Transformations and Coupling Relations for Affine Connections

James Tao
Harvard University, Cambridge, MA

Jun Zhang
University of Michigan, Ann Arbor, MI

Send correspondence to: junz@umich.edu

Abstract. The statistical structure on a manifold $\mathfrak{M}$ is predicated upon a special kind of coupling between the Riemannian metric g and a torsion-free affine connection ∇ on the $T\mathfrak{M}$, such that ∇g is totally symmetric, forming, by definition, a "Codazzi pair" $\{\nabla, g\}$. In this paper, we first investigate various transformations of affine connections, including additive translation (by an arbitrary (1,2)-tensor K), multiplicative perturbation (through an arbitrary invertible operator L on $T\mathfrak{M}$), and conjugation (through a non-degenerate two-form h). We then study the Codazzi coupling of ∇ with h and its coupling with L, and the link between these two couplings. We introduce, as special cases of K-translations, various transformations that generalize traditional projective and dual-projective transformations, and study their commutativity with L-perturbation and h-conjugation transformations. Our derivations allow affine connections to carry torsion, and we investigate conditions under which torsions are preserved by the various transformations mentioned above. While reproducing some known results regarding Codazzi transform, conformal-projective transformation, etc., we extend much of these geometric relations, and hence obtain new geometric insights, for the general case of a non-degenerate two-form h (instead of the symmetric g) and an affine connection with possibly non-vanishing torsion. Our systematic approach establishes a general setting for the study of Information Geometry based on transformations and coupling relations of affine connections.

1 Introduction

On the tangent bundle $T\mathfrak{M}$ of a differentiable manifold $\mathfrak{M}$, one can introduce two separate structures: affine connection ∇ and Riemannian metric g. The coupling of these two structures has been of great interest to, say, affine geometers and information geometers. When coupled, $\{\nabla, g\}$ is called a Codazzi pair e.g., [17, 20], which is an important concept in affine hypersurface theory, e.g., [18, 12], statistical manifolds [8], and related fields. To investigate the robustness of the Codazzi structure, one would perturb the metric and perturb the affine connection, and examine whether, after perturbation, the resulting metric and connection will still maintain Codazzi coupling [15].

Codazzi transform is a useful concept in coupling projective transform of a connection and conformal transformation of the Riemannian metric: the pair $\{\nabla, g\}$ is jointly transformed in such a way that Codazzi coupling is preserved, see [20]. This is done through an arbitrary function that transforms both the metric and connection. A natural question to ask is whether there is a more general transformation of the metric and of the connection that preserves the Codazzi coupling, such that Codazzi transform (with the freedom of one function) is a special case. In this paper, we provide a positive answer to this question. The second goal of this paper is to investigate the role of torsion in affine connections and their transformations. Research on this topic is isolated, and the general importance has not been appreciated.

In this paper, we will collect various results on transformations on affine connection and classify them through one of the three classes, L-perturbation, h-conjugation, and the more general K-translation. They correspond to transforming ∇ via a (1,1)-tensor, (0,2)-tensor, or (1,2)-tensor. We will investigate the interactions between these transformations, based on known results but generalizing them to more arbitrary and less restrictive conditions. We will show how a general transformation of a non-degenerate two-form and a certain transformation of the connection are coupled; here transformation of a connection can be through L-perturbation, h-conjugation, and K-translation which specialzes to various projective-like transformations.

We will show how they are linked in the case when they are Codazzi coupled to a same connection ∇. The outcome are depicted in commutative diagrams as well as stated as Theorems.

Among transformations of affine connections, projective transformation and projective equivalence has long been characterized and understood. Researchers have introduced, progressively, the notions of 1-conformal transformation and α-conformal transformation in general [6], dual-projective transformation which is essentially (-1)-conformal transformation [5], and conformal-projective transformation [7], which encompass all previous cases.

Two statistical manifolds $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are called *α-conformally equivalent* [6] if there exists a function ϕ such that

$$g'(u,v) = e^{\phi} g(u,v),$$
$$\nabla'_u v = \nabla_u v - \frac{1+\alpha}{2} g(u,v)\mathrm{grad}_g \phi + \frac{1-\alpha}{2}\{d\phi(u)v + d\phi(v)u\},$$

where $\mathrm{grad}_g \phi$ is the gradient vector field of ϕ with respect to g, namely, $g(u, \mathrm{grad}_g \phi) = d_u \phi$. When $\alpha = -1$, it describes projective equivalency. When $\alpha = 1$, it describes dual-projective equivalency. It is easily seen [22] that

1. $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are α-conformally equivalent iff $(\mathfrak{M}, \nabla^*, g)$ and $(\mathfrak{M}, \nabla'^*, g')$ are $(-\alpha)$-conformally equivalent.
2. If $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are α-conformally equivalent, then $(\mathfrak{M}, \nabla^{(\beta)}, g)$ and $(\mathfrak{M}, \nabla'^{(\beta)}, g')$ are $(\alpha\beta)$-conformally equivalent.

Two statistical manifolds $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are said to be *conformally-projectively equivalent* [7] if there exist two functions ϕ and ψ such that

$$\bar{g}(u,v) = e^{\phi+\psi} g(u,v),$$
$$\nabla'_u v = \nabla_u v - g(u,v)\mathrm{grad}_g \psi + \{d\phi(u)v + d\phi(v)u\}.$$

Note: $\phi = \psi$ yields conformal equivalency; $\phi = const$ yields 1-conformal (i.e., dual projective) equivalency; $\psi = const$ yields (-1)-conformal (i.e., projective) equivalency. It is shown [11] that when two statistical manifolds $(\mathfrak{M}, \nabla, g)$ and $(\mathfrak{M}, \nabla', g')$ are conformally-projectively equivalent, then $(\mathfrak{M}, \nabla^{(\alpha)}, g)$ and $(\mathfrak{M}, \nabla'^{(\alpha)}, g')$ are also conformally-projectively equivalent, with inducing functions $\phi^{(\alpha)} = \frac{1+\alpha}{2}\phi + \frac{1-\alpha}{2}\psi, \psi^{(\alpha)} = \frac{1-\alpha}{2}\psi + \frac{1+\alpha}{2}\phi$.

Our paper will provide a generalization of the conformal-projective transformation mentioned above, with an additional degree of freedom, and specify the conditions under which such transformation preserves Codazzi pairing of g and ∇.

2 Transformations of Affine Connections

2.1 Affine connections

An affine (linear) connection ∇ is an endomorphism of $T\mathfrak{M}$: $\nabla : (X, Y) \in T\mathfrak{M} \times T\mathfrak{M} \mapsto \nabla_X Y \in T\mathfrak{M}$ that is bilinear in the vector fields X, Y and that satisfies the *Leibniz rule*

$$\nabla_X(\phi Y) = X(\phi)Y + \phi \nabla_X Y,$$

for any smooth function ϕ on $\mathfrak{M}$. The torsion of a connection ∇ is characterized by the *torsion tensor*

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

whereas its curvature is given by the *curvature tensor*:

$$R^{\nabla}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} X.$$

Torsion and curvature are two fundamental invariants of an affine connection; they are (1,2)-tensor and (1,3)-tensor, respectively.

Torsion-free connections enjoy the nice property that at each point on the manifold, one can find a local coordinate system such that the coefficients to express the affine connection is zero. When such a connection, in addition to being torsion-free, is curvature-free, then the above coordinate system globally exists.

An affine connection specifies the manner parallel transport of tangent vectors is performed on a manifold. Associated with any affine connection is a system of auto-parallel curves (also called geodesics): the family of auto-parallel curves passing through any point on the manifold is called the *geodesic spray*.

Torsion describes the manner of twist or screw of a moving frame around a curve about the tangent direction of the curve, as one travels along the curve. The torsion tensor characterizes how tangent spaces twist about a curve when they are parallel transported. As a comparison, curvature describes how the moving frame rolls along a curve without twisting. Torsion-free connections with non-zero curvatures are encountered in classical Information Geometry. Curvature-free connections with non-zero torsion have been encountered in theoretical physics known as Weitzenböck connection. Torsion-free and curvature-free connections are, of course, well-known as Hessian manifolds [16], the starting point for deformation and cohomology approach to affine connections [2].

