

$$r_t(x,t) = u(r(x,t),t) \ \text{ for } \ t > 0 \ \text{ and } \ r(x,0) = r_0(x), \quad x \in I := \left(0, \bar R\right).$$

Here $r_0(x)$ is the initial position which maps $\bar I \to [0, R_0]$ satisfying

$$\int_0^{r_0(x)} \rho_0(s)s^2 ds = \int_0^x \bar\rho(s)s^2 ds, \quad x \in \bar I, \qquad (2.1.4)$$

so that

$$\rho_0(r_0(x))r_0^2(x)r_0'(x) = \bar\rho(x)x^2, \ x \in \bar I. \qquad (2.1.5)$$

(Indeed, (2.1.4) means that the initial mass in the ball with the radius $r_0(x)$ is the same as that of the Lane-Emden solution in the ball with the radius x. Then smoothness of $r_0(x)$ at $x = \bar R$ is equivalent to that the initial density ρ_0 has the same behavior near R_0 as that of $\bar\rho$ near $\bar R$.) The choice of r_0 can be described by

$$r_0(x) = \psi^{-1}(\xi(x)), \quad 0 \le x \le \bar R; \qquad (2.1.6)$$

where ξ and ψ are one-to-one mappings, defined by

$$\xi : (0, \bar R) \to (0, M) : \ x \mapsto \int_0^x s^2\bar\rho(s)ds \ \text{ and } \ \psi : (0, R_0) \to (0, M) : \ z \mapsto \int_0^z s^2\rho_0(s)ds.$$

Moreover $r_0(x)$ is an increasing function and the initial total mass has to be the same as that for $\bar{\rho}$, that is,

$$\int_0^{R_0} 4\pi\rho_0(s)s^2 ds = \int_0^{r_0(\bar{R})} 4\pi\rho_0(s)s^2 ds = \int_0^{\bar{R}} 4\pi\bar{\rho}(s)s^2 ds = M, \tag{2.1.7}$$

to ensure that r_0 is a diffeomorphism from $\bar{I}$ to $[0, R_0]$. In view of $(1.1.2)_1$, we see

$$\int_0^{r(x,t)} \rho(s,t)s^2 ds = \int_0^{r_0(x)} \rho_0(s)s^2 ds, \quad x \in I. \tag{2.1.8}$$

Define the Lagrangian density and velocity respectively by

$$f(x,t) = \rho(r(x,t),t) \ \text{ and } \ v(x,t) = u(r(x,t),t).$$

Then the Lagrangian version of $(1.1.2)_{1,2}$ can be written on the reference domain I as

$$\begin{aligned} &(r^2 f)_t + r^2 f \frac{v_x}{r_x} = 0 && \text{in } I \times (0,T], \\ &f v_t + \frac{(f^\gamma)_x}{r_x} + 4\pi f r^{-2} \int_0^{r_0(x)} \rho_0(s)s^2 ds = \frac{\mu}{r_x}\left(\frac{(r^2 v)_x}{r^2 r_x}\right)_x && \text{in } I \times (0,T]. \end{aligned} \tag{2.1.9}$$

Solving $(2.1.9)_1$ gives that

$$f(x,t)r^2(x,t)r_x(x,t) = \rho_0(r_0(x))r_0^2(x)r_{0x}(x), \quad x \in I.$$

Therefore,

$$f(x,t) = \frac{x^2\bar{\rho}(x)}{r^2(x,t)r_x(x,t)} \quad \text{for } \ x \in I,$$

due to (2.1.5). So, (1.1.2) can be written on the reference domain $I = \big(0, \bar{R}\big)$ as

$$\begin{aligned} &\bar{\rho}\left(\frac{x}{r}\right)^2 v_t + \left[\left(\frac{x^2}{r^2}\frac{\bar{\rho}}{r_x}\right)^\gamma\right]_x + \frac{x^2}{r^4}\bar{\rho}\int_0^x 4\pi\bar{\rho}y^2 dy = \mu\left(\frac{(r^2 v)_x}{r^2 r_x}\right)_x && \text{in } I \times (0,T], \\ &v(0,t) = 0, \quad \mathfrak{B}(\bar{R},t) = 0 && \text{on } (0,T], \\ &(r, \ v)(x,0) = (r_0(x), \ u_0(r_0(x))) && \text{on } I \times \{t=0\}, \end{aligned} \tag{2.1.10}$$

where $\mathfrak{B}$ is the normal stress at the boundary given by

$$\mathfrak{B} := \frac{4}{3}\lambda_1\left(\frac{v_x}{r_x} - \frac{v}{r}\right) + \lambda_2\left(\frac{v_x}{r_x} + 2\frac{v}{r}\right) = \frac{4}{3}\lambda_1\frac{r}{r_x}\left(\frac{v}{r}\right)_x + \lambda_2\frac{(r^2 v)_x}{r_x r^2}. \tag{2.1.11}$$

To obtain the higher-order estimates, we define

$$\mathcal{G} := \ln r_x + 2\ln\left(\frac{r}{x}\right).$$

This transformation between $\mathcal{G}$ and r is one-to-one, and we can solve for r in terms of $\mathcal{G}$ by

$$r(x,t) = \left(3\int_0^x y^2 \exp\left[\mathcal{G}(y,t)\right] dy\right)^{1/3} \quad \text{for } \ x \in \bar{I} \ \text{ and } \ t \geq 0. \tag{2.1.12}$$

Then equation $(2.1.10)_1$ and the normal stress at the boundary, $\mathfrak{B}$, can also be written in the form of

$$\frac{x^2}{r^2}\bar\rho v_t - \gamma\left(\frac{x^2\bar\rho}{r^2 r_x}\right)^\gamma \mathcal{G}_x - \left[\left(\frac{x^2}{r^2 r_x}\right)^\gamma - \left(\frac{x}{r}\right)^4\right] x\phi\bar\rho = \mu\mathcal{G}_{xt} \tag{2.1.13}$$

and

$$\mathfrak{B} = \mu\mathcal{G}_t - 4\lambda_1 v/r.$$

We will work on (2.1.10) and (2.1.13) alternatively. (2.1.10) is used for the lower-order estimates, while (2.1.13) is important for the higher-order estimates. (2.1.10) and (2.1.13) can be understood as either for r or $\mathcal{G}$ through (2.1.12). The adoption of this point of view is helpful to establishing both lower-order and higher-order estimates.

2.2 Main theorems and remarks

Throughout the rest of this paper, c and C will be used to denote generic positive constants which are independent of time t but may depend on γ, λ_1, λ_2, M and the bounds of $\bar\rho$ such as $\bar\rho(0)$ and $\bar\rho(\bar R/2)$; and we will use the following notations:

$$\int := \int_I, \quad \|\cdot\| := \|\cdot\|_{L^2(I)} \text{ and } \|\cdot\|_{L^p} := \|\cdot\|_{L^p(I)} \text{ with } p = 1, \infty.$$

A strong solution to problem (2.1.10) is defined as follows.

Definition 2.1 $v \in C\left([0,T]; H^2_{loc}([0,\bar R))\right) \cap C\left([0,T]; W^{1,\infty}(I)\right)$ *with*

$$r(x,t) = r_0(x) + \int_0^t v(x,s)ds \quad \text{for} \quad (x,t) \in I \times [0,T] \tag{2.2.1}$$

satisfying the initial condition $(2.1.10)_3$ *is called a strong solution of problem* (2.1.10) *in* $[0,T]$, *if*

1) $r_x(x,t) > 0$ *for* $(x,t) \in I \times [0,T]$;

2) $\bar\rho^{\gamma-1/2}\left(v_x,\ x^{-1}v\right)_x \in C([0,T]; L^2(I))$, $\bar\rho^{-1/2}\left[(r^2 r_x)^{-1}(r^2 v)_x\right]_x \in C([0,T]; L^2(I))$ *and* $\mathfrak{B} \in C([0,T]; H^1(I))$ *with* $\mathfrak{B}$ *is defined in* (2.1.11);

3) $\bar\rho^{1/2} v \in C^1([0,T]; L^2(I))$;

4) $v(0,t) = 0$ *and* $\mathfrak{B}(\bar R, t) = 0$ *hold in the sense of* $W^{1,\infty}$*-trace and* H^1*-trace, respectively, for* $t \in [0,T]$;

5) $(2.1.10)_1$ *holds for* $(x,t) \in I \times [0,T]$, *a.e..*

Let $\alpha \in [0, \gamma - 1)$ be any fixed constant. Denote

$$\mathfrak{E}(t) = \|(r_x - 1,\ v_x)(\cdot,t)\|^2_{L^\infty} + \left\|\bar\rho^{\gamma-\frac12}\mathcal{G}_x(\cdot,t)\right\|^2 + \left\|\bar\rho^{-1/2}\mathcal{G}_{xt}(\cdot,t)\right\|^2, \tag{2.2.2}$$

$$\mathfrak{F}(t) = \mathfrak{E}(t) + \left\|\bar\rho^{\gamma-1-\alpha/2}\mathcal{G}_x(\cdot,t)\right\|^2.$$

Then the functional $\mathfrak{E}(t)$ contains the L^2-norms of all terms in equation (2.1.13). It will be shown that the finiteness of $\mathfrak{E}(t)$ for all $t \geq 0$ ensures the global existence of strong solutions. Assuming the compatibility condition of the initial data with the boundary conditions:

$$v(0,0) = 0 \quad \text{and} \quad \mathfrak{B}(\bar{R}, 0) = 0, \tag{2.2.3}$$

we then have the following theorem of the global existence of strong solutions, which also gives the strong Lyaprov stability of the Lane-Emden solution:

Theorem 2.2 *Let* $\gamma \in (4/3,\ 2)$ *and* $\bar{\rho}$ *be the Lane-Emden solution satisfying* (2.1.1)-(2.1.2). *Assume that* (2.2.3) *holds and the initial density* ρ_0 *satisfies* (1.1.3) *and* (2.1.7). *There exists a constant* $\bar{\delta} > 0$ *such that if*

$$\mathfrak{E}(0) \leq \bar{\delta},$$

then the problem (2.1.10) *admits a unique strong solution in* $I \times [0, \infty)$ *with*

$$\mathfrak{E}(t) \leq C\mathfrak{E}(0), \quad t \geq 0,$$

for some constant C *independent of* t.

For any $t \geq 0$, since $r_x(x,t) > 0$ for $x \in \bar{I}$, $r(x,t)$ defines a diffeomorphism from the reference domain $\bar{I}$ to the changing domain $\{0 \leq r \leq R(t)\}$ with the boundary

$$R(t) = r\left(\bar{R}, t\right). \tag{2.2.4}$$

It also induces a diffeomorphism from the initial domain, $\bar{B}_{R_0}(0)$, to the evolving domain, $\bar{B}_{R(t)}(0)$, for all $t \geq 0$:

$$\mathbf{x} \neq \mathbf{0} \in \bar{B}_{R_0}(0) \to r\left(r_0^{-1}(|\mathbf{x}|), t\right) \frac{\mathbf{x}}{|\mathbf{x}|} \in \bar{B}_{R(t)}(0),$$

where r_0^{-1} is the inverse map of r_0 defined in (2.1.6). Here

$$\bar{B}_{R_0}(0) := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq R_0\} \quad \text{and} \quad \bar{B}_{R(t)}(0) := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq R(t)\}.$$

Denote the inverse of the map $r(x,t)$ by $\mathcal{R}_t$ for $t \geq 0$ so that

$$\text{if} \quad r = r(x,t) \quad \text{for} \quad 0 \leq r \leq R(t), \quad \text{then} \quad x = \mathcal{R}_t(r).$$

For the strong solution (r, v) obtained in Theorem 2.2, we set for $0 \leq r \leq R(t)$ and $t \geq 0$,

$$\rho(r,t) = \frac{x^2 \bar{\rho}(x)}{r^2(x,t) r_x(x,t)} \quad \text{and} \quad u(r,t) = v(x,t) \quad \text{with} \quad x = \mathcal{R}_t(r). \tag{2.2.5}$$

Then the triple $(\rho(r,t), u(r,t), R(t))$ $(t \geq 0)$ defines a global strong solution to the free boundary problem (1.1.2). Furthermore, we have the strong nonlinear asymptotic stability of the Lane-Emden solution as follows.

Theorem 2.3 *Under the assumptions in Theorem 2.2. Then the triple* $(\rho, u, R(t))$ *defined by* (2.2.4) *and* (2.2.5) *is the unique global strong solution to the free boundary problem* (1.1.1) *satisfying* $R \in W^{1,\infty}([0, +\infty))$. *Moreover, the solution satisfies the following estimates.*

i) For any $0 < \theta < \min\{2(\gamma-1)/(3\gamma),\ (4-2\gamma)/\gamma\}$, *there exists a positive constant* $C(\theta)$ *independent of* t *such that for all* $t \geq 0$,

$$\sup_{0\le x\le \bar{R}} |r(x,t)-x| \le C(\theta)(1+t)^{-\frac{\gamma-1}{\gamma}+\frac{\theta}{2}}\sqrt{\mathfrak{E}(0)}, \tag{2.2.6}$$

$$\sup_{0\le r\le R(t)} |u(r,t)| \le C(\theta)(1+t)^{-\frac{2\gamma-1}{4\gamma}+\frac{\theta}{4}}\sqrt{\mathfrak{E}(0)}, \tag{2.2.7}$$

$$\sup_{0\le r\le R(t)} \left|\left(u_r,\ r^{-1}u\right)(r,t)\right| \le C(\theta)(1+t)^{-\frac{\gamma-1}{2\gamma}+\frac{\theta}{2}}\sqrt{\mathfrak{E}(0)}, \tag{2.2.8}$$

$$\sup_{0\le x\le \bar{R}} |\rho(r(x,t),t)-\bar{\rho}(x)| \le C(\theta)(1+t)^{-\frac{2-\gamma}{4\gamma}+\frac{\theta}{8}\frac{\gamma+1-\gamma\theta}{2\gamma-1-\frac{\gamma\theta}{2}}}\sqrt{\mathfrak{E}(0)}. \tag{2.2.9}$$

ii) Suppose that $\mathfrak{F}(0) < \infty$ *for some* $\alpha \in [0, \gamma-1)$. *Let* θ *be any constant satisfying* $0 < \theta < \min\{2(\gamma-1)/(3\gamma), 2(\gamma-1-\alpha)/\gamma, (4-2\gamma)/\gamma\}$. *Set* $\kappa = 0$ *when* $\alpha = 0$ *and* $\kappa = \alpha/\gamma - \theta$ *when* $\alpha > 0$. *Then there exists a positive constant* $C(\theta)$ *such that for all* $t \geq 0$,

$$\sup_{0\le r\le R(t)} |u(r,t)| \le C(\theta)(1+t)^{-\frac{3}{8}\left(\frac{2\gamma-1}{\gamma}-\theta+\kappa\right)}\sqrt{[\mathfrak{F}(0)+\mathfrak{F}^2(0)]}, \tag{2.2.10}$$

$$\sup_{0\le r\le R(t)} \left|(u_r,\ r^{-1}u)(r,t)\right| \le C(\theta)(1+t)^{-\frac{3}{16}\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)}\sqrt{[\mathfrak{F}(0)+\mathfrak{F}^2(0)]}, \tag{2.2.11}$$

$$\begin{aligned}&\sup_{0\le x\le \bar{R}} \left|\bar{\rho}^{\gamma/2-1}(x)\left[\rho(r(x,t),t)-\bar{\rho}(x)\right]\right| \\ &\le C(\theta)(1+t)^{-\min\left\{\frac{1}{4}\left(\frac{2\gamma-1}{\gamma}-\theta+\kappa\right),\ \frac{3}{16}\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)\right\}}\sqrt{[\mathfrak{F}(0)+\mathfrak{F}^2(0)]}.\end{aligned} \tag{2.2.12}$$

Furthermore, if $\|x\bar{\rho}^{-1/2}\mathcal{G}_{xtt}(\cdot,0)\| + |v_t(\bar{R},0)| < \infty$, *then* $R \in W^{2,\infty}([0,+\infty))$ *and*

$$|\ddot{R}(t)| \le |v_t(\bar{R},0)| + C(\mathfrak{E}(0))^{1/4}\left((\mathfrak{F}(0))^{1/4} + \|x\bar{\rho}^{-\frac{1}{2}}\mathcal{G}_{xtt}(\cdot,0)\|^{1/2}\right), \quad t \ge 0. \tag{2.2.13}$$

Remark 2.4 *The estimate in* (2.2.12) *yields the uniform convergence with rates of the density to* (1.1.1) *to that of the Lane-Emden solution for both large time and near the vacuum boundary since* $\gamma < 2$.

Remark 2.5 *The initial perturbation here includes three parts: the deviation of the initial domain from that of the Lane-Emden solution, the difference of initial density from that of the Lane-Emden solution, and the velocity. Since the Lane-Emden solution is completely determined by the total mass* M, *our nonlinear asymptotic stability result shows that the time asymptotic state of the free boundary problem is determined by the total mass which is conserved in the time evolution.*

Remark 2.6 *We make comments on the finiteness of functionals $\mathfrak{E}$ and $\mathfrak{F}$ at $t = 0$ in terms of the initial data $\rho_0(r)$ and $u_0(r)$. Note that*

$$\mathcal{G}(x,0) = \ln\left(\frac{\bar\rho(x)}{\rho_0(r_0(x))}\right), \quad x \in I,$$

and near the vacuum boundary the Lane Emden solution, $\bar\rho$, behaves as

$$\bar\rho(x) \sim (\bar R - x)^{1/(\gamma-1)}, \quad \bar\rho'(x) \sim (\bar R - x)^{(2-\gamma)/(\gamma-1)}, \quad \text{as} \quad x \to \bar R^-.$$

If the initial density, ρ_0, obeys the same behavior near the vacuum boundary as the Lame-Emden solution, $\bar\rho$, then

$$|\bar\rho^{\gamma-\frac12}\mathcal{G}_x(x,0)| \le C(\bar R - x)^{\frac{1}{2(\gamma-1)}} \quad \text{and} \quad |\bar\rho^{\gamma-1-\frac{\alpha}{2}}\mathcal{G}_x(x,0)| \le C(\bar R - x)^{-\frac{\alpha}{2(\gamma-1)}} \quad \text{for} \quad x \in I.$$

So, $\|\bar\rho^{\gamma-1/2}\mathcal{G}_x(\cdot,0)\|$ and $\left\|\bar\rho^{\gamma-1-\alpha/2}\mathcal{G}_x(\cdot,0)\right\|^2$ are finite for $\alpha \in [0,\gamma-1)$. The finiteness of $\left\|\bar\rho^{-1/2}\mathcal{G}_{xt}(\cdot,0)\right\|$ is a requirement on the initial velocity $v(x,0)$ (and thus u_0).

Remark 2.7 *The condition $\|x\bar\rho^{-1/2}\mathcal{G}_{xtt}(\cdot,0)\| + |v_t(\bar R,0)| < \infty$ in ii) of Theorem 2.3 to ensure $R \in W^{2,\infty}([0,+\infty))$ (uniform boundedness of the acceleration of the vacuum boundary) is a higher-order compatibility condition of the initial data with the vacuum boundary. Indeed, one may check from the proof that every particle moving with the fluid has the bounded acceleration for $t \in [0,\infty)$ if it does so initially.*

3 Proof of main results

3.1 A theorem with detailed estimates

For the convenience of presentation, we set $\bar R = 1$ and

$$I = (0, \bar R) = (0,1).$$

Indeed, we will prove the following results for the global strong solutions obtained in Theorem 2.2, which gives not only the nonlinear asymptotic stability results stated in Theorem 2.3, but also detailed behavior of the solutions both in large time and near the vacuum boundary and the origin.

Theorem 3.1 *Let v be the global strong solution to the problem* (2.1.10) *with r given by* (2.2.1) *as obtained in Theorem 2.2.*

i) Let θ and δ be any constants satisfying

$$0 < \theta < \min\{2(\gamma-1)/(3\gamma),\ (4-2\gamma)/\gamma\} \quad \text{and} \quad \delta \in (0,1).$$

Then there exist positive constants $C(\theta)$ and $C(\theta,\delta)$ independent of t such that for all $t \ge 0$,

$$\begin{aligned}
&\left\|\bar\rho^{\frac{\gamma\theta}{4}-\frac{\gamma-1}{2}}\left(r-x, xr_x - x\right)(\cdot,t)\right\|^2 + (1+t)^{\frac{2(\gamma-1)}{\gamma}-\theta}\left\|(r-x)(\cdot,t)\right\|_{L^\infty}^2 \\
&+ (1+t)^{\frac12\left(\frac{2\gamma-1}{\gamma}-\theta\right)}\left\|(v, xv_x)(\cdot,t)\right\|_{L^\infty}^2 + (1+t)^{\frac{2-\gamma}{2\gamma}-\frac{\theta}{4}\frac{\gamma+1-\gamma\theta}{2\gamma-1-\frac{\gamma\theta}{2}}}\left\|\bar\rho\mathcal{G}(\cdot,t)\right\|_{L^\infty}^2 \\
&+ (1+t)^{\frac{2\gamma-1}{\gamma}-\theta}\left(\left\|\left(x\bar\rho^{\frac12}v_t, v, xv_x\right)(\cdot,t)\right\|^2 + \left\|\bar\rho^{\frac{\gamma}{2}}\left(r-x, xr_x - x\right)(\cdot,t)\right\|^2\right) \\
&+ (1+t)^{\frac{\gamma-1}{\gamma}-\theta}\left\|\left(r_x - 1, \frac{r}{x}-1, \bar\rho^{\frac12}v_t\right)(\cdot,t)\right\|^2 \le C(\theta)\mathfrak{E}(0)
\end{aligned} \tag{3.1.1}$$

and

$$(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\left(\left\|\left(r_x-1,\frac{r}{x}-1,v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{H^1([0,1-\delta])}+\|(\mathcal{G}_x,\mathcal{G}_{xt})(\cdot,t)\|^2_{L^2([0,1-\delta])}\right)$$
$$\leq C(\theta,\delta)\mathfrak{E}(0). \tag{3.1.2}$$

ii) Suppose that the assumptions in ii) of Theorem 2.3 hold and $\delta\in(0,1)$. Then there exist positive constants $C(\theta)$ and $C(\theta,\delta)$ such that for all $t\geq 0$,

$$\mathfrak{F}(t)+(1+t)^{\kappa}\left\|\bar\rho^{\gamma-1}\mathcal{G}_x(\cdot,t)\right\|^2+(1+t)^{\min\left\{\frac12\left(\frac{2\gamma-1}{\gamma}-\theta+\kappa\right),\ \frac38\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)\right\}}\left\|\bar\rho^{\frac{\gamma}{2}}\mathcal{G}(\cdot,t)\right\|^2_{L^\infty}$$
$$+(1+t)^{\frac12\left(\frac{2\gamma-1}{\gamma}-\theta+3\kappa\right)}\left\|\left(v_x,\frac{v}{x},x\bar\rho^{\frac32\gamma-1}\mathcal{G}_x,\ x\bar\rho^{\frac{\gamma}{2}-1}\mathcal{G}_{xt},\ \bar\rho^{\frac12}v_t\right)(\cdot,t)\right\|^2$$
$$+(1+t)^{\frac38\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)}\left\|\left(v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{L^\infty}+(1+t)^{\frac34\left(\frac{2\gamma-1}{\gamma}-\theta+\kappa\right)}\|(xv_x,v)(\cdot,t)\|^2_{L^\infty}$$
$$\leq C(\theta)\left[\mathfrak{F}(0)+\mathfrak{F}^2(0)\right] \tag{3.1.3}$$

and

$$(1+t)^{\frac38\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)}\left\|\left(r_x-1,\frac{r}{x}-1\right)(\cdot,t)\right\|^2_{L^\infty([0,1-\delta])}$$
$$+(1+t)^{\frac12\left(\frac{2\gamma-1}{\gamma}-\theta+3\kappa\right)}\left\|\left(r_x-1,\ \frac{r}{x}-1\right)(\cdot,t)\right\|^2_{L^2([0,1-\delta])}$$
$$+(1+t)^{\frac14\left(\frac{2\gamma-1}{\gamma}-\theta+9\kappa\right)}\left[\left\|\left(r_x,\ \frac{r}{x},v_x,\frac{v}{x}\right)_x(\cdot,t)\right\|^2_{L^2([0,1-\delta])}+\|(\mathcal{G}_x,\mathcal{G}_{xt})(\cdot,t)\|^2_{L^2([0,1-\delta])}\right]$$
$$\leq C(\theta,\delta)\left[\mathfrak{F}(0)+\mathfrak{F}^2(0)\right]. \tag{3.1.4}$$

Moreover, if $\|x\bar\rho^{-1/2}\mathcal{G}_{xtt}(\cdot,0)\|<\infty$, then

$$\int x^2\bar\rho v_{tt}^2(x,t)dx+\int_0^\infty\int(v_{tt}^2+x^2v_{xtt}^2)dxds\leq C(\mathfrak{F}(0)+\|x\bar\rho^{-\frac12}\mathcal{G}_{xtt}(\cdot,0)\|^2). \tag{3.1.5}$$

iii) Suppose that $\|\mathcal{G}_x(\cdot,0)\|^2$ is finite and $0<\theta<\min\{2(\gamma-1)/(3\gamma),(4-2\gamma)/\gamma\}$. Then there exist positive constants C and $C(\theta)$ independent of t such that for all $t\geq 0$,

$$\|\mathcal{G}_x(\cdot,t)\|^2+\|(r_x,\ r/x)_x(\cdot,t)\|^2\leq C\|\mathcal{G}_x(\cdot,0)\|^2+C(\theta)\mathfrak{E}(0)(1+t)^{\frac12+\theta\frac{\gamma}{2\gamma-2}} \tag{3.1.6}$$

and

$$\|(v_x,\ v/x)_x(\cdot,t)\|^2\leq C(\theta)\left(\|\mathcal{G}_x(\cdot,0)\|^2+\mathfrak{E}(0)\right)(1+t)^{-\frac{7\gamma-6}{4\gamma}+\theta\left(4+\frac{\gamma}{2\gamma-2}\right)}. \tag{3.1.7}$$

