SPECTRAL ASYMPTOTICS OF ONE-DIMENSIONAL FRACTAL LAPLACIANS IN THE ABSENCE OF SECOND-ORDER IDENTITIES

SZE-MAN NGAI, WEI TANG, AND YUANYUAN XIE

Abstract. We observe that some self-similar measures defined by finite or infinite iterated function systems with overlaps satisfy certain "bounded measure type condition", which allows us to extract useful measure-theoretic properties of iterates of the measure. We develop a technique to obtain a closed formula for the spectral dimension of the Laplacian defined by self-similar measure satisfying this condition. For Laplacians defined by fractal measures with overlaps, spectral dimension has been obtained earlier only for a small class of one-dimensional self-similar measures satisfying Strichartz' second-order self-similar identities. The main technique we use relies on the vector-valued renewal theorem proved by Lau, Wang and Chu [16].

Contents

1. Introduction

The origin of spectral asymptotics can be traced back to the work of Hermann Weyl. Let $U \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and with volume $|U|$, Δ be the

Date: November 12, 2016.

2010 *Mathematics Subject Classification.* Primary: 28A80, 35P20; Secondary: 35J05, 43A05, 47A75.

Key words and phrases. Fractal, Laplacian, spectral dimension, self-similar measures with overlaps, bounded measure type condition.

The authors are supported in part by the National Natural Science Foundation of China, Grant 11271122 and Construct Program of the Key Discipline in Hunan Province. The first author is also supported in part by the Hunan Province Hundred Talents Program and the Center of Mathematical Sciences and Applications of Harvard University.

Dirichlet Laplacian on U, $\{\lambda_n\}$ be the eigenvalues, and $N(\lambda, \Delta) := \#\{n : \lambda_n \leq \lambda\}$ be the eigenvalue counting function. In a seminal work started in 1911, Weyl [28] proved the following asymptotic formula, known as the *Weyl law*:

$$N(\lambda, \Delta) = \frac{\omega_d}{(4\pi)^{d/2}} |U| \lambda^{d/2} + o(\lambda^{d/2}), \tag{1.1}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Using this formula he proved a conjecture posed independently by A. Sommerfeld and physicist H. A. Lorentz, which states that the density of standing electromagnetic waves in a bounded cavity U is, at high frequencies, independent of the shape of U. There is an enormous amount of work originated from this formula, both in Euclidean domains and manifolds, most notably the work concerning Weyl's conjecture on the remainder estimate [2, 17, 26, 4, 10]. For domains with fractal boundaries, remainder estimate, in terms of the Minkowski dimension of the boundary, was obtained by Lapidus [13].

For Dirichlet Laplacians Δ_μ on domains defined by a measure, we would like to obtain a crude analogue of (1.1) of the form

$$C_1 \lambda^{\dim_s(\mu)/2} \leq N(\lambda, -\Delta_\mu) \leq C_2 \lambda^{\dim_s(\mu)/2},$$

where $\dim_s(\mu)$ is the spectral dimension of Δ_μ (see definition below). Spectral dimension has been computed by McKean and Ray [22] for the Cantor measure, by Fujita [6] and Naimark and M. Solomyak [20] for self-similar measures satisfying the *open set condition (OSC)* (see [9]), and by Freiberg [7] for generalized measure geometric Laplacians on Cantor like sets. Kigami and Lapidus [12] computed the spectral dimension of Laplacians on postcritically finite self-similar sets with a harmonic structure.

If (OSC) fails, we say that the IFS, as well as any associate self-similar measure, has overlaps. In this case, it is much harder to compute the spectral dimension. The first author [23] obtained the spectral dimension for a class of one-dimensional self-similar measures satisfying second-order identities. These identities were first introduced by Strichartz and are used in [27] to approximate the density of the infinite Bernoulli convolution associated with the golden ratio. However, very few self-similar measures are known to satisfy second-order identities. In fact, for the class of symmetric infinite Bernoulli convolutions with overlaps, only the one associated with the golden ratio has been verified rigorously to satisfy this condition. Other examples are all defined by iterated function systems with contraction ratios equal to the reciprocal of an integer. This includes a class of convolutions of the Cantor measure. To the best of the authors' knowledge, in the absence of second-order identities, the spectral dimension of Laplacians defined by iterated function systems with overlaps has not been obtained before, and this is a main motivation of this paper.

For convenience, we summarize the definition of the Dirichlet Laplacian on a bounded domain defined by a measure; details can be found in [8]. Let $U \subseteq \mathbb{R}^d$ be a bounded open subset and μ be a positive finite Borel measure with $\mathrm{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that μ satisfies the *Poincaré inequality for measures (MPI)*: There exists a constant $C > 0$ such that

$$\int_U |u|^2 \, d\mu \leq C \int_U |\nabla u|^2 \, dx \qquad \text{for all } u \in C_c^\infty(U) \tag{1.2}$$

(see, e.g., [19, 20, 8]). (MPI) implies that each equivalence class $u \in H_0^1(U)$ contains a unique (in the $L^2(U,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(U,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(U)$ such that $u_n \to \hat{u}$ in $H_0^1(U)$ and $u_n \to \hat{u}$ in $L^2(U,\mu)$;
(2) $\hat{u}$ satisfies inequality (1.2).

We call $\hat{u}$ the *$L^2(U,\mu)$-representative* of u. Define a mapping $\iota : H_0^1(U) \to L^2(U,\mu)$ by

$$\iota(u) = \hat{u}.$$

ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(U)$ defined as

$$\mathcal{N} := \bigl\{ u \in H_0^1(U) : \|\iota(u)\|_{L^2(U,\mu)} = 0 \bigr\}.$$

Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(U)$. Then $\iota : \mathcal{N}^\perp \to L^2(U,\mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(U,\mu)$-representative $\hat{u}$ simply by u.

Consider a non-negative bilinear form $\mathscr{E}(\cdot,\cdot)$ on $L^2(U,\mu)$ given by

$$\mathscr{E}(u,v) := \int_U \nabla u \cdot \nabla v \, dx \tag{1.3}$$

with domain $\mathrm{dom}\,\mathscr{E} = \mathcal{N}^\perp$, or more precisely, $\iota(\mathcal{N}^\perp)$. (MPI) implies that $(\mathscr{E}, \mathrm{dom}\,\mathscr{E})$ is a closed quadratic form on $L^2(U,\mu)$. Hence there exists a non-negative self-adjoint operator in $L^2(U,\mu)$, which we denote by $-\Delta_\mu$ and call the *(Dirichlet) Laplacian* with respect to μ, such that $\mathrm{dom}\,\mathscr{E} = \mathrm{dom}\,(-\Delta_\mu)^{1/2}$ and $\mathscr{E}(u,v) = \langle (-\Delta_\mu)^{1/2}u, (-\Delta_\mu)^{1/2}v \rangle_{L^2(U,\mu)}$ for all $u, v \in \mathrm{dom}\,\mathscr{E}$. For $u \in \mathrm{dom}\,\mathscr{E}$, we have $u \in \mathrm{dom}\,\Delta_\mu$ if and only if there exists $f \in L^2(U,\mu)$ such that $\mathscr{E}(u,v) = \langle f, v \rangle_{L^2(U,\mu)}$ for all $v \in \mathrm{dom}\,\mathscr{E}$. In this case, $-\Delta_\mu u = f$. We remark that if $d = 1$, then (MPI) holds for any such μ, and thus Δ_μ is well-defined.

We assume $L^2(U,\mu)$ is infinite dimensional. It is well known that there exists an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U,\mu)$ consisting of the eigenfunctions of $-\Delta_\mu$. The eigenvalues $\lambda_n = \lambda_n(-\Delta_\mu)$ satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$. The *eigenvalue counting function* for $-\Delta_\mu$ is defined as

$$N(\lambda, -\Delta_\mu) := \#\{n : \lambda_n \leq \lambda\},$$

where $\#A$ denotes the cardinality of a set A. Define *lower and upper spectral dimensions* of μ, respectively, as

$$\underline{\dim}_s(\mu) := \varliminf_{\lambda\to\infty} \frac{2\ln N(\lambda, -\Delta_\mu)}{\ln\lambda} \qquad \text{and} \qquad \overline{\dim}_s(\mu) := \varlimsup_{\lambda\to\infty} \frac{2\ln N(\lambda, -\Delta_\mu)}{\ln\lambda}.$$

If $\underline{\dim}_s(\mu) = \overline{\dim}_s(\mu)$, the common value, denoted by $\dim_s(\mu)$, is called the *spectral dimension* of μ; it measures the asymptotic growth rate of the eigenvalue counting function.

Throughout this paper an *iterated function system (IFS)* refers to a finite or countably infinite family of contractive similitudes defined on a compact subset X of $\mathbb{R}^d$. If necessary, we use FIFS and IIFS respectively to distinguish between finite and infinite IFSs. We introduce condition (B), which describes the uniform boundedness of nonbasic measure types on all levels of iteration. It is a key assumption in computing spectral dimension and is formulated in Sections 2 and 5 for FIFSs and IIFSs, respectively.

Let μ be a self-similar measure defined by a finite type IFSs on $\mathbb{R}^d$. In Section 2, we define the set of all level-k islands (see Definition 2.6) $\mathfrak{I}_k$. Roughly speaking, two islands $\mathcal{I}_1$ and $\mathcal{I}_2$ are of the same *measure type* (with respect to μ) if $\mu|_{I_2} = w\mu|_{I_1} \circ S^{-1}$ for some $w > 0$ and some similitude $S : I_1 \to I_2$, where I_j $(j = 1, 2)$ is the open interval corresponding to $\mathcal{I}_j$, and $\mu|_F$ denotes the restriction of the measure μ to $F \subseteq \mathbb{R}^d$.

Intuitively, for FIFSs, μ satisfies condition (B) if for some $k \geq 1$, there is a uniform bound on those level-m $(m > k)$ islands whose measure types, as well the measure types of their predecessors up to level $k + 1$, are different from those of islands in $\mathfrak{I}_k$. In this case, if $k =: k_b$ is the minimum number satisfying this condition, then we call the corresponding $\mathfrak{I}_{k_b} =: \mathbf{I}_b$ the *basic set of islands*. The precise statements are given in Definition 2.10.

Assume $\{S_i\}_{i\in\Lambda}$ is an FIFSs on $\mathbb{R}$ and condition (B) holds with $\mathbf{I}_b$ being the basic set of islands. Let $\mathbf{I} := \{\mathcal{I}_{1,i}\}_{i\in\Gamma} \subseteq \mathbf{I}_b$ be a minimal subset such that the measure type of any island in $\mathbf{I}_b$ equals that of some island in $\mathbf{I}$. Then we can derive renewal equations for the eigenvalue counting functions, and express them in vector form as:

$$\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}, \tag{1.4}$$

where $\alpha \geq 0$, and

$$\begin{aligned} &\mathbf{f} := [f_i^{(\alpha)}(t)]_{i\in\Gamma}, \quad t \in \mathbb{R}; \\ &\mathbf{M}_\alpha := [\mu_{ij}^{(\alpha)}]_{i,j\in\Gamma} \quad \text{is some finite matrix of Borel measures on } \mathbb{R}; \\ &\mathbf{z} := \mathbf{z}^{(\alpha)}(t) = [z_i^{(\alpha)}(t)]_{i\in\Gamma} \quad \text{is an error function.} \end{aligned} \tag{1.5}$$

Let

$$\mathbf{M}_\alpha(\infty) := \Big[\mu_{ij}^{(\alpha)}(\mathbb{R})\Big]_{i,j\in\Gamma}. \tag{1.6}$$

For each $i \in \Gamma$ and $\alpha \geq 0$, define

$$F_i(\alpha) := \sum_{j\in\Gamma} \mu_{ij}^{(\alpha)}(\mathbb{R}), \quad D_i := \{\alpha \geq 0 : F_i(\alpha) < \infty\}, \quad \widetilde{\alpha}_i := \inf D_i. \tag{1.7}$$

If the error functions decay exponentially to 0 as $t \to \infty$, then the spectral dimension of $\dim_s(\mu)$ is given by the unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1. The following is the main result for FIFSs.

Theorem 1.1. *Let μ be a self-similar measure defined by a finite type FIFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}$. Assume that μ satisfies condition (B). Let $\mathbf{M}_\alpha(\infty)$, $F_i(\alpha)$, and $\widetilde{\alpha}_i$ be defined as in* (1.6) *and* (1.7). *Assume that for each $i \in \Gamma$, $\lim_{\alpha\to\widetilde{\alpha}_i^+} F_i(\alpha) > 1$.*

(a) *There exists a unique $\alpha > 0$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to* 1.
(b) *If we assume, in addition, that for the unique α in (a), there exists $\sigma > 0$ such that for all $i \in \Gamma$, $z_i^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$. Then $\dim_s(\mu) = 2\alpha$. Moreover, if $\mathbf{M}_\alpha(\infty)$ is irreducible, then there exist constants $C_1, C_2 > 0$ such that for λ sufficiently large,*

$$C_1\lambda^\alpha \leq N(\lambda, -\Delta_\mu) \leq C_2\lambda^\alpha.$$

In Section 4, we illustrate Theorem 1.1 by the following family of FIFSs:

$$S_1(x) = r_1x, \quad S_2(x) = r_2x + r_1(1 - r_2), \quad S_3(x) = r_2x + 1 - r_2, \tag{1.8}$$

where the contraction ratios $r_1, r_2 \in (0, 1)$ satisfy $r_1 + 2r_2 - r_1r_2 \leq 1$, i.e., $S_2(1) \leq S_3(0)$ (see Figure 1). The Hausdorff dimension of the self-similar sets is computed in [15]. This family is also used as basic examples of IFSs of general finite type [11, 14]. The multifractal properties of the corresponding self-similar measures are recently studied by Deng and the first author [3].

Theorem 1.2. *Let μ be a self-similar measure defined by an FIFS in* (1.8) *together with a probability vector $(p_i)_{i=1}^3$. Then there exists a unique positive real number α satisfying*

$$\big(1 - (r_2p_2)^\alpha\big)\big(1 - (r_2p_3)^\alpha\big) \sum_{k=0}^{\infty} \big(r_1r_2^kw_1(k)\big)^\alpha + r_2^\alpha\big(p_2^\alpha + p_3^\alpha\big) = 1, \tag{1.9}$$

where $w_1(k) := p_1\sum_{i=0}^k p_2^{k-i}p_3^i$. Moreover, $\dim_s(\mu) = 2\alpha$, and there exist positive constants C_1, C_2 such that for all λ sufficiently large,

$$C_1\lambda^\alpha \leq N(\lambda, -\Delta_\mu) \leq C_2\lambda^\alpha.$$

Numerical approximations by taking $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 1/4$, $r_1 = 1/3$, $r_2 = 2/7$, and taking k up to 500 yield $\dim_s(\mu) \approx 0.871431\ldots$.

In Section 5, we extend condition (B) to IIFSs on $\mathbb{R}$ (see Definition 5.4). IIFSs are more complicated because of the presence of the so-called "tails"(see Definition 5.1). Assume that condition (B) holds for an IIFS on $\mathbb{R}$ with $\mathbf{I}_b$ and $\mathbf{T}_b$ being the basic set of islands and the basic set of tails, respectively. Let $\mathbf{I} := \{\mathcal{I}_{1,i}\}_{i\in\Gamma_1} \subseteq \mathbf{I}_b$ be a minimal subset such that the measure type of any island in $\mathbf{I}_b$ equals that of some island in $\mathbf{I}$. Analogously, we choose $\mathbf{T} := \{\mathcal{T}_{1,i}\}_{i\in\Gamma_2} \subseteq \mathbf{T}_b$. Let $\Gamma := \Gamma_1 \cup \Gamma_2$. Then we also can derive renewal equations for the eigenvalue counting functions, and express them in vector form as (1.4) and (1.5). With these modifications, we can now state the main theorem for IIFSs.