2.2 Three kinds of transformations

The space of affine connections is convex in the following sense: if $\nabla, \tilde{\nabla}$ are affine connections, then so is $\alpha\nabla + \beta\tilde{\nabla}$ for any $\alpha, \beta \in \mathfrak{R}$ so long as $\alpha + \beta = 1$. This normalization condition is needed to ensure that the Leibniz rule holds. For example, $\frac{1}{2}\nabla_X Y + \frac{1}{2}\tilde{\nabla}_X Y$ is a connection, whereas $\frac{1}{2}\nabla_X Y + \frac{1}{2}\nabla_Y X$ is not; both are bilinear forms of X, Y.

Definition 1. *A* transformation of affine connections *is an arbitrary map from the set $\mathfrak{D}$ of affine connections ∇ of some differentiable manifold $\mathfrak{M}$ to $\mathfrak{D}$ itself.*

In this following, we investigate three kinds of transformations of affine connections:

(i) *translation* by a (1,2)-tensor;
(ii) *perturbation* by an invertible operator or (1,1)-tensor;
(iii) *conjugation* by a non-degenerate two-form or (0,2)-tensor.

Additive transformation: *K*-translation

Proposition 1. *Given two affine connections ∇ and $\tilde{\nabla}$, then their difference $K(X,Y) := \tilde{\nabla}_X Y - \nabla_X Y$ is a $(1,2)$-tensor. Conversely, any affine connection $\tilde{\nabla}$ arises this way as an additive transformation by a (1,2)-tensor $K(X,Y)$ from ∇:*

$$\tilde{\nabla}_X Y = \nabla_X Y + K(X,Y).$$

Proof. When scaling Y by an arbitrary smooth function f, the terms $X(f)Y$ obtained from the Leibniz rule for $\tilde{\nabla}$ and ∇ cancel out, so $K(X,Y)$ is in fact $C^\infty(\mathfrak{M})$-linear in each argument, hence a $(1,2)$-tensor, since each of $\tilde{\nabla}$ and ∇ are additive in each argument. For the converse, take K to be the $(1,2)$-tensor given by $\tilde{\nabla}_X Y - \nabla_X Y$.

It follows that a transformation T of affine connections is equivalent to a choice of $(1,2)$-tensor T_∇ for every affine connection ∇. Stated otherwise, given a connection, any other connection can be obtained in this way, i.e. by adding an appropriate (1,2)-tensor, which may or may not depend on ∇. When the (1,2)-tensor $K(X,Y)$ is independent of ∇, we say that $\tilde{\nabla}_X Y$ is a *K-translation* of ∇.

Additive transformations obviously commute with each other, since tensor addition is commutative. So additive transformations from a given affine connection form a group.

For any two connections ∇ and $\tilde{\nabla}$, their difference tensor $K(X,Y)$ decomposes in general as $\frac{1}{2}A(X,Y) + \frac{1}{2}B(X,Y)$ where A is symmetric and B is anti-symmetric. Since the difference between the torsion tensors of $\tilde{\nabla}$ and ∇ is given by

$$T^{\tilde{\nabla}}(X,Y) - T^{\nabla}(X,Y) = K(X,Y) - K(Y,X) = B(X,Y),$$

we have the following:

Proposition 2. *K-translaton of an affine connection preserves torsion if and only if K is symmetric: $K(X,Y) = K(Y,X)$.*

The symmetric part, $A(X,Y)$, of the difference tensor $K(X,Y)$ reflects a difference in the geodesic spray associated with each affine connection: $\widetilde{\nabla}$ and ∇ have the same families of geodesic spray if and only if $A(X,Y) = 0$.

The following examples are K-translations that will be discussed in great length later on:

(i) $\mathsf{P}^\vee(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(X)Y$, called $\mathsf{P}^\vee$*-transformation*;
(ii) $\mathsf{P}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X$, called P-*transformation*;
(iii) $\mathsf{Proj}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X + \tau(X)Y$, called *projective transformation*;
(iv) $\mathsf{D}(h,\xi) : \nabla_X Y \mapsto \nabla_X Y - h(Y,X)\xi$, called D-*transformation*, or dual-projective transformation.

Here, τ is an arbitrary one-form or (0,1)-tensor, h is a non-degenerate two-form or (0,2)-tensor, X, Y, ξ are all vector fields. From Proposition 2, $\mathsf{Proj}(\tau)$ is always torsion-preserving, while $\mathsf{D}(h,\xi)$ is torsion-preserving when h is symmetric.

It is obvious that $\mathsf{Proj}(\tau)$ is the composition of $\mathsf{P}(\tau)$ and $\mathsf{P}^\vee(\tau)$ for any τ. This may be viewed as follows: the P-transformation introduces torsion in one direction, i.e. it adds $B(X,Y) := \tau(Y)X - \tau(X)Y$ to the torsion tensor, but the $\mathsf{P}^\vee$ transformation cancels out this torsion, by adding $-B(X,Y) = \tau(X)Y - \tau(Y)X$, resulting in a torsion-preserving transformation of $\mathsf{Proj}(\tau)$.

Any affine connection ∇ on $T\mathfrak{M}$ induces an action on $T^*\mathfrak{M}$. The action of ∇ on a one-form ω is defined as:

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y).$$

When ∇ undergoes a K-translation, $\nabla_X Y \mapsto \nabla_X Y + K(X,Y)$ for a (1,2)-tensor K, then

$$(\nabla_X \omega)(Y) \mapsto (\nabla_X \omega)(Y) - \omega(K(X,Y)).$$

In particular, the transformation $\mathsf{P}^\vee(\tau)$ of a connection acting on $T\mathfrak{M}$ induces a change of $\mathsf{P}^\vee(-\tau)$ when the connection acts on $T^*\mathfrak{M}$.

Multiplicative transformation: *L*-pertubation Complementing the additive transformation, we define a "multiplicative" transformation of affine connections through an invertible operator $L : T\mathfrak{M} \to T\mathfrak{M}$.

Proposition 3. *([15]) Given an affine connection ∇ and an invertible operator L on $T\mathfrak{M}$, then $L^{-1}(\nabla_X(L(Y)))$ is also an affine connection.*

Proof. Additivity and $C^\infty(\mathfrak{M})$-linearity in X follow from the same properties for L and $\nabla_X Y$. Furthermore, for any scalar function ϕ,

$$\begin{aligned} L^{-1}(\nabla_X(L(\phi Y))) &= L^{-1}\big[X(\phi)L(Y) + \phi\nabla_X(L(Y))\big] \\ &= X(\phi)Y + \phi L^{-1}(\nabla_X(L(Y))), \end{aligned}$$

so the Leibniz rule holds.

Definition 2. *Given a connection ∇, the L-perturbation of ∇, denoted variously ∇^L, $L(\nabla)$, or $\Gamma_L(\nabla)$, is an endomorphism of $T\mathfrak{M}$ defined as:*

$$\Gamma_L(\nabla) \equiv L(\nabla) \equiv \nabla^L_X Y \equiv L^{-1}(\nabla_X LY).$$

Proposition 4. *([15]) The L-perturbations form a group such that group composition is simply operator concatenation: $\Gamma_K \circ \Gamma_L = \Gamma_{LK}$ for invertible operators K and L.*

Proof. We have

$$\begin{aligned} (\Gamma_K(\Gamma_L(\nabla)))_X Y &= K^{-1}\big[(\Gamma_L(\nabla))_X(K(Y))\big] \\ &= K^{-1}\big[L^{-1}(\nabla_X(L(K(Y))))\big] \\ &= (K^{-1}L^{-1})\big[\nabla_X((LK)(Y))\big] \\ &= (\Gamma_{LK}(\nabla))_X Y, \end{aligned}$$

as desired.

Conjugation transformation by h If h is any non-degenerate $(0,2)$-tensor, it induces isomorphisms $h(X,-)$ and $h(-,X)$ from vector fields X to one-forms. When h is not symmetric, these two isomorphisms are different. Given an affine connection ∇, we can take the covariant derivative of the one-form $h(Y,-)$ with respect to X, and obtain a corresponding one-form ω such that, when fixing Y,

$$\omega_X(Z) = X(h(Y,Z)) - h(Y, \nabla_X Z).$$

Since h is non-degenerate, there exists a U such that $\omega_X = h(U,-)$ as one-forms, so that

$$X(h(Y,Z)) = h(U(X,Y),Z) + h(Y,\nabla_X Z).$$

Defining $D(X,Y) := U(X,Y)$ gives a map from $T\mathfrak{M} \times T\mathfrak{M} \to T\mathfrak{M}$.