Remark 3.2 *It should be noted that $\|\mathcal{G}_x(\cdot,t)\|$ and $\|r_{xx}(\cdot,t)\|$ may grow in time, while $\|v_{xx}(\cdot,t)\|$ and $\|(v/x)_x(\cdot,t)\|$ decays provided that $\|\mathcal{G}_x(\cdot,0)\|$ is finite. (Indeed, the finiteness of $\|\mathcal{G}_x(\cdot,0)\|$ can be verified, for example, the initial density, ρ_0, is a compact perturbation of the Lame-Emden solution, $\bar\rho$.)*

Remark 3.3 *We will prove (3.1.2) and (3.1.4) when $\delta=1/2$ for the simplicity of the presentation. The proof works for any $\delta\in(0,1)$.*

3.2 Outline and main steps of the proofs

The main steps for the proofs of Theorems 2.2 and 3.1 are outlined in this subsection. The local existence and uniqueness of strong solutions to (2.1.10) can be obtained as in [12]. In order to prove the global existence of the strong solution, we need to obtain the uniform-in-time *a priori* estimates on any given time interval $[0,T]$ satisfying

$$\sup_{t\in[0,T]} \mathfrak{E}(t) < \infty.$$

To this end, we use a bootstrap argument by making the following *a priori* assumptions. Let v be a strong solution to (2.1.10) on $[0,T]$ with

$$r(x,t) = r_0(x) + \int_0^t v(x,\tau)d\tau, \quad (x,t) \in [0,\ 1] \times [0,T].$$

The basic *a priori* assumption is that there exist suitably small fixed constants $\epsilon_0 \in (0,1/2]$ and $\epsilon_1 \in (0,1]$ such that for $(x,t) \in I \times [0,T]$,

$$|r_x - 1| \le \epsilon_0 \ \text{ and } \ |v_x| \le \epsilon_1. \tag{3.2.1}$$

It follows from (3.2.1) and the boundary condition $v(0,t) = 0$ (so $r(0,t) = 0$) that

$$|r_x - 1| + |r/x - 1| \le 2\epsilon_0 \ \text{ for } \ (x,t) \in [0,\ 1] \times [0,T], \tag{3.2.2}$$

$$|v_x| + |v/x| \le 2\epsilon_1 \ \text{ for } \ (x,t) \in [0,\ 1] \times [0,T]. \tag{3.2.3}$$

In particular, it holds that

$$1/2 \le r_x,\ r/x \le 3/2 \ \text{ for } \ (x,t) \in I \times [0,T]. \tag{3.2.4}$$

To prove Theorem 2.2, one of the key issues is to estimate $\|r_x - 1\|_{L^\infty}$ and $\|v_x\|_{L^\infty}$. $\|r_x - 1\|_{L^\infty}$ can be achieved by a combination of the local L^2-estimate for r_{xx} away from the vacuum boundary, $\|r_{xx}\|_{L^2([0,1/2])}$, and the pointwise estimate away from the origin, $\|r_x - 1\|_{L^\infty([1/2,1])}$. The bound for r_{xx} can be obtained by the L^2-estimates of $\mathcal{G}_x$, given by (3.5.1), while $\|r_x - 1\|_{L^\infty([1/2,1])}$ is estimated by noting the fact that the viscosity term can be written as the space-time derivative of $\mathcal{G}$ so that one can integrate equation (2.1.13) with respect to both x and t to get the desired estimates (see Lemma 3.14, where the monotonicity of the Lane-Emden density plays an important role). Similarly, we estimate $\|v_x\|_{L^\infty}$ by considering two cases. Away from the vacuum boundary, the desired estimate can follow from the estimate

$$\left\|\bar\rho^{\gamma-1/2}\left(v_x, v/x\right)_x(\cdot,t)\right\|^2 \le C\left\|\bar\rho^{\gamma-1/2}\mathcal{G}_{xt}(\cdot,t)\right\|^2 + C\left\|\bar\rho^{\gamma-1/2}\mathcal{G}_x(\cdot,t)\right\|^2$$

and the estimate on $\|(xv_x, v)\|$; and away from the origin, we use (3.3.20):

$$\|xv_x\|_{L^\infty}^2 \le C\|xv_x\|\left(\|x\mathcal{G}_{tx}\| + \|v_x\| + \|v/x\|\right)$$

and (3.3.30):

$$\|(v_x, v/x)(\cdot,t)\|^2 \le C\left(\|xv_x\|^2 + \|v\|^2 + \|x\mathcal{G}_{xt}\|^2\right)$$

to get the desired estimates.

We define a new functional $\mathcal{E}$ as follows:

$$\mathcal{E}(t) := \|(r-x, xr_x - x)(\cdot,t)\|^2 + \|(v, xv_x)(\cdot,t)\|^2 + \|(r_x-1)(\cdot,t)\|^2_{L^\infty([1/2,\ 1])}$$
$$+ \left\|\bar\rho^{\gamma-\frac{1}{2}}\mathcal{G}_x(\cdot,t)\right\|^2 + \left\|\bar\rho^{-1/2}\mathcal{G}_{xt}(\cdot,t)\right\|^2. \tag{3.2.5}$$

(Indeed, $\mathcal{E}$ is equivalent to $\mathfrak{E}$ under the a priori assumption (3.2.1), which will be verifed in Lemma 3.10.) It suffices to show the higher-order energy functional $\mathcal{E}(t)$ is bounded uniformly by the initial data, i.e.,

$$\mathcal{E}(t) \le C\mathcal{E}(0) \quad \text{for all} \quad t \in [0,T].$$

For our purpose, the key elements in the analysis are the weighted estimates by applying various multipliers to the following equation:

$$\bar\rho\left(\frac{x}{r}\right)^2 v_t + \left[\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma\right]_x - \frac{x^4}{r^4}\left(\bar\rho^\gamma\right)_x = \mathfrak{B}_x + 4\lambda_1\left(\frac{v}{r}\right)_x, \tag{3.2.6}$$

which is equivalent to $(2.1.10)_1$. Here $\mathfrak{B}$ is defined in (2.1.11). Lemma 3.11 yields the bound for the basic energy

$$\left\|x\bar\rho^{1/2}v(\cdot,t)\right\|^2 + \left\|x\bar\rho^{\gamma/2}\left(r/x-1, r_x-1\right)(\cdot,t)\right\|^2 + \int_0^t \|(v, xv_x)(\cdot,s)\|^2\, ds.$$

In Lemma 3.12, a bound is obtained for

$$\|x\left(r/x-1, r_x-1\right)(\cdot,t)\|^2 + \int_0^t \left\|x\bar\rho^{\gamma/2}\left(r/x-1, r_x-1\right)(\cdot,s)\right\|^2 ds,$$

which refines the weighted estimate of $\|x\bar\rho^{\gamma/2}\left(r/x-1, r_x-1\right)\|$ obtained in the basic energy estimates. We also show the decay estimates for the basic energy by establishing a bound for

$$(1+t)\left(\left\|x\bar\rho^{1/2}v(\cdot,t)\right\|^2 + \left\|\bar\rho^{\gamma/2}\left(r-x, xr_x-x\right)(\cdot,t)\right\|^2\right) + \int_0^t (1+s)\,\|(v, xv_x)(\cdot,s)\|^2\, ds.$$

Here the multiplier $r^3 - x^3$ plays an important role. With those estimates, we are able to bound $|r_x - 1|$ away from the origin in Lemma 3.14. A bound for

$$(1+t)\left(\left\|x\bar\rho^{1/2}v_t(\cdot,t)\right\|^2 + \left\|\bar\rho^{\gamma/2}\left(v, xv_x\right)(\cdot,t)\right\|^2\right) + \int_0^t (1+s)\,\|(v_t, xv_{tx})(\cdot,s)\|^2\, ds$$

is given in Lemma 3.15. Further decay estimates are given in Lemma 3.16, which is important to the derivation of the decay for $\|r-x\|_{L^\infty}$ in (3.1.1). This in particular implies the convergence of the evolving boundary $r = R(t)$ to that of the Lane-Emden stationary solution. Lemma 3.16 also shows that rates of time decay in various norms depend on the behavior of the initial data near the vacuum boundary. It should be noticed that these further decay estimates are derived from the following two multipliers:

$$\int_0^x \bar\rho^{-\beta}(y)(r^3-y^3)_y dy \quad\text{and}\quad \int_0^x \bar\rho^{-\beta}(y)(r^2v)_y dy \quad\text{for}\quad 0<\beta<\gamma-1.$$

A crucial point for higher-order estimates is to link the pressure and viscosity together by introducing the quantity $\mathcal{G}$. In this way, one may write $(2.1.10)_1$ as

$$\mu\mathcal{G}_{xt}+\gamma\left(\frac{x^2\bar\rho}{r^2r_x}\right)^\gamma\mathcal{G}_x=\frac{x^2}{r^2}\bar\rho v_t-\left[\left(\frac{x^2}{r^2r_x}\right)^\gamma-\left(\frac{x}{r}\right)^4\right]x\phi\bar\rho. \tag{3.2.7}$$

The lower-order estimates give only the estimates related to $\mathcal{G}$ and its t-derivatives, $\mathcal{G}_t$ and $\mathcal{G}_{tt}$, but not to the terms involving x-derivatives (the derivatives in the normal direction to the boundary in the original Eulerian coordinates). One may obtain the higher-order estimates and the sup-norm estimates (3.2.2) and (3.2.3) by obtaining the weighted L^2-estimate on terms involving the x-derivative of $\mathcal{G}$ such as $\mathcal{G}_x$ and $\mathcal{G}_{xt}$. It should be noted that the principal part of (3.2.7) is an ODE for $\mathcal{G}_x$, with a degenerate damping because the coefficient appearing in front of $\mathcal{G}_x$ is non-negative. With the lower-order estimates obtained already, we can derive in Lemma 3.17 the uniform bound for

$$\left\|\left(\bar\rho^{\gamma-1/2}\mathcal{G}_x,\bar\rho^{-1/2}\mathcal{G}_{xt}\right)(\cdot,t)\right\|^2$$

and the decay estimates for

$$\left\|\bar\rho^{1/2}v_t(\cdot,t)\right\|^2+\left\|\left(\mathcal{G}_x,\mathcal{G}_{xt}\right)(\cdot,t)\right\|^2_{L^2([0,1/2])}.$$

This completes the proof of the uniform-in-time bounds for the higher-order energy functional $\mathcal{E}(t)$, which also verifies the *a priori* assumptions (3.2.2) and (3.2.3) due to the equivalence of $\mathcal{E}(t)$ and $\mathfrak{E}(t)$, and consequently, the global existence of the strong solution is obtained. With the decay estimates for the lower-order norms in Lemma 3.16 and the higher-order estimates in Lemma 3.17, we prove the decay estimates of

$$\|(r_x-1,\ r/x-1)(\cdot,t)\|^2,\quad \|(v,\ v_x)(\cdot,t)\|^2_{L^\infty}\quad\text{and}\quad \|(r-x)(\cdot,t)\|^2_{L^\infty}$$

in Lemma 3.18, with which part *i)* of Theorem 3.1 is proved.

The second part of the higher-order estimates will be given in section 3.5.2, in which the faster decay estimates are given under the assumption of the finiteness of $\mathfrak{F}(0)$. In Lemma 3.19, we prove the bound for $\left\|\bar\rho^{\gamma-1-\alpha/2}\mathcal{G}_x(\cdot,t)\right\|^2$ and the decay estimates for

$$\left\|\bar\rho^{\gamma-1}\mathcal{G}_x(\cdot,t)\right\|^2\quad\text{and}\quad\left\|x\left(\bar\rho^{3\gamma/2-1}\mathcal{G}_x,\ \bar\rho^{\gamma/2-1}\mathcal{G}_{xt}\right)(\cdot,t)\right\|^2.$$

A key ingredient in the proof is to use the new multiplier $x^2\bar\rho^{\gamma-2}\mathfrak{P}_t$, where

$$\mathfrak{P}(x,t):=\gamma\left(\frac{x^2\bar\rho}{r^2r_x}\right)^\gamma\mathcal{G}_x+\left[\left(\frac{x^2}{r^2r_x}\right)^\gamma-\left(\frac{x}{r}\right)^4\right]x\phi\bar\rho,$$

the sum of the gradient of the pressure and gravitation force. The decay for

$$\left\|\left(\bar\rho^{3\gamma/2-1}\mathcal{G}_x,\ \bar\rho^{\gamma/2-1}\mathcal{G}_{xt}\right)(\cdot,t)\right\|^2\quad\text{and}\quad\left\|\bar\rho^{1/2}v_t(\cdot,t)\right\|^2$$

is derived in Lemma 3.20. With those estimates, we finish the proof of *ii)* of Theorem 3.1 by Lemma 3.21. Part *iii)* of Theorem 3.1 on the further regularity of solutions is proved in section 3.5.3.

3.3 Preliminaries

The main goal of this subsection is to give some preliminary inequalities which will be used later, and prove the equivalence of the functionals $\mathfrak{E}(t)$ and $\mathcal{E}(t)$ under the *a priori* assumptions (3.2.2) and (3.2.3).

Lemma 3.4 (Weighted Sobolev Embedding) *Let d denote the distance function to the boundary ∂I. Then the weighted Sobolev space $H^1_d(I)$, given by*

$$H^1_d(I) := \left\{ dF \in L^2(I) : \quad \int_I d^2 \left(|F|^2 + |F_x|^2 \right) dx < \infty \right\},$$

satisfies the embedding:

$$H^1_d(I) \hookrightarrow L^2(I). \tag{3.3.1}$$

The proof of this lemma can be found in [8], Section 8. We will use the following general version of the Hardy inequality whose proof can be found also in [8].

Lemma 3.5 (Hardy Inequality) *Let $k > 1$ be a given real number and g be a function satisfying*

$$\int_0^{1/2} x^k \left(g^2 + g_x^2 \right) dx < \infty,$$

then it holds that

$$\int_0^{1/2} x^{k-2} g^2 dx \le c \int_0^{1/2} x^k \left(g^2 + g_x^2 \right) dx,$$

where c is a generic constant independent of g.

As a consequence, one has

$$\int_{1/2}^1 (1-x)^{k-2} g^2 dx \le c \int_0^1 (1-x)^k \left(g^2 + g_x^2 \right) dx, \tag{3.3.2}$$

provided that the right-hand side is finite.

In what follows, we show how to use $\mathcal{G}$ and its derivatives to control r and v and their derivatives by identifying the principal parts of $\mathcal{G}$, $\mathcal{G}_t$, $\mathcal{G}_x$, $\mathcal{G}_{tt}$ and $\mathcal{G}_{xt}$. Note that

$$\mathcal{G} = (r_x - 1) + 2\left(\frac{r}{x} - 1 \right) + O\left(|r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right), \tag{3.3.3}$$

$$\mathcal{G}_x = \left[r_{xx} + 2\left(\frac{r}{x} \right)_x \right] + \left(\frac{1}{r_x} - 1 \right) r_{xx} + 2\left(\frac{r}{x} - 1 \right)\left(\frac{r}{x} \right)_x, \tag{3.3.4}$$

$$\mathcal{G}_t = \left(v_x + 2\frac{v}{x} \right) + \left(\frac{1}{r_x} - 1 \right) v_x + 2\left(\frac{r}{x} - 1 \right)\left(\frac{v}{x} \right), \tag{3.3.5}$$

$$\begin{aligned} \mathcal{G}_{xt} = {} & \left(v_x + 2\frac{v}{x} \right)_x + \left[\left(\frac{1}{r_x} - 1 \right) v_{xx} + 2\left(\frac{r}{x} - 1 \right)\left(\frac{v}{x} \right)_x \right] \\ & - \left[\frac{r_{xx}}{r_x^2} v_x + 2\left(\frac{x}{r} \right)^2 \left(\frac{r}{x} \right)_x \frac{v}{x} \right]. \end{aligned} \tag{3.3.6}$$

Then we have the following lemmas.

Lemma 3.6 *Suppose* (3.2.2) *holds for a suitable small* ϵ_0. *Then,*

$$\|(v_x, v/x)\|^2 \le c\,\|\mathcal{G}_t\|^2 + c\,\|v\|^2\,, \tag{3.3.7}$$

$$\left\|\left(r_x - 1, \frac{r}{x} - 1\right)\right\|^2 \le c\,\|\mathcal{G}\|^2 + c\,\|r - x\|^2\,, \tag{3.3.8}$$

$$\|(r_x,\ r/x)_x\|^2 \le c\,\|\mathcal{G}_x\|^2 + c\,\|(xr_x - x, r - x)\|^2\,, \tag{3.3.9}$$

$$\begin{aligned}\|(v_x,\ v/x)_x\|^2 \le{}& c\,\|\mathcal{G}_{xt}\|^2 + c\,\|(xv_x, v)\|^2 \\ &+ c\,\|(v_x, v/x)\|^2_{L^\infty}\left(\|\mathcal{G}_x\|^2 + \|(xr_x - x, r - x)\|^2\right).\end{aligned} \tag{3.3.10}$$

Proof. We prove (3.3.7) first . Let $\chi_1 \in [0, 1]$ be a non-increasing smooth function satisfying

$$\chi_1 = 1 \ \text{ on } \ [0, 1/4], \quad \chi_1 = 0 \ \text{ on } \ [1/2, 1] \ \text{ and } \ \chi_1' \le 0.$$

Note that

$$v_x + 2\frac{v}{x} = \mathcal{G}_t - \left(\frac{1}{r_x} - 1\right) v_x - 2\left(\frac{x}{r} - 1\right)\frac{v}{x}$$

and

$$\int \chi_1 \left|v_x + 2\frac{v}{x}\right|^2 dx = \int \chi_1 \left[v_x^2 + 6\left(\frac{v}{x}\right)^2\right] dx + 2\int |\chi_1' x| \left(\frac{v}{x}\right)^2 dx,$$

where one has used the integration by parts and the fact $\chi_1' \le 0$. Thus,

$$\begin{aligned}&\int \chi_1 \left[v_x^2 + 6\left(\frac{v}{x}\right)^2\right] dx + 2\int |\chi_1'| x \left(\frac{v}{x}\right)^2 dx \\ \le& \int \chi_1 \left|v_x + 2\frac{v}{x}\right|^2 dx \le 2\int \chi_1 \mathcal{G}_t^2 dx + c\epsilon_0^2 \int \chi_1 \left[v_x^2 + \left(\frac{v}{x}\right)^2\right] dx;\end{aligned}$$

which implies that

$$\int \chi_1 \left[v_x^2 + 6\left(\frac{v}{x}\right)^2\right] dx + 4\int |\chi_1'| x \left(\frac{v}{x}\right)^2 dx \le 4\int \chi_1 \mathcal{G}_t^2 dx, \tag{3.3.11}$$

provided that (3.2.2) holds for a suitably small number ϵ_0. This gives the bounds away from the boundary. Away from the origin, it is easy to see that

$$\begin{aligned}\int (1 - \chi_1)\left[v_x^2 + \left(\frac{v}{x}\right)^2\right] dx =& \int (1 - \chi_1)\left[r_x^2\left(\mathcal{G}_t - 2\frac{v}{r}\right)^2 + \left(\frac{v}{x}\right)^2\right] dx \\ \le& c\int \mathcal{G}_t^2 dx + c\int_{1/4}^1 v^2 dx.\end{aligned}$$

So, we finally obtain

$$\int \left[v_x^2 + \left(\frac{v}{x}\right)^2\right] dx \le c\int \mathcal{G}_t^2 dx + c\int_{1/4}^1 v^2 dx, \tag{3.3.12}$$

which yields (3.3.7). Clearly, (3.3.8) follows from similar arguments.

For $\mathcal{G}_x$ defined in (3.3.4), a similar way as the derivation of (3.3.11) and (3.3.12) shows that

$$\int \chi_1 \left(r_{xx}^2 + 10 \left| \left(\frac{r}{x} \right)_x \right|^2 \right) dx + 4 \int |\chi_1'| x \left| \left(\frac{r}{x} \right)_x \right|^2 dx \le 4 \int \chi_1 \mathcal{G}_x^2 dx \tag{3.3.13}$$

and

$$\begin{aligned} \int \left(r_{xx}^2 + \left| \left(\frac{r}{x} \right)_x \right|^2 \right) dx \le & c \int \mathcal{G}_x^2 dx + c \int_{1/4}^1 \left| \left(\frac{r}{x} \right)_x \right|^2 dx \\ \le & c \int \mathcal{G}_x^2 dx + c \int_{1/4}^1 x^2 \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx. \end{aligned} \tag{3.3.14}$$

For $\mathcal{G}_{xt}$ given by (3.3.6), it follows from the Cauchy inequality and (3.2.2) that

$$\mathcal{G}_{xt}^2 \ge \frac{1}{2} \left| \left(v_x + 2 \frac{v}{x} \right)_x \right|^2 - c\epsilon_0^2 \left(v_{xx}^2 + \left| \left(\frac{v}{x} \right)_x \right|^2 \right) - c \left\| \left(v_x, \frac{v}{x} \right) \right\|_{L^\infty}^2 \left(r_{xx}^2 + \left| \left(\frac{r}{x} \right)_x \right|^2 \right).$$

In a similar way as to the derivation of (3.3.12), we can obtain, noting (3.3.14), that

$$\begin{aligned} & \int \left(v_{xx}^2 + \left| \left(\frac{v}{x} \right)_x \right|^2 \right) dx \\ \le & c \int \mathcal{G}_{xt}^2 dx + c \int_{1/4}^1 \left| \left(\frac{v}{x} \right)_x \right|^2 dx + c \left\| \left(v_x, \frac{v}{x} \right) \right\|_{L^\infty}^2 \int \left(r_{xx}^2 + \left| \left(\frac{r}{x} \right)_x \right|^2 \right) dx \\ \le & c \int \mathcal{G}_{xt}^2 dx + c \int_{1/4}^1 \left(x^2 v_x^2 + v^2 \right) dx + c \left\| \left(v_x, \frac{v}{x} \right) \right\|_{L^\infty}^2 \int \mathcal{G}_x^2 dx \\ & + c \left\| \left(v_x, \frac{v}{x} \right) \right\|_{L^\infty}^2 \int_{1/4}^1 x^2 \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx. \end{aligned}$$

This finishes the proof of (3.3.7)-(3.3.10). □

Lemma 3.7 *Let $\delta > 0$ be any fixed constant. Suppose that* (3.2.2) *holds for a suitable small* ϵ_0, *then*

$$\left\| \bar\rho^\delta \left(r_x, r/x \right)_x \right\|^2 + \left\| x \bar\rho^{\delta - (\gamma-1)/2} \left(r/x \right)_x \right\|^2 \le c \left\| \bar\rho^\delta \mathcal{G}_x \right\|^2, \tag{3.3.15}$$

$$\left\| \bar\rho^\delta \left(v_x, v/x \right)_x \right\|^2 \le c \left\| \bar\rho^\delta \mathcal{G}_{xt} \right\|^2 + c \left\| \left(v_x, \frac{v}{x} \right) \right\|_{L^\infty}^2 \left\| \bar\rho^\delta \mathcal{G}_x \right\|^2, \tag{3.3.16}$$

$$\left\| \bar\rho^\delta \left(r_x - 1, \frac{r}{x} - 1 \right) \right\|^2 + \left\| \bar\rho^{\delta - (\gamma-1)/2} (r - x) \right\|^2 \le c \left\| \bar\rho^\delta \mathcal{G} \right\|^2, \tag{3.3.17}$$

$$\left\| \bar\rho^\delta \left(v_x, v/x \right) \right\|^2 + \left\| \bar\rho^{\delta - (\gamma-1)/2} v \right\|^2 \le c \left\| \bar\rho^\delta \mathcal{G}_t \right\|^2. \tag{3.3.18}$$

Proof. In the derivation of (3.3.13), one can replace χ_1 by $\bar\rho^{2\delta}$ to obtain

$$\int \bar\rho^{2\delta}\left(r_{xx}^2+10\left|\left(\frac{r}{x}\right)_x\right|^2\right)dx+8\frac{\delta}{\gamma}\int x^2\phi\bar\rho^{2\delta-(\gamma-1)}\left|\left(\frac{r}{x}\right)_x\right|^2dx\le 4\int\bar\rho^{2\delta}\mathcal{G}_x^2dx$$

which verifies (3.3.15). Similarly, it can be shown that

$$\begin{aligned}&\int \bar\rho^{2\delta}\left(v_{xx}^2+10\left|\left(\frac{v}{x}\right)_x\right|^2\right)dx+8\frac{\delta}{\gamma}\int x^2\phi\bar\rho^{2\delta-(\gamma-1)}\left|\left(\frac{v}{x}\right)_x\right|^2dx\\ \le& 4\int\bar\rho^{2\delta}\mathcal{G}_{xt}^2dx+c\left\|\left(v_x,\frac{v}{x}\right)\right\|_{L^\infty}^2\int\bar\rho^{2\delta}\left(r_{xx}^2+\left|\left(\frac{r}{x}\right)_x\right|^2\right)dx,\end{aligned}$$

which yields (3.3.16) immediately due to (3.3.15). Clearly, (3.3.17) and (3.3.18) are consequences of (3.3.15) and (3.3.16) . This finishes the proof of Lemma 3.7. □