Theorem 1.3. *Let μ be a self-similar measure defined by a finite type IIFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}$. Assume that condition (B) holds. Let $\mathbf{M}_\alpha(\infty)$, $F_i(\alpha)$, and $\widetilde{\alpha}_i$ be defined as in* (1.6) *and* (1.7)*. Assume that for each $i \in \Gamma$, $\lim_{\alpha\to\widetilde{\alpha}_i^+} F_i(\alpha) > 1$. Then conclusions (a) and (b) of Theorem 1.1 hold.*

We illustrate Theorem 1.3 by the following family, which is studied in [24]:

$$\begin{aligned} &S_1(x) = rx + 1 - r, \qquad S_{2k} = r^k x + t(1 - r^{k-1}),\\ &S_{2k+1} = r^k x + t(1 - r^{k-1}) + r^k(1 - r), \qquad \text{for } k \geq 1, \end{aligned} \tag{1.10}$$

where $0 < r < (2 - \sqrt{2})/2 \approx 0.292893\ldots$, $r(2 - r)/(1 - r) < t < 1 - r$ (see Figure 4).

Theorem 1.4. *Let μ be a self-similar measure defined by an IIFS as in* (1.10) *together with probability vector $(p_i)_{i=1}^\infty$. Assume that there exists some integer $L \geq 2$, which is chosen to be the minimal one, such that*

$$\Big\{\frac{p_{2i}}{p_{2i+1}} : i \geq L\Big\} = \Big\{\frac{p_{2j}}{p_{2j+1}} : 1 \leq j \leq L - 1\Big\}. \tag{1.11}$$

Let $\mathbf{M}_\alpha(\infty)$ is defined as in (1.6)*.*

(a) *There exists a unique positive real number α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1. Moreover, if*

$$\frac{p_{2k}}{p_{2L}} \geq \frac{p_{2(k+i)}}{p_{2(L+i)}} \qquad \textit{for all } k \geq L \textit{ and } i \geq 0, \tag{1.12}$$

then $\dim_s(\mu) = 2\alpha$*.*

(b) *If, in addition, $\mathbf{M}_\alpha(\infty)$ is irreducible, then there exist constants $C_1, C_2 > 0$ such that for λ sufficiently large,*

$$C_1\lambda^\alpha \leq N(\lambda, -\Delta_\mu) \leq C_2\lambda^\alpha.$$

Numerical approximations by taking $r = 1/4$, $t = 2/3$, $p_1 = 1/3$, and $p_{2k} = p_{2k+1} = 1/4^k$for all $k \geq 1$ yield $\dim_s(\mu) \approx 0.93168$ (see Example 5.25).

We state some open problems in Section 6. Finally, we include the vector-valued renewal theorem in the Appendix for convenience.

2. Self-similar measures of bounded measure type

We first extend the finite type condition [25, 11, 14] to IIFSs and then introduce the concept of measure type of an island. We remark that the term island is adopted from [1].

Let X be a compact subset of $\mathbb{R}^d$ with nonempty interior, and $\{S_i\}_{i\in\Lambda}$ be an IFS of contractive similitudes on X with limit set $K\subseteq\mathbb{R}^d$. If Λ is finite, K is the unique compact subset satisfying $K=\bigcup_{i\in\Lambda}S_i(K)$. For IIFSs, K need not be compact (see [21]). To each probability vector $(p_i)_{i\in\Lambda}$ (i.e., $p_i>0$ and $\sum_{i\in\Lambda}p_i=1$), there corresponds a unique probability measure, called a *self-similar measure*, satisfying the *self-similar identity*

$$\mu=\sum_{i\in\Lambda}p_i\mu\circ S_i^{-1}. \tag{2.1}$$

Moreover, $\operatorname{supp}(\mu)=K$. An analogous result, with $\operatorname{supp}(\mu)=\overline{K}$, holds for IIFSs under additional assumptions (see [21] for IIFSs satisfying (OSC) and Proposition 5.17 for IIFSs studied in this paper).

2.1. **Finite type condition and measure type.** We extend the finite type condition to include IIFSs. Define the following sets of indices

$$\Lambda^k:=\{(i_1,\cdots,i_k):i_j\in\Lambda\text{ for }j=1,\dots,k\},\quad k\geq 1,\qquad\text{and}\qquad\Lambda^*:=\bigcup_{k\geq 0}\Lambda^k$$

(with $\Lambda^0:=\{\emptyset\}$). We call $\boldsymbol{i}=(i_1,\dots,i_k)\in\Lambda^k$ a *word* of length k, and denote its length by $|\boldsymbol{i}|$. If no confusion is possible, we will denote $\boldsymbol{i}=(i_1,\dots,i_k)$ simply by $\boldsymbol{i}:=i_1\cdots i_k$; in particular, if $i_j=i_1$ for all $j=1,\dots,k$, we write $\boldsymbol{i}=:i_1^k$. For $k\geq 0$ and $\boldsymbol{i}=(i_1,\dots,i_k)\in\Lambda^k$, we use the standard notation

$$S_{\boldsymbol{i}}:=S_{i_1}\circ\cdots\circ S_{i_k},\qquad r_{\boldsymbol{i}}:=r_{i_1}\cdots r_{i_k},\qquad p_{\boldsymbol{i}}:=p_{i_1}\cdots p_{i_k}$$

with $S_\emptyset:=id, r_\emptyset=p_\emptyset:=1$, where id is the identity map on $\mathbb{R}^d$.

For two indices $\boldsymbol{i},\boldsymbol{j}\in\Lambda^*$, we write $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$ if $\boldsymbol{i}$ is a prefix of $\boldsymbol{j}$ or $\boldsymbol{i}=\boldsymbol{j}$, and denote by $\boldsymbol{i}\not\preccurlyeq\boldsymbol{j}$ if $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$ does not hold. Let $\{\mathcal{M}_k\}_{k=1}^\infty$ be a sequence of index sets, where $\mathcal{M}_k\subseteq\Lambda^*$. Let

$$\underline{m}_k=\underline{m}_k(\mathcal{M}_k):=\min\{|\boldsymbol{i}|:\boldsymbol{i}\in\mathcal{M}_k\}\quad\text{and}\quad\overline{m}_k=\overline{m}_k(\mathcal{M}_k):=\max\{|\boldsymbol{i}|:\boldsymbol{i}\in\mathcal{M}_k\}.$$

We also let $\mathcal{M}_0:=\{\emptyset\}$.

Definition 2.1. *We say that $\{\mathcal{M}_k\}_{k=0}^\infty$ is a* sequence of nested index sets *if it satisfies the following conditions:*

(1) *both $\{\underline{m}_k\}$ and $\{\overline{m}_k\}$ are nondecreasing, and $\lim_{k\to\infty}\underline{m}_k=\lim_{k\to\infty}\overline{m}_k=\infty$;*
(2) *for each $k\geq 1$, $\mathcal{M}_k$ is an antichain in Λ^*;*
(3) *for each $\boldsymbol{j}\in\Lambda^*$ with $|\boldsymbol{j}|>\overline{m}_k$ or $\boldsymbol{j}\in\mathcal{M}_{k+1}$, there exists $\boldsymbol{i}\in\mathcal{M}_k$ such that $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$;*
(4) *for each $\boldsymbol{j}\in\Lambda^*$ with $|\boldsymbol{j}|<\underline{m}_k$ or $\boldsymbol{j}\in\mathcal{M}_{k-1}$, there exists $\boldsymbol{i}\in\mathcal{M}_k$ such that $\boldsymbol{j}\preccurlyeq\boldsymbol{i}$;*

(5) *there exists a positive integer L_0, independent of k, such that for all $\boldsymbol{i} \in \mathcal{M}_k$ and $\boldsymbol{j} \in \mathcal{M}_{k+1}$ with $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$, we have $|\boldsymbol{j}| - |\boldsymbol{i}| \le L_0$.*

Condition (2) means that the indices in $\mathcal{M}_k$ are incomparable. We also remark that (4) actually follows from (3). Clearly, by letting $\mathcal{M}_k = \Lambda^k$ for all $k \ge 0$, we obtain an example of a sequence of nested index sets.

To define neighborhood types, we fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$. For each integer $k \ge 0$, let $\mathcal{V}_k$ be the set of *level-k vertices* (with respect to $\{\mathcal{M}_k\}$) defined as

$$\mathcal{V}_0 := \big\{(id, 0)\big\} \quad \text{and} \quad \mathcal{V}_k := \big\{(S_{\boldsymbol{i}}, k) : \boldsymbol{i} \in \mathcal{M}_k\big\} \quad \text{for all } k \ge 1.$$

We call $(id, 0)$ the *root vertex* and denote it by $\boldsymbol{v}_{root}$. Let $\mathcal{V} := \bigcup_{k\ge 0} \mathcal{V}_k$ be the set of all vertices. For $\boldsymbol{v} = (S_{\boldsymbol{i}}, k) \in \mathcal{V}_k$, we use the convenient notation $S_{\boldsymbol{v}} := S_{\boldsymbol{i}}$ and $r_{\boldsymbol{v}} := r_{\boldsymbol{i}}$. Note that it is possible to have $\boldsymbol{v} = (S_{\boldsymbol{i}}, k) = (S_{\boldsymbol{j}}, k)$ with $\boldsymbol{i} \ne \boldsymbol{j}$. More generally, for any $k \ge 0$ and any subset $\mathcal{A} \subset \mathcal{V}_k$, we use the notation

$$S_{\mathcal{A}}(\Omega) := \bigcup_{\boldsymbol{v} \in \mathcal{A}} S_{\boldsymbol{v}}(\Omega). \tag{2.2}$$

Let $\Omega \subseteq X$ be a nonempty bounded open set which is *invariant* under $\{S_i\}_{i\in\Lambda}$, i.e., $\bigcup_{i\in\Lambda} S_i(\Omega) \subseteq \Omega$. Such an Ω exists by our assumption; in particular, X° is such a set. Two level-k vertices $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{V}_k$ (allowing $\boldsymbol{v} = \boldsymbol{v}'$) are said to be *neighbors* (with respect to Ω and $\{\mathcal{M}_k\}$) if $S_{\boldsymbol{v}}(\Omega) \cap S_{\boldsymbol{v}'}(\Omega) \ne \emptyset$. We call the set of vertices

$$\mathfrak{N}_\Omega(\boldsymbol{v}) := \{\boldsymbol{v}' : \boldsymbol{v}' \in \mathcal{V}_k \text{ is a neighbor of } \boldsymbol{v}\}$$

the *neighborhood* of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$). Note that $\boldsymbol{v} \in \mathfrak{N}_\Omega(\boldsymbol{v})$ by definition. If no confusion is possible, we denote $\mathfrak{N}_\Omega(\boldsymbol{v})$ simply by $\mathfrak{N}(\boldsymbol{v})$.

Let $\mathscr{S} := \{S_{\boldsymbol{j}} S_{\boldsymbol{i}}^{-1} : \boldsymbol{i}, \boldsymbol{j} \in \Lambda^*\}$. We define an equivalence relation on the set of vertices $\mathcal{V}$. Two vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ are said to be *equivalent*, denoted by $\boldsymbol{v} \sim_\tau \boldsymbol{v}'$ (or simply $\boldsymbol{v} \sim \boldsymbol{v}'$), if for $\tau := S_{\boldsymbol{v}'} S_{\boldsymbol{v}}^{-1} (\in \mathscr{S}) : \bigcup_{\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})} S_{\boldsymbol{u}}(X) \to X$, the following conditions hold:

(1) $\big\{S_{\boldsymbol{u}'} : \boldsymbol{u}' \in \mathfrak{N}(\boldsymbol{v}')\big\} = \big\{\tau S_{\boldsymbol{u}} : \boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})\big\}$; in particular, $\tau S_{\boldsymbol{u}}$ is defined for all $\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})$.
(2) for $\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})$ and $\boldsymbol{u}' \in \mathfrak{N}(\boldsymbol{v}')$ such that $S_{\boldsymbol{u}'} = \tau S_{\boldsymbol{u}}$, and for any positive integer $\ell \ge 1$, an index $\boldsymbol{i} \in \Lambda^*$ satisfies $(S_{\boldsymbol{u}} S_{\boldsymbol{i}}, k + \ell) \in \mathcal{V}_{k+\ell}$ if and only if it satisfies $(S_{\boldsymbol{u}'} S_{\boldsymbol{i}}, k' + \ell) \in \mathcal{V}_{k'+\ell}$.

It is straightforward to show that $\sim$ is an equivalence relation. We denote the equivalence class containing $\boldsymbol{v}$ by $[\boldsymbol{v}]$ and call it the *(neighborhood) type* of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$). Condition (1) is needed in showing that equivalent vertices generate the same number of offspring of each neighborhood type, as shown in Proposition 2.3.

We define an infinite graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and directed edges defined as follows. Let $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{u} \in \mathcal{V}_{k+1}$. Suppose there exists $\boldsymbol{i} \in \mathcal{M}_k$, $\boldsymbol{j} \in \mathcal{M}_{k+1}$, and $\boldsymbol{l} \in \Lambda^*$ such that

$$\boldsymbol{v} = (S_{\boldsymbol{i}}, k), \qquad \boldsymbol{u} = (S_{\boldsymbol{j}}, k+1), \qquad \boldsymbol{j} = (\boldsymbol{i}, \boldsymbol{l}).$$

Then we connect a directed edge $\boldsymbol{l} : \boldsymbol{v} \to \boldsymbol{u}$. We call $\boldsymbol{v}$ a *parent* of $\boldsymbol{u}$ and $\boldsymbol{u}$ an *offspring* of $\boldsymbol{v}$. We write $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of all directed edges defined above.

Remark 2.2. *Only vertices in $\mathfrak{N}(\boldsymbol{v})$ can be parents of any offspring of $\boldsymbol{v}$ in $\mathcal{G}$. (see* [14, Remark 2.3]*).*

Proposition 2.3. *For two equivalent vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$, let $\{\boldsymbol{u}_i\}_{i\in\Lambda_1}$ and $\{\boldsymbol{u}'_i\}_{i\in\Lambda'_1}$ be the offspring of $\boldsymbol{v}$ and $\boldsymbol{v}'$ in $\mathcal{G}$, respectively. Then, counting multiplicity,*

$$\{[\boldsymbol{u}_i] : i \in \Lambda_1\} = \{[\boldsymbol{u}'_i] : i \in \Lambda'_1\}.$$

In particular, $\#\Lambda_1 = \#\Lambda'_1$.