Proposition 5. *Taking $\tilde{\tilde{\nabla}}_X Y := D(X,Y)$ gives an affine connection $\tilde{\tilde{\nabla}}$ as induced from ∇.*

Proof. Linearity in X follows because $\nabla_X Z$ and $X(h(Y,Z))$ are both linear in X, for fixed Y and Z. Linearity in Y and the Leibniz rule are checked as follows: for vector fields Y, W and scalar function ϕ, we have

$$\begin{aligned} h(D(X,\phi Y + W), Z) &= X(h(\phi Y + W, Z)) - h(\phi Y + W, \nabla_X Z) \\ &= X(\phi)h(Y,Z) + \phi X(h(Y,Z)) + X(h(W,Z)) \\ &\quad - \phi h(Y,\nabla_X Z) - h(W,\nabla_X Z) \\ &= h(X(\phi)Y, Z) + \phi h(D(X,Y),Z) + h(D(X,W),Z). \end{aligned}$$

Since h is non-degenerate, $D(X,\phi Y + W) = X(\phi)Y + \phi D(X,Y) + D(X,W)$, as desired.

Definition 3. *This $\tilde{\tilde{\nabla}}$ is called the* left-conjugate *of ∇ with respect to h. The map taking ∇ to $\tilde{\tilde{\nabla}}$ will be denoted* $\mathsf{Left}(h)$. *Similarly, we have a right-conjugate of ∇ and an associated map* $\mathsf{Right}(h)$.

If $\tilde{h}(X,Y) := h(Y,X)$, then exchanging the first and second arguments of each h in the above derivation shows that $\mathsf{Left}(h) = \mathsf{Right}(\tilde{h})$ and $\mathsf{Right}(h) = \mathsf{Left}(\tilde{h})$. When h is symmetric or anti-symmetric, the left- and right-conjugates are equal; both reduce to the special case of the usual conjugate connection ∇^* with respect to h. In this case, conjugation is involutive: $(\nabla^*)^* = \nabla$.

For a non-degenerate (but not necessarily symmetric or anti-symmetric) h, if there exists a ∇ such that

$$Z(h(X,Y)) = h(\nabla_Z X, Y) + h(X, \nabla_Z Y),$$

then $\nabla = \mathsf{Left}(h)(\nabla) = \mathsf{Right}(h)(\nabla)$; in this case, ∇ is said to be parallel to the two-form h. Because in this case, $\nabla = \mathsf{Left}(\tilde{h})(\nabla) = \mathsf{Right}(\tilde{h})(\nabla)$, ∇ is also parallel to the two-form $\tilde{h}$.

2.3 Codazzi coupling and torsion preservation

Codazzi coupling of ∇ with operator L Let L be an isomorphism of the tangent bundle $T\mathfrak{M}$ of a smooth manifold $\mathfrak{M}$, i.e. L is a smooth section of the bundle $\mathrm{End}(T\mathfrak{M})$ such that it is invertible everywhere, i.e. an invertible $(1,1)$-tensor.

Definition 4. *Let L be an operator, and ∇ an affine connection. We call $\{\nabla, L\}$ a Codazzi pair if $(\nabla_X L)Y$ is symmetric in X and Y. In other words, the following identity holds*

$$(\nabla_X L)Y = (\nabla_Y L)X. \tag{1}$$

Here $(\nabla_X L)Y$ is, by definition,

$$(\nabla_X L)Y = \nabla_X(L(Y)) - L(\nabla_X Y).$$

We have the following characterization of Codazzi relations between an invertible operator and a connection:

Proposition 6. *([15]) Let ∇ and $\tilde{\tilde{\nabla}}$ be arbitrary affine connections, and L an invertible operator. Then the following statements are equivalent:*

1. *$\{\nabla, L\}$ is a Codazzi pair.*
2. *∇ and $\Gamma_L(\nabla)$ have equal torsions.*
3. *$\{\Gamma_L(\nabla), L^{-1}\}$ is a Codazzi pair.*

Proof. Since

$$L^{-1}\big[(\nabla_X L)(Y) - (\nabla_Y L)(X)\big] = \big[L^{-1}\nabla_X(L(Y)) - \nabla_X Y\big] - \big[L^{-1}\nabla_Y(L(X)) - \nabla_Y X\big],$$

the symmetry of $(\nabla_X L)(Y)$ in X and Y is equivalent to the equality of the torsion tensors of $\Gamma_L(\nabla)$ and ∇, and (1) ⇔ (2). As for (3), note that $\Gamma_{L^{-1}}(\Gamma_L(\nabla)) = \nabla$ by Proposition 4, so that ∇ and $\Gamma_L(\nabla)$ have equal torsions precisely when $\Gamma_L(\nabla)$ and $\Gamma_{L^{-1}}(\Gamma_L(\nabla))$ do.

Proposition 7. *Let $\{\nabla, L\}$ be a Codazzi pair. Let A be a symmetric (1,2)-tensor, and $\tilde{\nabla} = \nabla + A$. Then $\{\tilde{\nabla}, L\}$ forms a Codazzi pair if and only if L is self-adjoint with respect to A:*

$$A(L(X), Y) = A(X, L(Y))$$

for all vector fields X and Y.

In other words, A-translation preserves the Codazzi pair relationship of ∇ with L iff L is a *self-adjoint* operator with respect to A.

Proof. If $\tilde{\nabla}$ is the A-perturbation of ∇, then

$$\begin{aligned}(\tilde{\nabla}_X L)Y - (\tilde{\nabla}_Y L)X &= \tilde{\nabla}_X(L(Y)) - L(\tilde{\nabla}_X Y) - \tilde{\nabla}_Y(L(X)) + L(\tilde{\nabla}_Y X)\\ &= \nabla_X(L(Y)) - L(\nabla_X Y) - \nabla_Y(L(X)) + L(\nabla_Y X)\\ &\quad + A(X, L(Y)) - L(A(X,Y)) - A(L(X), Y) + L(A(X,Y))\\ &= (\nabla_X L)Y - (\nabla_Y L)X + A(X, L(Y)) - A(L(X), Y),\end{aligned}$$

and the result follows.

Therefore, for a fixed operator L, the Codazzi coupling relation can be interpreted as a quality of *equivalence classes* of connections modulo translations by symmetric (1,2)-tensors A with respect to which L is self-adjoint.

Codazzi coupling of $\boldsymbol{\nabla}$ with (0,2)-tensor $\boldsymbol{h}$ Now we investigate Codazzi coupling of ∇ with a non-degenerate (0,2)-tensor h. We introduce the (0,3)-tensor C defined by:

$$C(X, Y, Z) \equiv (\nabla_Z h)(X, Y) = Z(h(X, Y)) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y). \tag{2}$$

The tensor C is called the *cubic form* associated with $\{\nabla, h\}$ pair. When $C = 0$, then we say that h is parallel with respect to ∇.

Recall the definition of left-conjugate $\tilde{\nabla}$ with respect to a non-degenerate two-form h:

$$Z(h(X, Y)) = h(\tilde{\nabla}_Z X, Y) + h(X, \nabla_Z Y). \tag{3}$$

Using this relation in (2) gives

$$\begin{aligned}C(X, Y, Z) &\equiv (h(\tilde{\nabla}_Z X, Y) + h(X, \nabla_Z Y)) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y)\\ &= h((\tilde{\nabla} - \nabla)_Z X, Y),\end{aligned}$$

so that

$$C(X, Y, Z) - C(Z, Y, X) = h(T^{\tilde{\nabla}}(Z, X) - T^{\nabla}(Z, X), Y),$$

or

$$(\nabla_Z h)(X, Y) - (\nabla_X h)(Z, Y) = h(T^{\tilde{\nabla}}(Z, X) - T^{\nabla}(Z, X), Y).$$

The non-degeneracy of h implies that $C(X, Y, Z) = C(Z, Y, X)$ if and only if ∇ and $\tilde{\nabla}$ have equal torsions. This motivates the following definition, in analogy with the previous subsection.