Lemma 3.8 *Suppose that* (3.2.2) *holds for a suitable small* ϵ_0*, then for any* $a\in(0,1]$*,*

$$\|v\|_{L^\infty}^2\le 2\|v\|\,\|v_x\|, \tag{3.3.19}$$

$$\|xv_x\|_{L^\infty}^2\le c\|xv_x\|\left(\|x\mathcal{G}_{tx}\|+\|v_x\|+\|v/x\|\right), \tag{3.3.20}$$

$$\left\|\left(v_x,\frac{v}{x}\right)\right\|_{L^\infty([0,a])}^2\le\frac{1}{a}\left\|\left(v_x,\frac{v}{x}\right)\right\|_{L^2([0,a])}^2+2\left\|\left(v_x,\frac{v}{x}\right)\right\|_{L^2([0,a])}\left\|\left(v_x,\frac{v}{x}\right)_x\right\|_{L^2([0,a])}. \tag{3.3.21}$$

Proof. Clearly, (3.3.19) follows from the boundary condition $v(0,t)=0$ and the Hölder inequality. For xv_x, notice that

$$(xv_x)^2=r_x^2\left(x\frac{v_x}{r_x}\right)^2=2r_x^2\int_0^x\left(y\frac{v_y}{r_y}\right)\left(y\frac{v_y}{r_y}\right)_ydy\le c\left\|x\frac{v_x}{r_x}\right\|\left(\left\|x\left(\frac{v_x}{r_x}\right)_x\right\|+\left\|\frac{v_x}{r_x}\right\|\right)$$

and

$$\left\|x\left(\frac{v_x}{r_x}\right)_x\right\|=\left\|x\left(\mathcal{G}_t-2\frac{v}{r}\right)_x\right\|\le\|x\mathcal{G}_{tx}\|+2\left\|x\left(\frac{v}{r}\right)_x\right\|.$$

Thus,

$$\|xv_x\|_{L^\infty}^2\le c\|xv_x\|\left(\|x\mathcal{G}_{tx}\|+\|v_x\|+\|v/x\|\right),$$

which verifies (3.3.20). (3.3.21) follows from simple calculations. □

Lemma 3.9 *Let* δ *be a fixed positive constant. Then for any* $a\in(0,1]$*,*

$$\|r-x\|_{L^\infty}^2\le 2\|r-x\|\,\|r_x-1\|, \tag{3.3.22}$$

$$\left\|x^{\frac{3}{2}}\bar\rho^{\delta}\mathcal{G}\right\|_{L^\infty}^2+\left\|x^2\bar\rho^{\delta-\frac{\gamma-1}{2}}\mathcal{G}\right\|^2\le\left\|x\bar\rho^\delta\mathcal{G}\right\|^2+\left\|x^3\bar\rho^{2\delta}\mathcal{G}\mathcal{G}_x\right\|_{L^1}, \tag{3.3.23}$$

$$\begin{aligned}\left\|\left(r_x-1,\frac{r}{x}-1\right)\right\|_{L^\infty([0,a])}^2\le&\frac{1}{a}\left\|\left(r_x-1,\frac{r}{x}-1\right)\right\|_{L^2([0,a])}^2\\&+2\left\|\left(r_x-1,\frac{r}{x}-1\right)\right\|_{L^2([0,a])}\left\|\left(r_x,\frac{r}{x}\right)_x\right\|_{L^2([0,a])}.\end{aligned} \tag{3.3.24}$$

Proof. Recall that $r_0(0)=0$ and $v(0,t)=0$. It follows that $r(0,t)=0$ and thus (3.3.22) holds. Notice that for any $x\in[0,1]$, one has from (2.1.2) that

$$\begin{aligned} x^3\bar\rho^{2\delta}\mathcal{G}^2 &= 2\int_0^x \left[y^{3/2}\bar\rho^{\delta}\mathcal{G}\right]\left[y^{3/2}\bar\rho^{\delta}\mathcal{G}\right]_y dy \\ =&3\int_0^x y^2\bar\rho^{2\delta}\mathcal{G}^2dy - 2\frac{\delta}{\gamma}\int_0^x y^4\phi\bar\rho^{2\delta-(\gamma-1)}\mathcal{G}^2dy + 2\int_0^x y^3\bar\rho^{2\delta}\mathcal{G}\mathcal{G}_ydy. \end{aligned}$$

Then,

$$\begin{aligned} &\left\|x^3\bar\rho^{2\delta}\mathcal{G}^2\right\|_{L^\infty} + \int y^4\bar\rho^{2\delta-(\gamma-1)}\mathcal{G}^2dy \\ \le& c\int y^2\bar\rho^{2\delta}\mathcal{G}^2dy + C\int y^3\bar\rho^{2\delta}\left|\mathcal{G}\mathcal{G}_y\right|dy, \end{aligned}$$

which yields (3.3.23). (3.3.24) follows from a simple calculation. □

The following lemma is on the equivalence of the functionals $\mathcal{E}(t)$ and $\mathfrak{E}(t)$ and is the key to the verification of the *a priori* assumptions (3.2.2) and (3.2.3).

Lemma 3.10 *Suppose that* (3.2.2) *and* (3.2.3) *hold for suitably small numbers* ϵ_0 *and* ϵ_1*. Then,*

$$\left\|\left(r_x-1,\frac{r}{x}-1,v_x,\frac{v}{x}\right)(\cdot,t)\right\|_{L^\infty}^2 \le C\mathcal{E}(t), \tag{3.3.25}$$

$$c\mathcal{E}(t)\le\mathfrak{E}(t)\le C\mathcal{E}(t). \tag{3.3.26}$$

Proof. The proof of (3.3.25) consists of two steps, in which the L^∞-bounds on the intervals $I_1=[0,1/2]$ and $I_2=[1/2,1]$ will be shown, respectively. Once (3.3.25) is proved, (3.3.26) follows then from the definitions of $\mathcal{E}(t)$ and $\mathfrak{E}(t)$ by noticing that $v(0,t)=0$.

Step 1 (away from the boundary). Taking $\delta=\gamma-\frac{1}{2}$ in (3.3.15) yields that

$$\left\|(r_x,r/x)_x(\cdot,t)\right\|_{L^2(I_1)}^2 \le C\left\|\bar\rho^{\gamma-\frac{1}{2}}(r_x,r/x)_x(\cdot,t)\right\|^2 \le C\left\|\bar\rho^{\gamma-\frac{1}{2}}\mathcal{G}_x(\cdot,t)\right\|^2.$$

This, together with the weighted Sobolev embedding (3.3.1), implies that

$$\left\|\left(r_x-1,\frac{r}{x}-1\right)(\cdot,t)\right\|_{L^2(I_1)}^2 \le c\left\|x\left(r_x-1,\frac{r}{x}-1\right)(\cdot,t)\right\| + C\left\|\bar\rho^{\gamma-\frac{1}{2}}\mathcal{G}_x(\cdot,t)\right\|^2,$$

which gives

$$\left\|\left(r_x-1,\frac{r}{x}-1\right)(\cdot,t)\right\|_{L^\infty(I_1)}^2 \le c\left\|\left(r_x-1,\frac{r}{x}-1\right)(\cdot,t)\right\|_{H^1(I_1)}^2 \le C\mathcal{E}(t).$$

As for v, it follows from (3.3.16) and (3.2.3) that

$$\left\|\bar\rho^{\gamma-\frac{1}{2}}(v_x,v/x)_x(\cdot,t)\right\|^2 \le c\left\|\bar\rho^{\gamma-\frac{1}{2}}\mathcal{G}_{xt}(\cdot,t)\right\|^2 + c\left\|\bar\rho^{\gamma-\frac{1}{2}}\mathcal{G}_x(\cdot,t)\right\|^2 \le c\mathcal{E}(t); \tag{3.3.27}$$

hence,

$$\|(v_x, v/x)_x(\cdot,t)\|^2_{L^2(I_1)} \le C\mathcal{E}(t).$$

Similar to (3.3.27), one can obtain

$$\|(v_x, v/x)(\cdot,t)\|^2_{L^\infty(I_1)} \le C\mathcal{E}(t). \tag{3.3.28}$$

Step 2 (away from the origin). It follows from the definition of $\mathcal{E}$ and the Sobolev embedding that

$$\|(r_x-1)(\cdot,t)\|^2_{L^\infty(I_2)} \le \mathcal{E}(t),$$
$$\|(r/x-1)(\cdot,t)\|_{L^\infty(I_2)} \le c\,\|(r-x)(\cdot,t)\|_{H^1(I_2)} \le c\mathcal{E}(t).$$

It remains to show the L^∞-bounds for v_x and v/x away from the origin. Since

$$c_1 d(x,\partial I) \le x\bar\rho^{\gamma-1}(x) \le c_2 d(x,\partial I)$$

for some positive constants c_1 and c_2, it follows from (3.3.1) that

$$\|\mathcal{G}_t\|^2 \le C\int x^2\bar\rho^{2(\gamma-1)}(\mathcal{G}_t^2+\mathcal{G}_{xt}^2)dx \le C\int x^2(\mathcal{G}_t^2+\mathcal{G}_{xt}^2)dx. \tag{3.3.29}$$

In view of (3.3.7), (3.3.5) and the definition of $\mathcal{E}(t)$, one has that

$$\begin{aligned}\|(v_x, v/x)(\cdot,t)\|^2 \le& C(\|\mathcal{G}_t\|^2+\|v\|^2)\\ \le& C\int x^2\left[v_x^2+(v/x)^2+\mathcal{G}_{xt}^2\right](x,t)+C\|v\|^2 \le C\mathcal{E}(t);\end{aligned} \tag{3.3.30}$$

which implies, with the aid of (3.3.19) and (3.3.20), that

$$\|(v_x, v/x)(\cdot,t)\|^2_{L^\infty(I_2)} \le 2\,\|(xv_x, v)(\cdot,t)\|^2_{L^\infty} \le C\mathcal{E}(t).$$

This finishes the proof of (3.3.25). □

3.4 Lower-order estimates

In this and the next subsections, we derive the *a priori* estimates for the strong solution in the time interval $[0,T]$ satisfying

$$\sup_{t\in[0,T]} \mathfrak{E}(t) < +\infty$$

under the assumption (3.2.2) and (3.2.3). We start with the lower-order estimates in this subsection. First, we estimate the basic energy, for which the condition $\gamma > 4/3$ is crucial.

Lemma 3.11 *Suppose that* (3.2.2) *holds for a suitably small positive number* ϵ_0. *Then,*

$$\begin{aligned}&\left\|x\bar\rho^{\frac12}v(\cdot,t)\right\|^2+(3\gamma-4)\left\|x\bar\rho^{\frac{\gamma}{2}}\left(\frac{r}{x}-1, r_x-1\right)(\cdot,t)\right\|^2+\sigma\int_0^t\|(v,xv_x)(\cdot,s)\|^2ds\\ \le& c\left(\left\|x\bar\rho^{\frac12}v(\cdot,0)\right\|^2+\left\|x\bar\rho^{\frac{\gamma}{2}}\left(\frac{r_0}{x}-1, r_{0x}-1\right)\right\|^2\right)\end{aligned} \tag{3.4.1}$$

holds for $0\le t\le T$, *where* $\sigma = \min\{2\lambda_1/3,\ \lambda_2\}$.

Proof. Define a weighted nonlinear functional density as

$$\tilde{\eta}(x,t) := \frac{1}{2}x^2\bar{\rho}v^2 + x^2\bar{\rho}^\gamma\left[\frac{1}{\gamma-1}\left(\frac{x}{r}\right)^{2\gamma-2}\left(\frac{1}{r_x}\right)^{\gamma-1} + \left(\frac{x}{r}\right)^2 r_x - 4\frac{x}{r}\right]. \tag{3.4.2}$$

It then follows from (3.2.6) and the boundary conditions $(2.1.10)_2$ and (2.1.1) that

$$\frac{d}{dt}\int \tilde{\eta}(x,t)dx = -\int \mathfrak{B}\left(r^2 v\right)_x dx + 4\lambda_1\int r^2 v\left(\frac{v}{r}\right)_x dx. \tag{3.4.3}$$

By the Taylor expansion, the quantity $[\cdot]$ in $\tilde{\eta}$ can be rewritten as

$$\frac{4-3\gamma}{\gamma-1} + (2-\gamma)\left(\frac{r}{x}-r_x\right)^2 + \frac{3\gamma-4}{2}\left[2\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] + \widetilde{Q},$$

where $\widetilde{Q}$ represents the cubic terms which can be bounded by

$$|\widetilde{Q}| \le c\left(|r_x-1|^3 + \left|\frac{r}{x}-1\right|^3\right) \le c\epsilon_0\left(|r_x-1|^2 + \left|\frac{r}{x}-1\right|^2,\right)$$

due to (3.2.4) and (3.2.2). This implies that for $\gamma\in(4/3,2]$,

$$\frac{1}{\gamma-1}\left(\frac{x}{r}\right)^{2\gamma-2}\left(\frac{1}{r_x}\right)^{\gamma-1} + \left(\frac{x}{r}\right)^2 r_x - 4\frac{x}{r} \ge \frac{4-3\gamma}{\gamma-1} + \frac{3\gamma-4}{4}\left[2\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right],$$

provided that ϵ_0 is less than a constant depending on $3\gamma-4$. Set

$$\eta(x,t) := \tilde{\eta}(x,t) - \frac{4-3\gamma}{\gamma-1}x^2\bar{\rho}^\gamma. \tag{3.4.4}$$

Then the above calculations imply that

$$\eta(x,t) \ge \frac{1}{2}x^2\bar{\rho}v^2 + \frac{3\gamma-4}{4}x^2\bar{\rho}^\gamma\left[2\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right], \tag{3.4.5}$$

$$\eta(x,t) \le \frac{1}{2}x^2\bar{\rho}v^2 + cx^2\bar{\rho}^\gamma\left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]. \tag{3.4.6}$$

Clearly, (3.4.3) and (2.1.11) show that

$$\frac{d}{dt}\int \eta(x,t)dx = -\frac{4}{3}\lambda_1\int \frac{r^4}{r_x}\left|\left(\frac{v}{r}\right)_x\right|^2 dx - \lambda_2\int \frac{1}{r_x r^2}\left|\left(r^2 v\right)_x\right|^2 dx. \tag{3.4.7}$$

Note that

$$\frac{r^4}{r_x}\left|\left(\frac{v}{r}\right)_x\right|^2 = \frac{r^2}{r_x}v_x^2 + r_x v^2 - 2rvv_x \quad\text{and}\quad \frac{|(r^2v)_x|^2}{r_x r^2} = \frac{r^2}{r_x}v_x^2 + 4r_x v^2 + 4rvv_x.$$

We obtain

$$\frac{d}{dt}\int \eta(x,t)dx \le -3\sigma\int\left[\frac{r^2}{r_x}v_x^2 + 2r_x v^2\right]dx, \tag{3.4.8}$$

where $\sigma=\min\{2\lambda_1/3,\ \lambda_2\}$; and

$$\int\eta(x,t)dx+3\sigma\int_0^t\int\left[\frac{r^2}{r_x}v_x^2+2r_xv^2\right]dxds\le\int\eta(x,0)dx,\quad t\in[0,T]. \tag{3.4.9}$$

This, together with (3.4.5), (3.4.6) and (3.2.4), implies (3.4.1). □

In the following Lemma, we will construct a nonlinear functional with a weight motivated by the virial equations to refine the weighted estimate of $\|x\bar\rho^{\gamma/2}(r/x-1,r_x-1)\|$ in Lemma 3.11 by improving the estimates near the vacuum, and gives the decay estimates for the basic energy.

Lemma 3.12 *Under the same assumptions as in Lemma 3.11, it holds that for* $0\le t\le T$*,*

$$\begin{aligned}&\sigma\left\|x\left(\frac{r}{x}-1,r_x-1\right)(\cdot,t)\right\|^2+(3\gamma-4)\int_0^t\left\|x\bar\rho^{\frac{\gamma}{2}}\left(\frac{r}{x}-1,r_x-1\right)(\cdot,s)\right\|^2ds\\ \le&C\left(\left\|x\left(\frac{r_0}{x}-1,r_{0x}-1\right)\right\|^2+\left\|x\bar\rho^{\frac12}v(\cdot,0)\right\|^2\right)\end{aligned} \tag{3.4.10}$$

and

$$\begin{aligned}&(1+t)\left\|x\bar\rho^{\frac12}v(\cdot,t)\right\|^2+(3\gamma-4)(1+t)\left\|x\bar\rho^{\frac{\gamma}{2}}\left(\frac{r}{x}-1,r_x-1\right)(\cdot,t)\right\|^2\\&+(1+t)^{\frac{2\gamma-2}{\gamma}}\|(r-x)(\cdot,t)\|^2+\sigma\int_0^t(1+s)\|(v,xv_x)(\cdot,s)\|^2ds\\ \le&\ C\left(\left\|x\left(\frac{r_0}{x}-1,r_{0x}-1\right)\right\|^2+\left\|x\bar\rho^{\frac12}v(\cdot,0)\right\|^2\right),\end{aligned} \tag{3.4.11}$$

where $\sigma=\min\{2\lambda_1/3,\ \lambda_2\}$.

Proof. The proof consists of two steps. With the basic energy estimate obtained in the previous lemma, we can achieve the estimate for $x(r_x-1,r/x-1)$ by a moment argument in Step 1. It should be pointed out the double integral obtained in Step 1 will play a crucial role in the derivation of higher-order estimates later. In Step 2, we show the time decay estimates for the basic energy.

Step 1. Set

$$\tilde\eta_0:=4\lambda_1\Phi_1\left(\frac{r}{xr_x}\right)+3\lambda_2\Phi_2\left(\frac{r^2}{x^2}r_x\right),\ \eta_0:=x^2\tilde\eta_0+x^3\bar\rho v\left(\frac{r}{x}-\frac{x^2}{r^2}\right), \tag{3.4.12}$$

where

$$\Phi_1(z):=\ln z+z^{-1}-1\ \text{ and }\ \Phi_2(z):=z-\ln z-1. \tag{3.4.13}$$

By virtue of (3.2.6), $(2.1.10)_2$ and (2.1.1), we can obtain by direct calculations that

$$\begin{aligned}&\frac{d}{dt}\int\eta_0(x,t)dx+\int\bar\rho^\gamma\left\{\left[\frac{x^4}{r^4}\left(r^3-x^3\right)\right]_x-\left(\frac{x^2}{r^2r_x}\right)^\gamma\left(r^3-x^3\right)_x\right\}dx\\ =&\int x^2\bar\rho v^2\left(1+2\frac{x^3}{r^3}\right)dx.\end{aligned} \tag{3.4.14}$$

Noting that the quantity $\{\cdot\}$ on the left-hand side of (3.4.14) can be rewritten as

$$x^2\left[3\left(\frac{x^2}{r^2r_x}\right)^\gamma - 3\left(\frac{x^2}{r^2r_x}\right)^{\gamma-1} - \left(\frac{x}{r}\right)^2 r_x + 4\left(\frac{x}{r}\right)^5 r_x - 7\left(\frac{x}{r}\right)^4 + 4\frac{x}{r}\right],$$

we can then show, using a similar way as to the derivation of (3.4.5), that

$$\begin{aligned}&\int \bar\rho^\gamma\left\{\left[\frac{x^4}{r^4}\left(r^3-x^3\right)\right]_x - \left(\frac{x^2}{r^2r_x}\right)^\gamma \left(r^3-x^3\right)_x\right\}dx\\ &\geq \frac{3(3\gamma-4)}{2}\int x^2\bar\rho^\gamma\left[2\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]dx,\end{aligned} \tag{3.4.15}$$

when (3.2.2) holds for a small ϵ_0. It follows from (3.4.14) and (3.4.15) that

$$\frac{d}{dt}\int \eta_0(x,t)dx + \frac{3(3\gamma-4)}{4}\int x^2\bar\rho^\gamma\left[2\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]dx \leq C\int v^2dx. \tag{3.4.16}$$

This implies, with the aid of (3.4.9) and (3.2.4), that

$$\begin{aligned}&\int \eta_0(x,t)dx + \frac{3(3\gamma-4)}{4}\int_0^t\int x^2\bar\rho^\gamma\left[2\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]dxds\\ \leq &C\int(\eta+\eta_0)(x,0)dx.\end{aligned} \tag{3.4.17}$$

It remains to analyze $\tilde\eta_0$. By the Taylor expansion and (3.2.2), one may get that

$$\tilde\eta_0 \geq \frac{1}{4}\left[4\lambda_1\left(\frac{r}{xr_x}-1\right)^2 + 3\lambda_2\left(\frac{r^2}{x^2}r_x-1\right)^2\right] \geq \frac{3}{4}\sigma\left[2\left(\frac{r}{xr_x}-1\right)^2 + \left(\frac{r^2}{x^2}r_x-1\right)^2\right],$$

where $\sigma = \min\{2\lambda_1/3,\ \lambda_2\}$; and

$$\tilde\eta_0 \leq 2\left[4\lambda_1\left(\frac{r}{xr_x}-1\right)^2 + 3\lambda_2\left(\frac{r^2}{x^2}r_x-1\right)^2\right] \leq C\left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]; \tag{3.4.18}$$

provided that ϵ_0 in (3.2.2) is suitably small. Notice that

$$\begin{aligned}\left(\frac{r}{x}-1\right)^2 \leq \left(\frac{r^3}{x^3}-1\right)^2 = &\left(\frac{r}{xr_x}\frac{r^2}{x^2}r_x-1\right)^2\\ \leq &C\left[\left(\frac{r}{xr_x}-1\right)^2 + \left(\frac{r^2}{x^2}r_x-1\right)^2\right] \leq C\sigma^{-1}\tilde\eta_0\end{aligned}$$

and also

$$(r_x-1)^2 \leq C\tilde\eta_0 + C\left(\frac{r}{x}-1\right)^2 \leq C\sigma^{-1}\tilde\eta_0.$$

We then achieve, with the help of (3.4.17) and (3.4.9), that

$$\sigma \int x^2 \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx + (3\gamma - 4) \int_0^t \int x^2 \bar{\rho}^\gamma \left[\left(\frac{r}{x} - 1 \right)^2 \right.$$
$$\left. + (r_x - 1)^2 \right] dxds \le C \int (\eta + \eta_0)(x, 0) dx. \tag{3.4.19}$$

This, together with (3.4.6) and (3.4.18), implies (3.4.10).

Step 2. We are ready to show the time decay of the basic energy. Let η be given by (3.4.4). It follows from (3.4.8) that

$$\begin{aligned} &(1+t) \int \eta(x,t) dx + 3\sigma \int_0^t (1+s) \int \left(\frac{r^2}{r_x} v_x^2 + 2 r_x v^2 \right) dxds \\ \le &\int \eta(x, 0) dx + \int_0^t \int \eta(x,s) dxds. \end{aligned}$$

In view of (3.4.6), (3.4.1) and (3.4.10), one has that

$$\begin{aligned} \int_0^t \int \eta(x,s) dxds \le &C \int_0^t \int v^2 dxds + C \int_0^t \int x^2 \bar{\rho}^\gamma \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dxds \\ \le &C \int (\eta_0 + \eta)(x, 0) dx. \end{aligned}$$

So, it holds that

$$(1+t) \int \eta(x,t) dx + 3\sigma \int_0^t (1+s) \int \left(\frac{r^2}{r_x} v_x^2 + 2 r_x v^2 \right) dxds \le C \int (\eta_0 + \eta)(x, 0) dx. \tag{3.4.20}$$

This, together with (3.4.6) and (3.4.18), implies

$$\begin{aligned} &(1+t) \left\| x \bar{\rho}^{\frac{1}{2}} v(\cdot, t) \right\|^2 + (3\gamma - 4)(1+t) \left\| x \bar{\rho}^{\frac{\gamma}{2}} \left(\frac{r}{x} - 1, r_x - 1 \right) (\cdot, t) \right\|^2 \\ &+ \sigma \int_0^t (1+s) \left\| (v, x v_x)(\cdot, s) \right\|^2 ds \le C \left(\left\| x \left(\frac{r_0}{x} - 1, r_{0x} - 1 \right) \right\|^2 + \left\| x \bar{\rho}^{\frac{1}{2}} v(\cdot, 0) \right\|^2 \right). \end{aligned}$$

Since $x\bar{\rho}^{\gamma-1}$ is equivalent to the distance function, $dist(x, \partial I)$, it then follows from the Sobolev embedding (3.3.1), (2.1.3) and the Hölder inequality that

$$\begin{aligned} &\int (r-x)^2(x,t) dx \le \int x^2 \bar{\rho}^{2(\gamma-1)} \left((r-x)^2 + (r_x - 1)^2 \right)(x,t) dx \\ \le &\left(\int x^2 \left((r-x)^2 + (r_x-1)^2 \right)(x,t) dx \right)^{\frac{2-\gamma}{\gamma}} \left(\int x^2 \bar{\rho}^\gamma \left((r-x)^2 + (r_x - 1)^2 \right) dx \right)^{\frac{2\gamma-2}{\gamma}} \\ \le &C(1+t)^{-\frac{2\gamma-2}{\gamma}} \left(\left\| x \left(\frac{r_0}{x} - 1, r_{0x} - 1 \right) \right\|^2 + \left\| x \bar{\rho}^{\frac{1}{2}} v(\cdot, 0) \right\|^2 \right). \end{aligned} \tag{3.4.21}$$

This finishes the proof of (3.4.11). □

Remark 3.13 *The construction of functionals $\int \eta_0 dx$ and $\int \tilde{\eta}$ in the proof of Lemma 3.12 is motivated by the virial equations for stellar dynamics (cf. [1, 39]). Indeed,* (3.4.10) *can be proved by taking inner product of* (3.2.6) *with $r^3 - x^3$.*

With the estimates obtained so far, we are able to derive a pointwise bound for $|r/x-1|$ and $|r_x-1|$ away from the origin, by realizing that the viscosity term is $\mathcal{G}_{xt}$ so that equation $(2.1.10)_1$ can be integrated with respect to both x and t. It should be noted that the monotonicity of the Lane-Emden density which decreases in the radial direction outward plays an important role for this estimate.