Proof. The proof is similar to that of [14, Proposition 2.4(b)]. □

Definition 2.4. *Let $\{S_i\}_{i\in\Lambda}$ be an IFS of contractive similitudes on a compact subset $X \subseteq \mathbb{R}^d$. We say that $\{S_i\}_{i\in\Lambda}$ is of* finite type *(or that it satisfies the* finite type condition*) if there exists a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$ and a nonempty bounded invariant open set $\Omega \subseteq X$ such that, with respect to Ω and $\{\mathcal{M}_k\}$, the set of equivalence classes $\mathcal{V}/_\sim := \big\{[\boldsymbol{v}] : \boldsymbol{v} \in \mathcal{V}\big\}$ is finite. We call such an Ω a* finite type condition set *(or FTC set).*

Example 2.5. *If $\{S_i\}_{i\in\Lambda}$ satisfies (OSC), then it is of finite type. In fact, $\mathcal{V}/_\sim$ consists of just one element (see,* [14, Example 2.5]*).*

Definition 2.6. *A subset $\mathcal{I} \subseteq \mathcal{V}_k$ is called a* level-k island *(with respect to Ω and $\{\mathcal{M}_k\}$) if the following conditions hold:*

(1) *for any two vertices $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{I}$ (allowing $\boldsymbol{v} = \boldsymbol{v}'$), there exists a finite sequence of vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ such that $\boldsymbol{v}_0 = \boldsymbol{v}$, $\boldsymbol{v}_n = \boldsymbol{v}'$, and $S_{\boldsymbol{v}_i}(\Omega) \cap S_{\boldsymbol{v}_{i+1}}(\Omega) \neq \emptyset$ for all $i = 0, \dots, n-1$;*
(2) *for any $\boldsymbol{u} \in \mathcal{V}_k \setminus \mathcal{I}$ and any $\boldsymbol{v} \in \mathcal{I}$, $S_{\boldsymbol{u}}(\Omega) \cap S_{\boldsymbol{v}}(\Omega) = \emptyset$.*

Intuitively, for each level-k island $\mathcal{I}$, $S_{\mathcal{I}}(\Omega)$ is a connected component of $S_{\mathcal{V}_k}(\Omega)$. Note that for each $\boldsymbol{v} \in \mathcal{V}_k$, there exists a unique island, denoted by $\mathcal{I}(\boldsymbol{v})$, containing $\boldsymbol{v}$ and, moreover, $\mathfrak{N}(\boldsymbol{v}) \subseteq \mathcal{I}(\boldsymbol{v})$. Clearly, if $\{S_i\}_{i\in\Lambda}$ satisfies (OSC) with Ω being an OSC set, then $\mathcal{I}(\boldsymbol{v}) = \{\boldsymbol{v}\}$ for all $\boldsymbol{v} \in \mathcal{V}$. Let

$$\mathfrak{I}_k := \{\mathcal{I} : \mathcal{I} \text{ is a level-}k \text{ island}\} \qquad \text{and} \qquad \mathfrak{I} := \bigcup_{k\geq 0} \mathfrak{I}_k$$

be the collection of all level-k islands and the collection of all islands, respectively. Generalizing (2.2), for any $k \geq 0$ and any subset $\mathcal{B} \subset \mathfrak{I}_k$, we use the notation

$$S_{\mathcal{B}}(\Omega) := \bigcup_{\mathcal{I} \in \mathcal{B}} S_{\mathcal{I}}(\Omega). \tag{2.3}$$

We say that two islands $\mathcal{I} \in \mathfrak{I}_k$ and $\mathcal{I}' \in \mathfrak{I}_{k'}$ are *equivalent*, and denote it by $\mathcal{I} \approx_\tau \mathcal{I}'$ (or simply, $\mathcal{I} \approx \mathcal{I}'$), if there exists some $\tau \in \mathscr{S}$ such that $\{S_{\boldsymbol{v}'} : \boldsymbol{v}' \in \mathcal{I}'\} = \{\tau S_{\boldsymbol{v}} : \boldsymbol{v} \in \mathcal{I}\}$ and, moreover, $\boldsymbol{v} \sim_\tau \boldsymbol{v}'$ for any $\boldsymbol{v} \in \mathcal{I}$ and $\boldsymbol{v}' \in \mathcal{I}'$ satisfying $S_{\boldsymbol{v}'} = \tau S_{\boldsymbol{v}}$. We denote the equivalence class of $\mathcal{I}$ by $[\mathcal{I}]$ and we call $[\mathcal{I}]$ the *(island) type* of $\mathcal{I}$.

For $\mathcal{I} \in \mathfrak{I}_k$, $\mathcal{I}' \in \mathfrak{I}_{k+1}$, $\mathcal{I}$ is said to be a *parent* of $\mathcal{I}'$ and $\mathcal{I}'$ an *offspring* of $\mathcal{I}$ if for any $\boldsymbol{v} \in \mathcal{I}'$, $\mathcal{I}$ contains some parent of $\boldsymbol{v}$. For any $k \geq 0$ and $\mathcal{I} \in \mathfrak{I}_k$, let

$$O(\mathcal{I}) := \{\mathcal{J} : \mathcal{J} \text{ is an offspring of } \mathcal{I}\} \tag{2.4}$$

be the collection of all offspring of $\mathcal{I}$.

Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i \in \Lambda}$ of finite type with Ω being an FTC set. Two equivalent vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ are *μ-equivalent*, denoted by $\boldsymbol{v} \sim_{\mu,\tau,w} \boldsymbol{v}'$ (or simply $\boldsymbol{v} \sim_\mu \boldsymbol{v}'$) if for $\tau = S_{\boldsymbol{v}'} \circ S_{\boldsymbol{v}}^{-1}$, there exists a number $w > 0$ such that

$$\mu|_{S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)} = w \cdot \mu|_{S_{\mathfrak{N}(\boldsymbol{v})}(\Omega)} \circ \tau^{-1}.$$

As $\sim$ is an equivalence relation, so is $\sim_\mu$. Denote the μ-equivalence class of $\boldsymbol{v}$ by $[\boldsymbol{v}]_\mu$ and call it the *(neighborhood) measure type* of $\boldsymbol{v}$ (with respect to Ω, $\{\mathcal{M}_k\}$ and μ). Intuitively, $\boldsymbol{v} \sim_\mu \boldsymbol{v}'$ means that the measures $\mu|_{S_{\mathfrak{N}(\boldsymbol{v})}(\Omega)}$ and $\mu|_{S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)}$ have the same structure. The following proposition shows that μ-equivalent vertices generate the same number of offspring of each neighborhood measure type.

Proposition 2.7. *Assume the hypothese of Proposition 2.3. If $[\boldsymbol{v}]_\mu = [\boldsymbol{v}']_\mu$, then, counting multiplicity,$\{[\boldsymbol{u}_i]_\mu : i \in \Lambda_1\} = \{[\boldsymbol{u}'_i]_\mu : i \in \Lambda'_1\}$.*

Proof. Let $\boldsymbol{u}$ and $\boldsymbol{u}'$ be offspring of $\boldsymbol{v}$ and $\boldsymbol{v}'$ in $\mathcal{G}$ by an edge $\boldsymbol{i}$, respectively. By the proof of Proposition 2.3, we have $\boldsymbol{u} \sim_\tau \boldsymbol{u}'$, where $\tau := S_{\boldsymbol{u}'} \circ S_{\boldsymbol{u}}^{-1} = S_{\boldsymbol{v}'} \circ S_{\boldsymbol{v}}^{-1} \in \mathscr{S}$. Thus it suffices to show that $\boldsymbol{u} \sim_\mu \boldsymbol{u}'$. Since $[\boldsymbol{v}]_\mu = [\boldsymbol{v}']_\mu$, there exists $w > 0$ such that $\mu|_{S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)} = w \cdot \mu|_{S_{\mathfrak{N}(\boldsymbol{v})}(\Omega)} \circ \tau^{-1}$. It follows that

$$\mu|_{S_{\mathfrak{N}(\boldsymbol{u}')}(\Omega)} = w \cdot \mu|_{S_{\mathfrak{N}(\boldsymbol{u})}(\Omega)} \circ \tau^{-1},$$

since $S_{\mathfrak{N}(\boldsymbol{u}')}(\Omega) \subseteq S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)$ and $\boldsymbol{u} \sim_\tau \boldsymbol{u}'$. Hence, $[\boldsymbol{u}]_\mu = [\boldsymbol{u}']_\mu$, completing the proof. $\square$

Definition 2.8. *Let μ be a self-similar measure defined by a finite type IFS $\{S_i\}_{i \in \Lambda}$ on $\mathbb{R}^d$ with Ω being an FTC set. Two islands $\mathcal{I} \in \mathfrak{I}_k$ and $\mathcal{I}' \in \mathfrak{I}_{k'}$ are said to be* μ-equivalent*,* *denoted* $\mathcal{I} \approx_{\mu,\tau,w} \mathcal{I}'$ *(or simply* $\mathcal{I} \approx_\mu \mathcal{I}'$*), if* $\mathcal{I} \approx_\tau \mathcal{I}'$ *and there exists some* $w > 0$ *such that*

$$\mu|_{S_{\mathcal{I}'}(\Omega)} = w \cdot \mu|_{S_{\mathcal{I}}(\Omega)} \circ \tau^{-1}. \tag{2.5}$$

We remark that (2.5) holds if and only if $\boldsymbol{v} \sim_{\mu,\tau,w} \boldsymbol{v}'$ for any $\boldsymbol{v} \in \mathcal{I}$ and $\boldsymbol{v}' \in \mathcal{I}'$ satisfying $S_{\boldsymbol{v}'} = \tau S_{\boldsymbol{v}}$. We note that $\approx_\mu$ is an equivalence relation. We denote the μ-equivalence class of $\mathcal{I}$ by $[\mathcal{I}]_\mu$, and call $[\mathcal{I}]_\mu$ the *(island) measure type* of $\mathcal{I}$ (with respect to Ω, $\{\mathcal{M}_k\}$ and μ). From the definition of $\approx_\mu$, we obtain an analog of Proposition 2.7 concerning $\approx_\mu$. That is, μ-equivalent islands generate the same number of offspring of each island measure type.

Definition 2.9. *Let μ be a self-similar measure defined by a finite type IFS. Let $\mathbf{B} \subseteq \mathfrak{I}_k$ for $k \geq 0$ and $\mathcal{B}_\mu := \{[\mathcal{I}]_\mu : \mathcal{I} \in \mathbf{B}\}$. We call $\mathcal{I}$ a* level-2 nonbasic island *with respect to* $\mathbf{B}$ *if $\mathcal{I} \in O(\mathcal{J})$ for some $\mathcal{J} \in \mathbf{B}$ and $[\mathcal{I}]_\mu \notin \mathcal{B}_\mu$. Inductively, for $\ell \geq 3$, we call $\mathcal{I}$ a* level-ℓ nonbasic island *with respect to* $\mathbf{B}$ *if $\mathcal{I}$ is an offspring of some level-$(\ell-1)$ nonbasic island with respect to* $\mathbf{B}$ *and $[\mathcal{I}]_\mu \notin \mathcal{B}_\mu$.*

We remark that, by definition, for any $\ell \geq 2$, $\mathcal{I}$ is a level-ℓ nonbasic island with respect to $\mathbf{B}$ if and only if there exists a finite sequence of islands $\mathcal{I}_1, \dots, \mathcal{I}_\ell$ such that $\mathcal{I}_1 \in \mathbf{B}$, $\mathcal{I}_\ell = \mathcal{I}$, $[\mathcal{I}_i]_\mu \notin \mathcal{B}_\mu$, and $\mathcal{I}_i$ is an offspring of $\mathcal{I}_{i-1}$ for all $i = 2, \dots, \ell$. In particular, $\mathcal{I}_i$ is a level-i nonbasic island with respect to $\mathbf{B}$ for all $i = 2, \dots, \ell$.

Analogously, we define the equivalence and μ-equivalence of two subsets $\mathcal{B} \subseteq \mathfrak{I}_k$ and $\mathcal{B}' \subseteq \mathfrak{I}_{k'}$, denoted by $\mathcal{B} \mathrel{\dot{\approx}_\tau} \mathcal{B}'$ (or simply, $\mathcal{B} \mathrel{\dot{\approx}} \mathcal{B}'$) and $\mathcal{B} \mathrel{\dot{\approx}_{\mu,\tau,w}} \mathcal{B}'$ (or simply, $\mathcal{B} \mathrel{\dot{\approx}_\mu} \mathcal{B}'$), respectively. Moreover, we denote the equivalence and μ-equivalence class of $\mathcal{B}$ by $[\mathcal{B}]$ and $[\mathcal{B}]_\mu$, respectively.

2.2. **Condition (B) for FIFSs.**

Definition 2.10. *Let μ be a self-similar measure defined by a finite type FIFS on $\mathbb{R}^d$. We say that μ satisfies* condition (B) *if there exists some $k \geq 1$ such that the number of level-ℓ nonbasic island with respect to $\mathfrak{I}_k$ is uniformly bounded for all $\ell \geq 2$. If $k := k_b$ is the minimum non-negative integer satisfying this condition, then we call the corresponding $\mathfrak{I}_k =: \mathbf{I}_b$ the* basic set of islands.

We remark that not all FIFSs with exact overlaps satisfy condition (B); a simple example is the IFS defining the infinite Bernoulli convolution associated with the golden ratio. We illustrate condition (B) with the following two classes of examples.

Example 2.11. *Let μ be a self-similar measure defined by an FIFS $\{S_i\}_{i\in\Lambda}$ in $\mathbb{R}^d$ satisfying (OSC). Then μ satisfies condition (B).*

Proof. By Example 2.5, $\mathcal{V}/_\sim = \{[\boldsymbol{v}_{\mathrm{root}}]\}$. Since $\mathcal{I}(\boldsymbol{v}) = \{\boldsymbol{v}\}$ for any $\boldsymbol{v} \in \mathfrak{I}$, $\mathfrak{I}/_{\approx_\mu} = \{[\mathcal{I}(\boldsymbol{v}_{root})]_\mu\}$. It follows that the set of all level-ℓ nonbasic islands with respect to $\mathfrak{I}_1$ is the empty set for any $\ell \geq 2$. Thus μ satisfies condition (B) with $\mathbf{I}_b = \mathfrak{I}_1$. □

We now consider the family of IFS in (1.8). It is known that each IFS in the family is of finite type with FTC set $\Omega = (0,1)$ and $\mathcal{M}_k = \Lambda^k$ [11, 14].

Example 2.12. *Let μ be the self-similar measure defined by the IFS $\{S_i\}_{i=1}^3$ in* (1.8) *(see Figure 1) and a probability vector $(p_i)_{i=1}^3$. Then μ satisfies condition (B).*

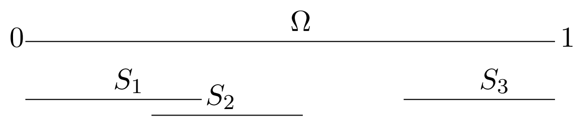

FIGURE 1. The first iteration of the IFS $\{S_i\}_{i=1}^3$ defined in (1.8). The figure is drawn with $r_1 = 1/3$ and $r_2 = 2/7$.

To prove Example 2.12, we need some propositions and lemmas. We first summarize without proof some elementary properties.

Proposition 2.13. *Let $\{S_i\}_{i=1}^3$ be defined as in* (1.8) *and $\Omega = (0,1)$.*

(a) *Then $S_{13} = S_{21}$. Moreover, define $W_n := \{2^{n-i}13^i : i = 0,1,\ldots,n\}$ for any $n \geq 1$, we have $S_{\boldsymbol{i}} = S_{\boldsymbol{j}}$ for any $\boldsymbol{i},\boldsymbol{j} \in W_n$.*

(b) *Define*

$$\mathcal{I}_{1,0} := \{(S_3, 1)\} \quad \textit{and} \quad \mathcal{I}_{1,1} := \{(S_1,1),(S_2,1)\} \tag{2.6}$$

(see Figure 2). Then $\mathfrak{I}_1 = \{\mathcal{I}_{1,0}, \mathcal{I}_{1,1}\}$.