Definition 5. *Let h be a two-form, and ∇ an affine connection. We call $\{\nabla, h\}$ a* Codazzi pair *if $(\nabla_Z h)(X,Y)$ is symmetric in X and Z.*

The cubic form associated with the pair $\{\widetilde{\nabla}, h\}$, denoted as $\widetilde{C}$, is:

$$\widetilde{C}(X,Y,Z) \equiv (\widetilde{\nabla}_Z h)(X,Y) = Z(h(X,Y)) - h(\widetilde{\nabla}_Z X, Y) - h(X, \widetilde{\nabla}_Z Y).$$

We derive, analogously,

$$\widetilde{C}(X,Y,Z) = h(X, (\nabla - \widetilde{\nabla})_Z Y),$$

from which we obtain

$$\widetilde{C}(X,Y,Z) - \widetilde{C}(Z,Y,X) = h(X, T^{\nabla}(Y,Z) - T^{\widetilde{\nabla}}(Y,Z)).$$

Summarizing the above results, we have, in analogy with Proposition 6:

Proposition 8. *Let ∇ be an arbitrary affine connection, h be an arbitrary non-degenerate two-form, and $\widetilde{\nabla}$ denotes the left-conjugate of ∇ with respect to h. Then the following statements are equivalent:*

1. *$\{\nabla, h\}$ is a Codazzi pair.*
2. *∇ and $\widetilde{\nabla}$ have equal torsions.*
3. *$\{\widetilde{\nabla}, h\}$ is a Codazzi pair.*

This proposition says that an arbitrary affine connection ∇ and an arbitrary non-degenerate two-form h form a Codazzi pair precisely when ∇ and its left-conjugate $\widetilde{\nabla}$ with respect to h have equal torsions.

Note that the definition of Codazzi pairing of ∇ with h is with respect to the first slot of h, left-conjugate is a more useful concept. The left- and right-conjugate of a connection ∇ with respect to h can become one and the same, when (i) h is symmetric; or (ii) ∇ is parallel to h: $\nabla h = 0$. These scenarios will be discussed next.

From the definition of the cubic form (2), it holds that

$$(\nabla_Z \tilde{h})(X,Y) = C(Y,X,Z) = (\nabla_Z h)(Y,X)$$

where $\tilde{h}(X,Y) = h(Y,X)$. So $C(X,Y,Z) = C(Y,X,Z)$ holds for any vector fields X, Y, Z if and only if $h = \tilde{h}$, that is, h is symmetric.

Proposition 9. *For a non-degenerate two-form h, $\nabla h = 0$ if and only if ∇ equals its left (equivalently, right) conjugate with respect to h.*

Proof. Since $\nabla h = 0$ means that

$$Z(h(X,Y)) = h(\nabla_Z X, Y) + h(X, \nabla_Z Y)$$

for all vector fields X, Y, Z, the result follows from definition of left and right conjugate.

Note that in the above proof, we do not require h to be symmetric.

The following standard definition is a special case:

Definition 6. *If g is a Riemannian metric, and ∇ an affine connection, the* conjugate connection *∇^* is the left-conjugate (or equivalently, right-conjugate) of ∇ with respect to g. Denote $\mathsf{C}(g)$ as the involutive map that sends ∇ to ∇^*.*

This leads to the well-known result:

Corollary 7. *$\{\nabla, g\}$ is a Codazzi pair if and only if $\mathsf{C}(g)$ preserves the torsion of ∇.*

Proof. Immediate from Proposition 8.

2.4 Linking two Codazzi couplings

In order to relate these two notions of Codazzi pairs, one involving perturbations via a operator L, and one involving conjugation with respect to a two-form h, we need the following definition:

Definition 8. *The* left L-perturbation *of a $(0,2)$-tensor h is the $(0,2)$-tensor $h_L(X,Y) := h(L(X),Y)$. Similarly, the* right L-perturbation *is given by $h^L(X,Y) := h(X,L(Y))$.*

Proposition 10. *Let h be a non-degenerate $(0,2)$-tensor. If $\tilde{\nabla}$ is the left-conjugate of ∇ with respect to h, then the left-conjugate of ∇ with respect to h_L is $\Gamma_L(\tilde{\nabla})$. Analogously, if $\hat{\nabla}$ is the right-conjugate of ∇ with respect to h, then the right-conjugate of ∇ with respect to h^L is $\Gamma_L(\hat{\nabla})$.*

Proof. The following equations are equivalent:

$$\begin{aligned} X(h(Y,Z)) &= h(\tilde{\nabla}_X Y, Z) + h(Y, \nabla_X Z) \\ X(h(L(Y),Z)) &= h(\tilde{\nabla}_X (L(Y)), Z) + h(L(Y), \nabla_X Z) \\ X(h_L(Y,Z)) &= h_L((\Gamma_L(\tilde{\nabla}))_X Y, Z) + h_L(Y, \nabla_X Z). \end{aligned}$$

The analogous statement follows in exactly the same way:

$$\begin{aligned} X(h(Y,Z)) &= h(\nabla_X Y, Z) + h(Y, \hat{\nabla}_X Z) \\ X(h(Y,L(Z)) &= h(\nabla_X Y, L(Z)) + h(Y, \hat{\nabla}_X (L(Z))) \\ X(h^L(Y,Z)) &= h^L(\nabla_X Y, Z) + h^L(Y, (\Gamma_L(\hat{\nabla}))_X Z). \end{aligned}$$

Corollary 9. *Let $\tilde{\nabla}$ be the left-conjugate of ∇ with respect to h. If $\{\nabla, h\}$ and $\{\tilde{\nabla}, L\}$ are Codazzi pairs, then $\{\nabla, h_L\}$ is a Codazzi pair.*

Proof. By Proposition 6, we need only show that ∇ has equal torsions with its left-conjugate with respect to h_L. The above proposition says that this left-conjugate is $\Gamma_L(\tilde{\nabla})$, and Propositions 6 and 8 imply that ∇ has equal torsions with $\tilde{\nabla}$ and $\Gamma_L(\tilde{\nabla})$.

The following result describes how L-perturbation of a two-form (i.e., a (0,2)-tensor) induces a corresponding "L-perturbation" on the cubic form $C(X,Y,Z)$ as defined in the previous subsubsection.

Proposition 11. *Let $h(X,Y)$ be a non-degenerate two-form and L be an invertible operator. Write $f := h_L$ for notational convenience. Then, for any connection ∇,*

$$C_f(X,Y,Z) = C_h(L(X),Y,Z) + h((\nabla_Z L)X, Y),$$

where C_f and C_h are the cubic tensors of ∇ with respect to f and h.

Proof. By direct calculation,

$$\begin{aligned} (\nabla_Z f)(X,Y) &= Z(f(X,Y)) - f(\nabla_Z X, Y) - f(X, \nabla_Z Y) \\ &= Z(h(L(X),Y)) - h(L(\nabla_Z X), Y) - h(L(X), \nabla_Z Y) \\ &= (\nabla_Z h)(L(X),Y) + h(\nabla_Z(L(X)), Y) - h(L(\nabla_Z X), Y) \\ &= (\nabla_Z h)(L(X),Y) + h((\nabla_Z L)X, Y). \end{aligned}$$

With the notion of L-perturbation of a two-form, we can now state our main theorem describing the relation between L-perturbation of an affine connection and h-conjugation of that connection.

Theorem 10. *Fix a non-degenerate (0, 2)-tensor h, denote its L-perturbations $h_L(X,Y) = h(L(X),Y)$ and $h^L(X,Y) = h(X,L(Y))$ as before. For an arbitrary connection ∇, denote its left-conjugate (respectively, right-conjugate) of ∇ with respect to h as $\tilde{\nabla}$ (respectively, $\hat{\nabla}$). Then:*

(i) $\nabla h_L = 0$ *if and only if* $\Gamma_L(\tilde{\nabla}) = \nabla$.

(ii) $\nabla h^L = 0$ *if and only if* $\Gamma_L(\widehat{\nabla}) = \nabla$.