Lemma 3.14 *Let $I_2 = [1/2, 1]$. For a suitably small constant ϵ_0 in* (3.2.2)*, it holds that for $(x,t) \in I_2 \times [0,T]$,*

$$\begin{aligned} &\left|x^{-1}r(x,t) - 1\right| + |r_x(x,t) - 1| \\ \le & C\left(\left\|x\bar{\rho}^{\frac{1}{2}}v(\cdot,0)\right\| + \left\|x\left(\frac{r_0}{x} - 1, r_{0x} - 1\right)\right\| + \|r_{0x} - 1\|_{L^\infty(I_2)}\right). \end{aligned} \tag{3.4.22}$$

Proof. The proof consists two steps.

Step 1 (bound for $r/x - 1$). Notice that

$$x(r-x)^2 = \int_0^x \left[y(r(y,t)-y)^2\right]_y dy \le \|r-x\|^2 + 2\|r-x\|\,\|x(r_x-1)\|.$$

This, together with (3.4.10), yields that for $x \in I_2$,

$$\left|\frac{r}{x} - 1\right|^2 \le 8x(r-x)^2 \le C\left(\left\|x\bar{\rho}^{\frac{1}{2}}v(\cdot,0)\right\|^2 + \left\|x\left(\frac{r_0}{x} - 1, r_{0x} - 1\right)\right\|^2\right). \tag{3.4.23}$$

Step 2 (bound for $r_x - 1$). Integrating equation (3.2.6) over $[x,1]$ and using the boundary conditions $(2.1.10)_2$ and (2.1.1), one gets

$$\int_x^1 \bar{\rho}\left(\frac{y}{r}\right)^2 v_t dy - \left(\frac{x^2}{r^2}\frac{\bar{\rho}}{r_x}\right)^\gamma - \int_x^1 \frac{y^4}{r^4}\left(\bar{\rho}^\gamma\right)_y dy = -\mathfrak{B} + 4\lambda_1 \int_x^1 \left(\frac{v}{r}\right)_y dy; \tag{3.4.24}$$

where $\mathfrak{B}$, defined by (2.1.11), can be rewritten as

$$\mathfrak{B} = \mu\left(\ln r_x\right)_t - \left(\frac{4}{3}\lambda_1 - 2\lambda_2\right)\left(\ln r\right)_t.$$

So, (3.4.24) is equivalent to

$$\begin{aligned} \mu\left(\ln r_x\right)_t = & \left(\frac{x^2}{r^2}\frac{\bar{\rho}}{r_x}\right)^\gamma - \left(\int_x^1 \bar{\rho}\frac{y^2}{r^2}v dy\right)_t - 2\int_x^1 \bar{\rho}\frac{y^2}{r^3}v^2 dy + \int_x^1 \frac{y^4}{r^4}\left(\bar{\rho}^\gamma\right)_y dy \\ & + \left(\frac{4}{3}\lambda_1 - 2\lambda_2\right)\left(\ln r\right)_t + 4\lambda_1 \int_x^1 \left(\ln r\right)_{yt} dy. \end{aligned}$$

Integrate it with respect to the temporal variable to obtain

$$\mu \ln\left(\frac{r_x}{r_{0x}}\right) = \int_0^t \left(\frac{x^2}{r^2}\frac{\bar{\rho}}{r_x}\right)^\gamma ds + \mathfrak{L},$$

where

$$\mathfrak{L} = -\int_x^1 \bar\rho \frac{y^2}{r^2} v dy \Big|_0^t - 2\int_0^t\int_x^1 \bar\rho \frac{y^2}{r^3} v^2 dyds + \int_0^t \int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds$$
$$+ \left(\frac{4}{3}\lambda_1 - 2\lambda_2\right)\ln\left(\frac{r}{r_0}\right) + 4\lambda_1 \ln\left(\frac{r(1,t)}{r_0(1)}\frac{r_0(x)}{r(x,t)}\right);$$

which implies that

$$r_x = r_{0x}\exp\left\{\frac{1}{\mu}\mathfrak{A}\right\}\exp\left\{\frac{1}{\mu}\mathfrak{L}\right\}, \quad \text{where } \mathfrak{A} = \int_0^t \left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma ds. \tag{3.4.25}$$

On the other hand, direct calculations show, by virtue of (3.4.25), that

$$\mathfrak{A}_t = \left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma = \left(\frac{x^2}{r^2}\frac{\bar\rho}{r_{0x}}\right)^\gamma \exp\left\{-\frac{\gamma}{\mu}\mathfrak{A}\right\}\exp\left\{-\frac{\gamma}{\mu}\mathfrak{L}\right\},$$

so that

$$\exp\left\{\frac{\gamma}{\mu}\mathfrak{A}\right\} = 1 + \int_0^t \frac{\gamma}{\mu}\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_{0x}}\right)^\gamma \exp\left\{-\frac{\gamma}{\mu}\mathfrak{L}\right\}d\tau.$$

It then follows from (3.4.25) that

$$\begin{aligned} r_x =& r_{0x}\left[1 + \int_0^t \frac{\gamma}{\mu}\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_{0x}}\right)^\gamma \exp\left\{-\frac{\gamma}{\mu}\mathfrak{L}_1\right\}\exp\left\{-\frac{\gamma}{\mu}\int_0^\tau\int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds\right\}d\tau\right]^{1/\gamma} \\ &\times \exp\left\{\frac{1}{\mu}\mathfrak{L}_1\right\}\exp\left\{\frac{1}{\mu}\int_0^t\int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds\right\}, \end{aligned} \tag{3.4.26}$$

where

$$\mathfrak{L}_1 = \mathfrak{L} - \int_0^t\int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds.$$

In view of (3.4.1) and (3.4.23), one can get that for $x \ge 1/2$,

$$\begin{aligned} |\mathfrak{L}_1| \le& C\left(\int x^2\bar\rho v^2 dx \int \bar\rho dx\right)^{1/2} + C\left(\int x^2 \bar\rho u_0^2(r_0(x))dx\int\bar\rho dx\right)^{1/2} \\ &+ C\int_0^t\int v^2 dyds + C\left\|\frac{r}{x} - 1\right\|_{L^\infty(I_2\times[0,T])} \le C\tilde{\mathfrak{e}} \end{aligned}$$

where

$$\tilde{\mathfrak{e}} = \left\|x\bar\rho^{\frac{1}{2}}v(\cdot,0)\right\| + \left\|x\left(\frac{r_0}{x} - 1, r_{0x} - 1\right)\right\|.$$

It therefore follows from (3.4.26) and (3.4.23) that

$$\begin{aligned} r_x \le& r_{0x}\left[1 + (1 + C\mathfrak{e})\int_0^t \frac{\gamma}{\mu}\bar\rho^\gamma \exp\left\{-\frac{\gamma}{\mu}\int_0^\tau\int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds\right\}d\tau\right]^{1/\gamma} \\ &\times (1 + C\mathfrak{e})\exp\left\{\frac{1}{\mu}\int_0^t\int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds\right\} \\ \le& r_{0x}(1 + C\mathfrak{e})\left[\exp\left\{\frac{\gamma}{\mu}\int_0^t\int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds\right\}\right. \\ &\left. + (1 + C\mathfrak{e})\int_0^t \frac{\gamma}{\mu}\bar\rho^\gamma \exp\left\{\frac{\gamma}{\mu}\int_\tau^t\int_x^1 \frac{y^4}{r^4}\left(\bar\rho^\gamma\right)_y dyds\right\}d\tau\right]^{1/\gamma}, \end{aligned}$$

where $\mathfrak{e} = \tilde{\mathfrak{e}} + \|r_{0x} - 1\|_{L^\infty(I_2)}$. Observe that $(\bar\rho^\gamma)_x < 0$. So, one can derive from (3.4.23) that

$$
\begin{aligned}
r_x \le & r_{0x}(1+C\mathfrak{e})\left[\exp\left\{\frac{\gamma}{\mu}\int_0^t\int_x^1 (1-C\mathfrak{e})(\bar\rho^\gamma)_y\, dyds\right\}\right. \\
&\left. + (1+C\mathfrak{e})\int_0^t \frac{\gamma}{\mu}\bar\rho^\gamma \exp\left\{\frac{\gamma}{\mu}\int_\tau^t\int_x^1 (1-C\mathfrak{e})(\bar\rho^\gamma)_y\, dyds\right\} d\tau\right]^{1/\gamma} \\
\le & r_{0x}(1+C\mathfrak{e})\left[\exp\left\{-\frac{\gamma}{\mu}(1-C\mathfrak{e})\bar\rho^\gamma t\right\}\right. \\
&\left. + (1+C\mathfrak{e})\int_0^t \frac{\gamma}{\mu}\bar\rho^\gamma \exp\left\{-\frac{\gamma}{\mu}(1-C\mathfrak{e})\bar\rho^\gamma (t-\tau)\right\} d\tau\right]^{1/\gamma} \\
\le & r_{0x}(1+C\mathfrak{e})\left[\exp\left\{-\frac{\gamma}{\mu}(1-C\mathfrak{e})\bar\rho^\gamma t\right\}\right. \\
&\left. + \frac{1+C\mathfrak{e}}{1-C\mathfrak{e}}\exp\left\{-\frac{\gamma}{\mu}(1-C\mathfrak{e})\bar\rho^\gamma (t-\tau)\right\}\Big|_{\tau=0}^t\right]^{1/\gamma} \\
\le & r_{0x}(1+C\mathfrak{e})\left\{\exp\left\{-\frac{\gamma}{\mu}(1-C\mathfrak{e})\bar\rho^\gamma t\right\}\left[1-\frac{1+C\mathfrak{e}}{1-C\mathfrak{e}}\right] + \frac{1+C\mathfrak{e}}{1-C\mathfrak{e}}\right\}^{1/\gamma} \\
\le & r_{0x}(1+C\mathfrak{e}).
\end{aligned}
$$

Similarly,

$$
r_x \ge r_{0x}(1-C\mathfrak{e}).
$$

These two estimates, together with (3.4.23), imply (3.4.22). □

The following lemma gives the decay estimates for the weighted norms of both the time and spatial derivatives of v.

Lemma 3.15 *Let* (3.2.2) *and* (3.2.3) *be true. Then it holds that, for* $0 \le t \le T$,

$$
\begin{aligned}
&(1+t)\left\|\left(x\bar\rho^{\frac12}v_t, v, xv_x\right)(\cdot,t)\right\|^2 + \int_0^t (1+s)\left\|(v_t, xv_{tx},)(\cdot,s)\right\|^2 ds \\
&\le C\left(\left\|\left(v, xv_x, x\bar\rho^{-\frac12}\mathcal{G}_{xt}, x\bar\rho^{\gamma-\frac12}\mathcal{G}_x\right)(\cdot,0)\right\|^2 + \left\|x\left(\frac{r_0}{x}-1, r_{0x}-1\right)\right\|^2\right).
\end{aligned}
\tag{3.4.27}
$$

Proof. Multiplying equation (3.2.6) by r^2 and differentiating the resulting equation with respect to t, we obtain

$$
\begin{aligned}
&\bar\rho x^2 v_{tt} - \gamma r^2\left[\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma\left(2\frac{v}{r}+\frac{v_x}{r_x}\right)\right]_x + 2rv\left[\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma\right]_x + 2\frac{x^4}{r^3}v(\bar\rho^\gamma)_x \\
=& r^2\left[\mathfrak{B}_{xt} + 4\lambda_1\left(\frac{v}{r}\right)_{xt}\right] + 2rv\left[\mathfrak{B}_x + 4\lambda_1\left(\frac{v}{r}\right)_x\right].
\end{aligned}
\tag{3.4.28}
$$

Let

$$
\begin{aligned}
\eta_1(x,t) :=& \frac12 x^2\bar\rho v_t^2 + \left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma\left[(2\gamma-1)r_x v^2 + 2(\gamma-1)rvv_x + \frac{\gamma}{2}\frac{r^2}{r_x}v_x^2\right] \\
&- \bar\rho^\gamma\left[\left(4\frac{x^3}{r^3} - 3\frac{x^4}{r^4}r_x\right)v^2 + 2\frac{x^4}{r^3}vv_x\right].
\end{aligned}
\tag{3.4.29}
$$

Following the estimates for η defined in (3.4.4), we can show that, for $\gamma \in (4/3, 2]$,

$$\eta_1(x,t) \geq \frac{1}{2}x^2\bar{\rho}v_t^2 + \frac{3\gamma-4}{4}x^2\bar{\rho}^\gamma\left[2\left(\frac{v}{x}\right)^2 + v_x^2\right], \tag{3.4.30}$$

$$\eta_1(x,t) \leq \frac{1}{2}x^2\bar{\rho}v_t^2 + cx^2\bar{\rho}^\gamma\left[\left(\frac{v}{x}\right)^2 + v_x^2\right], \tag{3.4.31}$$

provided that (3.2.2) holds with ϵ_0 being suitably small. It yields from (3.4.28), $(2.1.10)_2$ and (2.1.1) that

$$\frac{d}{dt}\int \eta_1(x,t)dx + \int \left[\mathfrak{B}_t\left(r^2v_t\right)_x - 4\lambda_1 r^2 v_t\left(\frac{v}{r}\right)_{xt}\right]dx = \mathfrak{I}_1 + \mathfrak{I}_2, \tag{3.4.32}$$

where

$$\begin{aligned}
\mathfrak{I}_1 :=& -\int\left[\mathfrak{B}(2rvv_t)_x - 8\lambda_1\left(\frac{v}{r}\right)_x rvv_t\right]dx,\\
\mathfrak{I}_2 :=&(2\gamma-1)\int\left[\left(\frac{x^2}{r^2}\frac{\bar{\rho}}{r_x}\right)^\gamma r_x\right]_t v^2dx + 2(\gamma-1)\int\left[\left(\frac{x^2}{r^2}\frac{\bar{\rho}}{r_x}\right)^\gamma r\right]_t vv_xdx\\
&+\frac{\gamma}{2}\int\left[\left(\frac{x^2}{r^2}\frac{\bar{\rho}}{r_x}\right)^\gamma\frac{r^2}{r_x}\right]_t v_x^2dx - \int\bar{\rho}^\gamma\left[\left(4\frac{x^3}{r^3} - 3\frac{x^4}{r^4}r_x\right)_t v^2 + 2\left(\frac{x^4}{r^3}\right)_t vv_x\right]dx.
\end{aligned}$$

The second term on the left-hand side of (3.4.32) can be estimated as follows. Notice that

$$\mathfrak{B}_t = \frac{4}{3}\lambda_1\frac{r}{r_x}\left(\frac{v_t}{r}\right)_x + \lambda_2\frac{(r^2v_t)_x}{r_xr^2} + \bar{\mathfrak{B}},$$

where

$$\bar{\mathfrak{B}} :=\frac{4}{3}\lambda_1\left[\left(\frac{v}{r}\right)^2 - \left(\frac{v_x}{r_x}\right)^2\right] - \lambda_2\left[2\left(\frac{v}{r}\right)^2 + \left(\frac{v_x}{r_x}\right)^2\right]. \tag{3.4.33}$$

Thus,

$$\begin{aligned}
&\int\left[\mathfrak{B}_t\left(r^2v_t\right)_x - 4\lambda_1r^2v_t\left(\frac{v}{r}\right)_{xt}\right]dx\\
=&\int\left\{\frac{4}{3}\lambda_1\left(\frac{v_t}{r}\right)_x\left[\frac{r}{r_x}\left(r^2v_t\right)_x - 3r^2v_t\right] + \lambda_2\frac{\left|(r^2v_t)_x\right|^2}{r_xr^2}\right\}dx - \mathfrak{I}_3\\
\geq&3\sigma\int\left[\frac{r^2}{r_x}v_{tx}^2 + 2r_xv_t^2\right]dx - \mathfrak{I}_3,
\end{aligned} \tag{3.4.34}$$

where $\sigma = \min\{2\lambda_1/3,\ \lambda_2\}$ and

$$\mathfrak{I}_3 := -\int\bar{\mathfrak{B}}\left(r^2v_t\right)_x dx - 4\lambda_1\int r^2v_t\left(\frac{v^2}{r^2}\right)_x dx.$$

(3.4.32) implies that

$$\frac{d}{dt}\int \eta_1(x,t)dx + 3\sigma\int\left(\frac{r^2}{r_x}v_{tx}^2 + 2r_xv_t^2\right)dx \le \mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3.$$

For $\mathfrak{I}_1$ and $\mathfrak{I}_3$, it follows from (3.2.4), (3.2.3) and the Cauchy inequality that

$$\mathfrak{I}_1+\mathfrak{I}_3 \le \sigma\int\left(\frac{r^2}{r_x}v_{tx}^2 + 2r_xv_t^2\right)dx + C\sigma^{-1}\epsilon_1^2\int\left(x^2v_x^2+v^2\right)dx.$$

Similarly, $\mathfrak{I}_2$ can be bounded by

$$\mathfrak{I}_2 \le C\epsilon_1\int\left(x^2v_x^2+v^2\right)dx.$$

So, we arrive at the following estimate

$$\frac{d}{dt}\int \eta_1(x,t)dx + 2\sigma\int\left(\frac{r^2}{r_x}v_{tx}^2 + 2r_xv_t^2\right)dx \le C\int\left(x^2v_x^2+v^2\right)dx, \tag{3.4.35}$$

provided that (3.2.3) holds for $\epsilon_1 \le 1$. This, together with (3.4.9), implies that

$$\int \eta_1(x,t)dx + \sigma\int_0^t\int\left(x^2v_{tx}^2+v_t^2\right)dxds \le \int\eta_1(x,0)dx + C\int\eta(x,0)dx \tag{3.4.36}$$

and

$$\begin{aligned}&(1+t)\int\eta_1(x,t)dx + \sigma\int_0^t(1+s)\int\left(x^2v_{tx}^2+v_t^2\right)dxds\\ \le&\int\eta_1(x,0)dx + \int_0^t\int\eta_1(x,s)dxds + C\int_0^t(1+s)\int\left(x^2v_x^2+v^2\right)dxds\\ \le&\int\eta_1(x,0)dx + C\int_0^t\int v_t^2dxds + C\int_0^t(1+s)\int\left(x^2v_x^2+v^2\right)dxds.\end{aligned}$$

Here (3.4.31) has been used. This, together with (3.4.11), (3.4.36) and (3.4.31), implies

$$\begin{aligned}&(1+t)\left(\left\|x\bar\rho^{\frac12}v_t(\cdot,t)\right\|^2 + \left\|\bar\rho^{\frac{\gamma}{2}}\left(v,xv_x\right)(\cdot,t)\right\|^2\right) + \int_0^t(1+s)\left\|\left(v_t,xv_{tx},\right)(\cdot,s)\right\|^2ds\\ \le& C\left(\left\|x\bar\rho^{\frac12}(v_t,v)(\cdot,0)\right\|^2 + \left\|\bar\rho^{\frac{\gamma}{2}}\left(v,xv_x\right)(\cdot,0)\right\|^2 + \left\|x\left(\frac{r_0}{x}-1,r_{0x}-1\right)\right\|^2\right).\end{aligned} \tag{3.4.37}$$

Observe that

$$(1+t)v^2(x,t) \le v^2(x,0) + \int_0^t(1+s)\left[2v^2(x,s)+v_s^2(x,s)\right]ds.$$

Integrate the above inequality with respect to the spatial variable to give

$$(1+t)\int v^2(x,t)dx \le \int v^2(x,0)dx + \int_0^t(1+s)\int\left[2v^2(x,s)+v_s^2(x,s)\right]dxds.$$

Similarly, it holds that

$$(1+t)\int x^2 v_x^2(x,t)dx \le \int x^2 v^2(x,0)dx + \int_0^t (1+s)\int \left[2x^2v^2(x,s)+x^2v_s^2(x,s)\right]dxds.$$

This, together with (3.4.11) and (3.4.37), implies that

$$\begin{aligned}&(1+t)\left(\left\|x\bar\rho^{\frac12}v_t(\cdot,t)\right\|^2+\|(v,xv_x)(\cdot,t)\|^2\right)+\int_0^t(1+s)\|(v_t,xv_{tx},)(\cdot,s)\|^2ds\\ &\le C\left(\left\|x\bar\rho^{\frac12}v_t(\cdot,0)\right\|^2+\|(v,xv_x)(\cdot,0)\|^2+\left\|x\left(\frac{r_0}{x}-1,r_{0x}-1\right)\right\|^2\right).\end{aligned} \tag{3.4.38}$$

The term $x\bar\rho^{1/2}v_t(x,0)$ is determined by the initial data $r_0(x)$ and $v(x,0)$ through the equation. Indeed, it follows from (3.2.7) that

$$\begin{aligned}&\left\|x\bar\rho^{1/2}v_t(\cdot,0)\right\|^2=\left\|x\bar\rho^{-1/2}\left\{\mu\left(\frac{(r^2v)_x}{r^2r_x}\right)_x-\left[\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma\right]_x+\frac{x^4}{r^4}\left(\bar\rho^\gamma\right)_x\right\}(\cdot,0)\right\|^2\\ \le&C\left\|x\bar\rho^{-1/2}\mathcal{G}_{xt}(\cdot,0)\right\|^2+C\left\|x\bar\rho^{\gamma-\frac12}\mathcal{G}_x(\cdot,0)\right\|^2+C\left\|x^2\bar\rho^{1/2}\left(\frac{r_0}{x}-1,r_{0x}-1\right)\right\|^2.\end{aligned}$$

This finishes the proof of (3.4.27). □

Next, we derive further time decay estimates based on Lemmas 3.11, 3.12 and 3.15 by using two multipliers

$$\int_0^x \bar\rho^{-\beta}(y)(r^3-y^3)_y dy \ \text{ and } \ \int_0^x \bar\rho^{-\beta}(y)(r^2v)_y dy \ \text{ for } \ 0<\beta<\gamma-1.$$

The key is to deal with the behavior of solutions near both the boundary and geometrical singularity at the origin simultaneously. The improved decay estimates obtained in this lemma are important to the derivation of the decay of $\|r-x\|_{L^\infty(I)}$ in (3.1.1) which in particular implies the convergence of the evolving boundary $r=R(t)$ to that of the Lane-Emden stationary solution.

Lemma 3.16 *Suppose that* (3.2.2) *and* (3.2.3) *hold. Then for any* $\theta\in(0,2(\gamma-1)/(3\gamma))$, *there exists a constant* $C(\theta)$ *independent of* t *such that*

$$\begin{aligned}&\left\|\bar\rho^{\frac{\gamma\theta}{4}-\frac{\gamma-1}{2}}(r-x,xr_x-x)(\cdot,t)\right\|^2+(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\|(xr_x-x)(\cdot,t)\|^2+(1+t)^{\frac{3(\gamma-1)}{\gamma}-\theta}\\ &\times\|(r-x)(\cdot,t)\|^2+(1+t)^{\frac{2\gamma-1}{\gamma}-\theta}\left(\left\|\left(x\bar\rho^{\frac12}v_t,v,xv_x\right)(\cdot,t)\right\|^2+\left\|\bar\rho^{\frac\gamma2}(r-x,xr_x-x)(\cdot,t)\right\|^2\right)\\ &+\int_0^t\left[\left\|\bar\rho^{\frac{\theta\gamma+2}{4}}(r-x,xr_x-x)(\cdot,s)\right\|^2+(1+s)^{\frac{\gamma-1}{\gamma}-\theta}\left\|\bar\rho^{\frac\gamma2}(r-x,xr_x-x)(\cdot,s)\right\|^2\right]ds\\ &+\int_0^t\left[(1+s)^{\frac{2\gamma-1}{\gamma}-\theta}\|(v,xv_x,v_s,xv_{sx})(\cdot,s)\|^2+(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac\theta2}\left\|\bar\rho^{\frac{\gamma\theta}{4}-\frac{\gamma-1}{2}}(v,xv_x)(\cdot,s)\right\|^2\right]ds\\ &\le C(\theta)\left(\left\|\left(v,xv_x,x\bar\rho^{-\frac12}\mathcal{G}_{xt},x\bar\rho^{\gamma-\frac12}\mathcal{G}_x\right)(\cdot,0)\right\|^2+\|r_{0x}-1\|_{L^\infty}^2\right),\quad t\in[0,T].\end{aligned} \tag{3.4.39}$$

Proof. For any given $\theta \in (0,\ 2(\gamma-1)/(3\gamma))$, we set

$$\beta := \gamma - 1 - \frac{\gamma\theta}{2}, \quad \iota := \frac{\theta}{2}, \quad \kappa := \frac{\beta}{\gamma} - \iota, \quad \nu := \frac{1}{2}\left(1 + \frac{\beta}{\gamma} - \iota\right), \tag{3.4.40}$$

so that

$$0 < \beta < \gamma - 1 \ \text{ and } \ 0 < \iota < \frac{\beta}{2\gamma}.$$

The proof of this lemma consists of the following five steps.