(c) *$S_1(\Omega) \cap S_{2^n}(\Omega) = S_{2^n1}(\Omega)$ for $n \geq 1$.*

Let $\{S_i\}_{i=1}^3$ be defined as in (1.8) and μ be the self-similar measure associated with a probability vector $(p_i)_{i=1}^3$. Define

$$w_1(k) := p_1 \sum_{i=0}^{k} p_2^{k-i} p_3^i \qquad \text{for } k \geq 0. \tag{2.7}$$

We remark that for $k \geq 0$,

$$p_1 p_3^{k+1} + p_2 w_1(k) = p_1 p_2^{k+1} + p_3 w_1(k) = w_1(k+1) \quad \text{and} \quad w_1(k+1) \leq w_1(k) \leq p_1. \tag{2.8}$$

Lemma 2.14. *Assume the hypotheses of Example 2.12 and let $\Omega = (0,1)$. Define*

$$I_{1,0} := S_{\mathcal{I}_{1,0}}(\Omega) = S_3(\Omega) \qquad \textit{and} \qquad I_{1,1} := S_{\mathcal{I}_{1,1}}(\Omega) = S_1(\Omega) \cup S_2(\Omega), \tag{2.9}$$

where $\mathcal{I}_{1,i}$, $i = 0,1$, is defined in (2.6). *Let $w_1(k)$ be defined as in* (2.7). *Then*

(a) *for* $k \geq 1$, $\mu|_{S_{2^k}(I_{1,0})} = p_2^k \cdot \mu|_{I_{1,0}} \circ S_{2^k}^{-1}$;

(b) *for* $k \geq 0$,

$$\mu|_{S_{2^k 1}(I_{1,1})} = w_1(k) \cdot \mu|_{I_{1,1}} \circ S_{2^k 1}^{-1}; \tag{2.10}$$

(c) *for* $k \geq 1$, $\mu|_{S_{2^k}(I_{1,1})} = w_1(k-1)\mu|_{I_{1,0}} \circ S_{2^{k-1}1}^{-1} + p_2^k \mu|_{I_{1,1}} \circ S_{2^k}^{-1}$.

Proof. (a) Let $A \subseteq S_{2^k}(I_{1,0})$ for some $k \geq 1$. Using Proposition 2.13(b,c), we have $S_{2^i}^{-1}(A) \subseteq S_{2^{k-i}}(I_{1,0}) \subseteq I_{1,1} \setminus S_1(\Omega)$ for any $0 \leq i \leq k-1$. Hence

$$\mu(A) = p_2 \mu \circ S_2^{-1}(A) = \cdots = p_2^k \mu \circ S_{2^k}^{-1}(A).$$

(b) Using Proposition 2.13(a,b), we have $\mu(A) = p_1 \mu \circ S_1^{-1}(A)$ for any $A \subseteq S_1(I_{1,1})$, i.e., (2.10) holds for $k = 0$. Assume that (2.10) holds for $k = m$. For $k = m+1$, let $A \subseteq S_{2^{m+1}1}(I_{1,1})$. Then $S_2^{-1}(A) \subseteq S_{2^m 1}(I_{1,1})$. By assumption,

$$\mu(S_2^{-1}(A)) = w_1(m)\mu \circ S_{2^{m+1}1}^{-1}(A). \tag{2.11}$$

Proposition 2.13(a) implies $S_1(\Omega) \cap S_{2^{m+1}1}(I_{1,1}) = S_{13^{m+1}}(I_{1,1})$. Thus $S_1^{-1}(A) \cap \Omega \subseteq S_{3^{m+1}}(\Omega)$. It follows that

$$\mu(S_1^{-1}(A)) = p_3 \mu \circ S_3^{-1}(S_1^{-1}(A)) = \cdots = p_3^{m+1} \mu \circ S_{13^{m+1}}^{-1}(A).$$

Combining this with (2.11), we have

$$\begin{aligned}
\mu(A) &= p_1 \mu \circ S_1^{-1}(A) + p_2 \mu \circ S_2^{-1}(A) \\
&= p_1 p_3^{m+1} \mu \circ S_{13^{m+1}}^{-1}(A) + p_2 w_1(m) \mu \circ S_{2^{m+1}1}^{-1}(A) \\
&= w_1(m+1)\mu \circ S_{2^{m+1}1}^{-1}(A),
\end{aligned}$$

where $S_{13^m} = S_{2^m 1}$ and (2.8) are used in the last equality. This proves part (b).

(c) The proof is similar to that of (b). □

Proof of Example 2.12. Let $\Omega = (0,1)$ and for each $i \geq 0$, let $\mathcal{M}_i = \{1,2,3\}^i$. Next we show that μ satisfies condition (B) with $\mathbf{I}_b := \mathfrak{I}_1$ being the basic set of islands. Let $\mathcal{I}_{1,i}$ be defined as in (2.6). Thus $\mathbf{I}_b = \{\mathcal{I}_{1,0}, \mathcal{I}_{1,1}\}$. Let $\mathbf{I}_{b,\mu} := \{[\mathcal{I}_{1,0}]_\mu, [\mathcal{I}_{1,1}]_\mu\}$. It suffices to show that $\mathcal{I}_{k,1,2} := \{(S_{2^{k-1}1}, k), (S_{2^k}, k)\}$ is the only level-k nonbasic island with respect to $\mathbf{I}_b$ for any $k \geq 2$ (see Figure 2). Since $[\mathcal{I}(\boldsymbol{v}_{\rm root})]_\mu = [\mathcal{I}_{1,0}]\mu$, $\mathcal{I}$ is not a nonbasic island with respect to $\mathbf{I}_b$ for any $\mathcal{I} \in O(\mathcal{I}_{1,0})$. Upon iterating the IFS once, $\mathcal{I}_{1,1}$ generates three islands:

$$\mathcal{I}_{2,1,0} := \{(S_{23}, 2)\}, \quad \mathcal{I}_{2,1,1} := \{(S_1, 2), (S_{12}, 2)\}, \text{ and } \mathcal{I}_{2,1,2} := \{(S_{21}, 2), (S_{22}, 2)\}. \tag{2.12}$$

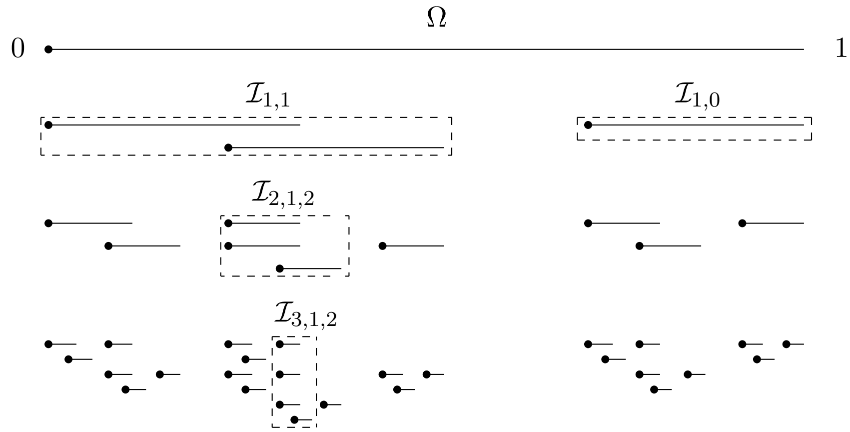

FIGURE 2. Islands $\mathfrak{I}_k$ for $k = 0, 1, 2, 3$ in Example 2.12, using the notation in the proof of Example 2.12. $\mathbf{I}_b := \{\mathcal{I}_{1,0}, \mathcal{I}_{1,1}\}$ is the basic set of islands and $\mathcal{I}_{k,1,2}$ is the unique level-k nonbasic island with respect to $\mathbf{I}_b$ for $k \geq 2$. Islands that are labeled consist of vertices enclosed by a box. The figure is drawn with $r_1 = 1/3$ and $r_2 = 2/7$.

Lemma 2.14 implies that $\mathcal{I}_{2,1,i} \in [\mathcal{I}_{1,i}]_\mu$ for $i = 0, 1$, and $[\mathcal{I}_{2,1,2}]_\mu \notin \mathbf{I}_{b,\mu}$. Thus $\mathcal{I}_{2,1,2}$ is the only level-2 nonbasic island with respect to $\mathbf{I}_b$. Assume that $\mathcal{I}_{k,1,2} := \{(S_{2^{k-1}1}, k), (S_{2^k}, k)\}$ is the only level-k nonbasic island with respect to $\mathbf{I}_b$. Similarly, $\mathcal{I}_{k,1,2}$ generates three islands:

$$\begin{aligned}
\mathcal{I}_{k+1,1,0} &:= \{(S_{2^k 3}, k+1)\} \\
\mathcal{I}_{k+1,1,1} &:= \{(S_{2^{k-1}11}, k+1), (S_{2^{k-1}12}, k+1)\}, \\
\mathcal{I}_{k+1,1,2} &:= \{(S_{2^k 1}, k+1), (S_{2^{k+1}}, k+1)\}.
\end{aligned} \tag{2.13}$$

Lemma 2.14 again implies that $\mathcal{I}_{k+1,1,i} \in [\mathcal{I}_{1,i}]_\mu$ for $i = 0, 1$, and $[\mathcal{I}_{k+1,1,2}]_\mu \notin \mathbf{I}_{b,\mu}$. Thus, $\mathcal{I}_{k+1,1,2}$ is the only level-$(k+1)$ nonbasic island with respect to $\mathbf{I}_b$, completing the proof. □

3. Renewal equation and proof of Theorem 1.1

In order to derive renewal equations and prove our main theorems, we will only consider the one-dimensional case.

3.1. **Eigenvalue counting function.** Let $(\mathscr{E}, \operatorname{dom} \mathscr{E})$ be defined as in (1.3) with $U = (a, b)$ and let $-\Delta_\mu$ be the associated Dirichlet Laplacian on $L^2((a, b), \mu)$. Let $\mathcal{P} = \{a_i\}_{i=0}^{n+1}$ be a partition of $[a, b]$ satisfying

$$a_0 := a < a_1 < \cdots < a_{n+1} =: b \quad \text{and} \quad a_i \in \operatorname{supp}(\mu) \quad \text{for } i \in \{0, \dots, n+1\}.$$

Define $\mathcal{F} := \mathcal{F}(\mathcal{P}) = \{u \in \operatorname{dom} \mathscr{E} : u(a_i) = 0 \text{ for all } i = 0 \dots, n+1\}$. Then $\mathcal{F}$ is a closed subspace of $\operatorname{dom} \mathscr{E}$. Define a relation $\sim_{\mathscr{E}}$ on $\operatorname{dom} \mathscr{E}$, induced by $\mathcal{F}$, by $u \sim_{\mathscr{E}} v$ if and only if

$u-v\in\mathcal{F}$. Then $\sim_{\mathscr{E}}$ is an equivalence relation on $\operatorname{dom}\mathscr{E}$. Define the quotient space

$$\operatorname{dom}\mathscr{E}/\mathcal{F} := \{[u]_{\mathscr{E}} : u\in\operatorname{dom}\mathscr{E}\},$$

where $[u]_{\mathscr{E}}$ is the equivalence class of u. Define addition and scalar multiplication on $\operatorname{dom}\mathscr{E}/\mathcal{F}$ as usual. For each $i=1,\dots,n$, let f_i be a function in $\operatorname{dom}\mathscr{E}$ that satisfies

$$f_i(a_j)=\delta_{ij},\qquad i,j=1,\dots,2n,$$

where δ_{ij} is the Kronecker delta. Such an f_i clearly exists. It is easy to prove that

$$\operatorname{dom}\mathscr{E}/\mathcal{F} = \text{ span }\{[f_i]_{\mathscr{E}} : i=1,\dots,n\}\qquad\text{and}\qquad \dim(\operatorname{dom}\mathscr{E}/\mathcal{F})=n.$$

Let $-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}$ be the Laplacian defined by the Dirichlet form (1.3) with $\operatorname{dom}\mathcal{E}=\mathcal{F}$, and let $N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}) := \#\{n : \lambda_n(-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})\le\lambda\}$ be the associated *eigenvalue counting function*. If $\mathcal{F}=\mathcal{N}^{\perp}$, $N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})$ reduces to $N(\lambda,-\Delta_{\mu|_{(a,b)}})$. If $\operatorname{supp}(\mu)=[a,b]$, then $N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})=\sum_{i=0}^{n}N(\lambda,-\Delta_{\mu|_{(a_i,a_{i+1})}})$. We prove a similar formula.

Proposition 3.1. *If there exists a subset $S\subseteq\{0,\dots,n\}$ such that $\mu(a_i,a_{i+1})>0$ for any $i\in S$ and $\mu(a_j,a_{j+1})=0$ for any $j\notin S$. Then*

$$N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})=\sum_{i\in S}N(\lambda,-\Delta_{\mu|_{(a_i,a_{i+1})}}).\tag{3.1}$$

Proof. Let λ be an eigenvalue of $-\Delta_{\mu|_{(a_i,a_{i+1})}}$ with an eigenfunction u_i for $i\in S$. Define $\widetilde{u}_i := u_i$ on (a_i,a_{i+1}) and $\widetilde{u}_i := 0$ otherwise. Then $\widetilde{u}\in\mathcal{F}$. Thus for all $v\in\mathcal{F}$,

$$\int_a^b \widetilde{u}_i'v'\,dx=\int_{a_i}^{a_{i+1}}u_i'v'\,dx=\lambda\int_{a_i}^{a_{i+1}}u_iv\,d\mu=\lambda\int_a^b\widetilde{u}_iv\,d\mu.$$

It follows that λ is also an eigenvalue of $-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}}$ with $\widetilde{u}$ being an eigenfunction. Thus $N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})\ge\sum_{i\in S}N(\lambda,-\Delta_{\mu|_{(a_i,a_{i+1})}})$. On the other hand, since the collection of all eigenfunctions of $\{-\Delta_{\mu|_{(a_i,a_{i+1})}}\}_{i\in S}$ spans $\mathcal{F}$, (3.1) holds. $\square$

It follows from the variational formula that

$$N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})\le N(\lambda,-\Delta_{\mu|_{(a,b)}})\le N(\lambda,-\Delta^{\mathcal{F}}_{\mu|_{(a,b)}})+\#\mathcal{P}-2.\tag{3.2}$$

3.2. **Renewal equation for FIFSs.** Let $\{S_i\}_{i\in\Lambda}$ be a finite type FIFS on a compact subset $X\subseteq\mathbb{R}$ with FTC set $\Omega\subseteq X$ and let μ be an associated self-similar measure. For $\mathcal{I}\in\mathfrak{I}$, let $S_{\mathcal{I}}(\Omega)$ and $O(\mathcal{I})$ be defined as in (2.2) and (2.4), respectively. We denote the contraction ratio of a contractive similitude τ by r_τ.

The following proposition follows directly from [23, Proposition 2.2 (b)].

Proposition 3.2. (a) *Assume that $\mathcal{I} \approx_{\mu,\tau,w} \mathcal{I}'$ for $\mathcal{I}, \mathcal{I}' \in \mathfrak{I}$. Then $(wr_\tau)\cdot(-\Delta_{\mu|_{S_{\mathcal{I}}'(\Omega)}})$ and $-\Delta_{\mu|_{S_{\mathcal{I}}(\Omega)}}$ have the same set of eigenvalues.*

(b) *For two subsets $\mathcal{B} \subseteq \mathfrak{I}_k$ and $\mathcal{B}' \subseteq \mathfrak{I}_{k'}$, let B and B' be the minimal open intervals containing $S_{\mathcal{B}}(\Omega)$ and $S_{\mathcal{B}'}(\Omega)$, respectively. Assume that $\mathcal{B} \dot{\approx}_{\mu,\tau,w} \mathcal{B}'$ and $\mu|_{B'} = w\mu|_B \circ \tau^{-1}$. Then $(wr_\tau)\cdot(-\Delta_{\mu|_{B'}})$ and $-\Delta_{\mu|_B}$ have the same set of eigenvalues.*

Proof. (a) $\mathcal{I} \approx_{\mu,\tau,w} \mathcal{I}'$ implies that $\mu|_{S_{\mathcal{I}'}(\Omega)} = w\cdot\mu|_{S_{\mathcal{I}}(\Omega)} \circ \tau^{-1}$ and $\tau : S_{\mathcal{I}}(\Omega) \to S_{\mathcal{I}'}(\Omega)$ is a contractive similitude. Thus the result follows from [23, Proposition 2.2 (b)].