Proof. This follows from Propositions 9 and 10 and Corollary 9,

This Theorem means that ∇ is parallel to h_L (respectively, h^L) if and only if the left (respectively, right) h-conjugate of the L-perturbation of ∇ is ∇ itself. In this case, L-perturbation of ∇ and h-conjugation of ∇ can be coupled to render the perturbed two-form parallel with respect to ∇. Note that in the above Theorem, there is no torsion-free assumption about ∇, no symmetry assumption about h, and no Codazzi pairing assumption of $\{\nabla, h\}$.

2.5 Commutation relations between transformations

Definition 11. *Given a one-form τ, we define the following transformation of an affine connection ∇:*

(i) $\mathsf{P}^\vee$*-transformation, denoted* $\mathsf{P}^\vee(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(X)Y$*;*
(ii) P*-transformation, denoted* $\mathsf{P}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X$*.*
(iii) projective transformation, denoted $\mathsf{Proj}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X + \tau(X)Y$*.*

All these are "translations" of an affine connection (see Section 2). The first two transformations, (i) and (ii), are "half" of the projective transformation in (iii). While the projective transformation Proj of ∇ preserves its torsion, both $\mathsf{P}^\vee$-transformation and P-transformation introduce torsion (in opposite amounts).

Definition 12. *Given a vector field ξ and a non-degenerate 2-form h, we define the* D*-transformation of an affine connection ∇ as*

$$\mathsf{D}(h, \xi) : \nabla_X Y \mapsto \nabla_X Y - h(Y, X)\xi.$$

Furthermore, the transformation $\tilde{\mathsf{D}}(h, \xi)$ *is defined to be* $\mathsf{D}(\tilde{h}, \xi)$*.*

These transformations behave very nicely with respect to left and right h-conjugation, as well as L-perturbation. More precisely, we make the following definition:

Definition 13. *We call left (respectively right) h-*image *of a transformation of a connection the induced transformation on the left (respectively right) h-conjugate of that connection. Similarly, we call L-*image *of a transformation of a connection the induced transformation on the L-perturbation of that connection.*

Proposition 12. *The left and right h-images of* $\mathsf{P}^\vee(\tau)$ *are both* $\mathsf{P}^\vee(-\tau)$*.*

Proof. Let ∇^{left} and ∇^{right} denote left and right h-conjugates, for notational convenience. The following equations, quantified over all vector fields $X, Y,$ and Z, are equivalent, since h is non-degenerate:

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \tau(X)Y \\ h(\tilde{\nabla}_X Y, Z) &= h(\nabla_X Y, Z) + h(\tau(X)Y, Z) \\ X(h(Y,Z)) - h(Y, \tilde{\nabla}^{\text{right}}_X Z) &= X(h(Y,Z)) - h(Y, \nabla^{\text{right}}_X Z) + h(Y, \tau(X)Z) \\ \tilde{\nabla}^{\text{right}}_X Z &= \nabla^{\text{right}}_X Z - \tau(X)Z. \end{aligned}$$

Therefore, the right h-image of $\mathsf{P}^\vee(\tau)$ is a translation by $K(X,Z) = -\tau(X)Z$, i.e. a $\mathsf{P}^\vee(-\tau)$ transformation. Similarly, the following equations are equivalent:

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \tau(X)Y \\ h(Z, \tilde{\nabla}_X Y) &= h(Z, \nabla_X Y) + h(Z, \tau(X)Y) \\ X(h(Z,Y)) - h(\tilde{\nabla}^{\text{left}}_X Z, Y) &= X(h(Z,Y)) - h(\nabla^{\text{left}}_X Z, Y) + h(\tau(X)Z, Y) \\ \tilde{\nabla}^{\text{left}}_X Z &= \nabla^{\text{left}}_X Z - \tau(X)Z, \end{aligned}$$

so the left h-image is also $\mathsf{P}^\vee(-\tau)$, as desired.

Proposition 13. *The L-image of* $\mathsf{P}^\vee(\tau)$ *is* $\mathsf{P}^\vee(\tau)$ *itself.*

Proof. The following equations are equivalent:

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + \tau(X)Y \\
\tilde{\nabla}_X(L(Y)) &= \nabla_X(L(Y)) + \tau(X)L(Y) \\
L^{-1}(\tilde{\nabla}_X(L(Y))) &= L^{-1}(\nabla_X(L(Y))) + L^{-1}(\tau(X)L(Y)) \\
(\mathit{\Gamma}_L\tilde{\nabla})_X Y &= (\mathit{\Gamma}_L\nabla)_X Y + \tau(X)Y,
\end{aligned}$$

so the L-image of $\mathsf{P}^\vee(\tau)$ is a translation by $K(X,Y) = \tau(X)Y$, as desired.

Proposition 14. *If V is a vector field, so that $h(V,-)$ is a one-form, then the left h-image of $\mathsf{P}(h(V,-))$ is $\mathsf{D}(h,V)$, while the right h-image of $\mathsf{P}(h(-,V))$ is $\tilde{\mathsf{D}}(h,V)$.*

Proof. Keep the notations $\nabla^{\text{left/right}}$ from before. The following equations are equivalent:

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + h(V,Y)X \\
h(Z,\tilde{\nabla}_X Y) &= h(Z,\nabla_X Y) + h(Z,h(V,Y)X) \\
X(h(Z,Y)) - h(\tilde{\nabla}^{\text{left}}_X Z, Y) &= X(h(Z,Y)) - h(\nabla^{\text{left}}_X Z, Y) + h(h(Z,X)V,Y) \\
\tilde{\nabla}^{\text{left}}_X Z &= \nabla^{\text{left}}_X Z - h(Z,X)V,
\end{aligned}$$

as desired. Symmetrically, the following equations are equivalent:

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + h(Y,V)X \\
h(\tilde{\nabla}_X Y, Z) &= h(\nabla_X Y, Z) + h(h(Y,V)X, Z) \\
X(h(Y,Z)) - h(Y,\tilde{\nabla}^{\text{right}}_X Z) &= X(h(Y,Z)) - h(Y,\nabla^{\text{right}}_X Z) + h(Y,h(X,Z)V) \\
\tilde{\nabla}^{\text{right}}_X Z &= \nabla^{\text{right}}_X Z - h(X,Z)V,
\end{aligned}$$

as desired.

Proposition 15. *The L-image of $\mathsf{D}(h,V)$ is $\mathsf{D}(h_L, L^{-1}(V))$, whereas the L-image of $\tilde{\mathsf{D}}(h,V)$ is $\tilde{\mathsf{D}}(h^L, L^{-1}(V))$.*

Proof. The following equations are equivalent:

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y - h(Y,X)V \\
L^{-1}(\tilde{\nabla}_X(L(Y))) &= L^{-1}\Big(\nabla_X(L(Y)) - h(L(Y),X)V\Big) \\
(\mathit{\Gamma}_L\tilde{\nabla})_X Y &= (\mathit{\Gamma}_L\nabla)_X Y - h_L(Y,X)L^{-1}(V),
\end{aligned}$$