Step 1. In this step, we prove that

$$\begin{aligned}
&\int \left[(1+t)^{\nu}\bar{\rho}^{\gamma} + 1\right] x^2 \bar{\rho}^{-\beta} \left[\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2\right](x,t)dx \\
&+ \int_0^t (1+s)^{\nu} \int \bar{\rho}^{-\beta}\left(x^2 v_x^2 + v^2\right) dxds + \int_0^t \int x^2 \bar{\rho}^{\gamma-\beta} \left[\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2\right] dxds \\
&\le C\left\|r_{0x} - 1\right\|_{L^\infty}^2 + C\sum_{i=1}^3 \int_0^t (1+s)^{\nu}|K_i|ds + C\sum_{i=1}^3 \int_0^t |L_i|ds,
\end{aligned} \tag{3.4.41}$$

where

$$\begin{aligned}
L_1 =& - \int \bar{\rho}\frac{x^2}{r^2} v_t \left(\int_0^x \bar{\rho}^{-\beta}(r^3 - y^3)_y dy\right) dx, \\
L_2 =& \int \bar{\rho}^{\gamma} \left(\frac{x^4}{r^4}\right)_x \left[\bar{\rho}^{-\beta}\left(r^3 - x^3\right) - \int_0^x \bar{\rho}^{-\beta}(r^3 - y^3)_y dy\right] dx, \\
L_3 =& 4\lambda_1 \int \left(\frac{v}{r}\right)_x \left[\int_0^x \bar{\rho}^{-\beta}(r^3 - y^3)_y dy - \bar{\rho}^{-\beta}\left(r^3 - x^3\right)\right] dx;
\end{aligned} \tag{3.4.42}$$

and

$$\begin{aligned}
K_1 =& - \int \bar{\rho}\frac{x^2}{r^2} v_t \left(\int_0^x \bar{\rho}^{-\beta}(r^2 v)_y dy\right) dx, \\
K_2 =& \int \bar{\rho}^{\gamma} \left(\frac{x^4}{r^4}\right)_x \left[\bar{\rho}^{-\beta} r^2 v - \int_0^x \bar{\rho}^{-\beta}(r^2 v)_y dy\right] dx, \\
K_3 =& 4\lambda_1 \int \left(\frac{v}{r}\right)_x \left[\left(\int_0^x \bar{\rho}^{-\beta}(r^2 v)_y dy\right) - \bar{\rho}^{-\beta} r^2 v\right] dx.
\end{aligned} \tag{3.4.43}$$

To this end, we multiply (3.2.6) by the multiplier $\int_0^x \bar{\rho}^{-\beta}(y)(r^3 - y^3)_y dy$ and integrate the resulting equation with respect to the spatial variable to obtain, after the integration by parts and using $(2.1.10)_2$ and (2.1.1), that

$$\begin{aligned}
&\int \bar{\rho}^{\gamma-\beta}\left\{\left[\frac{x^4}{r^4}\left(r^3 - x^3\right)\right]_x - \left(\frac{x^2}{r^2 r_x}\right)^{\gamma}\left(r^3 - x^3\right)_x\right\} dx \\
&+ \int \bar{\rho}^{-\beta}\left[\mathfrak{B}\left(r^3 - x^3\right)_x - 4\lambda_1\left(\frac{v}{r}\right)_x\left(r^3 - x^3\right)\right] dx = \sum_{i=1}^3 L_i.
\end{aligned}$$

Noticing that

$$\begin{aligned}
&\int x^2\bar\rho^{-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right](x,0)dx\\
&\le C\left\|(r_0-x,\ xr_{0x}-x)\right\|_{L^\infty}^2\int_0^1(1-x)^{-\frac{\beta}{\gamma-1}}dx\le C\frac{\|r_{0x}-1\|_{L^\infty}^2}{(\gamma-1)-\beta}
\end{aligned}$$

due to (2.1.3) and $r_0(0)=0$, one can obtain, following the derivation of (3.4.19), that

$$\begin{aligned}
&\int x^2\bar\rho^{-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right](x,t)dx+\int_0^t\int x^2\bar\rho^{\gamma-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dxds\\
&\le C\left\|r_{0x}-1\right\|_{L^\infty}^2+C\sum_{i=1}^3\int_0^t|L_i|ds.
\end{aligned}\tag{3.4.44}$$

Next, multiplying equation (3.2.6) by $\int_0^x\bar\rho^{-\beta}(y)(r^2v)_ydy$ and integrating the product with respect to spatial variable, one obtains that, by integrating by parts based on $(2.1.10)_2$ and (2.1.1),

$$\begin{aligned}
&\int\bar\rho^{\gamma-\beta}\left[\left(\frac{x^4}{r^2}v\right)_x-\left(\frac{x^2}{r^2r_x}\right)^\gamma\left(r^2v\right)_x\right]dx\\
&+\int\bar\rho^{-\beta}\left[\mathfrak{B}\left(r^2v\right)_xdx-4\lambda_1r^2v\left(\frac{v}{r}\right)_x\right]dx=\sum_{i=1}^3K_i,
\end{aligned}$$

Following the derivation of (3.4.8), one can then obtain

$$\frac{d}{dt}\int\eta_2(x,t)dx+3\sigma\int\bar\rho^{-\beta}\left[\frac{r^2}{r_x}v_x^2+2r_xv^2\right]dx\le\sum_{i=1}^3K_i,$$

where

$$\eta_2(x,t):=\bar\rho^{-\beta}\left(\eta(x,t)-\frac{1}{2}x^2\bar\rho v^2\right)\approx x^2\bar\rho^{\gamma-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right].$$

Here and thereafter, $f\approx g$ means that $C^{-1}g\le f\le Cg$ with a generic positive constant C. Multiplying the equation above by $(1+t)^\nu$ and integrating the product with respect to the temporal variable lead to

$$\begin{aligned}
&(1+t)^\nu\int x^2\bar\rho^{\gamma-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right](x,t)dx+\int_0^t(1+s)^\nu\int\bar\rho^{-\beta}\left(x^2v_x^2+v^2\right)dxds\\
&\le C\left\|r_{0x}-1\right\|_{L^\infty}^2+C\sum_{i=1}^3\int_0^t(1+s)^\nu|K_i|ds+C\int_0^t\int x^2\bar\rho^{\gamma-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dxds,
\end{aligned}\tag{3.4.45}$$

due to the fact $\beta/\gamma<1$. So, estimate (3.4.41) follows by a suitable combination of (3.4.45) and (3.4.44).

Step 2. In this step, we show that

$$\begin{aligned}
&\int \left[(1+t)^{2\nu}\bar\rho^{\gamma}+(1+t)^{\kappa}+\bar\rho^{-\beta}\right]x^2\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right](x,t)dx\\
&+(1+t)^{2\nu}\int x^2\bar\rho v^2(x,t)dx+\int_0^t\int\left[(1+s)^{\kappa}+\bar\rho^{-\beta}\right]x^2\bar\rho^{\gamma}\\
&\times\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dxds+\int_0^t\int\left[(1+s)^{2\nu}+(1+s)^{\nu}\bar\rho^{-\beta}\right]\left(x^2v_x^2+v^2\right)dxds\\
&\le C\left(\left\|x\bar\rho^{\frac12}v(\cdot,0)\right\|^2+\|r_{0x}-1\|_{L^\infty}^2\right)+C\sum_{i=1}^3\int_0^t(1+s)^{\nu}|K_i|ds+C\sum_{i=1}^3\int_0^t|L_i|ds,
\end{aligned}\tag{3.4.46}$$

where L_i and K_i ($i=1,2,3$) are given by (3.4.42) and (3.4.43), respectively.

To prove (3.4.46), one can integrate the product of $(1+t)^{2\nu}$ and (3.4.8) with respect to the temporal variable to get

$$\begin{aligned}
&(1+t)^{2\nu}\int\left\{x^2\bar\rho v^2+x^2\bar\rho^{\gamma}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]\right\}(x,t)dx\\
&+\int_0^t(1+s)^{2\nu}\int\left(x^2v_x^2+v^2\right)dxds\le C\left\|x\bar\rho^{\frac12}v(\cdot,0)\right\|^2+C\|r_{0x}-1\|_{L^\infty}^2\\
&+C\int_0^t(1+s)\int v^2dxds+C\int_0^t(1+s)^{\frac{\beta}{\gamma}-\iota}\int\ x^2\bar\rho^{\gamma}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dxds,
\end{aligned}$$

since $\beta/\gamma<1$. Integrate the product of $(1+t)^{\kappa}$ and (3.4.16) with respect to the temporal variable to give

$$\begin{aligned}
&(1+t)^{\kappa}\int x^2\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right](x,t)dx\\
&+\int_0^t(1+s)^{\kappa}\int x^2\bar\rho^{\gamma}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dxds\\
&\le C\|r_{0x}-1\|_{L^\infty}^2+C\int(1+s)\int v^2dxds\\
&+C\int(1+s)^{\frac{\beta}{\gamma}-\iota-1}\int x^2\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right](x,t)dxds.
\end{aligned}$$

The last term on the right-hand side of the inequality above is estimated as follows. It follows from the Hölder inequality and the Young inequality that

$$\begin{aligned}
\int x^2\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dx\le&\left(\int x^2\bar\rho^{\gamma-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dx\right)^{\frac{\beta}{\gamma}}\\
&\times\left(\int x^2\bar\rho^{-\beta}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dx\right)^{\frac{\gamma-\beta}{\gamma}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t (1+s)^{\beta/\gamma-\iota-1} \int x^2 \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] dxds \\
&\le C\int_0^t \int x^2 \bar{\rho}^{\gamma-\beta} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] dxds \\
&+ C\int_0^t (1+s)^{(\beta/\gamma-\iota-1)\frac{\gamma}{\gamma-\beta}} ds \sup_{s\in[0,t]} \int x^2 \bar{\rho}^{-\beta} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right](x,s)dx \\
&\le C\frac{\gamma-\beta}{\iota\gamma} \sup_{s\in[0,t]} \int x^2 \bar{\rho}^{-\beta} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right](x,s)dx \\
&+ C\int_0^t \int x^2 \bar{\rho}^{\gamma-\beta} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] dxds.
\end{aligned}$$

In a similar way as to deriving (3.4.41), we then have, noting (3.4.11), that

$$\begin{aligned}
&(1+t)^{2\nu} \int \left\{ x^2\bar{\rho}v^2 + x^2\bar{\rho}^{\gamma} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]\right\}(x,t)dx \\
&+ (1+t)^{\kappa} \int x^2 \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right](x,t)dx + \int_0^t (1+s)^{2\nu} \int \left(x^2v_x^2+v^2\right) dxds \\
&+ \int_0^t (1+s)^{\kappa} \int x^2\bar{\rho}^{\gamma} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] dxds \le C\left\|x\bar{\rho}^{\frac{1}{2}}v(\cdot,0)\right\|^2 \\
&+ C\left\|r_{0x}-1\right\|_{L^\infty}^2 + \sup_{s\in[0,t]} \int x^2\bar{\rho}^{-\beta} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right](x,s)dx \\
&+ C\int_0^t \int x^2 \bar{\rho}^{\gamma-\beta} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] dxds.
\end{aligned} \tag{3.4.47}$$

Make a summation of $k \times$ (3.4.41) and (3.4.47) with suitable large k to give (3.4.46).

Step 3. We claim that

$$\begin{aligned}
&\int \left[(1+t)^{\frac{2\gamma-1}{\gamma}-\theta}\bar{\rho}^{\gamma} + (1+t)^{\frac{\gamma-1}{\gamma}-\theta} + \bar{\rho}^{-\left(\gamma-1-\frac{1}{2}\gamma\theta\right)}\right] x^2 \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right](x,t)dx \\
&+ \int_0^t \int \left[(1+s)^{\frac{2\gamma-1}{\gamma}-\theta} + (1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}}\bar{\rho}^{-\left(\gamma-1-\frac{1}{2}\gamma\theta\right)}\right] \left(x^2v_x^2+v^2\right) dxds \\
&+ \int_0^t \int \left[(1+s)^{\frac{\gamma-1}{\gamma}-\theta} + \bar{\rho}^{-\left(\gamma-1-\frac{1}{2}\gamma\theta\right)}\right] x^2\bar{\rho}^{\gamma} \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] dxds \\
&+ (1+t)^{\frac{2\gamma-1}{\gamma}-\theta} \int x^2\bar{\rho}v^2(x,t)dx \le CQ(0),
\end{aligned} \tag{3.4.48}$$

where and in the following

$$Q(0) := \left\|\left(v, xv_x, x\bar{\rho}^{-\frac{1}{2}}\mathcal{G}_{xt}, x\bar{\rho}^{\gamma-\frac{1}{2}}\mathcal{G}_x\right)(\cdot,0)\right\|^2 + \left\|r_{0x}-1\right\|_{L^\infty}^2.$$

To prove this claim, it remains to estimate K_i and L_i in (3.4.46). First, it follows from (2.1.3) that for any given constants $\delta \in (0,1]$ and $\beta \in (0, \gamma-1)$,

$$\begin{aligned}\int_{1-\delta}^1 \left(\int_0^x \bar\rho^{-\beta} dy\right) dx \le & C \int_{1-\delta}^1 \left(1-\frac{\beta}{\gamma-1}\right)^{-1} \left[1-(1-x)^{1-\frac{\beta}{\gamma-1}}\right] dx \\ \le & C\frac{\gamma-1}{(\gamma-1)-\beta}\delta = C\left[(\gamma-1)-\beta\right]^{-1}\delta.\end{aligned} \tag{3.4.49}$$

Let $\omega \in (0,1/2)$ be a small constant to be determined at the end of this step. It follows from the Cauchy inequality, (3.4.27), the Hölder inequality and (3.4.49) that

$$\begin{aligned}&\int_0^t (1+s)^\nu |K_1| ds \\ \le & C\omega^{-1}\int_0^t (1+s)\int v_s^2 dxds + \omega\int_0^t (1+s)^\nu \int\left(\int_0^x \bar\rho^{-\beta} y(|v|+|yv_y|)dy\right)^2 dxds, \\ \le & C\omega^{-1}Q(0) + C\omega\int_0^t (1+s)^\nu \int_0^1 \bar\rho^{-\beta}(|v|^2+|yv_y|^2)dyds,\end{aligned} \tag{3.4.50}$$

since

$$\begin{aligned}\int\left(\int_0^x \bar\rho^{-\beta} y(|v|+|yv_y|)dy\right)^2 dx \le & \int_0^1 \bar\rho^{-\beta}(|v|^2+|yv_y|^2)dy \int\left(\int_0^x \bar\rho^{-\beta}y^2 dy\right)dx \\ \le & C\int_0^1 \bar\rho^{-\beta}(|v|^2+|yv_y|^2)dy.\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}&\int_0^t |L_1| ds \\ \le & C\omega^{-1}\int_0^t\int v_s^2 dxds + \omega\int_0^t\int \bar\rho^2\left(\int_0^x \bar\rho^{-\beta}y^2\left(\left|\frac{r}{y}-1\right| + |r_y-1|\right)dy\right)^2 dxds \\ \le & C\omega^{-1}Q(0) + C\omega\int_0^t\int x^2\bar\rho^{\gamma-\beta}\left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]dxds,\end{aligned} \tag{3.4.51}$$

since

$$\int \bar\rho^2\left(\int_0^x \bar\rho^{-\gamma-\beta}dy\right)dx \le C\int \bar\rho^2\bar\rho^{-\gamma-\beta+(\gamma-1)}dx = C\int \bar\rho^{2-\gamma}\bar\rho^{-\beta+(\gamma-1)}dx \le C.$$

K_2 can be rewritten as

$$\begin{aligned}K_2 = & \int_0^{1-\omega} \bar\rho^\gamma \left(\frac{x^4}{r^4}\right)_x \int_0^x \left(\bar\rho^{-\beta}\right)_y r^2 v dydx \\ & + \int_{1-\omega}^1 \bar\rho^\gamma\left(\frac{x^4}{r^4}\right)_x\left[\bar\rho^{-\beta}r^2 v - \int_0^x \bar\rho^{-\beta}(r^2 v)_y dy\right]dx =: K_{21}+K_{22}.\end{aligned}$$

Note that

$$
\begin{aligned}
|K_{21}| = & \left| \frac{\beta}{\gamma} \int_0^{1-\omega} \bar{\rho}^{\gamma} \left(\frac{x^4}{r^4} \right)_x \int_0^x \bar{\rho}^{-\beta-(\gamma-1)} y \phi r^2 v dy dx \right| \\
\leq & C \int_0^{1-\omega} \bar{\rho}^{\gamma} x \left(|r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) x^{-2} \left(\int_0^x y^6 dy \right)^{1/2} dx \left(\int_0^1 v^2 dy \right)^{1/2} \\
\leq & C \left(\int x^2 \bar{\rho}^{\gamma} \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx \right)^{1/2} \left(\int_0^1 v^2 dy \right)^{1/2} \\
\leq & C\omega^{-1}(1+t)^{-\nu} \int x^2 \bar{\rho}^{\gamma} \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx + C\omega(1+t)^{\nu} \int v^2 dx
\end{aligned}
$$

and

$$
\begin{aligned}
& |K_{22}| \leq \int_{1-\omega}^1 \bar{\rho}^{\gamma} \left(\frac{x^4}{r^4} \right)_x \left[\bar{\rho}^{-\beta} r^2 v - \int_0^x \bar{\rho}^{-\beta} (r^2 v)_y dy \right] dx \\
& \leq C \int_{1-\omega}^1 x \bar{\rho}^{\gamma-\beta} \left(|r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| dx \\
& + C \int_{1-\omega}^1 x \bar{\rho}^{\gamma} \left(|r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) \left(\int_0^x \bar{\rho}^{-\beta} \left(|v| + |y v_y| \right) dy \right) dx \\
& \leq C\omega^{\frac{\gamma-\beta}{2(\gamma-1)}} \left[(1+t)^{-\nu} \int_{1-\omega}^1 x^2 \bar{\rho}^{\gamma-\beta} \left(|r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx + (1+t)^{\nu} \int v^2 dx \right] \\
& + C\omega^{\frac{\gamma}{4(\gamma-1)}} \left[(1+t)^{-\nu} \int_{1-\omega}^1 x^2 \bar{\rho}^{\gamma-\beta} \left(|r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx + (1+t)^{\nu} \int_0^1 (v^2 + x^2 v_x^2) dx \right],
\end{aligned}
$$

due to

$$
\begin{aligned}
& \int_{1-\omega}^1 \bar{\rho}^{\frac{\gamma}{2}+\beta} \left(\int_0^x \bar{\rho}^{-\beta} (|v|^2 + |y v_y|^2) dy \right)^2 dx \\
\leq & \left(\int_{1-\omega}^1 \bar{\rho}^{\frac{\gamma}{2}+\beta} \int_0^x \bar{\rho}^{-2\beta} dy dx \right) \int_0^1 (|v|^2 + |y v_y|^2) dy \leq C \int_0^1 (|v|^2 + |y v_y|^2) dy.
\end{aligned}
$$

Then, one gets, using (3.4.10), that

$$
\begin{aligned}
& \int_0^t (1+s)^{\nu} |K_2| ds \leq C\omega^{-1} Q(0) + C \left(\omega + \omega^{\frac{\gamma}{4(\gamma-1)}} + \omega^{\frac{\gamma-\beta}{2(\gamma-1)}} \right) \\
& \times \left[\int_0^t (1+s)^{2\nu} \int_0^1 (v^2 + x^2 v_x^2) dx ds + \int_0^t \int x^2 \bar{\rho}^{\gamma-\beta} \left(|r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx ds \right].
\end{aligned}
\tag{3.4.52}
$$

Similarly, L_2 can be rewritten as

$$
\begin{aligned}
L_2 = & \int_0^{1-\omega} \bar{\rho}^{\gamma} \left(\frac{x^4}{r^4} \right)_x \int_0^x \left(\bar{\rho}^{-\beta} \right)_y (r^3 - y^3) dy dx \\
& + \int_{1-\omega}^1 \bar{\rho}^{\gamma} \left(\frac{x^4}{r^4} \right)_x \left[\bar{\rho}^{-\beta} (r^3 - x^3) - \int_0^x \bar{\rho}^{-\beta} (r^3 - y^3)_y dy \right] dx =: L_{21} + L_{22}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|L_{21}| = & \left| \frac{\beta}{\gamma} \int_0^{1-\omega} \bar\rho^\gamma \left(\frac{x^4}{r^4} \right)_x \int_0^x \bar\rho^{-\beta-(\gamma-1)} y \phi (r^3 - y^3) dy dx \right| \\
\le & C \int_0^{1-\omega} \bar\rho^\gamma x \left(|r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) x^{-2} \left(\int_0^x \bar\rho^{-\beta-(\gamma-1)} y^2 |r - y| dy \right) dx \\
\le & C(\omega) \int x^2 \bar\rho^\gamma \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx
\end{aligned}
$$

and

$$
\begin{aligned}
|L_{22}| \le & C \int_{1-\omega}^1 x \bar\rho^{\gamma-\beta} \left(|r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |r - x| dx \\
& + C \int_{1-\omega}^1 x \bar\rho^\gamma \left(|r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) \left(\int_0^x \bar\rho^{-\beta} y^2 \left(|r_y - 1| + \left| \frac{r}{y} - 1 \right| \right) dy \right) dx \\
\le & \omega \int x^2 \bar\rho^{\gamma-\beta} \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx + C\omega^{-1} \int_{1-\omega}^1 \bar\rho^{\gamma-\beta} (r - x)^2 dx \\
& + C \left(\int x^2 \bar\rho^{\gamma-\beta} \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx \right)^{1/2} \\
& \times \left(\int_{1-\omega}^1 \bar\rho^{\gamma+\beta} \left(\int_0^x \bar\rho^{-\beta} y^2 \left(|r_y - 1| + \left| \frac{r}{y} - 1 \right| \right) dy \right)^2 dx \right)^{1/2} \\
\le & C \int \left(\omega \bar\rho^{\gamma-\beta} + \omega^{-1} \bar\rho^\gamma \right) x^2 \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx,
\end{aligned}
$$

where we have used the following simple estimates due to (3.3.2) and (2.1.3):

$$
\begin{aligned}
\int_{1-\omega}^1 \bar\rho^{\gamma-\beta} (r - x)^2 dx \le & C \int_{1/2}^1 \bar\rho^{\gamma-\beta+2(\gamma-1)} \left[(r - x)^2 + (r_x - 1)^2 \right] \\
\le & C \int_{1/2}^1 \bar\rho^\gamma \left[(r - x)^2 + x^2 (r_x - 1)^2 \right]
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{1-\omega}^1 \bar\rho^{\gamma+\beta} \left(\int_0^x \bar\rho^{-\beta} y^2 \left(|r_y - 1| + \left| \frac{r}{y} - 1 \right| \right) dy \right)^2 dx \\
\le & \, C \int_{1-\omega}^1 \bar\rho^{\gamma+\beta} \int_0^x \bar\rho^{-2\beta-\gamma}(y) dy \int_0^x \bar\rho^\gamma y^2 \left(|r_y - 1|^2 + \left| \frac{r}{y} - 1 \right|^2 \right) dy dx \\
\le & \, C \int_{1-\omega}^1 \bar\rho^{-\beta} dx \int_0^1 x^2 \bar\rho^\gamma \left(|r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx \\
\le & \, C \int_0^1 x^2 \bar\rho^\gamma \left(|r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx.
\end{aligned}
$$

Thus, it follows from these and (3.4.10) that

$$\int_0^t |L_2| ds \le C(\omega)Q(0) + C\omega \int_0^t \int x^2 \bar{\rho}^{\gamma-\beta} \left(|r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dxds. \tag{3.4.53}$$

Rewrite K_3 as

$$\begin{aligned} K_3 = &- 4\lambda_1 \int_0^{1-\omega} \left(\frac{v}{r}\right)_x \int_0^x \left(\bar{\rho}^{-\beta}\right)_y (r^2 v) dydx \\ &- 4\lambda_1 \int_{1-\omega}^1 \left(\frac{v}{r}\right)_x \left[\bar{\rho}^{-\beta} r^2 v - \left(\int_0^x \bar{\rho}^{-\beta} (r^2 v)_y dy \right) \right] dx =: K_{31} + K_{32}. \end{aligned}$$

K_{31} can be bounded by

$$\begin{aligned} |K_{31}| = &\left| 4\frac{\beta}{\gamma}\lambda_1 \int_0^{1-\omega} \left(\frac{v}{r}\right)_x \int_0^x \bar{\rho}^{-\beta-(\gamma-1)} y\phi r^2 v dydx \right| \\ \le &C \int_0^{1-\omega} (|xv_x| + |v|) x^{-2} \left(\int_0^x y^3 |v| dy \right) dx \\ \le &C \int_0^{1-\omega} (|xv_x| + |v|) \left(\int_0^1 v^2 dy \right)^{1/2} dx \le C \int (v^2 + x^2 v_x^2) dx, \end{aligned}$$

and K_{32} can be bounded by

$$\begin{aligned} |K_{32}| \le &C \int_{1-\omega}^1 \bar{\rho}^{-\beta} (|xv_x| + |v|) |v| dx + \int_{1-\omega}^1 (|xv_x| + |v|) \left(\int_0^x \bar{\rho}^{-\beta} \left(|v| + |yv_y|\right) dy \right) dx \\ \le &\omega \int_{1-\omega}^1 \bar{\rho}^{-\beta} (|xv_x|^2 + |v|^2) dx + \omega^{-1} \int_{1-\omega}^1 \bar{\rho}^{-\beta} |v|^2 dx \\ &+ \omega^{-1} \int_{1-\omega}^1 (|xv_x|^2 + |v|^2) dx + \omega \int_0^1 \bar{\rho}^{-\beta} (|v|^2 + |yv_y|^2) dy, \end{aligned}$$

due to (3.4.49). Since $\beta < \gamma - 1$, the Hardy inequality (3.3.2) implies that

$$\int_{1-\omega}^1 \bar{\rho}^{-\beta} |v|^2 dx \le C \int_{1/2}^1 \bar{\rho}^{-\beta+2(\gamma-1)} (v^2 + v_x^2) dx \le C \int (v^2 + v_x^2) dx.$$