(b) The result follows as in (a). $\square$

In the rest of this subsection, we assume that μ satisfies condition (B). Let $\mathbf{I}_b := \mathfrak{I}_{k_b}$ be the basic set of islands and $\mathbf{I}_{b,\mu} := \big\{[\mathcal{I}]_\mu : \mathcal{I} \in \mathbf{I}_b\big\}$. Using (3.2), we have

$$\sum_{\mathcal{I}\in\mathbf{I}_b} N(\lambda, -\Delta_{\mu|_{S_{\mathcal{I}}(\Omega)}}) \le N(\lambda,-\Delta_\mu) \le \sum_{\mathcal{I}\in\mathbf{I}_b} N(\lambda, -\Delta_{\mu|_{S_{\mathcal{I}}(\Omega)}}) + 2\#\mathbf{I}_b - 2. \tag{3.3}$$

We choose a subset $\mathbf{I} := \{\mathcal{I}_{1,\ell}\}_{\ell\in\Gamma} \subseteq \mathbf{I}_b$ such that for any $\mathcal{I} \in \mathbf{I}_b$, there exists a unique $\ell \in \Gamma$ satisfying $\mathcal{I} \in [\mathcal{I}_{1,\ell}]_\mu$. In view of Proposition 3.2 and (3.3), to study $N(\lambda, \Delta_\mu)$, it suffices to study $N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}})$ for $\ell \in \Gamma$, where $I_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega)$.

Step 1. Derivation of a functional equation for $N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}})$ for $\ell \in \Gamma$. For $\ell \in \Gamma$, define

$$\mathbf{I}_{2,\ell} := \{\mathcal{I} : \mathcal{I} \in O(\mathcal{I}_{1,\ell}) \text{ and } [\mathcal{I}]_\mu \in \mathbf{I}_{b,\mu}\} \quad \text{and} \quad \mathbf{I}'_{2,\ell} := \{\mathcal{I} : \mathcal{I} \in O(\mathcal{I}_{1,\ell}) \text{ and } [\mathcal{I}]_\mu \notin \mathbf{I}_{b,\mu}\}.$$

Thus $O(\mathcal{I}_{1,\ell}) = \mathbf{I}_{2,\ell} \cup \mathbf{I}'_{2,\ell}$. For $k \ge 3$, if $\mathbf{I}'_{k-1,\ell} \ne \emptyset$, we define

$$\begin{aligned}
\mathbf{I}_{k,\ell} &:= \{\mathcal{I} : \mathcal{I} \in O(\mathcal{J}) \text{ for some } \mathcal{J} \in \mathbf{I}'_{k-1,\ell} \text{ and } [\mathcal{I}]_\mu \in \mathbf{I}_{b,\mu}\},\\
\mathbf{I}'_{k,\ell} &:= \{\mathcal{I} : \mathcal{I} \in O(\mathcal{J}) \text{ for some } \mathcal{J} \in \mathbf{I}'_{k-1,\ell} \text{ and } [\mathcal{I}]_\mu \notin \mathbf{I}_{b,\mu}\}.
\end{aligned}$$

We remark that for any $k \ge 2$, $\cup_{\ell\in\Gamma}\mathbf{I}'_{k,\ell}$ is the set of all level-k nonbasic islands with respect to $\mathbf{I}$.

Without loss of generality, we assume that Γ can be partitioned into two sub-collections, Γ_* and Γ'_*, defined as follows. For $\ell \in \Gamma$, we say $\ell \in \Gamma_*$ if there exists some $\kappa_\ell \ge 2$, depending on ℓ, such that κ_ℓ is the minimal number satisfying $\mathbf{I}'_{\kappa_\ell,\ell} = \emptyset$; otherwise, $\ell \in \Gamma'_*$. Define $\kappa_\ell := \infty$ for $\ell \in \Gamma'_*$.

Condition (B) implies that $\sum_{\ell\in\Gamma} \#\mathbf{I}'_{k,\ell}$ is uniformly bounded for any $k \ge 2$. Fix $\ell \in \Gamma$. Then for any $2 \le k \le \kappa_\ell$, there exist two finite disjoint subsets $G_{k,\ell}, G'_{k,\ell} \subseteq \mathbb{Z}$ such that

$$\mathbf{I}_{k,\ell} = \{\mathcal{I}_{k,\ell,i} : i \in G_{k,\ell}\}, \quad \mathbf{I}'_{k,\ell} = \{\mathcal{I}_{k,\ell,i} : i \in G'_{k,\ell}\},$$

and $0 \le \#G'_{k,\ell} \le M$, where $M > 0$ is a constant. We remark that $G'_{\kappa_\ell,\ell} = \emptyset$. Define

$$I_{k,\ell,i} := S_{\mathcal{I}_{k,\ell,i}}(\Omega) \qquad \text{for } 2 \le k \le \kappa_\ell \text{ and } i \in G_{k,\ell} \cup G'_{k,\ell}.$$

Proposition 3.3. *Assume that condition (B) holds. For $\ell \in \Gamma$, $2 \le k \le \kappa_\ell$, and $i \in G_{k,\ell}$, there exist unique $\xi(k,\ell,i) > 0$ and $c(k,\ell,i) \in \Gamma$ such that*

$$N(\lambda, -\Delta_{\mu|_{I_{k,\ell,i}}}) = N\big(\xi(k,\ell,i)\lambda, (-\Delta_{\mu|_{I_{1,c(k,\ell,i)}}})\big). \tag{3.4}$$

Proof. By the definition of $\mathbf{I}_{k,\ell}$ and Γ, for any $i \in D_{k,\ell}$, there exists unique $\tau(k,\ell,i) \in \mathscr{S}$, $w(k,\ell,i) > 0$, and $c(k,\ell,i) \in \Gamma$ such that $\mathcal{I}_{1,c(k,\ell,i)} \approx_{\mu,\tau(k,\ell,i),w(k,\ell,i)} \mathcal{I}_{k,\ell,i}$. Combining this with Proposition 3.2, we have (3.4) holds with $\xi(k,\ell,i) := r_{\tau(k,\ell,i)} w(k,\ell,i)$. $\square$

For each integer $n \ge 1$ and $\ell \in \Gamma$, we define a partition $\mathcal{P}_{n,\ell}$ of $I_{1,\ell}$ as follows:

$$\begin{aligned}
&\mathcal{P}_{1,\ell} := \big\{x : x \text{ is an end-point of } I_{1,\ell}\big\};\\
&\mathcal{P}_{n,\ell} := \mathcal{P}_{n-1,\ell} \cup \big\{x : x \text{ is an end-point of some interval of the form}\\
&\qquad\qquad I_{n,\ell,i},\ i \in G_{n,\ell} \cup G'_{n,\ell}\big\} \quad \text{for } 2 \le n \le \kappa_\ell.
\end{aligned}$$

Then $\mathcal{P}_{n,\ell}$, $\ell \in \Gamma$, $2 \le n \le \kappa_\ell$, are end-points of subintervals generated by the following procedure. First, replace $I_{1,\ell}$ by the subintervals of the form $I_{2,\ell,i}$, $i \in G_{2,\ell} \cup G'_{2,\ell}$. If $\kappa_\ell = 2$, i.e., $G'_{2,\ell} = \emptyset$, we stop this procedure. Otherwise, keep intervals $\{I_{2,\ell,i}\}_{i \in G_{2,\ell}}$, and replace intervals $\{I_{2,\ell,i}\}_{i \in G'_{2,\ell}}$ by subintervals of the form $I_{3,\ell,j}$, $j \in G_{3,\ell} \cup G'_{3,\ell}$. Continue.

We note that $\mathcal{P}_{n,\ell} \subseteq \operatorname{supp}(\mu)$ and $\#\mathcal{P}_{n,\ell} = \sum_{i=2}^{n} \#G_{i,\ell} + \#G'_{n,\ell}$ for all $\ell \in \Gamma$ and all $2 \le n \le \kappa_\ell$.

For $\ell \in \Gamma$, let $\mathcal{F}_{n,\ell} := \mathcal{F}(\mathcal{P}_{n,\ell})$ for $1 \le n \le \kappa_\ell$. Proposition 3.1 implies, for $\ell \in \Gamma$ and $2 \le n \le \kappa_\ell$, that

$$N(\lambda, -\Delta^{\mathcal{F}_{n,\ell}}_{\mu|_{I_{1,\ell}}}) = \sum_{j=2}^{n} \sum_{i \in G_{j,\ell}} N(\lambda, -\Delta_{\mu|_{I_{j,\ell,i}}}) + \sum_{i \in G'_{n,\ell}} N(\lambda, -\Delta_{\mu|_{I_{n,\ell,i}}}).$$

Thus using (3.2) and (3.4) , we have

$$\begin{aligned}
N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}}) &= N(\lambda, -\Delta^{\mathcal{F}_{\kappa_\ell,\ell}}_{\mu|_{I_{1,\ell}}}) + \varepsilon(\kappa_\ell,\ell) = \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} N(\lambda, -\Delta_{\mu|_{I_{j,\ell,i}}}) + \varepsilon(\kappa_\ell,\ell)\\
&= \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} N\big(\xi(j,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c(j,\ell,i)}}}\big) + \varepsilon(\kappa_\ell,\ell) \quad \text{for } \ell \in \Gamma_*,
\end{aligned} \tag{3.5}$$

where $0 \le \varepsilon(\kappa_\ell,\ell) \le \#\mathcal{P}_{\kappa_\ell,\ell} - 2$. Similarly, we have

$$\begin{aligned}
N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}}) &= \sum_{j=2}^{n} \sum_{i \in G_{j,\ell}} N\big(\xi(j,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c(j,\ell,i)}}}\big) + \varepsilon(n,\ell)\\
&\quad + \sum_{i \in G'_{n,\ell}} N(\lambda, -\Delta_{\mu|_{I_{n,\ell,i}}}) + \varepsilon(n,\ell) \qquad \text{for } n \ge 2, \ell \in \Gamma'_*,
\end{aligned} \tag{3.6}$$

where $0 \le \varepsilon(n,\ell) \le \#\mathcal{P}_{n,\ell} - 2$.

Step 2. Derivation of the vector-valued equation. For each $\ell \in \Gamma$, define

$$f_\ell(t) = f_\ell^{(\alpha)}(t) := e^{-\alpha t} N(e^t, -\Delta_{\mu|_{I_{1,\ell}}}).$$

If we let $\lambda = e^t$, then $e^{-\alpha t} N(\beta\lambda, -\Delta_{\mu|_{I_{1,\ell}}}) = \beta^\alpha f_\ell(t + \ln\beta)$ for any $\beta > 0$. Multiplying both sides of (3.5) and (3.6) by $e^{-\alpha t}$, we have

$$f_\ell(t) = \sum_{j=2}^{k_\ell} \sum_{i \in G_{j,\ell}} \xi(j,\ell,i)^\alpha f_{c(j,\ell,i)}\big(t + \ln(\xi(j,\ell,i))\big) + z_\ell^{(\alpha)}(t) \quad \text{for } \ell \in \Gamma_*, \tag{3.7}$$

where $z_\ell^{(\alpha)}(t) := e^{-\alpha t}\varepsilon(\kappa_\ell, \ell)$; while for $\ell \in \Gamma'_*$ and $n \geq 2$,

$$\begin{aligned} f_\ell(t) &= \sum_{j=2}^{\infty} \sum_{i \in G_{j,\ell}} \xi(j,\ell,i)^\alpha f_{c(j,\ell,i)}\big(t + \ln(\xi(j,\ell,i))\big) \\ &\quad - \sum_{j=n+1}^{\infty} \sum_{i \in G_{j,\ell}} \xi(j,\ell,i)^\alpha f_{c(j,\ell,i)}\big(t + \ln(\xi(j,\ell,i))\big) + z_\ell^{(\alpha)}(t), \end{aligned} \tag{3.8}$$

where $z_\ell^{(\alpha)}(t) := e^{-\alpha t}\big(\sum_{i \in G'_{n,\ell}} N(e^t, -\Delta_{\mu|_{I_{n,\ell,i}}}) + \varepsilon(n,\ell)\big)$. Since $\lambda_1(-\Delta_{\mu|_{I_{1,\ell}}}) > 0$ for all $\ell \in \Gamma$, there exists $t_0 \in \mathbb{R}$ such that $f_\ell(t) = 0$ for any $t < t_0$ and any $\ell \in \Gamma$. For each $t \in \mathbb{R}$, let n_t be the smallest integer such that

$$t + \max\big\{\ln(\xi(n_t,\ell,i)) : \ell \in \Gamma'_*,\ i \in G_{n_t,\ell}\big\} < t_0. \tag{3.9}$$

Then the second summation in (3.8) vanishes and thus we get

$$f_\ell(t) = \sum_{j=2}^{\infty} \sum_{i \in G_{j,\ell}} \xi(j,\ell,i)^\alpha f_{c(j,\ell,i)}(t + \ln(\xi(j,\ell,i))) + z_\ell^{(\alpha)}(t) \qquad \text{for } \ell \in \Gamma'_*. \tag{3.10}$$

For $\ell, m \in \Gamma$, let $\mu_{\ell m}^{(\alpha)}$ be the discrete measure such that

$$\mu_{\ell m}^{(\alpha)}\big(-\ln\big(\xi(j,\ell,i)\big)\big) := \xi(j,\ell,i)^\alpha \qquad \text{for } 2 \leq j \leq \kappa_\ell,\ i \in G_{j,\ell},\ c(j,\ell,i) = m. \tag{3.11}$$

We summarize the above derivations in the following theorem.

Theorem 3.4. *Let μ be a self-similar measure defined by an FIFS $\{S_i\}_{i\in\Lambda}$ of finite type. Let $\mathbf{f}$, $\mathbf{M}_\alpha$, and $\mathbf{z}$ be defined as in* (1.5). *Assume that μ satisfies condition (B). Then $\mathbf{f}$ satisfies the vector-valued renewal equation $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$.*

3.3. **Proof of Theorem 1.1.** Use a similar argument as that in [23, Theorem 1.1]. The proof involves verifying a moment condition in the vector-valued renewal theorem by Lau *et al.* [16], which is stated in the Appendix for convenience.

Proof of Theorem 1.1. (a) Similar with the proof of that [23, Theorem 1.1(a)].

(b) For the rest of the proof we let α be the unique number in part (a). Let $\mathbf{m} := [m_{ij}^{(\alpha)}] = [\int_0^\infty x\, d\mu_{ij}^{(\alpha)}]$ be the moment matrix. Following the proof of [23, Theorem 1.1(b)], we need to show that the moment condition holds. It suffices to show that

$$0 < \sum_{k\in\Gamma} m_{\ell k}^{(\alpha)} < \infty. \tag{3.12}$$

It is easy to check that for $\ell \in \Gamma$, $\sum_{k\in\Gamma} m_{\ell k}^{(\alpha)}$ takes the following values:

$$\sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} \xi(j,\ell,i)^\alpha \big|\ln(\xi(j,\ell,i))\big|.$$

Hence, (3.12) follows from our assumption $\lim_{\alpha\to\widetilde{\alpha}_i^+} F_i(\alpha) > 1$. Moreover, it follows from the derivation of equations (3.7) and (3.10) that each column of $\mathbf{M}_\alpha$ is nondegenerate at 0. From Theorem 3.4, we have $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$. Hence the results follows by considering the two cases $\mathbf{M}_\alpha(\infty)$ is irreducible or reducible. □

4. A class of finite IFSs with overlaps

In this section, we derive renewal equations and compute the spectral dimension for the Laplacians defined by the family of self-similar measures associated with the IFSs in (1.8).