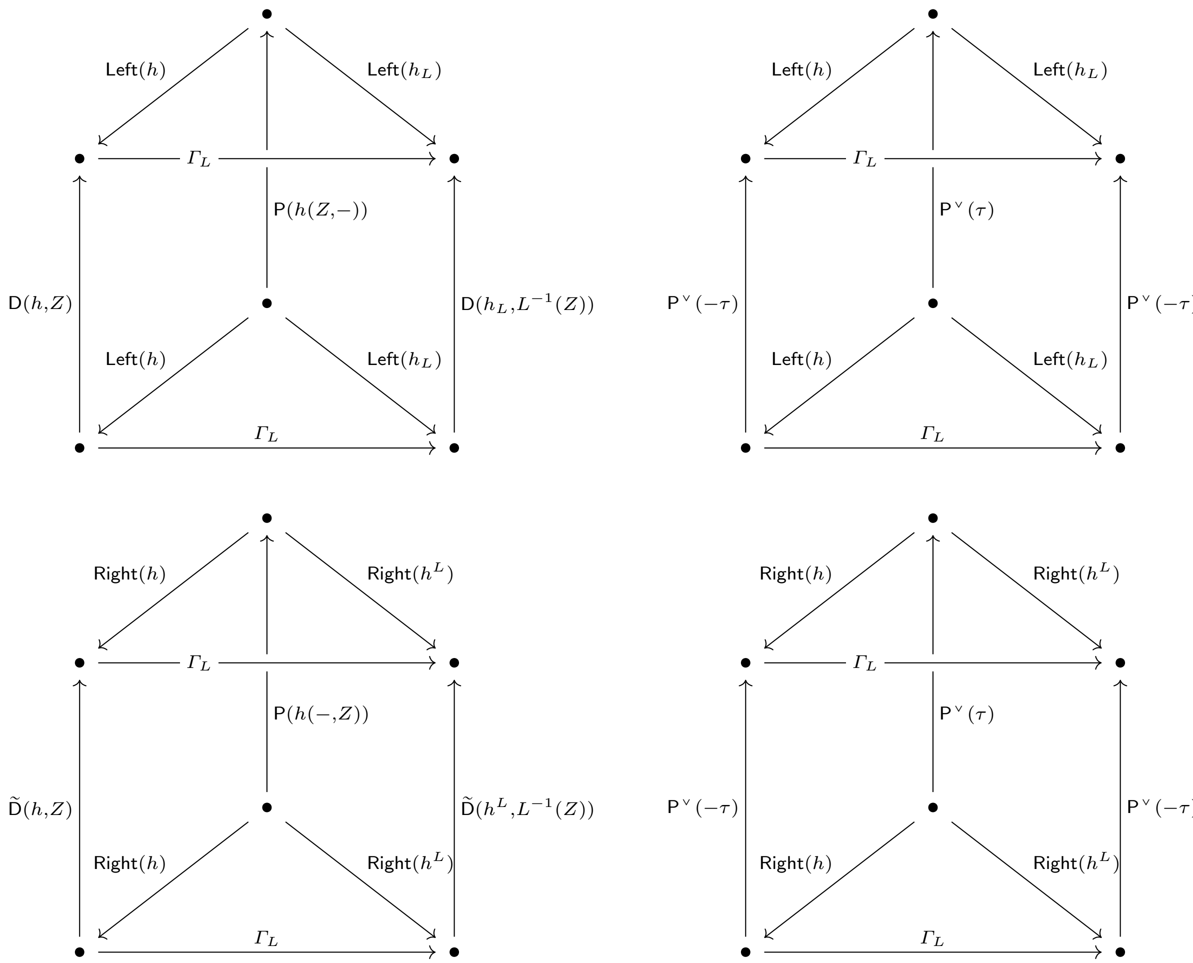
as desired. Symmetrically, the following equations are equivalent:

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y - h(X,Y)V \\
L^{-1}(\tilde{\nabla}_X(L(Y))) &= L^{-1}\Big(\nabla_X(L(Y)) - h(X,L(Y))V\Big) \\
(\mathit{\Gamma}_L\tilde{\nabla})_X Y &= (\mathit{\Gamma}_L\nabla)_X Y - h^L(Y,X)L^{-1}(V),
\end{aligned}$$

as desired.

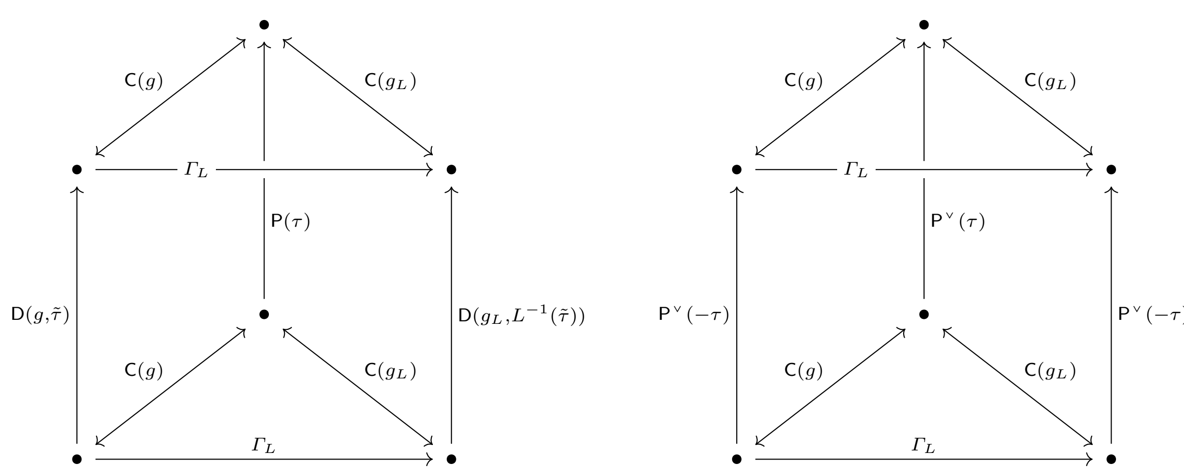
We summarize the above results in the following commutative prisms.

Theorem 14. *Let h be a non-degenerate two-form, L be an invertible operator, Z be a vector field, and τ be a one-form. Then we have four commutative prisms:*



Proof. For the triangles, Proposition 10 suffices. The above Propositions imply that the left and middle squares commute, which in turn implies that the right squares commute as well.

Corollary 15. *With respect to a Riemannian metric g, an invertible operator L, and an arbitrary one-form τ, we have the following commutative prisms:*



Each $\bullet$ represents the space of affine connections of some differentiable manifold $\mathfrak{M}$.

These commutative prisms are extremely useful in characterizing transformations that preserve Codazzi coupling. Indeed, Propositions 6 and 8 say that it is enough to characterize the torsion introduced by the various translations in Definitions 11 and 12. We have the following:

Proposition 16. *With respect to the transformation of connections:* $\nabla_X Y \mapsto \tilde{\nabla}_X Y$*, let* $I(X,Y)$ *denote the induced change in torsion, i.e.* $B(X,Y) := T^{\tilde{\nabla}}(X,Y) - T^{\nabla}(X,Y)$*. Then*

(i) For $\mathsf{P}^\vee(\tau)$*:* $B(X,Y) = \tau(X)Y - \tau(Y)X$*.*

(ii) For $\mathsf{P}(\tau)$*:* $B(X,Y) = \tau(Y)X - \tau(X)Y$*.*

(iii) For $\mathsf{Proj}(\tau)$*:* $B(X,Y) = 0$*. Projective transformations are torsion-preserving.*

(iv) For $\mathsf{D}(h,\xi)$*:* $B(X,Y) = \Big(h(X,Y) - h(Y,X)\Big)\xi$*.*

(v) For $\tilde{\mathsf{D}}(h,\xi)$*:* $B(X,Y) = \Big(h(Y,X) - h(X,Y)\Big)\xi$*.*

Proof. Clear from Definitions 11 and 12.

Note that the torsion change $B(X,Y)$ is same in amount but opposite in sign for cases (i) and (ii), and for cases (iv) and (v). $B(X,Y)$ is always zero for case (iii), and becomes zero for cases (iv) and (v) when h is symmetric.

Corollary 16. $\mathsf{P}^\vee$*-transformations* $\mathsf{P}^\vee(\tau)$ *preserve Codazzi pairing of* ∇ *with* L*: For arbitrary one-form* τ*, if* $\{\nabla, L\}$ *is a Codazzi pair, then* $\{\mathsf{P}^\vee(\tau)\nabla, L\}$ *is a Codazzi pair.*

Proof. By assumption, ∇ and $\Gamma_L\nabla$ have equal torsions. Since $\mathsf{P}^\vee(\tau)$ perturbs both torsions by the same amount $\tau(X)Y - \tau(Y)X$, it preserves the equality of the torsions.

Corollary 17. D*-transformations* $\mathsf{D}(g,\xi)$ *preserve Codazzi pairing of* ∇ *with* L*: For any symmetric two-form* g*, vector field* ξ*, and operator* L *that is self-adjoint with respect to* g*, if* $\{\nabla, L\}$ *is a Codazzi pair, then* $\{\mathsf{D}(g,\xi)\nabla, L\}$ *is a Codazzi pair.*

Proof. Under these conditions, g and g_L are both symmetric, so $\mathsf{D}(g,\xi)$ and its L-image $\mathsf{D}(g_L, L^{-1}V)$ induce no torsion, by Proposition 16. Therefore $\mathsf{D}(g,\xi)$ preserves the equality of torsions between ∇ and its L-perturbation.

Towards the end of the next section, we will show how composing the P, $\mathsf{P}^\vee$, and D transformations can cancel the induced torsions, resulting in interesting classes of transformations that preserve Codazzi pairs.