These, together with (3.4.11), yield

$$\begin{aligned} &\int_0^t (1+s)^\nu |K_3| ds \\ \le &C\omega^{-1} \int_0^t (1+s) \int (x^2 v_x^2 + v^2) dxds + C\omega \int_0^t (1+s)^\nu \int_{1-\omega}^1 \bar{\rho}^{-\beta} (|xv_x|^2 + |v|^2) dxds \quad (3.4.54) \\ \le &C(\omega)Q(0) + C\omega \int_0^t (1+s)^\nu \int_{1-\omega}^1 \bar{\rho}^{-\beta} (|xv_x|^2 + |v|^2) dxds. \end{aligned}$$

Similarly, L_3 is rewritten as

$$
\begin{aligned}
L_3 = & -4\lambda_1 \int_0^{1-\omega} \left(\frac{v}{r}\right)_x \int_0^x \left(\bar{\rho}^{-\beta}\right)_y (r^3 - y^3) dy dx \\
& -4\lambda_1 \int_{1-\omega}^1 \left(\frac{v}{r}\right)_x \left[\bar{\rho}^{-\beta}(r^3 - x^3) - \left(\int_0^x \bar{\rho}^{-\beta}(r^3 - y^3)_y dy\right)\right] dx =: L_{31} + L_{32}.
\end{aligned}
$$

Clearly, L_{31} and L_{32} can be bounded by

$$
|L_{31}| \leq C \int (v^2 + x^2 v_x^2) dx + C \int \bar{\rho}^{\gamma} (r - x)^2 dx.
$$

and

$$
|L_{32}| \leq C \int_{1-\omega}^1 \bar{\rho}^{-\beta} \left(|x v_x| + |v|\right) |r - x| dx + C \int_{1-\omega}^1 \left(|x v_x| + |v|\right) \left(\int_0^x \bar{\rho}^{-\beta} y |r_y - 1| dy\right) dx.
$$

The second term in L_{32} is bounded by

$$
\begin{aligned}
& \int_{1-\omega}^1 \left(|x v_x| + |v|\right) \left(\int_0^x \bar{\rho}^{-\beta} y |r_y - 1| dy\right) dx \\
\leq & C \omega^{\frac{1}{2}} (1+t)^{2\nu} \int_{1-\omega}^1 (|x v_x|^2 + |v|^2) dx \\
& + C \omega^{-\frac{1}{2}} (1+t)^{-1-\frac{\beta}{\gamma}+\iota} \int_0^1 \bar{\rho}^{-\beta} y^2 (r_y - 1)^2 dy \int_{1-\omega}^1 \int_0^x \bar{\rho}^{-\beta} dy dx \\
\leq & C \omega^{\frac{1}{2}} \left[(1+t)^{2\nu} \int_{1-\omega}^1 (|x v_x|^2 + |v|^2) dx + (1+t)^{-1-\frac{\beta}{\gamma}+\iota} \int_0^1 \bar{\rho}^{-\beta} y^2 (r_y - 1)^2 dy\right],
\end{aligned}
$$

due to (3.4.49). The first term in L_{32} can be bounded as follows. Since $\beta < \gamma - 1$, it follows from (2.1.3) and the Hardy inequality (3.3.2) that for $\gamma > 4/3$,

$$
\begin{aligned}
& \int_{1-\omega}^1 \bar{\rho}^{-\beta} \left(|x v_x| + |v|\right) |r - x| dx \\
\leq & C \omega^{\frac{h}{2(\gamma-1)}} \left[(1+t)^{\nu} \int_{1-\omega}^1 \bar{\rho}^{-\beta} \left(v_x^2 + v^2\right) dx + (1+t)^{-\nu} \int_{1-\omega}^1 \bar{\rho}^{-\beta-h} |r - x|^2 dx\right] \\
\leq & C \omega^{\frac{h}{2(\gamma-1)}} \left[(1+t)^{\nu} \int_{1-\omega}^1 \bar{\rho}^{-\beta} \left(v_x^2 + v^2\right) dx + \int_{1/2}^1 \bar{\rho}^{\gamma-\beta} \left[(r - x)^2 + x^2 (r_x - 1)^2\right] dx \right. \\
& \left. + (1+t)^{-\frac{\nu\gamma}{h+2-\gamma}} \int_{1/2}^1 \bar{\rho}^{-\beta} \left[(r - x)^2 + x^2 (r_x - 1)^2\right] dx\right],
\end{aligned}
$$

where $h = \min\{\beta/8, \quad (\gamma-1-\beta)/4\}$ (it should be noted that $\beta + h < \gamma - 1$), and we have used the estimate

$$\int_{1-\omega}^{1} \bar\rho^{-\beta-h}|r-x|^2 dx \le \int_{1/2}^{1} \bar\rho^{2(\gamma-1)-\beta-h}\left[|r-x|^2+(r_x-1)^2\right]dx$$
$$\le \left(\int_{1/2}^{1}\bar\rho^{-\beta}\left[(r-x)^2+x^2(r_x-1)^2\right]dx\right)^{\frac{h+2-\gamma}{\gamma}}$$
$$\times\left(\int_{1/2}^{1}\bar\rho^{\gamma-\beta}\left[(r-x)^2+x^2(r_x-1)^2\right]dx\right)^{\frac{2(\gamma-1)-h}{\gamma}}.$$

Consequently, taking into account of (3.4.1), (3.4.10) and (3.4.49), one gets that

$$\begin{aligned}\int_0^t |L_3|ds \le & C(\omega)Q(0) + C\left[\omega^{\frac12}+\omega^{\frac{h}{2(\gamma-1)}}\right]\left[\int_0^t\int\left[(1+s)^{2\nu}+(1+s)^{\nu}\bar\rho^{-\beta}\right]\right.\\ & \times(|xv_x|^2+|v|^2)dxds+\sup_{[0,t]}\int\bar\rho^{-\beta}\left[(r-x)^2+x^2(r_x-1)^2\right]dx \\ & \left.+\int_0^t\int\bar\rho^{\gamma-\beta}\left[(r-x)^2+x^2(r_x-1)^2\right]dx\right].\end{aligned} \tag{3.4.55}$$

We finally derive from (3.4.46) and (3.4.50)-(3.4.55), by choosing $\omega\in(0,1/2)$ suitably small, that

$$\begin{aligned}&\int\left[(1+t)^{2\nu}\bar\rho^{\gamma}+(1+t)^{\kappa}+\bar\rho^{-\beta}\right]x^2\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right](x,t)dx\\ &+(1+t)^{2\nu}\int x^2\bar\rho v^2(x,t)dx+\int_0^t\int\left[(1+s)^{\kappa}+\bar\rho^{-\beta}\right]x^2\bar\rho^{\gamma}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dxds\\ &+\int_0^t\int\left[(1+s)^{2\nu}+(1+s)^{\nu}\bar\rho^{-\beta}\right]\left(x^2v_x^2+v^2\right)dxds\le CQ(0).\end{aligned}$$

Due to (3.4.40), this completes the proof of (3.4.48).

Step 5. Multiply equation (3.4.35) by $(1+t)^{\frac{2\gamma-1}{\gamma}-\theta}$ and integrate the product to deduce that

$$(1+t)^{\frac{2\gamma-1}{\gamma}-\theta}\int\left[\bar\rho v_t^2+\bar\rho^{\gamma}\left(v^2+x^2v_x^2\right)\right]dx+\int_0^t(1+s)^{\frac{2\gamma-1}{\gamma}-\theta}\int\left(x^2v_{sx}^2+v_s^2\right)dxds\le CQ(0).$$

In a similar way to the derivation of (3.4.21) and (3.4.38), one can show

$$\int(r-x)^2(x,t)dx\le C(1+t)^{-\frac{3(\gamma-1)}{\gamma}+\theta}Q(0)$$

and

$$\int\left(v^2+x^2v_x^2\right)(x,t)dx\le C(1+t)^{-\frac{2\gamma-1}{\gamma}+\theta}Q(0).\quad\square$$

3.5 Higher-order estimates

3.5.1 Part I: global existence and decay of strong solutions

In this subsubsection, we prove the global existence and large time decay of the strong solution for suitably small $\mathfrak{E}(0)$.

Lemma 3.17 *Suppose that* (3.2.2) *and* (3.2.3) *hold. Then there exist positive constants* C *and* $C(\theta)$ *independent of* t *such that for any* $\theta \in (0,\ 2(\gamma-1)/(3\gamma))$,

$$\begin{aligned}&\left\|\left(\bar\rho^{\gamma-\frac12}\mathcal{G}_x, \bar\rho^{-\frac12}\mathcal{G}_{xt}\right)(\cdot,t)\right\|^2+\int_0^t\left\|\left(\bar\rho^{\frac{3\gamma-1}{2}}\mathcal{G}_x, \bar\rho^{\frac{\gamma-1}{2}}\mathcal{G}_{xs}, v_x, \frac{v}{x}\right)(\cdot,s)\right\|^2 ds\\&\le C\left(\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right),\quad t\in[0,T],\end{aligned} \tag{3.5.1}$$

$$\begin{aligned}&(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\left(\left\|\bar\rho^{\frac12}v_t(\cdot,t)\right\|^2+\|(\mathcal{G}_x,\mathcal{G}_{xt})(\cdot,t)\|^2_{L^2([0,\frac12])}\right)\\&+\int_0^t(1+s)^{\frac{\gamma-1}{\gamma}-\theta}\left[\left\|\left(v_{sx},\frac{v_s}{x}\right)(\cdot,s)\right\|^2+\left\|\left(\mathcal{G}_x,\mathcal{G}_{xs},v_x,\frac{v}{x}\right)(\cdot,s)\right\|^2_{L^2([0,\frac12])}\right]ds\\&\le C(\theta)\left(\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right),\quad t\in[0,T].\end{aligned} \tag{3.5.2}$$

Proof. We show the weighted estimate for $\mathcal{G}_x$ in the first step. With the space-time estimate of $\mathcal{G}_x$ at hand, the estimate for v_t away from the boundary can be proved. Finally, we use equation (3.2.7) and these estimates to obtain the L^2-estimates for $\mathcal{G}_{xt}$.

Step 1. We claim that

$$\int\bar\rho^{2\gamma-1}\mathcal{G}_x^2dx+\int_0^t\int\left(\bar\rho^{3\gamma-1}\mathcal{G}_x^2+\bar\rho^{\gamma-1}\mathcal{G}_{xs}^2+v_x^2+(v/x)^2\right)dxds\le C\mathcal{E}(0). \tag{3.5.3}$$

Multiplying equation (3.2.7) by $\bar\rho^{2\gamma-1}\mathcal{G}_x$ and integrating the product with respect to the spatial variable, one gets, using the Cauchy-Schwarz inequality, that

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int\bar\rho^{2\gamma-1}\mathcal{G}_x^2dx+\frac{\gamma}{2}\int\left(\frac{x^2}{r^2r_x}\right)^\gamma\bar\rho^{3\gamma-1}\mathcal{G}_x^2dx\\\le&C\int v_t^2dx+C\int x^2\bar\rho^\gamma\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dx.\end{aligned} \tag{3.5.4}$$

It follows from this, (3.4.10) and (3.4.27) that

$$\int\bar\rho^{2\gamma-1}\mathcal{G}_x^2dx+\int_0^t\int\bar\rho^{3\gamma-1}\mathcal{G}_x^2dxds\le C\mathcal{E}(0). \tag{3.5.5}$$

Multiplying the square of (3.2.7) by $\bar\rho^{\gamma-1}$ and integrating the product with respect to the spatial and temporal variables, we have, using (3.5.5), (3.4.10) and (3.4.27), that

$$\begin{aligned}\int_0^t\int\bar\rho^{\gamma-1}\mathcal{G}_{xs}^2dxds\le&C\int_0^t\int\bar\rho^{3\gamma-1}\mathcal{G}_x^2dxds+C\int_0^t\int v_s^2dxds\\&+C\int_0^t\int x^2\bar\rho^\gamma\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dxds\le C\mathcal{E}(0);\end{aligned} \tag{3.5.6}$$

which, together with (3.3.7) and (3.4.1), gives that

$$\int_0^t \int \left(v_x^2 + |v/x|^2\right) dxds \le c \int_0^t \int \left(\mathcal{G}_s^2 + v^2\right) dxds \le C\mathcal{E}(0), \tag{3.5.7}$$

where we have used the following estimate due to Lemma 3.4,

$$\int \mathcal{G}_t^2 dx \le C \int \left(x\bar{\rho}^{\gamma-1}\right)^2 \left(\mathcal{G}_t^2 + \mathcal{G}_{xt}^2\right) dx \le C \int (v^2 + x^2 v_x^2) dx + C \int x^2 \bar{\rho}^{2(\gamma-1)} \mathcal{G}_{xt}^2 dx. \tag{3.5.8}$$

So, the claim (3.5.3) follows from (3.5.5)-(3.5.7).

Step 2. In this step, we prove that

$$\begin{aligned} &\int_0^t (1+s)^{\frac{\gamma-1}{\gamma}-\theta} \left[\int_0^{3/4} \left(\mathcal{G}_x^2 + \mathcal{G}_{xs}^2\right) dx + \int_0^{1/2} \left(v_x^2 + |v/x|^2\right) dx\right] ds \\ &+ (1+t)^{\frac{\gamma-1}{\gamma}-\theta} \int_0^{3/4} \mathcal{G}_x^2 dx \le C\left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2\right]. \end{aligned} \tag{3.5.9}$$

Let $\bar{\psi}$ be a non-increasing function defined on $[0,1]$ satisfying

$$\bar{\psi} = 1 \ \text{ on } \ [0, 3/4], \quad \bar{\psi} = 0 \ \text{ on } \ [7/8, 1] \ \text{ and } \ |\bar{\psi}'| \le 32. \tag{3.5.10}$$

In a similar way to the derivation of (3.5.4), we can get

$$\begin{aligned} &\frac{\mu}{2}\frac{d}{dt} \int \bar{\psi} \bar{\rho}^{2\gamma-1} \mathcal{G}_x^2 dx + \frac{\gamma}{2} \int \bar{\psi} \left(\frac{x^2}{r^2 r_x}\right)^\gamma \bar{\rho}^{3\gamma-1} \mathcal{G}_x^2 dx \\ \le& C \int v_t^2 dx + C \int x^2 \bar{\rho}^\gamma \left[\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2\right] dx. \end{aligned}$$

Multiplying the above by $(1+t)^{(\gamma-1)/\gamma-\theta}$ and integrating the product, we have, using (3.4.39) and (3.5.5), that

$$\begin{aligned} &(1+t)^{\frac{\gamma-1}{\gamma}-\theta} \int \bar{\psi} \mathcal{G}_x^2 dx + \int_0^t (1+s)^{\frac{\gamma-1}{\gamma}-\theta} \int \bar{\psi} \mathcal{G}_x^2 dxds \\ \le& C\left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2\right] + C \int_0^t \int \bar{\psi} \bar{\rho}^{3\gamma-2} \mathcal{G}_x^2 dxds \le C\left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2\right]; \end{aligned}$$

which implies

$$(1+t)^{\frac{\gamma-1}{\gamma}-\theta} \int_0^{3/4} \mathcal{G}_x^2 dx + \int_0^t (1+s)^{\frac{\gamma-1}{\gamma}-\theta} \int_0^{3/4} \mathcal{G}_x^2 dxds \le C\left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2\right]. \tag{3.5.11}$$

Following the arguments for (3.5.6) in Step 1 by squaring (3.2.7), one can obtain that

$$\int_0^t (1+s)^{\frac{\gamma-1}{\gamma}-\theta} \int_0^{3/4} \mathcal{G}_{xs}^2 dxds \le C\left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2\right]. \tag{3.5.12}$$

Similar to (3.3.11) and (3.5.8), it holds that

$$\begin{aligned}\int_0^t (1+s)^{\frac{\gamma-1}{\gamma}-\theta}\int_0^{1/2}\left(v_x^2+(v/x)^2\right)dxds \le& 4\int_0^t (1+s)^{\frac{\gamma-1}{\gamma}-\theta}\int_0^{3/4}\mathcal{G}_s^2 dxds\\ \le& C\left[\mathcal{E}(0)+\|r_{0x}-1\|_{L^\infty}^2\right].\end{aligned}$$

This, together with (3.5.11) and (3.5.12), gives (3.5.9).

Step 3. In this step, we show that

$$\begin{aligned}&(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\left(\int\bar\rho v_t^2dx+\int_0^{3/4}\mathcal{G}_{xt}^2(x,t)dx\right)\\ &+\int_0^t\int(1+s)^{\frac{\gamma-1}{\gamma}-\theta}\left(v_{xt}^2+\frac{v_t^2}{x^2}\right)dxds\le C(\theta)\left[\mathcal{E}(0)+\|r_{0x}-1\|_{L^\infty}^2\right].\end{aligned}\tag{3.5.13}$$

Differentiating (3.2.6) with respect to t yields

$$\begin{aligned}&\bar\rho\left(\frac{x}{r}\right)^2v_{tt}-2\bar\rho\left(\frac{x}{r}\right)^3\frac{v}{x}v_t-\gamma\left[\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma\left(2\frac{v}{r}+\frac{v_x}{r_x}\right)\right]_x+4\left(\frac{x}{r}\right)^5\frac{v}{x}\left(\bar\rho^\gamma\right)_x\\ =&\mu\left(\frac{v_{xt}}{r_x}+2\frac{v_t}{r}\right)_x-\mu\left(\frac{v_x^2}{r_x^2}+2\frac{v^2}{r^2}\right)_x.\end{aligned}\tag{3.5.14}$$

Let ψ be a non-increasing function defined on $[0,1]$ satisfying

$$\psi=1\ \text{ on }\ [0,1/8],\quad \psi=0\ \text{ on }\ [1/2,1]\ \text{ and }\ |\psi'|\le 32.$$

Multiplying equation (3.5.14) by ψv_t and integrating the product with respect to the spatial variable, one has, using the integration by parts and boundary condition (2.1.1), that

$$\frac{d}{dt}\int\frac12\bar\rho\psi\left(\frac{x}{r}\right)^2v_t^2dx+\mu\int\left(\frac{v_{xt}}{r_x}+2\frac{v_t}{r}\right)(\psi v_t)_x\,dx=J_1+J_2+J_3,\tag{3.5.15}$$

where

$$\begin{aligned}J_1:=&\int\frac{v}{r}\bar\rho\psi\left(\frac{x}{r}\right)^2v_t^2dx+4\int\left(\frac{x}{r}\right)^5v\phi\bar\rho\psi v_tdx\le C\int_0^{1/2}\left(v^2+v_t^2\right)dx,\\ J_2:=&-\gamma\int\left(\frac{x^2}{r^2}\frac{\bar\rho}{r_x}\right)^\gamma\left(2\frac{v}{r}+\frac{v_x}{r_x}\right)(\psi v_t)_x\,dx,\\ J_3:=&\mu\int\left[\frac{v_x^2}{r_x^2}+2\frac{v^2}{r^2}\right](\psi v_t)_x\,dx.\end{aligned}$$

The 2nd term on the left-hand side of (3.5.15) can be estimated as follows:

$$\begin{aligned}&\int\left(\frac{v_{xt}}{r_x}+2\frac{v_t}{r}\right)(\psi v_t)_x\,dx\\ =&\int\psi\frac{v_{xt}^2}{r_x}dx+2\int\frac{\psi}{r}v_tv_{tx}dx+\int\psi'\left(\frac{v_{xt}}{r_x}+2\frac{v_t}{r}\right)v_tdx\\ \ge&\int\psi\frac{v_{xt}^2}{r_x}dx-\int\left(\frac{\psi}{r}\right)'v_t^2dx-c\int_0^{1/2}\left(x^2v_{xt}^2+v_t^2\right)dx\\ \ge&\int\psi\left[\frac{v_{xt}^2}{r_x}+\frac{r_xv_t^2}{r^2}\right]dx-c\int_0^{1/2}\left(x^2v_{xt}^2+v_t^2\right)dx.\end{aligned}$$

Then,

$$\frac{d}{dt}\int \frac{1}{2}\bar{\rho}\psi\left(\frac{x}{r}\right)^2 v_t^2 dx + \mu\int \psi\left[\frac{v_{xt}^2}{r_x} + \frac{r_x v_t^2}{r^2}\right]dx \le C\int_0^{1/2}\left(x^2 v_{xt}^2 + v_t^2 + v^2\right)dx + J_2 + J_3.$$

It therefore follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\frac{d}{dt}\int \frac{1}{2}\bar{\rho}\psi\left(\frac{x}{r}\right)^2 v_t^2 dx + \frac{\mu}{2}\int \psi\left[\frac{v_{xt}^2}{r_x} + \frac{r_x v_t^2}{r^2}\right]dx\\ \le & C\int_0^{1/2}\left(x^2 v_{xt}^2 + v_t^2 + (v/x)^2 + v_x^2\right)dx.\end{aligned} \tag{3.5.16}$$

This, together with (3.4.27) and (3.5.9), implies that

$$\begin{aligned}&(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\int_0^{1/8}\bar{\rho}v_t^2 dx + \int_0^t\int_0^{1/8}(1+s)^{\frac{\gamma-1}{\gamma}-\theta}\left(v_{xt}^2 + \frac{v_t^2}{x^2}\right)dxds\\ \le & C(\theta)\left[\mathcal{E}(0) + \|r_{0x}-1\|_{L^\infty}^2\right].\end{aligned}$$

Using (3.4.27) again, we conclude that

$$(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\int \bar{\rho}v_t^2 dx + \int_0^t\int(1+s)^{\frac{\gamma-1}{\gamma}-\theta}\left(v_{xt}^2 + \frac{v_t^2}{x^2}\right)dxds \le C(\theta)\left[\mathcal{E}(0) + \|r_{0x}-1\|_{L^\infty}^2\right].$$

It follows from (3.2.7), (3.5.5), (3.4.10), (3.4.39) and (3.5.11) that

$$\begin{aligned}&\int \bar{\rho}^{-1}\mathcal{G}_{xt}^2(x,t)dx \le C\int \bar{\rho}v_t^2(x,t)dx + C\int \bar{\rho}^{2\gamma-1}\mathcal{G}_x^2(x,t)dx\\ &+C\int x^2\bar{\rho}\left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right](x,t)dx \le C\left[\mathcal{E}(0) + \|r_{0x}-1\|_{L^\infty}^2\right]\end{aligned}$$

and

$$(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\int_0^{3/4}\mathcal{G}_{xt}^2(x,t)dx \le C\left[\mathcal{E}(0) + \|r_{0x}-1\|_{L^\infty}^2\right].$$

Collecting all these estimates proves (3.5.13). So, this lemma follows from Steps 1-3. □

Proof of Theorem 2.2. It should be noticed that the uniqueness of strong solutions follows from the same argument as in [12]. So, the global well-posedness theorem, Theorem 2.2, can be shown by Lemma 3.17, together with the equivalence of $\mathcal{E}(t)$ and $\mathfrak{E}(t)$ shown in (3.3.26) and the lower-order estimates obtained in subsection 3.4, through the standard continuation argument . □

To complete the proof of part i) of Theorem 3.1, it suffices to show the following Lemma.

Lemma 3.18 *For the global strong solution obtained in Theorem 2.2, it holds that*

$$
\begin{aligned}
&(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\left(\left\|\left(r_x-1,\ \frac{r}{x}-1\right)(\cdot,t)\right\|^2+\left\|\left(r_x-1,\ \frac{r}{x}-1,\ v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{H^1([0,\frac12])}\right)\\
&+(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}}\|(v,xv_x)(\cdot,t)\|^2_{L^\infty}+(1+t)^{\frac{2(\gamma-1)}{\gamma}-\theta}\|(r-x)(\cdot,t)\|^2_{L^\infty}\\
&+(1+t)^{\frac{2-\gamma}{2\gamma}-\frac{\theta}{4}\frac{\gamma+1-\gamma\theta}{2\gamma-1-\frac{\gamma\theta}{2}}}\|\bar\rho\mathcal{G}(\cdot,t)\|^2_{L^\infty}\le C(\theta)\left[\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right],\quad t\in[0,\infty),
\end{aligned}
\tag{3.5.17}
$$

for any $0<\theta<\min\{2(\gamma-1)/(3\gamma),\ (4-2\gamma)/\gamma\}$.