Let μ be the self-similar measure defined by the IFS $\{S_i\}_{i=1}^3$ in (1.8) together with a probability vector $(p_i)_{i=1}^3$. We use the notation defined Example 2.12. Let $\mathcal{I}_{1,i}$ be defined as in (2.6) for $i = 0, 1$. Example 2.12 shows that μ satisfies condition (B) with $\mathbf{I}_b := \{\mathcal{I}_{1,0}, \mathcal{I}_{1,1}\}$ being the basic set of islands. Moreover, $\Gamma = \{0, 1\}$, $\Gamma_* = \{0\}$ and $\Gamma'_* = \{1\}$. Let $\mathbf{I}_{k,\ell}$, $\mathbf{I}'_{k,\ell}$, $G_{k,\ell}$, and $G'_{k,\ell}$ be defined as in the Step 2 of Section 3.2 for $\ell \in \Gamma$ and $2 \le k \le \kappa_\ell$.

Define $\mathcal{I}_{2,0,0} := \{(S_{33}, 2)\}$ and $\mathcal{I}_{2,0,1} := \{(S_{31}, 2), (S_{32}, 2)\}$. Since $(S_3, 1) \sim_{\mu,S_3,p_3} (id, 0)$, we have $\mathbf{I}_{2,0} = \{\mathcal{I}_{2,0,0}, \mathcal{I}_{2,0,1}\}$ and $\mathbf{I}'_{2,0} = \emptyset$. Thus $G_{2,0} = \{0, 1\}$ and $G'_{2,0} = \emptyset$.

For $k \ge 2$ and $i = 0, 1, 2$, let $\mathcal{I}_{k,1,i}$ be defined as in (2.12) and (2.13). The proof of Example 2.12 yields $\mathbf{I}_{k,1} = \{\mathcal{I}_{k,1,0}, \mathcal{I}_{k,1,1}\}$ and $\mathbf{I}'_{k,1} = \{\mathcal{I}_{k,1,2}\}$. Thus $G_{k,1} = \{0, 1\}$ and $G'_{k,1} = \{2\}$. In the rest of this section, let $w_1(k)$ be defined as in (2.7).

Proposition 4.1. *Let $\xi(\cdot,\cdot,\cdot)$ and $c(\cdot,\cdot,\cdot)$ be defined as in Proposition 3.3. Then*

(a) $\xi(2, 0, i) = r_2 p_3$ *and* $c(2, 0, i) = i$ *for* $i = 0, 1$.

(b) *for* $k \ge 2$ *and* $i = 0, 1$,

$$\xi(k, 1, 0) = r_2^{k-1} p_2^{k-1}, \quad \xi(k, 1, 1) = r_1 r_2^{k-2} w_1(k-2), \quad \textit{and} \quad c(k, 1, i) = i. \tag{4.1}$$

Proof. (a) The result follows from the fact that $(S_3, 1) \sim_{\mu,S_3,p_3} (id, 0)$.

(b) Lemma 2.14 implies that for $k \ge 2$,

$$\mathcal{I}_{1,1} \approx_{\mu,S_{2^{k-2}1},w_1(k-2)} \mathcal{I}_{k,1,1} \quad \text{and} \quad \mathcal{I}_{1,0} \approx_{\mu,S_{2^{k-1}},p_2^{k-1}} \mathcal{I}_{k,1,0}.$$

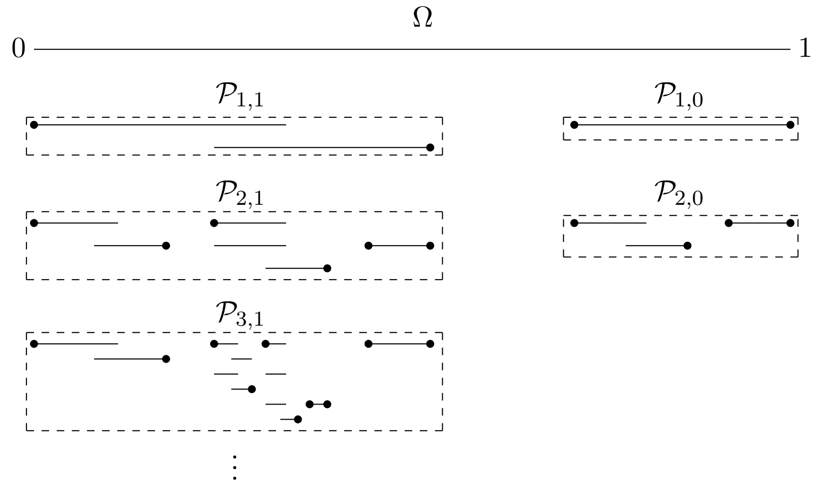

FIGURE 3. Iterates of the IFS $\{S_i\}_{i=1}^3$ with $r_1 = 1/3$ and $r_2 = 2/7$. $\{\mathcal{P}_{n,\ell}\}_{n=1}^{\kappa_\ell}$ is the partition of $I_{1,\ell}$ defined in Section 3.

Combining this with Proposition 3.3, we get (4.1). $\square$

Let $I_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega)$ for $\ell \in \Gamma$. For $k \geq 2$, define

$$I_{k,1,2} := S_{\mathcal{I}_{k,1,2}}(\Omega) = S_{2^{k-1}}(I_{1,1}).$$

Let $\{\mathcal{P}_{n,\ell}\}$ be the partition of $I_{1,\ell}$, $\ell \in \Gamma$, $2 \leq n \leq \kappa_\ell$, defined as in the Section 3 (see Figure 3). It is easy to check that $\#\mathcal{P}_{\kappa_0,0} = \mathcal{P}_{2,0} = 4$ and $\#\mathcal{P}_{n,1} = 4n - 2$ for $n \geq 2$.

Using Theorem 3.4, the the vector-valued renewal equation is

$$\begin{aligned}
f_0(t) &= (r_2p_3)^\alpha \sum_{i=0}^{1} f_i\big(t + \ln(r_2p_3)\big) + z_0^{(\alpha)}(t),\\
f_1(t) &= \sum_{j=1}^{\infty}(r_2p_2)^{j\alpha} f_0(t + \ln(r_2p_2)^j) + \sum_{j=0}^{\infty}\big(r_1r_2^jw_1(j)\big)^\alpha f_1\big(t + \ln\big(r_1r_2^jw_1(j)\big)\big) + z_1^\alpha(t),
\end{aligned}$$

where $z_0^{(\alpha)}(t) := e^{-\alpha t}\varepsilon(2,0)$ and $z_1^\alpha(t) := e^{-\alpha t}N(e^t, -\Delta_{\mu|_{I_{n_t,1,2}}}) + e^{-\alpha t}\varepsilon(n_t, 1)$.

Let $\{\mu_{\ell m}^{(\alpha)}\}_{\ell,m\in\Gamma}$ be the discrete measures defined as in (3.11). Since $\mathbf{M}_\alpha(\infty) := [\mu_{\ell m}^{(\alpha}(\mathbb{R})]_{\ell,m=0}^1$, it is irreducible.

Proposition 4.2. *For $\ell = 0, 1$, let $F_\ell(\alpha)$ and $\widetilde{\alpha}_\ell$ be defined as in* (1.7)*. Then $\widetilde{\alpha}_\ell = 0$ and $F_\ell(0) > 1$ for all $\ell \in \{0, 1\}$.*

Proof. By the definition of $F_\ell(\alpha)$, $\ell = 0, 1$, we see that

$$F_0(\alpha) = 2(r_2p_3)^\alpha \quad \text{and} \quad F_1(\alpha) = \frac{(r_2p_2)^\alpha}{1 - (r_2p_2)^\alpha} + \sum_{j=0}^{\infty}\big(r_2^jr_1w_1(j)\big)^\alpha.$$

We first note that $\tilde{\alpha}_0 = 0$ and $F_0(\tilde{\alpha}_0) = F_0(0) = 2$. Since for any $\alpha > 0$, $\sum_{j=0}^{\infty} \big(r_2^j r_1 w_1(j)\big)^{\alpha}$ converges, $F_1(\alpha) < \infty$ for any $\alpha > 0$. By the definition of $\tilde{\alpha}$, we have $\tilde{\alpha}_1 = 0$ and $F_1(0) = \infty$, which completes the proof. □

Proposition 4.2 implies that there exists a unique $\alpha > 0$ such that the spectral radius of $\mathbf{M}_{\alpha}(\infty)$ is 1. That is, α is the unique solution of $|I_2 - \mathbf{M}_{\alpha}(\infty)| = 0$, where I_2 is the identity $(2,2)$-matrix. Thus α is the unique number satisfying (1.9).

Now we need to show that there exists some $\sigma > 0$ such that $z_i^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$ for $i = 0, 1$. To this end, we will first show that $N(e^t, -\Delta_{\mu|_{I_{n_t,1,2}}})$ is bounded.

Proposition 4.3. *There exists $C > 0$ such that*

$$N(e^t, -\Delta_{\mu|_{I_{n_t,1,2}}}) \leq C.$$

Proof. Let $A \subseteq I_{n_t,1,2} = S_{2^{n_t}-1}(I_{1,1})$, we have $S_{2^{n_t}-1}^{-1}(A) \subseteq I_{1,1}$. Thus

$$\mu(S_{2^{n_t}-1}^{-1}(A)) \;=\; p_1 \mu \circ S_{2^{n_t-1}1}^{-1}(A) + p_2 \mu \circ S_{2^{n_t}}^{-1}(A). \tag{4.2}$$

Multiplying both sides of (4.2) by $w_1(n_t - 2)/p_1$ and using Lemma 2.14(c), we have

$$\begin{aligned}\frac{w_1(n_t-2)}{p_1}\mu\big(S_{2^{n_t}-1}^{-1}(A)\big) &= w_1(n_t-2)\mu\big(S_{2^{n_t-1}1}^{-1}(A)\big) + \frac{p_2 w_1(n_t-2)}{p_1}\mu\big(S_{2^{n_t}}^{-1}(A)\big)\\ &\geq w_1(n_t-2)\mu\big(S_{2^{n_t-1}1}^{-1}(A)\big) + p_2^{n_t-1}\mu\big(S_{2^{n_t}}^{-1}(A)\big) = \mu(A).\end{aligned}$$

Thus $\mu|_{I_{n_t,1,2}} \leq w_1(n_t-2)/p_1 \cdot \mu \circ S_{2^{n_t}-1}^{-1}$ on $I_{n_t,1,2}$. [23, Proposition 2.3] implies that

$$\lambda_n(-\Delta_{\mu|_{I_{n_t,1,2}}}) \geq p_1/w_1(n_t-2) \cdot \lambda_n\Big(-\Delta_{\mu\circ S_{2^{n_t}-1}^{-1}|_{I_{n_t,1,2}}}\Big).$$

Using [23, Proposition 2.2], we have

$$\begin{aligned}N(e^t, -\Delta_{\mu|_{I_{n_t,1,2}}}) &\leq N\Big(\frac{w_1(n_t-2)}{p_1}e^t, -\Delta_{\mu\circ S_{2^{n_t}-1}^{-1}|_{I_{n_t,1,2}}}\Big) + 1\\ &= N\Big(\frac{r_2^{n_t-1}w_1(n_t-2)}{p_1}e^t, -\Delta_{\mu|_{I_{1,1}}}\Big) + 1.\end{aligned} \tag{4.3}$$

By the definition of n_t, i.e., (3.9), we have $t + \ln(r_1 r_2^{n_t-1} w_1(n_t-2)) < t_0$. Combining this with (4.3), we obtain

$$N(e^t, -\Delta_{\mu|_{I_{n_t,1,2}}}) \leq N(r_1^{-1}p_1^{-1}e^{t_0}, -\Delta_{\mu|_{I_{1,1}}}) + 1 := C,$$

which proves the result. □

Proposition 4.4. *Let α be defined as in* (1.9). *Then there exists some $\sigma > 0$ such that for $i = 0, 1$, $z_i^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$.*

Proof. Proposition 4.3 implies that there exists some constant $c > 0$ such that

$$z_1^{(\alpha)}(t) = e^{-\alpha t} N(\lambda, -\Delta_{\mu|_{I_{n_t,1,2}}}) + e^{-\alpha t}\varepsilon(n_t, 1) \le (c + 4n_t - 2)e^{-\alpha t}.$$

Moreover, since $z_0^{(\alpha)}(t) = e^{-\alpha t}\varepsilon(2, 0) \le 2e^{-\alpha t}$, it suffices to show that for any $\sigma > 0$, $n_t e^{-\alpha t} = o(e^{-\sigma t})$ as $t \to \infty$. By the definition of n_t, i.e., (3.9), we have

$$t + \max\Big\{ \ln\big(r_2^{n_t-1} p_2^{n_t-1}\big),\ \ln\big(r_2^{n_t-2} r_1 w_1(n_t-2)\big)\Big\} \ge t_0.$$

Since $w_1(n_t - 2) \le 1$, we have $t + \ln(r_2^{n_t-2}) \ge t_0$ and hence, for any $\sigma < \alpha$, $n_t e^{-\alpha t} = o(e^{-\sigma t})$ as $t \to \infty$, which completes the proof. □

Proof of Theorem 1.2. Combine Propositions 4.3, and 4.4, and Theorem 1.1. □

5. IIFSs with overlaps

In this section we study condition (B) for IIFSs and prove Theorems 1.3 and 1.4. We will only consider IIFSs on $\mathbb{R}$. Let μ be a self-similar measure defined by a finite type IIFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}$ with Ω being an FTC set. In this section, we use the notation introduced in Section 2.

5.1. **Condition (B) for IIFSs.** We first introduce the definition of a tail.

Definition 5.1. *Let μ be a self-similar measure defined by a finite type IIFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}$ with Ω being an FTC set. For $k \ge 1$, we call a countably infinite sequence of islands $\mathcal{T} \subseteq \mathfrak{I}_k$ a* level-k tail *if it satisfies the following conditions:*

1. *if we let $U := (a, b) \subseteq \Omega$ be the minimal open interval containing $S_{\mathcal{T}}(\Omega)$, then for any $\mathcal{I} \in \mathfrak{I}_k \setminus \mathcal{T}$, $S_{\mathcal{I}}(\Omega) \cap U = \emptyset$;*
2. *for any $a_1 > a$ and $b_1 < b$, either $\#\{\mathcal{I} \in \mathcal{T} : S_{\mathcal{I}}(\Omega) \cap (a_1, b) \ne \emptyset\} < \infty$ or $\#\{\mathcal{I} \in \mathcal{T} : S_{\mathcal{I}}(\Omega) \cap (a, b_1) \ne \emptyset\} < \infty$;*
3. *there exists a finite subset $\mathcal{B} \subseteq \mathfrak{I}_k \setminus \mathcal{T}$ such that $\mathcal{B}$ contains all island measure types in $\mathcal{T}$ and $\mathcal{B} \cup \mathcal{T}$ satisfies conditions (1) and (2);*
4. *$\mathcal{T}$ is a maximal family satisfying conditions (1)–(3).*

Intuitively, condition (1) means that $\mu|_U$ and $\mu|_{S_{\mathcal{T}}(\Omega)}$ have similar measure structures; in particular, the closures of their supports in Ω are the same. Condition (2) implies that $\mathcal{T}$ can be expressed as a sequence of islands $\{\mathcal{I}_i\}_{i=1}^{\infty}$ with $S_{\mathcal{I}_i}(\Omega)$ converging to some point in ∂U as $i \to \infty$. Using conditions (3) and (4) we obtain the following remark.