3 Simultaneous transformation of metric and connection

3.1 Projective transformation and torsion preservation

The family of auto-parallel curves at any given point forms a geodesic spray. One can show that for any two connections $\nabla, \tilde{\nabla}$ with the same geodesic spray, i.e.,

$$\tilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma,$$

then $\nabla, \tilde{\nabla}$ can only differ by torsion. Hence, for any ∇, we can always obtain a torsion-free connection $\tilde{\nabla}$ that shares the same geodesic (auto-parallel curve) with ∇:

$$\tilde{\nabla}_X Y = -\frac{1}{2}T(X,Y) + \nabla_X Y.$$

We are interested in transformations of ∇ that preserve the geodesic arcs ("pre-geodesics"), i.e. preserve geodesics up to reparameterization. The following well-known result (see, e.g., [12]) characterizes what is known as a *projective transformation* in the study of torsion-free connections. The following is well-known:

Proposition 17. *Two connections* ∇ *and* $\tilde{\nabla}$ *have the same torsion and the same geodesic arcs if and only if there exists a one-form* τ *such that* $\tilde{\nabla}_X Y = \nabla_X Y + \tau(X)Y + \tau(Y)X$*.*

Proof. If the latter equation holds, and γ is a ∇-geodesic, then

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma} + 2\tau(\dot{\gamma})\dot{\gamma}$$

is proportional to $\dot{\gamma}$ so γ is also a $\widetilde{\nabla}$-geodesic, upon reparameterization. Replacing τ with $-\tau$, the statement holds with ∇ and $\widetilde{\nabla}$ reversed, so such a relation implies that they have the same geodesic arcs. Evidently it also implies that the two connections have the same torsion.

Conversely, Proposition 2 tells us that if ∇ and $\widetilde{\nabla}$ have the same torsion, then the difference tensor $K(X,Y)$ is symmetric. Furthermore, the requirement that $K(X,X)$ is proportional to X for all X (which follows easily from the requirement that the two connections have the same geodesic arcs) implies that

$$K(X+Y, X+Y) = K(X,X) + K(Y,Y) + 2K(X,Y)$$

is proportional to $X+Y$, and in particular that $K(X,Y)$ is in $\mathrm{span}(X,Y)$ for all X,Y. We may therefore define functions α, β from vector fields to scalar fields by

$$K(X,Y) = \alpha(X)Y + \beta(Y)X$$

for all X,Y. Simple linearity considerations imply that α and β are one-forms, and symmetry of K implies that $\alpha = \beta$. Then taking $\tau := \alpha = \beta$ suffices.

Traditionally, two connections whose difference tensor K is expressible by $K(X,Y) := \tau(X)Y + \tau(Y)X$ is called a *projective transformation*, which both preserves pre-geodesics and preserves torsion associated with two connections. There are many transformations of affine connections that preserve geodesic arcs (pre-geodesic), but not necessarily torsion. For example, the composition of any projective transformation with any additive transformation given by an anti-symmetric $(1,2)$-tensor B will be a geodesic transformation, since the addition of B affects the change in torsion but not the geodesic-arc-preserving relation.

3.2 Codazzi transformation

The Codazzi transformation for a metric g and affine connection ∇ has been defined as

$$\begin{aligned} g(X,Y) &\mapsto e^{\phi} g(X,Y) \\ \nabla_X Y &\mapsto \nabla_X Y + X(\phi)Y + Y(\phi)X \end{aligned}$$

for any smooth function ϕ. It is a known result that this preserves Codazzi pairs $\{\nabla, g\}$. Furthermore, this transformation can be described as follows: it is a (torsion-preserving) projective transformation $\mathsf{P}(d\phi)\mathsf{P}^{\vee}(d\phi)$ applied to ∇, and an L-perturbation $g \mapsto g_{e^{\phi}}$ applied to the metric, where e^{ϕ} is viewed as an invertible operator. This suggests the following generalization of Codazzi transformations:

Definition 18. *A generalized Codazzi transformation* $\mathsf{Cod}(\tau, L)$ *consists of an* L*-perturbation of the metric* g *and a torsion-preserving projective transformation* $\mathsf{P}(\tau)\mathsf{P}^{\vee}(\tau) = \mathsf{Proj}(\tau)$ *applied to the connection* ∇.

This is a very general transformation that includes arbitrary transformations of the Riemannian metric (including the more specialized conformal transformation of the metric):

Lemma 1. *If an invertible operator* L *is self-adjoint with respect to* g*, then* $g_L(X,Y) := g(L(X),Y) = g(X, L(Y))$ *is a symmetric two-form. Any other Riemannian metric can be obtained by perturbing* g *with some* g*-self-adjoint operator* L.

Proof. By taking a frame that is orthonormal with respect to g, an operator L is g-self-adjoint precisely when its matrix representation with respect to the orthonormal frame is symmetric. Any other Riemannian metric $\tilde{g}$ has some matrix $\tilde{G}$ with respect to this orthonormal frame, which is symmetric (and positive definite). Taking L to have that same matrix $\tilde{G}$ yields the result.

Our framework allows us to view the Codazzi-preserving property of the Codazzi transformation in a more general light:

Proposition 18. *A generalized Codazzi transformation* $\mathsf{Cod}(\tau, L)$ *induces the transformation* $\mathsf{P}^\vee(-\tau)\Gamma_L\mathsf{D}(g,\tilde{\tau})$ *on the conjugate connection. Here* $\tilde{\tau}$ *denotes the vector field corresponding to the one-form* τ *under the musical isomorphism of* g, *i.e.* $\tau(X) = g(X, \tilde{\tau})$ *for any vector field* X.

Proof. Clear from Corollary 15.

Proposition 19. *A generalized Codazzi transformation* $\mathsf{Cod}(\tau, L)$ *preserves Codazzi pairs* $\{\nabla, g\}$ *precisely when the torsion introduced by* Γ_L *cancels with that introduced by* $\mathsf{P}^\vee(-\tau)$. *In other words,*

$$L^{-1}(\nabla_X(L(Y))) - L^{-1}(\nabla_Y(L(X))) - \nabla_X Y + \nabla_Y X = Y\tau(X) - \tau(Y)X.$$

Proof. This follows from the previous proposition and the fact that $\mathsf{P}(\tau)\mathsf{P}^\vee(\tau)$ preserves torsion.

The fact that Codazzi transformations preserve Codazzi pairs follows from the fact that if ϕ is a smooth function viewed as an operator, $\Gamma_{e^\phi} = \mathsf{P}^\vee(d\phi)$ by the Leibniz rule, and therefore $\mathsf{P}^\vee(-d\phi)\Gamma_{e^\phi}$ is the identity. In fact, the converse holds:

Theorem 19. *If a generalized Codazzi transformation* $\mathsf{Cod}(\tau, L)$ *preserves Codazzi pairs* $\{\nabla, g\}$, *then it is in fact a Codazzi transformation, i.e.* $L = e^\phi$ *and* $\tau = d\phi$ *for some smooth function* ϕ. *(We assume for convenience that* $\dim \mathfrak{M} \geqslant 4$.*)*

Proof. By Proposition 19, it suffices to ask, for a fixed invertible operator L, when the following equation holds for all vector fields X and Y, and affine connections ∇:

$$\nabla_X(L(Y)) - \nabla_Y(L(X)) = L(\nabla_X Y - \nabla_Y X + \tau(X)Y - \tau(Y)X).$$

This applies (locally) to all ∇ because any ∇ locally admits a g such that $\{\nabla, g\}$ is Codazzi. For instance, take g such that $\nabla g = 0$, i.e. ∇ is metrical with respect to g. It is remarkable that this condition does not depend on g, even though the Codazzi condition on $\{\nabla, g\}$ certainly does. Within our framework, this is essentially because $\mathsf{D}(g, \tilde{\tau})$ is torsion-preserving.

If L is not a multiple of the identity at some point $p \in \mathfrak{M}$, then we may find $v \in T_p\mathfrak{M}$ such that v is not proportional to $L_p(v)$. Via a small perturbation of v (relying on the fact that $\dim \mathfrak{M} \geqslant 4$), we may find $w \in T_p\mathfrak{M}$ such that $v, w, L(v)$, and $L(w)$ are linearly independent. Now extend v and w to smooth vector fields X and Y such that $X_p = v$ and $Y_p = w$. In a sufficiently small neighborhood U of p, we have that X, Y, $L(X)$, and $L(Y)$ are linearly independent.