Proof. First, we prove that

$$
\|\bar\rho\mathcal{G}(\cdot,t)\|^2_{L^\infty}\le C(1+t)^{-\frac{2-\gamma}{2\gamma}+\frac{\theta}{4}\frac{\gamma+1-\gamma\theta}{2\gamma-1-\frac{\gamma\theta}{2}}}\left[\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right]. \tag{3.5.18}
$$

It follows from (3.3.23), the Hölder inequality, (3.5.1), (3.3.3) and (3.4.39) that

$$
\begin{aligned}
&\left\|x^{3/2}\bar\rho\mathcal{G}(\cdot,t)\right\|^2_{L^\infty}+\int x^4\bar\rho^{3-\gamma}\mathcal{G}^2(x,t)dx\le\left\|x\bar\rho^{\gamma/2}\mathcal{G}(\cdot,t)\right\|^2+\left\|x^3\bar\rho^2\mathcal{G}\mathcal{G}_x(\cdot,t)\right\|_{L^1}\\
\le& C\int x^2\bar\rho^\gamma\mathcal{G}^2(x,t)dx+\left(\int x^3\bar\rho^{3-2\gamma}\mathcal{G}^2(x,t)dx\right)^{1/2}\left(\int x^3\bar\rho^{2\gamma-1}\mathcal{G}_x^2(x,t)dx\right)^{1/2}\\
\le& C(1+t)^{-\frac{2-\gamma}{2\gamma}+\frac{\theta}{4}\frac{\gamma+1-\gamma\theta}{2\gamma-1-\frac{\gamma\theta}{2}}}\left[\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right],
\end{aligned}
\tag{3.5.19}
$$

where in the last inequality we have used the following estimate due to (3.3.3), (3.4.39) and $0<\theta<(4-2\gamma)/\gamma$,

$$
\begin{aligned}
\int x^2\bar\rho^{3-2\gamma}\mathcal{G}^2(x,t)dx\le&\left(\int x^2\bar\rho^{\frac{\gamma\theta}{2}-\gamma+1}\mathcal{G}^2(x,t)dx\right)^{\frac{3(\gamma-1)}{2\gamma-1-\frac{\gamma\theta}{2}}}\left(\int x^2\bar\rho^\gamma\mathcal{G}^2(x,t)dx\right)^{\frac{2-\gamma-\frac{\gamma\theta}{2}}{2\gamma-1-\frac{\gamma\theta}{2}}}\\
\le& C(1+t)^{-\frac{2-\gamma}{\gamma}+\frac{\theta}{2}\frac{\gamma+1-\gamma\theta}{2\gamma-1-\frac{\gamma\theta}{2}}}\left[\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right].
\end{aligned}
$$

This gives the desired bound near the boundary. For the origin, it follows from the weighted Sobolev embedding (3.3.1), (3.5.2) and (3.4.11) that

$$
\begin{aligned}
\int_0^{1/2}\mathcal{G}^2(x,t)dx\le& C\int_0^{1/2}x^2\left(\mathcal{G}^2+\mathcal{G}_x^2\right)dx\le C\int_0^{1/2}x^2\left(\bar\rho^\gamma\mathcal{G}^2+\mathcal{G}_x^2\right)dx\\
\le& C(1+t)^{-\frac{\gamma-1}{\gamma}+\theta}\left[\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right];
\end{aligned}
$$

which implies immediately that

$$
\|\mathcal{G}(\cdot,t)\|^2_{L^\infty([0,\frac12])}\le c\int_0^{1/2}\left(\mathcal{G}^2+\mathcal{G}_x^2\right)dx\le C(1+t)^{-\frac{\gamma-1}{\gamma}+\theta}\left[\mathcal{E}(0)+\|r_{0x}-1\|^2_{L^\infty}\right]. \tag{3.5.20}
$$

Then, (3.5.18) follows from (3.5.19) and (3.5.20).

Let $a \in (0,1)$ be a constant and χ be a non-increasing function defined on $[0,1]$ satisfying

$$\chi = 1 \ \text{ on } \ [0, a/2], \quad \psi = 0 \ \text{ on } \ [3a/4, 1] \ \text{ and } \ \chi' \le 0 \ \text{ on } \ [0,1].$$

Following the same argument for (3.3.13), we have

$$\int \chi \left(r_{xx}^2 + |(r/x)_x|^2 \right) dx \le C \int \chi \mathcal{G}_x^2 dx, \tag{3.5.21}$$

where C is a constant independent of a. Then the arguments for (3.5.2) yield

$$\int \chi \left(r_{xx}^2 + |(r/x)_x|^2 \right) dx \le C(a,\theta)(1+t)^{-\frac{\gamma-1}{\gamma}+\theta} \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right], \tag{3.5.22}$$

Since it follows from a similar argument as in the proof of Lemma 3.6 and (3.5.20) that

$$\left\| \left(r_x - 1, \ \frac{r}{x} - 1 \right) (\cdot, t) \right\|_{L^2\left([0, \frac{1}{2}] \right)}^2 \le C(1+t)^{-\frac{\gamma-1}{\gamma}+\theta} \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right]. \tag{3.5.23}$$

Then,

$$\left\| \left(r_x - 1, \ \frac{r}{x} - 1 \right) (\cdot, t) \right\|_{H^1\left([0, \frac{1}{2}] \right)}^2 \le C(1+t)^{-\frac{\gamma-1}{\gamma}+\theta} \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right]. \tag{3.5.24}$$

Recall (3.4.39), which implies that

$$\begin{aligned} &(1+t)^{\frac{\gamma-1}{\gamma}-\theta} \|(x(r_x - 1))(\cdot,t)\|^2 + (1+t)^{\frac{3(\gamma-1)}{\gamma}-\theta} \left\| x \left(\frac{r}{x} - 1 \right) (\cdot, t) \right\|^2 \\ \le & C \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right]. \end{aligned} \tag{3.5.25}$$

It follows from this and (3.5.24) that

$$\left\| \left(r_x - 1, \ \frac{r}{x} - 1 \right) (\cdot, t) \right\|^2 \le C(1+t)^{-\frac{\gamma-1}{\gamma}+\theta} \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right]. \tag{3.5.26}$$

As a consequence of (3.3.22), (3.5.25) and (3.5.26) , one then gets that

$$\|(r-x)(\cdot,t)\|_{L^\infty}^2 \le C(1+t)^{-\frac{2(\gamma-1)}{\gamma}+\theta} \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right].$$

To deal with v, one notes, due to (3.3.25) and Lemma 3.11-Lemma 3.17, that

$$\left\| \left(r_x - 1, \frac{r}{x} - 1, v_x, \frac{v}{x} \right) (\cdot, t) \right\|_{L^\infty}^2 \le C\mathcal{E}(t) \le C \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right];$$

which, together with (3.3.20), (3.3.19) and (3.4.39), yields that

$$\|(v, xv_x)(\cdot,t)\|_{L^\infty}^2 \le C(1+t)^{-\frac{2\gamma-1}{2\gamma}+\frac{\theta}{2}} \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right].$$

In a similar way to the derivation of (3.5.24), we can derive from (3.3.7), (3.3.10) (3.4.39), (3.5.1) and (3.5.2) that

$$\left\| \left(v_x, \ \frac{v}{x} \right) (\cdot, t) \right\|_{H^1\left([0, \frac{1}{2}] \right)}^2 \le C(1+t)^{-\frac{\gamma-1}{\gamma}+\theta} \left[\mathcal{E}(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right]. \ \square$$

3.5.2 Part II: faster decay

In this subsection, we prove part *ii*) of Theorem 3.1 under the assumption

$$\mathfrak{F}(0) < \infty. \tag{3.5.27}$$

Let $\alpha \in [0, \gamma - 1)$ be any fixed constant. For any

$$0 < \theta < \min\{2(\gamma-1)/(3\gamma), \quad 2(\gamma-1-\alpha)/\gamma, \quad (4-2\gamma)/\gamma\},$$

we set

$$\kappa = \alpha/\gamma - \theta \ \text{ when } \ \alpha > 0, \quad \kappa = 0 \ \text{ when } \ \alpha = 0. \tag{3.5.28}$$

The estimates in this subsection are for the global strong solution of (2.1.10) as stated in Theorem 2.2, in which the constant C may depend on α and θ, but not on t.

To obtain the faster time decay estimates of the higher-order norms, we rewrite equation (3.2.7) in the form of

$$\mathfrak{P}(x,t) := \gamma\left(\frac{x^2\bar{\rho}}{r^2 r_x}\right)^{\gamma}\mathcal{G}_x + \left[\left(\frac{x^2}{r^2 r_x}\right)^{\gamma} - \left(\frac{x}{r}\right)^4\right]x\phi\bar{\rho} = \frac{x^2}{r^2}\bar{\rho}v_t - \mu\mathcal{G}_{xt}. \tag{3.5.29}$$

This equation is convenient for us to derive the time decay estimates for $\|x\bar{\rho}^{\gamma/2-1}\mathfrak{P}\|$ in Lemma 3.19, which in turn gives the decay of $\|x\bar{\rho}^{3\gamma/2-1}\mathcal{G}_x\|$.

Lemma 3.19 *Under the same assumptions as in ii) of Theorem 2.3, it holds that*

$$\begin{aligned}
&(1+t)^{\kappa}\left\|\bar{\rho}^{\gamma-1}\mathcal{G}_x(\cdot,t)\right\|^2 + (1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\left\|x\left(\bar{\rho}^{\frac{3}{2}\gamma-1}\mathcal{G}_x,\ \bar{\rho}^{\frac{\gamma}{2}-1}\mathcal{G}_{xt}\right)(\cdot,t)\right\|^2\\
&+\left\|\bar{\rho}^{\gamma-1-\frac{\alpha}{2}}\mathcal{G}_x(\cdot,t)\right\|^2 + \int_0^t (1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\left\|\left(x\bar{\rho}^{\gamma-1}\mathcal{G}_{xt}, v_x, \frac{v}{x}\right)(\cdot,s)\right\|^2 ds\\
&+\int_0^t\left[(1+s)^{\kappa}\left\|\left(\bar{\rho}^{\frac{\gamma}{2}-1}\mathcal{G}_{xs},\ \bar{\rho}^{\frac{3}{2}\gamma-1}\mathcal{G}_x\right)(\cdot,s)\right\|^2 + \left\|\bar{\rho}^{\frac{3}{2}\gamma-1-\frac{\alpha}{2}}\mathcal{G}_x(\cdot,s)\right\|^2\right]ds\\
&\le C\widetilde{\mathfrak{F}}(0) + C\widetilde{\mathfrak{F}}^2(0), \quad t\in[0,\infty).
\end{aligned} \tag{3.5.30}$$

Proof. The proof consists of three steps.

Step 1. We claim that

$$\begin{aligned}
&\int_0^t (1+s)^{\kappa}\int\left[\bar{\rho}^{3\gamma-2}\mathcal{G}_x^2 + \bar{\rho}^{\gamma-2}\mathcal{G}_{xt}^2 + v_x^2 + \left(\frac{v}{x}\right)^2\right]dxds\\
&\qquad\qquad + (1+t)^{\kappa}\int\bar{\rho}^{2\gamma-2}\mathcal{G}_x^2(x,t)dx \le C\mathfrak{F}(0).
\end{aligned} \tag{3.5.31}$$

First, in a similar way to the derivation of (3.5.5)-(3.5.7), one can show that

$$\int\bar{\rho}^{2\gamma-2}\mathcal{G}_x^2dx + \int_0^t\int\left[\bar{\rho}^{3\gamma-2}\mathcal{G}_x^2 + \bar{\rho}^{\gamma-2}\mathcal{G}_{xt}^2 + v_x^2 + \left(\frac{v}{x}\right)^2\right]dxds \le C\mathfrak{F}(0), \tag{3.5.32}$$

which yields (3.5.31) for $\alpha = 0$.

When $\alpha > 0$, following the argument for (3.5.5), we can show by (3.4.39) that

$$\int \bar{\rho}^{2\gamma-2-\alpha}\mathcal{G}_x^2 dx + \int_0^t \int \bar{\rho}^{3\gamma-2-\alpha}\mathcal{G}_x^2 dxds \le C\mathfrak{F}(0), \tag{3.5.33}$$

due to $0 < \theta \le 2(\gamma-1-\alpha)/\gamma$. Indeed, one may multiply (3.2.7) by $\bar{\rho}^{2\gamma-2-\alpha}\mathcal{G}_x$ and integrate the resulting equation to obtain that

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int \bar{\rho}^{2\gamma-2-\alpha}\mathcal{G}_x^2 dx + \int \bar{\rho}^{3\gamma-2-\alpha}\mathcal{G}_x^2 dx\\ &\le C(\int \bar{\rho}^{\gamma-\alpha}v_t^2 dx + \int x^2\bar{\rho}^{\gamma-\alpha}(|r_x-1|^2 + |\frac{r}{x}-1|^2 dx),\end{aligned}$$

which, together with (3.4.39), yields (3.5.33). In a similar way to deriving (3.5.4), one gets

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int \bar{\rho}^{2\gamma-2}\mathcal{G}_x^2 dx + \frac{\gamma}{2}\int \left(\frac{x^2}{r^2 r_x}\right)^\gamma \bar{\rho}^{3\gamma-2}\mathcal{G}_x^2 dx\\ \le& C\int v_t^2 dx + C\int x^2\bar{\rho}^\gamma \left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right] dx.\end{aligned}$$

Multiplying the above inequality by $(1+t)^\kappa$ with $\kappa = \alpha/\gamma - \theta$, integrating the product and using (3.4.39) lead to

$$\begin{aligned}&(1+t)^\kappa \int \bar{\rho}^{2\gamma-2}\mathcal{G}_x^2(x,t)dx + \int_0^t (1+s)^\kappa \int \bar{\rho}^{3\gamma-2}\mathcal{G}_x^2 dxds\\ \le& C\int_0^t (1+s)^{\kappa-1}\int \bar{\rho}^{2\gamma-2}\mathcal{G}_x^2(x,s)dxds + C\mathfrak{F}(0).\end{aligned} \tag{3.5.34}$$

It follows from (3.5.33), Hölder and Young's inequalities that

$$\begin{aligned}&\int_0^t (1+s)^{\kappa-1}\int \bar{\rho}^{2\gamma-2}\mathcal{G}_x^2(x,s)dxds\\ \le& \int_0^t (1+s)^{\kappa-1}\left(\int \bar{\rho}^{2\gamma-2-\alpha}\mathcal{G}_x^2 dx\right)^{\frac{\gamma-\alpha}{\gamma}}\left(\int \bar{\rho}^{3\gamma-2-\alpha}\mathcal{G}_x^2 dx\right)^{\frac{\alpha}{\gamma}} ds\\ \le& C\int_0^t (1+s)^{(\kappa-1)\frac{\gamma}{\gamma-\alpha}} ds \sup_{[0,t]}\int \bar{\rho}^{2\gamma-2-\alpha}\mathcal{G}_x^2 dx + C\int_0^t \int \bar{\rho}^{3\gamma-2-\alpha}\mathcal{G}_x^2 dxds \le C\mathfrak{F}(0),\end{aligned} \tag{3.5.35}$$

if $\theta > 0$. Consequently, (3.5.34) and (3.5.35) yield that

$$(1+t)^\kappa \int \bar{\rho}^{2\gamma-2}\mathcal{G}_x^2(x,t)dx + \int_0^t (1+s)^\kappa \int \bar{\rho}^{3\gamma-2}\mathcal{G}_x^2 dxds \le C\mathfrak{F}(0), \tag{3.5.36}$$

for

$$\alpha > 0, \quad \kappa = \alpha/\gamma - \theta \quad \text{and} \quad 0 < \theta < \min\left\{2(\gamma-1-\alpha)/\gamma,\ 2(\gamma-1)/3\gamma\right\}.$$

We multiply the square of (3.2.7) by $(1+t)^\kappa \bar{\rho}^{\gamma-2}$ and integrate the product with respect to the spatial and temporal variables to obtain, using (3.5.36) and (3.4.39), that

$$\int_0^t \int (1+s)^\kappa \bar{\rho}^{\gamma-2} \mathcal{G}_{xs}^2 dxds \le C\mathfrak{F}(0). \tag{3.5.37}$$

Repeating the arguments in (3.5.7)-(3.5.8) yields that

$$\int_0^t \int (1+s)^\kappa \left(v_x^2 + |v/x|^2\right) dxds \le C\mathfrak{F}(0).$$

This completes the proof of (3.5.31) for $\alpha > 0$.

Step 2. In this step, we prove that

$$\begin{aligned} &(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa} \int x^2 \bar{\rho}^{3\gamma-2} \mathcal{G}_x^2 dx + \int_0^t (1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa} \int x^2 \bar{\rho}^{2\gamma-2} \mathcal{G}_{xs}^2 dxds \\ &\le C\widetilde{\mathfrak{F}}(0), \quad \text{ where } \widetilde{\mathfrak{F}}(0) := \mathfrak{F}(0) + (\mathfrak{F}(0))^2 . \end{aligned} \tag{3.5.38}$$

Multiplying (3.5.29) by $x^2\bar{\rho}^{\gamma-2}\mathfrak{P}_t$ and integrating the resulting equation give that

$$\frac{1}{2}\frac{d}{dt}\int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}^2 dx = \int x^2 \bar{\rho}^{\gamma-2}\mathfrak{P}_t \frac{x^2}{r^2}\bar{\rho} v_t dx - \mu \int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}_t \mathcal{G}_{xt} dx.$$

Note that

$$\mathfrak{P}_t = \gamma \bar{\rho}^\gamma \left(\frac{x^2}{r^2 r_x}\right)^\gamma \mathcal{G}_{xt} + \mathfrak{P}_1,$$

where

$$\mathfrak{P}_1 := \gamma\bar{\rho}^\gamma \left[\left(\frac{x^2}{r^2 r_x}\right)^\gamma\right]_t \mathcal{G}_x - \left[\left(\frac{x^2}{r^2 r_x}\right)^\gamma - \left(\frac{x}{r}\right)^4\right]_t x\phi\bar{\rho}.$$

Thus, one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int x^2 \bar{\rho}^{\gamma-2}\mathfrak{P}^2 dx + \mu\gamma \int x^2\bar{\rho}^{2\gamma-2}\left(\frac{x^2}{r^2 r_x}\right)^\gamma \mathcal{G}_{xt}^2 dx \\ =&\gamma \int x^2 \bar{\rho}^{2\gamma-1}\left(\frac{x^2}{r^2 r_x}\right)^\gamma \mathcal{G}_{xt}\frac{x^2}{r^2} v_t dx + \int x^2\bar{\rho}^{\gamma-1}\mathfrak{P}_1 \frac{x^2}{r^2} v_t dx - \mu \int x^2 \bar{\rho}^{\gamma-2}\mathfrak{P}_1 \mathcal{G}_{xt} dx; \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int x^2 \bar{\rho}^{\gamma-2}\mathfrak{P}^2 dx + \frac{\mu\gamma}{2} \int x^2\bar{\rho}^{2\gamma-2}\left(\frac{x^2}{r^2 r_x}\right)^\gamma \mathcal{G}_{xt}^2 dx &\le C\int \left(x^2\bar{\rho}^{-2}\mathfrak{P}_1^2 + v_t^2\right) dx \\ &\le C\int \left(x^2 v_x^2 + v^2\right)\bar{\rho}^{2\gamma-2}\mathcal{G}_x^2 dx + C\int \left(x^2v_x^2 + v^2 + v_t^2\right) dx. \end{aligned}$$

Combining this with (3.5.31) shows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int x^2 \bar{\rho}^{\gamma-2}\mathfrak{P}^2 dx + \frac{\mu\gamma}{2} \int x^2\bar{\rho}^{2\gamma-2}\left(\frac{x^2}{r^2 r_x}\right)^\gamma \mathcal{G}_{xt}^2 dx \\ \le& C\left(\|xv_x\|_{L^\infty}^2 + C\|v\|_{L^\infty}^2\right)\int \bar{\rho}^{2\gamma-2}\mathcal{G}_x^2 dx + C\int \left(x^2v_x^2 + v^2 + v_t^2\right) dx \\ \le& C\mathfrak{F}(0)(1+t)^{-\kappa}\left(\|xv_x\|_{L^\infty}^2 + \|v\|_{L^\infty}^2\right) + C\int\left(x^2v_x^2 + v^2 + v_t^2\right) dx. \end{aligned} \tag{3.5.39}$$

Then it follows from (3.3.20), (3.3.19) and the fact that $\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}-\frac{3}{2}\kappa>0$ that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}^2dx+\frac{\mu\gamma}{2}\int x^2\bar{\rho}^{2\gamma-2}\left(\frac{x^2}{r^2r_x}\right)^{\gamma}\mathcal{G}_{xt}^2dx\\
\leq & C\mathfrak{F}(0)(1+t)^{\frac{\theta}{2}-\frac{2\gamma-1}{2\gamma}-\frac{1}{2}\kappa}\int\left(\mathcal{G}_{tx}^2+v_x^2+\left(\frac{v}{x}\right)^2\right)dx\\
&+C\mathfrak{F}(0)\int\left[(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}-\frac{3}{2}\kappa}\left(x^2v_x^2+v^2\right)+v_t^2\right]dx.
\end{aligned}
$$

Multiplying the inequality above by $(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}$ and integrating the product give that

$$
\begin{aligned}
&(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}^2dx+\int_0^t(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int x^2\bar{\rho}^{2\gamma-2}\mathcal{G}_{xs}^2dxds\\
\leq & C\int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}^2(x,0)dx+C\mathfrak{F}(0)\int_0^t(1+s)^{\kappa}\int\left(\mathcal{G}_{sx}^2+v_x^2+(v/x)^2\right)dxds\\
&+C\left[\mathfrak{F}(0)+1\right]\int_0^t(1+s)^{\frac{2\gamma-1}{\gamma}-\theta}\int\left(x^2v_x^2+v^2+v_s^2\right)dxds\\
&+C\int_0^t(1+s)^{\frac{3}{2}\kappa-\frac{1}{2\gamma}-\frac{\theta}{2}}\int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}^2dxds.
\end{aligned}
$$

Since

$$\int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}^2dx\leq C\int x^2\bar{\rho}^{3\gamma-2}\mathcal{G}_x^2dx+C\int x^2\bar{\rho}^{\gamma}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dx$$

and

$$\frac{3}{2}\kappa-\frac{1}{2\gamma}-\frac{\theta}{2}\leq\min\left\{\kappa,\ \frac{\gamma-1}{\gamma}-\theta\right\},$$

it follows from (3.5.31) and (3.4.39) that

$$(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int x^2\bar{\rho}^{\gamma-2}\mathfrak{P}^2dx+\int_0^t(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int x^2\bar{\rho}^{2\gamma-2}\mathcal{G}_{xs}^2dxds\leq C\widetilde{\mathfrak{F}}(0).$$

This, together with (3.4.39), implies (3.5.38).

Step 3. Multiplying the square of (3.2.7) by $x^2\bar{\rho}^{\gamma-2}$ and integrating the product with respect to the spatial variable, one gets, by (3.5.38) and (3.4.39), that

$$
\begin{aligned}
&\int x^2\bar{\rho}^{\gamma-2}\mathcal{G}_{xt}^2(x,t)dx\leq C\left(\int x^2\bar{\rho}^{3\gamma-2}\mathcal{G}_x^2dx+\int x^2\bar{\rho}v_t^2dx\right.\\
&\left.+\int x^2\bar{\rho}^{\gamma}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dx\right)\leq C\widetilde{\mathfrak{F}}(0)(1+t)^{-\frac{2\gamma-1}{2\gamma}+\frac{\theta}{2}-\frac{3}{2}\kappa}.
\end{aligned}
$$

As a consequence of (3.3.7), (3.5.8), (3.4.39) and (3.5.38), one concludes that

$$
\begin{aligned}
&\int_0^t(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int\left(v_x^2+\left|\frac{v}{x}\right|^2\right)dxds\leq c\int_0^t(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int\left(\mathcal{G}_t^2+v^2\right)dxds\\
\leq & \int_0^t(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int\left(v^2+x^2v_x^2+x^2\bar{\rho}^{2(\gamma-1)}\mathcal{G}_{xt}^2\right)dxds\leq C\widetilde{\mathfrak{F}}(0).\quad\square
\end{aligned}
$$

Lemma 3.20 *Under the same assumptions as in ii) of Theorem 2.3, it holds that*

$$
\begin{aligned}
&(1+t)^{\frac{2\gamma-1}{4\gamma}-\frac{\theta}{4}+\frac{9}{4}\kappa}\left\|(\mathcal{G}_x,\mathcal{G}_{xt})(\cdot,t)\right\|^2_{L^2([0,\frac{1}{2}])}+\int_0^t(1+s)^{\frac{2\gamma-1}{4\gamma}-\frac{\theta}{4}+\frac{9}{4}\kappa}\left\|\mathcal{G}_{xs}(\cdot,s)\right\|^2_{L^2([0,\frac{1}{2}])}ds\\
&+(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\left\|\bar{\rho}^{\frac{1}{2}}v_t(\cdot,t)\right\|^2+\int_0^t(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\left\|\left(v_{sx},\frac{v_s}{x}\right)(\cdot,s)\right\|^2ds\\
&\le C\widetilde{\mathfrak{F}}(0),\quad t\in[0,\infty).
\end{aligned}
\tag{3.5.40}
$$

Proof. Let $\bar{\psi}$ be a non-increasing function defined on $[0,1]$ satisfying

$$\bar{\psi}=1\ \text{ on }\ [0,1/2],\quad \bar{\psi}=0\ \text{ on }\ [3/4,1]\ \text{ and }\ |\bar{\psi}'|\le 32.$$

Following the derivation of (3.5.39), one can obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\bar{\psi}\bar{\rho}^{\gamma-2}\mathfrak{P}^2dx+\frac{\mu\gamma}{2}\int\bar{\psi}\bar{\rho}^{2\gamma-2}\left(\frac{x^2}{r^2r_x}\right)^{\gamma}\mathcal{G}_{xt}^2dx\\
\le&C\mathfrak{F}(0)(1+t)^{-\kappa}\left(\|v_x\|^2_{L^\infty([0,\frac{3}{4}])}+\|v/x\|^2_{L^\infty([0,\frac{3}{4}])}\right)+C\int\left(v_x^2+v^2/x^2+v_t^2\right)dx.
\end{aligned}
$$