Remark 5.2. (a) *Any two distinct level-k tails are disjoint.*

(b) *For any level-k tail $\mathcal{T}$, there exists some $\mathcal{B} \subseteq \mathfrak{I}_k$, which is not contained in any level-k tail, and contains all island measure types in $\mathcal{T}$; in particular, $\mathcal{T}$ contains only a finite number of island measure types.*

We denote the collection of all level-k tails by $\mathfrak{T}_k$ and define $\mathfrak{T} := \bigcup_{k\geq 1}\mathfrak{T}_k$. For $\mathcal{T}\in\mathfrak{T}$, $[\mathcal{T}]_\mu$ is said to be the *(tail) measure type* of $\mathcal{T}$. We note that $\mathfrak{T}_k=\emptyset$ if $\#\mathfrak{I}_k<\infty$ for some $k\geq 1$.

Example 5.3. *Let $\{S_i\}_{i=0}^\infty$ be an IIFS on $\mathbb{R}$ satisfying (OSC) with OSC set $\Omega=(a,b)$ and μ be an associated self-similar measure. Define $\boldsymbol{v}_i := (S_i,1)$ for any $i\geq 0$. Assume that there exists some $x_0\in\Omega$ such that $S_{\boldsymbol{v}_i}(b)<S_{\boldsymbol{v}_{i+1}}(b)<x_0$ for any $i\geq 1$ and $\lim_{i\to\infty}S_{\boldsymbol{v}_i}(a)=\lim_{i\to\infty}S_{\boldsymbol{v}_i}(b)=x_0$. Then $\mathcal{T}:=\{\mathcal{I}(\boldsymbol{v}_i): i\geq 2\}$ is the only level-1 tail.*

Proof. It suffices to show that $\mathcal{T}$ satisfies conditions (1)–(4) of Definition 5.1. Example 2.5 implies that $\{S_i\}_{i=0}^\infty$ is of finite type with Ω being an FTC set. We note that $S_0(b)=b$ and $S_1(a)=a$. By assumption, $(S_{\boldsymbol{v}_2}(a),x_0)$ is the minimal open interval containing $S_{\mathcal{T}}(\Omega)$ and thus conditions (1) and (2) hold. Moreover, $\mathcal{T}\cup\{\mathcal{I}(\boldsymbol{v}_1)\}$ is the maximal family containing $\mathcal{T}$ and satisfying conditions (1) and (2) of Definition 5.1. Since $\mathfrak{I}/_{\approx_\mu}$ contains only one element, conditions (3) and (4) hold, which completes the proof. □

We will give another example of a tail in Lemma 5.12.

Definition 5.4. *Let μ be a self-similar measure defined by an IIFS on $\mathbb{R}$ of finite type. We say that μ satisfies* condition (B) *if there exists some $k\geq 1$ such that the following conditions hold:*

1. *the cardinalities of both $\mathfrak{T}_k$ and $\mathbf{I}_k := \{\mathcal{I}\in\mathfrak{I}_k : \mathcal{I}\notin\mathcal{T}$ for all $\mathcal{T}\in\mathfrak{T}_k\}$ are finite;*
2. *the number of level-ℓ nonbasic islands with respect to $\mathbf{I}_k$ is uniformly bounded for all $\ell\geq 2$;*
3. *for any $m\geq k$ and $\mathcal{I}\in\mathfrak{I}_m$, $O(\mathcal{I})$ can be expressed as the disjoint union of a finite subset of $\mathfrak{I}_{m+1}$ and a finite subset $\{\mathcal{T}_i\}_{i\in\Lambda_1}$ with the property that for any $i\in\Lambda_1$, there exists $\mathcal{T}\in\mathfrak{T}_k$ such that $[\mathcal{T}]_\mu=[\mathcal{T}_i]_\mu$.*

In this case, if $k:=k_b$ is the minimum non-negative integer satisfying all the above conditions, then we call the corresponding $\mathbf{I}_k =: \mathbf{I}_b$ and $\mathfrak{T}_{k_b} =: \mathbf{T}_b$ the basic set of islands *and the* basic set of tails*, respectively.*

Condition (1) and Remark 5.2(b) imply that $\mathbf{I}_k$ contains all level-k island measure types, i.e., $\{[\mathcal{I}]_\mu : \mathcal{I}\in\mathfrak{I}_k\}=\{[\mathcal{I}]_\mu : \mathcal{I}\in\mathbf{I}_k\}$. We remark that it is possible that for some $i\in\Lambda_1$, $\mathcal{T}_i\notin\mathfrak{T}_{m+1}$ in condition (3). (See Figure 5, $\mathcal{T}_{2,1,1}$ is not a tail.) We remark that under condition (B), there exists a unique basic set of islands $\mathbf{I}_b$ and a unique basic set of tails $\mathbf{T}_b$. Compared with that for FIFSs, condition (B) for IIFSs includes two additional assumptions, namely, conditions (1) and (3).

Remark 5.5. *Assume that condition (1) of Definition 5.4 holds with $\mathbf{I}_k\subseteq\mathfrak{I}_k$ for some $k\geq 1$. It follows that for any $m>k$,*

$$\{[\mathcal{I}]_\mu : \mathcal{I}\in\mathfrak{I}_m\}\subseteq\{[\mathcal{I}]_\mu : \mathcal{I}\in\mathbf{I}_k \text{ or } \mathcal{I} \text{ is a level-}(m-k+1) \text{ nonbasic island w.r.t. } \mathbf{I}_k\}.$$

Thus condition (3) of Definition 5.4 holds if for any $\mathcal{I} \in \mathbf{I}_k$, any $n \geq 2$, as well as any level-n nonbasic island $\mathcal{I}$ w.r.t. $\mathbf{I}_k$, $O(\mathcal{I})$ satisfies the property in condition (3) of Definition 5.4.

We illustrate condition (B) with the following two classes of examples.

Example 5.6. *Let μ be a self-similar measure defined by an IIFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}$ satisfying (OSC). Condition (1) of Definition 5.4 holds for $k = 1$ if and only if μ satisfies condition (B).*

Proof. Similar to Example 2.11, condition (2) of Definition 5.4 always holds. Condition (3) of Definition 5.4 follows from the fact that for any $m \geq 1$ and any $\mathcal{I} \in \mathfrak{I}_m$, $\mathcal{I} \approx_\mu \mathcal{I}(\boldsymbol{v}_{\text{root}})$. Thus μ satisfies condition (B). The converse is obvious. □

Example 5.7. *Let μ be the self-similar measure defined by an IIFS $\{S_i\}_{i=1}^\infty$ in* (1.10) *(see Figure 4) and a (positive) probability vector $(p_i)_{i=1}^\infty$. Assume that* (1.11) *holds. Then μ satisfies condition (B) with $\Omega = (0, 1)$.*

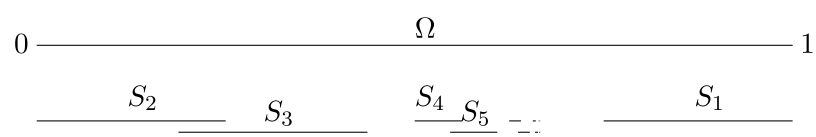

FIGURE 4. The first iteration of the IIFS $\{S_i\}_{i=1}^\infty$ defined in (1.10). The figure is drawn with $r = 1/4$ and $t = 2/3$.

We first summarize without proof some elementary properties.

Proposition 5.8. *Let $\{S_i\}_{i=1}^\infty$ be defined as in* (1.10) *and let $\Omega = (0, 1)$.*

(a) *Then $S_{(2k,1)} = S_{(2k+1,2)}$ for all $k \geq 1$. Moreover, for $k \geq 1$ and $m \geq 2$, define $W_{m,k} := \big\{(2k+1, 3^i, 2, 1^{m-2-i}) : i \in \{0, \dots, m-1\}\big\} \cup \{(2k, 1^{m-1})\}$. Then $S_{\boldsymbol{i}} = S_{\boldsymbol{j}}$ for any $\boldsymbol{i}, \boldsymbol{j} \in W_{m,k}$.*

(b) *Define*

$$\mathcal{I}_{1,0} := \{(S_1, 1)\} \quad \text{and} \quad \mathcal{I}_{1,k} := \{(S_{2k}, 1), (S_{2k+1}, 1)\} \quad \text{for all } k \geq 1. \tag{5.1}$$

Then $\mathfrak{I}_1 = \{\mathcal{I}_{1,k} : k \geq 0\}$ (See Figure 6).

(c) *For $k \geq 1$ and $m \geq 0$, $S_{2k}(\Omega) \cap S_{(2k+1,3^m)}(\Omega) = S_{(2k,1^{m+1})}(\Omega)$.*

Lemma 5.9. *The IIFS $\{S_i\}_{i=1}^\infty$ defined in* (1.10) *is of finite type with $\Omega = (0, 1)$ being an FTC set and with $\mathcal{M}_k = \Lambda^k$, where $\Lambda := \{i : i \in \mathbb{Z}_+\}$.*

Proof. We use the method in [14, Example 2.8]. Upon iterating the IFS once, the root vertex generates infinitely many vertices:

$$\boldsymbol{v}_{\text{root}} := \{id, 0\} \to \boldsymbol{v}_i := (S_i, 1) \qquad \text{for } i \in \Lambda.$$

Since $\mathfrak{N}(\boldsymbol{v}_1) = \{\boldsymbol{v}_1\}$, it follows that $\boldsymbol{v}_1 \sim \boldsymbol{v}_{\text{root}}$. Moreover, since $S_{\boldsymbol{v}_{2m}} \circ S_{\boldsymbol{v}_{2k}}^{-1} = S_{\boldsymbol{v}_{2m+1}} \circ S_{\boldsymbol{v}_{2k+1}}^{-1}$ for any $k, m \geq 1$, Proposition 5.8(b) implies that $\boldsymbol{v}_{2m} \sim \boldsymbol{v}_{2k}$ and $\boldsymbol{v}_{2m+1} \sim \boldsymbol{v}_{2k+1}$. Thus $\{[\boldsymbol{v}] : \boldsymbol{v} \in \mathcal{V}_1\} = \{[\boldsymbol{v}_1], [\boldsymbol{v}_2], [\boldsymbol{v}_3]\}$. Denote $\mathcal{T}_i := [\boldsymbol{v}_i]$ for $i = 1, 2, 3$. Thus

$$\begin{aligned}
\mathcal{T}_1 &= \{\boldsymbol{v} \in \mathcal{G} : \boldsymbol{v} \sim \boldsymbol{v}_{\text{root}}\} = \{v \in \mathcal{G} : \mathfrak{N}(\boldsymbol{v}) = \{\boldsymbol{v}\}\}, \\
\mathcal{T}_2 &= \{\boldsymbol{v} \in \mathcal{G} : \boldsymbol{v} \sim \boldsymbol{v}_2\} = \{v \in \mathcal{G} : \mathfrak{N}(\boldsymbol{v}) = \{\boldsymbol{v}, \boldsymbol{v}'\} \text{ and } S_{\boldsymbol{v}_2} \circ S_{\boldsymbol{v}}^{-1} \circ S_{\boldsymbol{v}'} = S_{\boldsymbol{v}_3}\}, \\
\mathcal{T}_3 &= \{\boldsymbol{v} \in \mathcal{G} : \boldsymbol{v} \sim \boldsymbol{v}_3\} = \{v \in \mathcal{G} : \mathfrak{N}(\boldsymbol{v}) = \{\boldsymbol{v}, \boldsymbol{v}'\} \text{ and } S_{\boldsymbol{v}_3} \circ S_{\boldsymbol{v}}^{-1} \circ S_{\boldsymbol{v}'} = S_{\boldsymbol{v}_2}\}.
\end{aligned} \tag{5.2}$$

Upon one more iteration, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ generate infinitely many offspring in $\mathcal{G}$:

$$\boldsymbol{v}_2 \to \boldsymbol{v}_{2,i} := (S_{(2,i)}, 2) \quad \text{and} \quad \boldsymbol{v}_3 \to \boldsymbol{v}_{3,i} := (S_{(3,i)}, 2) \qquad \text{for } i \in \Lambda.$$

Using (5.2), it is straightforward to verify that for all $k \geq 1$,

$$[\boldsymbol{v}_{2,2k}] = [\boldsymbol{v}_{3,2k}] = [\boldsymbol{v}_{2,1}] = [\boldsymbol{v}_2], \qquad [\boldsymbol{v}_{2,2k+1}] = [\boldsymbol{v}_{3,2k+1}] = [\boldsymbol{v}_3], \qquad [\boldsymbol{v}_{3,1}] = [\boldsymbol{v}_1].$$

Since no new neighborhood types are generated, Proposition 2.3 now implies that $\mathcal{V}/_\sim = \{[\boldsymbol{v}_1], [\boldsymbol{v}_2], [\boldsymbol{v}_3]\}$ and thus the finite type condition holds. $\square$

Proposition 5.10. *Let μ be the self-similar measure defined by the IIFS $\{S_i\}_{i=1}^\infty$ in* (1.10) *and a probability vector $(p_i)_{i=1}^\infty$. Let $\mathcal{I}_{1,i}$ be defined as in* (5.1) *for $i \geq 1$. Then for $m > k \geq 1$, $p_{2m}/p_{2m+1} = p_{2k}/p_{2k+1}$ if and only if $\mathcal{I}_{1,k} \approx_{\mu,\tau,w} \mathcal{I}_{1,m}$ with $\tau(x) := r^{m-k}x + t(1 - r^{m-k})$ and $w := p_{2m}/p_{2k}$.*

Proof. Fix any $m > k \geq 1$. Define $\tau(x) := r^{m-k}x + t(1 - r^{m-k})$ and $w := p_{2m}/p_{2k}$. We first note that $\tau \circ S_{2k} = S_{2m}$ and $\tau \circ S_{2k+1} = S_{2m+1}$. It follows that $\mathcal{I}_{1,k} \approx_\tau \mathcal{I}_{1,m}$. Let $A \subseteq S_{\mathcal{I}_{1,m}}(\Omega)$, we have $\tau^{-1}(A) \subseteq S_{\mathcal{I}_{1,k}}(\Omega)$. Thus

$$\begin{aligned}
&\mu(A) = p_{2m}\mu \circ S_{2k}^{-1}(\tau^{-1}(A)) + p_{2m+1}\mu \circ S_{2k+1}^{-1}(\tau^{-1}(A)), \quad \text{and} \\
&\mu(\tau^{-1}(A)) = p_{2k}\mu \circ S_{2k}^{-1}(\tau^{-1}(A)) + p_{2k+1}\mu \circ S_{2k+1}^{-1}(\tau^{-1}(A)).
\end{aligned}$$

Thus $\mu|_{S_{\mathcal{I}_{1,m}}(\Omega)} = w\mu|_{S_{\mathcal{I}_{1,k}}(\Omega)} \circ \tau^{-1}$ if and only if $p_{2m}/p_{2m+1} = p_{2k}/p_{2k+1}$. Hence, the result follows from the definition of $\approx_\mu$. $\square$

For $k \geq 1$, define

$$\begin{aligned}
&\widetilde{w}_1(0, k) := p_{2k}, \quad \widetilde{w}_1(m, k) := p_{2k+1}p_2\Big(\sum_{i=0}^{m-1} p_3^i p_1^{m-1-i}\Big) + p_{2k}p_1^m \quad \text{for } m \geq 1, \\
&\widetilde{w}_2(m, k) := p_{2k+1}p_3^m \qquad \text{for } m \geq 0.
\end{aligned} \tag{5.3}$$

We remark that for $m \geq 0$ and $k \geq 1$,

$$p_{2k}p_1^{m+1} + p_{2k+1}\widetilde{w}_1(m,1) = \widetilde{w}_1(m+1,k) \quad \text{and} \quad p_{2k+1}\widetilde{w}_2(m,1) = \widetilde{w}_2(m+1,k). \tag{5.4}$$

Define

$$I_{1,0} := S_{\mathcal{I}_{1,0}}(\Omega) = S_1(\Omega) \quad \text{and} \quad I_{1,k} := S_{\mathcal{I}_{1,k}}(\Omega) = S_{2k}(\Omega) \cup S_{2k+1}(\Omega) \quad \text{for } k \geq 1. \tag{5.5}$$

Proposition 5.11. *Assume the hypothese of Proposition 5.10. Let $\widetilde{w}_1(\cdot,\cdot)$ and $\widetilde{w}_2(\cdot,\cdot)$ be defined as in* (5.3)*, and $I_{1,i}$ be defined as in* (5.5) *for $i \geq 0$. Then for $k \geq 1$ and $m \geq 0$,*

(a) $\mu|_{S_{(2k+1,3^m)}(I_{1,1})} = \widetilde{w}_1(m,k)\mu|_{I_{1,0}} \circ S^{-1}_{(2k,1^m)} + \widetilde{w}_2(m,k)\mu|_{I_{1,1}} \circ S^{-1}_{(2k+1,3^m)}$;
(b) $\mu|_{S_{(2k,1^m)}(I_{1,\ell})} = \widetilde{w}_1(m,k) \cdot \mu|_{I_{1,\ell}} \circ S^{-1}_{(2k,1^m)}$ *for* $\ell \geq 1$*;*
(c) $\mu|_{S_{(2k+1,3^m)}(I_{1,\ell})} = \widetilde{w}_2(m,k) \cdot \mu|_{I_{1,\ell}} \circ S^{-1}_{(2k+1,3^m)}$ *for* $\ell = 0$ *or* $\ell \geq 2$.