An affine connection ∇ on an open subset $U \subset \mathfrak{M}$ can be defined by picking a frame $\{s_i\}_{i=1}^{\dim \mathfrak{M}}$ of vector fields defined on U, and arbitrarily defining $\nabla_{s_i} s_j$ for each i and j. By a partition of unity argument, we can find a smaller neighborhood U' such that $p \in U' \subset U$, and $\tilde{\nabla}$ is a connection on $\mathfrak{M}$ such that $\nabla = \tilde{\nabla}$ on U'. In particular, taking X, Y, $L(X)$, and $L(Y)$ as members of the frame, we find that $\nabla_X(L(Y))$, $\nabla_Y(L(X))$, $\nabla_X Y$, and $\nabla_Y X$ can be arbitrary vector fields, so the equation above cannot always hold.

This contradiction shows that L is everywhere a multiple of the identity, i.e. it is given by a smooth function. Since it is invertible (lest Γ_L be ill-defined), we may take this function to be e^ϕ for some ϕ. Then the equation above simplifies to

$$X(\phi)Y - Y(\phi)X = \tau(X)Y - \tau(Y)X,$$

which evidently implies that $X(\phi) = \tau(X)$ for all X, e.g. by taking Y linearly independent from X. Therefore $\tau = d\phi$, as desired.

Corollary 20. *The equations* $\Gamma_L = \mathsf{P}^\vee(\tau)$ *and* $\Gamma_L = \mathsf{P}(\tau)$ *have only the solutions* $L = e^\phi$ *and* $\tau = \pm d\phi$ *(respectively), where* ϕ *is any smooth function. (Again,* $\dim \mathfrak{M} \geqslant 4$ *for convenience.)*

Proof. In these cases, Γ_L will satisfy the torsion-canceling condition of Proposition 19, and the preceding theorem implies the result.

3.3 The conformal-projective transformation

As a generalization of the Codazzi transformation, we have the definition of a conformal-projective transformation [7]:

$$\begin{aligned} g(X,Y) &\mapsto e^{\psi+\phi} g(X,Y) \\ \nabla_X Y &\mapsto \nabla_X Y - g(X,Y)\operatorname{grad}_g \psi + X(\phi)Y + Y(\phi)X \end{aligned}$$

for any smooth functions ψ and ϕ. In our framework, we see that this transformation can be expressed as follows: it is an $e^{\psi+\phi}$-perturbation of the metric g, along with the affine connection transformation

$$\mathsf{D}(g, \operatorname{grad}_g \psi)\mathsf{Proj}(d\phi) = \mathsf{D}(g, \operatorname{grad}_g \psi)\mathsf{P}(d\phi)\mathsf{P}^\vee(d\phi).$$

The induced transformation on the conjugate connection will be

$$\begin{aligned} \mathit{\Gamma}_{e^{\psi+\phi}}\mathsf{P}(d\psi)\mathsf{D}(g, \operatorname{grad}_g \phi)\mathsf{P}^\vee(-d\phi) &= P^\vee(d\phi + d\psi)\mathsf{P}(d\psi)\mathsf{D}(g, \operatorname{grad}_g \phi)\mathsf{P}^\vee(-d\phi) \\ &= \mathsf{P}^\vee(d\psi)\mathsf{P}(d\psi)\mathsf{D}(g, \operatorname{grad}_g \phi) \\ &= \mathsf{D}(g, \operatorname{grad}_g \phi)\mathsf{Proj}(d\psi), \end{aligned}$$

which is a translation of the same form as before, but with ψ and ϕ exchanged. (The additional $\mathit{\Gamma}_{e^{\psi+\phi}}$ in front is induced by the $e^{\psi+\phi}$-perturbation of the metric.) In particular, this is a torsion-preserving transformation, because D and Proj are, which shows that conformal-projective transformations preserve Codazzi pairs $\{\nabla, g\}$. In analogy with the previous subsection, we have the following:

Definition 21. *Let V and W be vector fields, and L an invertible operator. A* generalized conformal-projective transformation $\mathsf{CP}(V,W,L)$ *consists of an L-perturbation of the metric g along with a torsion-preserving transformation $\mathsf{D}(g,W)\mathsf{Proj}(\tilde{V})$ of the connection, where $\tilde{V}$ is the one-form given by $\tilde{V}(X) := g(V,X)$ for any vector field X.*

Proposition 20. *A generalized conformal-projective transformation $\mathsf{CP}(V,W,L)$ induces the transformation $\mathit{\Gamma}_L\mathsf{P}(\tilde{W})\mathsf{D}(g,V)\mathsf{P}^\vee(-\tilde{V})$ on the conjugate connection.*

Proof. Clear from Corollary 15.

Proposition 21. *A generalized conformal-projective transformation $\mathsf{CP}(V,W,L)$ preserves Codazzi pairs $\{\nabla, g\}$ precisely when the torsion introduced by $\mathit{\Gamma}_L$ cancels with that introduced by $\mathsf{P}(\tilde{W})\mathsf{P}^\vee(-\tilde{V})$, i.e.*

$$L^{-1}(\nabla_X(L(Y))) - L^{-1}(\nabla_Y(L(X))) - \nabla_X Y + \nabla_Y X = (\tilde{W}+\tilde{V})(X)Y + (\tilde{W}+\tilde{V})(Y)X.$$

Proof. This follows from the fact that D and Proj preserve torsion.

Theorem 22. *A generalized conformal-projective transformation $\mathsf{CP}(V,W,L)$ preserves Codazzi pairs $\{\nabla, g\}$ if and only if $L = e^f$ for some smooth function f, and $\tilde{V} + \tilde{W} = df$. (Again, $\dim \mathfrak{M} \geqslant 4$.)*

Proof. The above proposition shows that the torsion cancellation condition is identical to that of Theorem 19, with τ replaced by $\tilde{W} + \tilde{V}$. The proof of that theorem implies the result.

This class of transformations is *strictly larger* than the class of conformal-projective transformations, since we may take $\tilde{V}$ to be an arbitrary one-form, not necessarily closed, and $\tilde{W} := df - \tilde{V}$ for some fixed smooth function f. The conformal-projective transformations result when f is itself the sum of two functions ϕ and ψ, in which case $df = d\phi + d\psi$ is a natural decomposition. Although Theorem 19 implies that the Codazzi transformation cannot be generalized as we had hoped, Theorem 22 shows that the conformal-projective transformation does admit interesting generalizations that preserve Codazzi pairs, by virtue of having an additional degree of freedom. This generalization demonstrates the utility of our "building block" transformations P, $\mathsf{P}^\vee$, D, and $\mathit{\Gamma}_L$ in investigating Codazzi pairing relationships under general transformations of affine connections. Furthermore, this analysis shows that even torsion-free transformations may be effectively studied by decomposing them into elementary transformations that induce nontrivial torsions.

4 Summary and Discussions

In this paper we first gave a catalogue of three types of transformations of affine connections, which appear to cover most commonly encountered in the literature; successive application these transformations can lead to more general kinds of transformations of affine connections, for instance, as studied in [3]. We then studied the Codazzi coupling of an affine connection with the tensor fields involved (i.e., (1,2)-tensor, (1,1)-tensor, (0,2)-tensor) in specifying the respective (translation, perturbation, conjugation) transformations, and their torsion-preserving nature as a consequence. We finally provided a complete commutative diagram for those transformations.

Codazzi structures, that is, an affine connection coupled with a tensor, are known to play important roles in PDEs and affine hypersurface theory [17, 13, 19, 9, 10, 14], etc. It also plays the fundamental role in the definition of "statistical structure" of a manifold [8] that emerges from the differential geometric study of statistical inference and probability functions [1]. The investigation in our current paper aims at illuminating the relationship between various transformations of and couplings with affine connection in the most general setting, i.e., without assuming torsion-free of the connection nor requiring the two-form to be a metric. Because affine connections maybe constructed by divergence ("contrast") functions [23] and pre-contrast functions [4], our investigation will shed light on application of geometric concept to statistical estimation and statistical inference.

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