In view of (3.3.21), we see

$$
\begin{aligned}
&\|v_x\|^2_{L^\infty([0,\frac{3}{4}])}+\|v/x\|^2_{L^\infty([0,\frac{3}{4}])}\\
\le&c\int\left(v_x^2+\left|\frac{v}{x}\right|^2\right)dx+\left[\int\left(v_x^2+\left|\frac{v}{x}\right|^2\right)dx\right]^{1/2}\left[\int_0^{3/4}\left(v_{xx}^2+\left|\left(\frac{v}{x}\right)_x\right|^2\right)dx\right]^{1/2}\\
\le&c\int\left(v_x^2+\left|\frac{v}{x}\right|^2\right)dx+\left[\int\left(v_x^2+\left|\frac{v}{x}\right|^2\right)dx\right]^{1/2}\left[\int\bar{\rho}^{3\gamma-2}\left(\mathcal{G}_{xt}^2+\mathcal{G}_x^2\right)dx\right]^{1/2},
\end{aligned}
$$

where one has used the following estimate

$$
\begin{aligned}
\int_0^{3/4}\left(v_{xx}^2+\left|\left(\frac{v}{x}\right)_x\right|^2\right)dx\le&C\int_0^{3/4}\bar{\rho}^{3\gamma-2}\left(v_{xx}^2+\left|\left(\frac{v}{x}\right)_x\right|^2\right)dx\\
\le&C\int\bar{\rho}^{3\gamma-2}\left(\mathcal{G}_{xt}^2+\mathcal{G}_x^2\right)dx,
\end{aligned}
\tag{3.5.41}
$$

which follows from (3.3.16). Similar to (3.5.38), one can obtain, by (3.5.30), that

$$(1+t)^{\frac{2\gamma-1}{4\gamma}-\frac{\theta}{4}+\frac{9}{4}\kappa}\int_0^{1/2}\mathcal{G}_x^2dx+\int_0^t(1+s)^{\frac{2\gamma-1-\alpha}{4\gamma}-\frac{\theta}{4}+\frac{5}{2}\kappa}\int_0^{1/2}\mathcal{G}_{xt}^2dxds\le C\widetilde{\mathfrak{F}}(0).\tag{3.5.42}$$

In view of (3.5.16), (3.4.39) and (3.5.30), it holds that

$$(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int\bar{\rho}v_t^2dx+\int_0^t(1+s)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\int\left(v_{xt}^2+v_t^2/x^2\right)dxds\le C\widetilde{\mathfrak{F}}(0).\tag{3.5.43}$$

Finally, with the aid of (3.4.11), (3.5.42) and (3.5.43), one can multiply the square of (3.2.7) by $\bar{\rho}^{\gamma-2}$ and integrate the product with respect to the spatial variable to get

$$
\begin{aligned}
\int\bar{\rho}^{\gamma-2}\mathcal{G}_{xt}^2dx\le&C\int\bar{\rho}^{3\gamma-2}\mathcal{G}_x^2dx+C\int\bar{\rho}^{\gamma}v_t^2dx+C\int x^2\bar{\rho}^{\gamma}\left[\left(\frac{r}{x}-1\right)^2+(r_x-1)^2\right]dx\\
\le&C(1+t)^{-\frac{2\gamma-1}{4\gamma}+\frac{\theta}{4}-\frac{9}{4}\kappa}\widetilde{\mathfrak{F}}(0).\quad\square
\end{aligned}
$$

Lemma 3.21 *Under the same assumptions as in ii) of Theorem 2.3, it holds that*

$$(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\left[\left\|\left(r_x-1,\ \frac{r}{x}-1\right)(\cdot,t)\right\|^2_{L^2([0,\frac{1}{2}])}+\left\|\left(v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2\right]$$
$$+(1+t)^{\frac{2\gamma-1}{4\gamma}-\frac{\theta}{4}+\frac{9}{4}\kappa}\left\|\left(r_x,\ \frac{r}{x},v_x,\frac{v}{x}\right)_x(\cdot,t)\right\|^2_{L^2([0,\frac{1}{2}])}+(1+t)^{\frac{3}{4}\left[\frac{2\gamma-1}{\gamma}-\theta+\kappa\right]}\|(xv_x,v)(\cdot,t)\|^2_{L^\infty}$$
$$+(1+t)^{\frac{3}{8}\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)}\left(\left\|\left(r_x-1,\frac{r}{x}-1\right)(\cdot,t)\right\|^2_{L^\infty([0,\frac{1}{2}])}+\left\|\left(v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{L^\infty}\right)$$
$$+(1+t)^{\min\left\{\frac{1}{2}\left(\frac{2\gamma-1}{\gamma}-\theta+\kappa\right),\ \frac{3}{8}\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)\right\}}\left\|\bar\rho^{\frac{\gamma}{2}}\mathcal{G}(\cdot,t)\right\|^2_{L^\infty}\le C\widetilde{\mathfrak{F}}(0),\quad t\in[0,\infty).\quad(3.5.44)$$

Proof. As proved in Lemma 3.18, we can show easily that

$$(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{1}{2}\kappa}\left\|x^{\frac{3}{2}}\bar\rho^{\frac{\gamma}{2}}\mathcal{G}(\cdot,t)\right\|^2_{L^\infty}+(1+t)^{\frac{2\gamma-1}{4\gamma}-\frac{\theta}{4}+\frac{9}{4}\kappa}\left\|\left(r_x,\ \frac{r}{x},v_x,\frac{v}{x}\right)_x(\cdot,t)\right\|^2_{L^2([0,\frac{1}{2}])}$$
$$+(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\left\|\left(r_x-1,\ \frac{r}{x}-1,v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{L^2([0,\frac{1}{2}])}\le C\widetilde{\mathfrak{F}}(0).\quad(3.5.45)$$

This, together with (3.4.39), implies that

$$(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}+\frac{3}{2}\kappa}\left\|\left(v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2\le C\widetilde{\mathfrak{F}}(0).$$

We can then obtain, with the aid of (3.3.19), (3.3.20) and (3.5.30), that

$$\|(xv_x,v)(\cdot,t)\|^2_{L^\infty}\le C(1+t)^{-\frac{3}{4}\left[\frac{2\gamma-1}{\gamma}-\theta+\kappa\right]}\widetilde{\mathfrak{F}}(0).\quad(3.5.46)$$

It follows from(3.3.21), (3.3.24) and (3.5.30) that

$$\left\|\left(r_x-1,\frac{r}{x}-1,v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{L^\infty([0,1/2])}\le C(1+t)^{-\frac{3}{8}\left(\frac{2\gamma-1}{\gamma}-\theta\right)-\frac{15}{8}\kappa}\widetilde{\mathfrak{F}}(0).\quad(3.5.47)$$

One has, in view of (3.5.45) and (3.5.47), that

$$\left\|\bar\rho^{\frac{\gamma}{2}}\mathcal{G}(\cdot,t)\right\|^2_{L^\infty}\le C\widetilde{\mathfrak{F}}(0)(1+t)^{-\min\left\{\frac{1}{2}\left(\frac{2\gamma-1}{\gamma}-\theta\right)+\frac{1}{2}\kappa,\ \frac{3}{8}\left(\frac{2\gamma-1}{\gamma}-\theta\right)+\frac{15}{8}\kappa\right\}}.$$

Finally, it follows from (3.5.46) and (3.5.47) that

$$\left\|\left(v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{L^\infty}\le(1+t)^{-\frac{3}{8}\left(\frac{2\gamma-1}{\gamma}-\theta+5\kappa\right)}.\ \square$$

Lemma 3.22 *Suppose that* $\|x\bar\rho^{-1/2}\mathcal{G}_{xtt}(\cdot,0)\|<\infty$ *and the assumptions of ii) in Theorem 2.3 hold. Then*

$$\int x^2\bar\rho v_{tt}^2(x,t)dx+\int_0^\infty\int(v_{ss}^2+x^2v_{xss}^2)dxds\le C\mathfrak{F}(0)+C\|x\bar\rho^{-\frac{1}{2}}\mathcal{G}_{xtt}(\cdot,0)\|^2,\ t\ge0.\ (3.5.48)$$

Proof. Multiplying $\partial_t^2(3.2.6)$ by $r^2 v_{tt}$ and integrating the resulting equation both in x and t, one can show that

$$\int x^2\bar{\rho}v_{tt}^2(x,t)dx + \int_0^\infty \int (v_{ss}^2 + x^2 v_{xss}^2)dxds \le C(\mathfrak{F}(0) + \|x\bar{\rho}^{\frac{1}{2}}v_{tt}(\cdot,0)\|^2). \tag{3.5.49}$$

Indeed, the derivation of (3.5.49) is similar to that of (3.4.36), so we omit the details here. It can be derived from (2.1.13) that

$$\|x\bar{\rho}^{\frac{1}{2}}v_{tt}\|^2(\cdot,0) \le C(\mathcal{E}(0) + \|x\bar{\rho}^{-\frac{1}{2}}\mathcal{G}_{xtt}(\cdot,0)\|^2).$$

This, together with (3.5.49), proves Lemma 3.22. Hence, we finish the proof of part *ii*) of Theorem 3.1. □

3.5.3 Part III: further regularity

In this subsection, we further study the higher regularity of the strong solution obtained in Theorem 2.2 and prove part *iii*) of Theorem 3.1.

Lemma 3.23 *Under the assumptions in iii) of Theorem 3.1, it holds that for all* $t \ge 0$ *and any* $0 < \theta < \min\{2(\gamma-1)/(3\gamma), (4-2\gamma)/\gamma\}$,

$$\|\mathcal{G}_x(\cdot,t)\|^2 + \|(r_x,\ r/x)_x(\cdot,t)\|^2 \le c\,\|\mathcal{G}_x(\cdot,0)\|^2 + C(\theta)\mathfrak{E}(0)(1+t)^{\frac{1}{2}+\theta\frac{\gamma}{2\gamma-2}},$$

$$\|(v_x,\ v/x)_x(\cdot,t)\|^2 \le C(\theta)\left(\|\mathcal{G}_x(\cdot,0)\|^2 + \mathfrak{E}(0)\right)(1+t)^{-\frac{7\gamma-6}{4\gamma}+\theta\left(4+\frac{\gamma}{2\gamma-2}\right)}.$$

Proof. In a similar way to the derivation of (3.5.5), we have

$$\begin{aligned}&\int \mathcal{G}_x^2 dx + \int_0^t \int \bar{\rho}^\gamma \mathcal{G}_x^2 dxds\\ &\le \int \mathcal{G}_x^2(0)dx + C\mathcal{E}(0) + C\int_0^t \int x^2\bar{\rho}^{2-\gamma}\left[\left(\frac{r}{x}-1\right)^2 + (r_x-1)^2\right]dxds.\end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned}&\int \bar{\rho}^{2-\gamma}\left[(r-x)^2 + (xr_x-x)^2\right]dx\\ &\le \left(\int \left[(r-x)^2 + (xr_x-x)^2\right]dx\right)^{\frac{2\gamma-2}{\gamma}}\left(\int \bar{\rho}^\gamma\left[(r-x)^2 + (xr_x-x)^2\right]dx\right)^{\frac{2-\gamma}{\gamma}}\\ &\le (1+t)^{-\frac{\gamma-1}{\gamma}+\theta}\left((1+t)^{\frac{\gamma-1}{\gamma}-\theta}\int \left[(r-x)^2 + (xr_x-x)^2\right]dx\right)^{\frac{2\gamma-2}{\gamma}}\\ &\times\left((1+t)^{\frac{\gamma-1}{\gamma}-\theta}\int \bar{\rho}^\gamma\left[(r-x)^2 + (xr_x-x)^2\right]dx\right)^{\frac{2-\gamma}{\gamma}},\end{aligned}$$

which, together with (3.4.39) and the Young inequality, implies that

$$\begin{aligned}
&\int_0^t \int x^2 \bar{\rho}^{2-\gamma} \left[\left(\frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dxds \\
&\leq C \int_0^t (1+s)^{\frac{\gamma-1}{\gamma} - \theta} \int \bar{\rho}^\gamma \left[(r-x)^2 + (xr_x - x)^2 \right] dxds \\
&+ C(1+t)^{1 - \left(\frac{\gamma-1}{\gamma} - \theta \right) \frac{\gamma}{2\gamma-2}} \sup_{[0,t]} \int (1+s)^{\frac{\gamma-1}{\gamma} - \theta} \left[(r-x)^2 + (xr_x - x)^2 \right] (x,s) dx \\
&\leq C \mathfrak{E}(0) (1+t)^{\frac{1}{2} + \theta \frac{\gamma}{2\gamma-2}}.
\end{aligned}$$

Due to (3.3.9) and (3.4.39), it holds that

$$\| \mathcal{G}_x(\cdot, t) \|^2 + \| (r_x, r/x)_x (\cdot, t) \|^2 \leq c \| \mathcal{G}_x(\cdot, 0) \|^2 + C \mathfrak{E}(0) (1+t)^{\frac{1}{2} + \theta \frac{\gamma}{2\gamma-2}}.$$

As shown in (3.5.44), one has the following bound away from the boundary:

$$(1+t)^{\frac{1}{4} \left(\frac{11\gamma - 10}{\gamma} - 19\theta \right)} \left\| \left(v_x, \frac{v}{x} \right)_x (\cdot, t) \right\|^2_{L^2\left(\left[0, \frac{1}{2} \right] \right)} \leq C \widetilde{\mathfrak{F}}(0).$$

Indeed, the above estimate holds if $\alpha = \gamma - 1 - \gamma\theta$ and $\kappa = (\gamma - 1)/\gamma - 2\theta$ in (3.5.44). Away from the origin, the same way as for (3.3.10) gives that

$$\begin{aligned}
\| (v_x, \ v/x)_x \|^2_{L^2\left(\left[\frac{1}{2}, 1 \right] \right)} &\leq c \| \mathcal{G}_{xt} \|^2_{L^2\left(\left[\frac{1}{4}, 1 \right] \right)} + c \| (xv_x, v) \|^2 \\
&+ c \| (v_x, v/x) \|^2_{L^\infty\left(\left[\frac{1}{4}, 1 \right] \right)} \left(\| \mathcal{G}_x \|^2 + \| (xr_x - x, r - x) \|^2 \right).
\end{aligned}$$

This, together with (3.1.1)-(3.1.4), implies that

$$\begin{aligned}
&\| (v_x, \ v/x)_x \|^2_{L^2\left(\left[\frac{1}{2}, 1 \right] \right)} \\
&\leq C \left(\| \mathcal{G}_x(\cdot, 0) \|^2 + \mathfrak{E}(0) \right) \left[(1+t)^{-\frac{1}{2} \left(\frac{5\gamma-4}{\gamma} - 7\theta \right)} + (1+t)^{-\frac{3}{4} \left(\frac{3\gamma-2}{\gamma} - 3\theta \right) + \frac{1}{2} + \theta \frac{\gamma}{2\gamma-2}} \right] \\
&\leq C \left(\| \mathcal{G}_x(\cdot, 0) \|^2 + \mathfrak{E}(0) \right) (1+t)^{-\frac{7\gamma-6}{4\gamma} + \theta \left(\frac{7}{2} + \frac{\gamma}{2\gamma-2} \right)}
\end{aligned}$$

We thus conclude that

$$\| (v_x, \ v/x)_x \|^2 \leq C \left(\| \mathcal{G}_x(\cdot, 0) \|^2 + \mathfrak{E}(0) \right) (1+t)^{-\frac{7\gamma-6}{4\gamma} + \theta \left(4 + \frac{\gamma}{2\gamma-2} \right)}. \quad \square$$

4 Proof of Theorem 2.3

Due to Theorem 3.1 and Theorem 2.2, the triple $(\rho, u, R(t))$ $(t \geq 0)$ defined by (2.2.4) and (2.2.5) gives the unique global strong solution to the free boundary problem (1.1.2). The decay estimates *i)* and *ii)* in Theorem 2.3, which follow from the corresponding ones in Theorem 3.1. Clearly, (2.2.7) and (2.2.6) comes directly from (3.1.1). Note that

$$\rho(r(x,t),t) - \bar{\rho}(x) = \bar{\rho}(x) \left[\exp\{ -\mathcal{G}(x,t) \} - 1 \right],$$

then (2.2.9) follows from (3.1.1). In view of (3.1.1), we see

$$(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\left\|\left(v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{H^1\left(\left[0,\frac{1}{2}\right]\right)}+(1+t)^{\frac{2\gamma-1}{2\gamma}-\frac{\theta}{2}}\left\|\left(\frac{v}{x},v_x\right)(\cdot,t)\right\|^2_{L^\infty\left(\left[\frac{1}{2},1\right]\right)}\le C\mathfrak{E}(0)$$

which implies

$$(1+t)^{\frac{\gamma-1}{\gamma}-\theta}\left\|\left(v_x,\frac{v}{x}\right)(\cdot,t)\right\|^2_{L^\infty}\le C\mathfrak{E}(0).$$

So that (2.2.8) is true, because of

$$u_r(r,t)=\frac{v_x(x,t)}{r_x(x,t)}\quad\text{and}\quad\frac{u(r,t)}{r}=\frac{x}{r(x,t)}\frac{v(x,t)}{x}.$$

Similarly, the decay estimates for ρ, u and u_r in part *ii*) follows from (3.1.3).

The $W^{2,\infty}$-estimate of $R(t)$ can be proved as follows. First, it follows from (3.4.27) that

$$\int_0^\infty v_t^2(1,t)dt\le C\int_0^\infty\int_{\frac{1}{2}}^1(v_t^2+v_{xt}^2)dxdt\le C\mathfrak{E}(0).$$

One the other hand, (3.1.5) implies that

$$\int_0^\infty v_{tt}^2(1,t)dt\le C\int_0^\infty\int_{\frac{1}{2}}^1(v_{tt}^2+v_{xtt}^2)dxdt\le C(\mathfrak{F}(0)+\|x\bar{\rho}^{-\frac{1}{2}}\mathcal{G}_{xtt}(\cdot,0)\|^2).$$

Combining these two estimates with the fact that

$$\ddot{R}^2(t)=v_t^2(1,t)\le v_t^2(1,0)+2\left(\int_0^\infty v_t^2(1,t)dt\right)^{1/2}\left(\int_0^\infty v_{tt}^2(1,t)dt\right)^{1/2}$$

gives (2.2.13) immediately. This finishes the proof of Theorem 2.3. □

Remark 4.1 *One may prove the boundedness of $r_{tt}(x,t)$ for any fixed $x\in[0,1]$ and $t\ge 0$ if $|v_t(x,0)|$ is finite by an argument similar to the above. This implies that every particle moving with the fluid has the bounded acceleration for $t\in(0,\infty)$ if it does so initially.*

Acknowledgement

This research was partially supported by the Zheng Ge Ru Foundation, and Hong Kong RGC Earmarked Research Grants CUHK-4041/11P, CUHK-4048/13P, a Focus Area Grant from The Chinese University of Hong Kong, and a grant from Croucher Foundation. Luo's research was supported in part by NSF under grant DMS-1408839. Zeng's research was supported in part by NSFC under grant 11301293.

References

[1] S. Chandrasekhar, An Introduction to the Study of Stellar Structures. University of Chicago Press, Chicago, 1938.

[2] Coutand, Daniel; Shkoller, Steve, Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible Euler equations in physical vacuum. Arch. Ration. Mech. Anal. 206 (2012), no. 2, 515-616.

[3] Coutand, D., Shkoller, S.: Well-posedness in smooth function spaces for the moving-boundary 1-D compressible Euler equations in physical vacuum. Commun. Pure Appl. Math. 64, 328-366 (2011)

[4] Coutand, Daniel; Shkoller, Steve; Well-Posedness in Smooth Function Spaces for the Moving-Boundary Three-Dimensional Compressible Euler Equations in Physical Vacuum. Arch. Ration. Mech. Anal. 206 (2012), no. 2, 515-616.

[5] Y. Deng, T.P. Liu, T. Yang, Z. Yao, Solutions of Euler-Poisson equations for gaseous stars. Arch. Ration. Mech. Anal. 164 (2002), no. 3, 261C285.

[6] B. Ducomet, A. Zlotnik: Stabilization and stability for the spherically symmetric Navier-Stokes-Poisson system, Appl.Math.Lett. 18 (2005), 1190-1198

[7] Q. Duan, On the dynamics of Navier-Stokes equations for a shallow water model, J. Differential Equations, 250 (2011), 2687-2714.

[8] A. Kufner, L. Maligranda, L.-E. Persson: The Hardy inequality. Vydavatelsksy Servis, PlzeTn, 2007. About its history and some related results.

[9] D.-Y. Fang and T. Zhang, Global behavior of compressible Navier-Stokes equations with a degenerate viscosity coefficient, Arch. Rational Mech. Anal., 182 (2006), 223-253.

[10] D.-Y. Fang and T. Zhang, Global behavior of spherically symmetric Navier-Stokes-Poisson system with degenerate viscosity coefficients, Arch. Rational Mech. Anal., 191 (2009), 195-243.

[11] Guo, Zhenhua; Li, Hai-Liang; Xin, Zhouping, Lagrange structure and dynamics for solutions to the spherically symmetric compressible Navier-Stokes equations. Comm. Math. Phys. 309 (2012), no. 2, 371-412

[12] Jang, J., Local well-posedness of dynamics of viscous gaseous stars. Arch. Rational Mech. Anal. 195, 797-863 (2010).

[13] J. Jang. Nonlinear instability in gravitational Euler-Poisson system for $\gamma = \frac{6}{5}$. Arch. Ration. Mech. Anal. 188 (2008), no. 2, 265C307.

[14] Jang, J., Masmoudi, N.:Well-posedness for compressible Euler with physical vacuum singularity. Commun. Pure Appl. Math. 62, 1327-1385 (2009)

[15] Jang, J., Masmoudi, N.: Well-posedness of compressible Euler equations in a physical vacuum, arXiv:1005.4441.

[16] J. Jang & I. Tice, Instability theory of the Navier-Stokes-Poisson equations, Anal. PDE 6 (2013), no. 5, 1121-1181.

[17] Jang, J. Nonlinear Instability Theory of Lane-Emden stars, arXiv:1211.2463.

[18] S. Jiang, Z. Xin, P. Zhang, Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity. Methods Appl. Anal. 12 (2005), no. 3, 239-251.

[19] Lebovitz, N.R., Lifschitz, A.: Short-wavelength instabilities of Riemann ellipsoids, Philos. Trans. Roy. Soc. London Ser. A 354(1709), 927-950 (1996).

[20] Lebovitz, N.R.: The virial tensor and its application to self-gravitating fluids. Astrophys. J. 134, (1961) 500-536.

[21] Lieb, E.H., Yau, H.T.: The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. Commun. Math. Phys. 112(1), 147-174 (1987)

[22] S.-S. Lin. Stability of gaseous stars in spherically symmetric motions. SIAM J. Math. Anal. 28 (1997), no. 3, 539-569.

[23] Liu, T.-P. Compressible flow with damping and vacuum. Japan J. Appl. Math. 13 (1996), 25-32.

[24] Liu, T.-P.; Yang, T., Compressible flow with vacuum and physical singularity. Methods Appl. Anal. 7 (2000), 495-509.

[25] T.-P. Liu, Z. Xin and T. Yang, Vacuum states of compressible flow, Discrete and Continuous Dynamical Systems, 4(1998), 1-32.

[26] T. Luo, Z. Xin and T. Yang, Interface behavior of compressible Navier-Stokes equations with vacuum, SIAM J. Math. Anal. 31 (6) (2000) 1175-1191.

[27] Luo, T.; Smoller, J.: Nonlinear dynamical stability of Newtonian rotating and non-rotating white dwarfs and rotating supermassive stars. Comm. Math. Phys. 284 (2008), no. 2, 425-457.

[28] Luo, T.; Smoller, J.: Existence and non-linear stability of rotating star solutions of the compressible Euler-Poisson equations. Arch. Ration. Mech. Anal. 191 (2009), no. 3, 447-496.

[29] T. Luo, Z. Xin & H. Zeng, Well-Posedness for the Motion of Physical Vacuum of the Three-dimensional Compressible Euler Equations with or without Self-Gravitation, Arch. Rational Mech. Anal, (DOI) 10.1007/s00205-014-0742-0 (published online).

[30] T. Luo, Z. Xin & H. Zeng, Nonlinear asymptotic stability of the Lane-Emden solutions for the viscous gaseous star problem with density dependent viscosity, in preparation.

[31] Makino, T.: On a local existence theorem for the evolution equation of gaseous stars. Patterns and Waves. Stud. Math. Appl., Vol. 18. North-Holland, Amsterdam, 459-479, 1986

[32] M. Okada, Free boundary value problems for the equation of one-dimensional motion of viscous gas, Japan J. Appl. Math., 6(1989), 161-177.

[33] M. Okada and T. Makino, Free boundary problem for the equations of spherically symmetrical motion of viscous gas, Japan J. Indust. Appl. Math., 10(1993), 219-235.

[34] S. Matusu-Necasova, M. Okada, T. Makino: Free boundary problem for the equation of spherically symmetric motion of viscous gas III, Japan J.Indust.Appl.Math. 14 (1997), 199-213

[35] Mari Okada, Free boundary problem for one-dimensional motions of compressible gas and vacuum, Japan J. Indust. Appl. Math. 21 (2) (2004) 109-128.

[36] G. Rein, Non-linear stability of gaseous stars. Arch. Ration. Mech. Anal. 168 (2003), no. 2, 115-130.

[37] P. Secchi, On the evolution equations of viscous gaseous stars. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 18 (1991), no. 2, 295–318.

[38] P. Secchi, On the uniqueness of motion of viscous gaseous stars. Math. Methods Appl. Sci. 13 (1990), no. 5, 391–404.

[39] S. H. Shapiro & S. A. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars, *WILEY-VCH*, (2004)

[40] Strohmer, G., Asymptotic estimates for a perturbation of the linearization of an equation for compressible viscous fluid flow. Studia Math. 185 (2008), no. 2, 99–125.

[41] S. Weinberg , Gravitation and Cosmology *John Wiley and Sons*, New York , 1972.

[42] T. Yang, Singular behavior of vacuum states for compressible fluids. J. Comput. Appl. Math. 190 (2006), no. 1-2, 211-231.

[43] Yang, Tong; Yao, Zheng-an; Zhu, Changjiang, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum. (English summary) Comm. Partial Differential Equations 26 (2001), no. 5-6, 965-981.

[44] Yang, Tong; Zhu, Changjiang, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum. Comm. Math. Phys. 230 (2002), no. 2, 329-363.

[45] Zhu, Changjiang; Zi, Ruizhao, Asymptotic behavior of solutions to 1D compressible Navier-Stokes equations with gravity and vacuum. Discrete Contin. Dyn. Syst. 30 (2011), no. 4, 1263-1283