Proof. (a) We only show that the equality holds for $k = 1$ and $m \geq 0$, i.e.,

$$\mu|_{S_{3^m}(I_{1,1})} = \widetilde{w}_1(m-1,1)\mu|_{I_{1,0}} \circ S^{-1}_{(2,1^{m-1})} + \widetilde{w}_2(m-1,1)\mu|_{I_{1,1}} \circ S^{-1}_{3^m} \quad \text{for } m \geq 1. \tag{5.6}$$

Proposition 5.8(b) implies that $\mu(A) = p_2\mu \circ S_2^{-1}(A) + p_3\mu \circ S_2^{-1}(A)$ for any $A \subseteq S_3(I_{1,1})$, i.e., (5.6) holds for $m = 1$. Assume that (5.6) holds for $m = \ell$. If $A \subseteq S_{3^{\ell+1}}(I_{1,1})$, then $S_3^{-1}(A) \subseteq S_{3^\ell}(I_{1,1})$. By assumption, we have

$$\mu(S_3^{-1}(A)) = \widetilde{w}_1(\ell-1,1)\mu \circ S^{-1}_{(3,2,1^{\ell-1})}(A) + \widetilde{w}_2(\ell-1,1)\mu \circ S^{-1}_{3^{\ell+1}}(A). \tag{5.7}$$

By Proposition 5.8(a,c) and the definition of $I_{1,1}$,

$$\begin{aligned} S_2(\Omega) \cap S_{3^{\ell+1}}(I_{1,1}) &= S_2(\Omega) \cap \big(S_{(3^{\ell+1},2)}(\Omega) \cup S_{3^{\ell+2}}(\Omega)\big) \\ &= S_2(\Omega) \cap \big(S_{(2,1^{\ell+1})}(\Omega) \cup S_{3^{\ell+2}}(\Omega)\big) \\ &= S_{(2,1^{\ell+1})}(\Omega) \cup S_{(2,1^{\ell+2})}(\Omega) = S_{(2,1^{\ell+1})}(\Omega). \end{aligned}$$

Thus $S_2^{-1}(A) \cap \Omega \subseteq S_{1^{\ell+1}}(\Omega) = S_{1^\ell}(I_{1,0})$. Hence, Proposition 5.8(b) implies

$$\mu(S_2^{-1}(A)) = p_1\mu \circ S^{-1}_{(2,1)}(A) = \cdots = p_1^\ell\mu \circ S^{-1}_{(2,1^\ell)}(A). \tag{5.8}$$

Combining (5.7) and (5.8), we have

$$\begin{aligned} \mu(A) &= p_2\mu \circ S_2^{-1}(A) + p_3\mu \circ S_3^{-1}(A) \\ &= p_2p_1^\ell\mu \circ S^{-1}_{(2,1^\ell)}(A) + p_3\Big(\widetilde{w}_1(\ell-1,1)\mu \circ S^{-1}_{(3,2,1^{\ell-1})}(A) + \widetilde{w}_2(\ell-1,1)\mu \circ S^{-1}_{3^{\ell+1}}(A)\Big) \\ &= \big(p_2p_1^\ell + p_3\widetilde{w}_1(\ell-1,1)\big)\mu \circ S^{-1}_{(2,1^\ell)}(A) + p_3\widetilde{w}_2(\ell-1,1)\mu \circ S^{-1}_{3^{\ell+1}}(A) \\ &= \widetilde{w}_1(\ell,1)\mu \circ S^{-1}_{(2,1^\ell)}(A) + \widetilde{w}_2(\ell,1)\mu \circ S^{-1}_{3^{\ell+1}}(A), \end{aligned}$$

where the third equality uses the fact $S_{(3,2)} = S_{(2,1)}$ and the last equality uses (5.4). By induction, (5.6) holds.

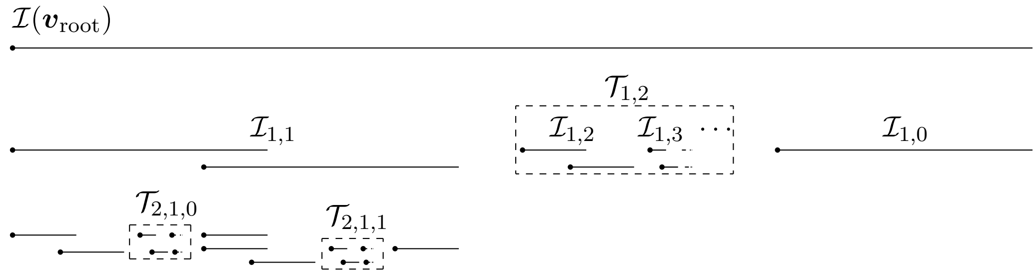

FIGURE 5. Let $\{S_i\}_{i=1}^{\infty}$ be defined as in (1.10) and assume (1.11) holds with $L = 2$. $\mathcal{T}_{1,2}$, $\mathcal{T}_{2,1,0}$, and $\mathcal{T}_{2,1,1}$ are defined in Lemma 5.12 and the proof of Example 5.7. They consist of islands enclosed by a box. We remark that $\mathcal{T}_{2,1,1}$ is not a tail, since it does not satisfy condition (3) of Definition 5.1. The figure is drawn with $r = 1/4$ and $t = 2/3$.

(b) The proof is similar to that of (a). (c) Fix any $k \geq 1, \ell \geq 2$ or $\ell = 0$, let $A \subseteq S_{(2k+1,3^m)}(I_{1,\ell})$. Proposition 5.8(a,c) imply

$$\begin{aligned} S_{(2k+1,3^m)}(I_{1,\ell}) \cap S_{2k}(\Omega) &= S_{(2k+1,3^m)}(I_{1,\ell}) \cap S_{(2k,1^{m+1})}(\Omega) \\ &= S_{(2k+1,3^m)}(I_{1,\ell}) \cap S_{(2k+1,3^m,2)}(\Omega) = \emptyset \quad \text{for } m \geq 0. \end{aligned}$$

In particular, $S_{3^m}(I_{1,\ell}) \cap S_2(\Omega) = \emptyset$ for $m \geq 1$. Thus

$$\mu(A) = p_{2k+1}\mu \circ S_{2k+1}^{-1}(A) = \widetilde{w}_2(m,k)\mu \circ S_{(2k+1,3^m)}^{-1}(A).$$

□

Lemma 5.12. *Assume the hypothese of Example 5.7. Let $\mathcal{I}_{1,i}$ be defined as in* (5.5) *for $i \in \mathbb{Z}_+$. Then $\mathcal{T}_{1,L} := \{\mathcal{I}_{1,i} : i \geq L\}$ is the only level-*1 *tail (see Figure 5).*

Proof. It suffices to show that $\mathcal{T}$ satisfies conditions (1)–(4) of Definition 5.1. We first note that condition (1) holds with $(S_{2L}(0), t)$ being the minimal open interval. We note that $\lim_{k\to\infty} S_{2k}(x) = \lim_{k\to\infty} S_{2k+1}(x) \searrow t$ for any $x \in (0,1)$. Thus condition (2) of Definition 5.1 holds. Similarly, $\{\mathcal{I}_{1,i} : 1 \leq i \leq L-1\} \cup \mathcal{T}_{1,L}$ also satisfies conditions (1) and (2) of Definition 5.1. Moreover, by our assumption and Proposition 5.10, we have $\{[\mathcal{I}_{1,i}]_\mu : 1 \leq i \leq L-1\} = \{[\mathcal{I}_{1,i}]_\mu : i \geq L\}$ and $\{[\mathcal{I}_{1,i}]_\mu : 1 \leq i \leq L-2\} \neq \{[\mathcal{I}_{1,i}]_\mu : i \geq L-1\}$. It follows that conditions (3) and (4) of Definition 5.1 hold. □

Proof of Example 5.7. For each $k \geq 0$, let $\mathcal{M}_k = \{i : i \in \mathbb{Z}_+\}^k$. Let $\mathcal{I}_{1,i}$ be defined as in (5.1) for $i \geq 0$. By assumption and Lemma 5.12, $\mathcal{T}_{1,L} := \{\mathcal{I}_{1,i} : i \geq L\}$ is the only level-1 tail. Choose $\mathbf{I}_b = \{\mathcal{I}_{1,i} : i = 0, \ldots, L-1\}$, $\mathbf{T}_b := \{\mathcal{T}_{1,L}\}$, and let $\mathbf{I}_{b,\mu} := \{[\mathcal{I}]_\mu : \mathcal{I} \in \mathbf{I}_b\}$. Then condition (1) of Definition 5.4 holds.

Since $\mathcal{I}_{1,0} \approx_\mu \mathcal{I}(\boldsymbol{v}_{\rm root})$, $\mathcal{I}_{1,0}$ does not generate any level-2 nonbasic island with respect to $\mathbf{I}_b$. Let $\Gamma'_{1,*} := \{1, \dots, L-1\}$. For $\ell \in \Gamma'_{1,*}$, $i \geq 1$, and $j = 1, 2$, define

$$\begin{aligned} &\mathcal{I}^1_{2,\ell,i} := \{(S_{(2\ell,2i)}, 2), (S_{(2\ell,2i+1)}, 2)\}, \quad \mathcal{I}^2_{2,\ell,i} := \{(S_{(2\ell+1,2i)}, 2), (S_{(2\ell+1,2i+1)}, 2)\}, \\ &\mathcal{I}^2_{2,\ell,0} := \{(S_{(2\ell+1,1)}, 2)\}, \quad \text{and} \quad \mathcal{T}_{2,\ell,j} := \{\mathcal{I}^j_{2,\ell,m} : m \geq L\}. \end{aligned} \tag{5.9}$$

(See Figure 6). Here for an island $\mathcal{I}^j_{2,\ell,i}$, the subscript 2 denotes the level $\mathcal{V}_2$ that the island belongs, ℓ indicates that $\mathcal{I}^j_{2,\ell,i}$ is an offspring of $\mathcal{I}_{1,\ell}$, and i indexes the islands according to the iterations of the IFS maps. The superscript j labels the parent of a vertex $\boldsymbol{v} \in \mathcal{I}^j_{2,\ell,i}$. If $j = 1$, $\boldsymbol{v}$ is an offspring of $(S_{2l}, 1)$; if $j = 2$, $\boldsymbol{v}$ is an offspring of $(S_{2l+1}, 1)$. Fix $\ell \in \Gamma'_{1,*}$, the set of all offspring of $\mathcal{I}_{1,\ell}$ is

$$\begin{aligned} O(\mathcal{I}_{1,\ell}) &= \{\mathcal{I}^1_{2,\ell,i} : i \geq 1\} \cup \{\mathcal{I}^2_{2,\ell,i} : i \geq 0\} \\ &= \{\mathcal{I}^j_{2,\ell,i} : i = 1, \dots, L-1, j = 1, 2\} \cup \{\mathcal{I}^2_{2,\ell,0}\} \cup \mathcal{T}_{2,\ell,1} \cup \mathcal{T}_{2,\ell,2}. \end{aligned} \tag{5.10}$$

Let $\widetilde{w}_1(\cdot,\cdot)$ and $\widetilde{w}_2(\cdot,\cdot)$ be defined as in (5.3). Proposition 5.11 implies that $\mathcal{I}_{1,i} \approx_{\mu,S_{2\ell},\widetilde{w}_1(0,\ell)} \mathcal{I}^1_{2,\ell,i}$ for $i \geq 1$, and $\mathcal{I}_{1,i} \approx_{\mu,S_{2\ell+1},\widetilde{w}_2(0,\ell)} \mathcal{I}^2_{2,\ell,i}$ for $i \geq 2$ and $i = 0$; moreover, $[\mathcal{I}^2_{2,\ell,1}]_\mu \notin \mathbf{I}_{b,\mu}$. Thus $\mathcal{I}_{1,\ell}$ generates only one level-2 nonbasic island, namely, $\mathcal{I}^2_{2,\ell,1}$, with respect to $\mathbf{I}_b$. Also, $\mathcal{T}_{1,L} \approx_{\mu,S_{2\ell},\widetilde{w}_1(0,\ell)} \mathcal{T}_{2,\ell,1}$, and $\mathcal{T}_{1,L} \approx_{\mu,S_{2\ell+1},\widetilde{w}_2(0,\ell)} \mathcal{T}_{2,\ell,2}$.

Similarly, for $\ell \in \Gamma'_{1,*}$, $i \geq 1$, $j = 1, 2$ and $k \geq 3$, define

$$\begin{aligned} &\mathcal{I}^1_{k,\ell,i} := \{(S_{(2\ell+1,3^{k-3},2,2i)}, k), (S_{(2\ell+1,3^{k-3},2,2i+1)}, k)\}, \\ &\mathcal{I}^2_{k,\ell,i} := \{(S_{(2\ell+1,3^{k-2},2i)}, k), (S_{(2\ell+1,3^{k-2},2i+1)}, k)\}, \\ &\mathcal{I}^2_{k,\ell,0} := \{(S_{(2\ell+1,3^{k-2},1)}, k)\}, \quad \text{and} \quad \mathcal{T}_{k,\ell,j} := \{\mathcal{I}^j_{k,\ell,m} : m \geq L\}. \end{aligned}$$

Assume that $\mathcal{I}^2_{m,\ell,1} := \{(S_{(2\ell+1,3^{m-2},2)}, m), (S_{(2\ell+1,3^{m-1})}, m)\}$ is a level-m nonbasic island with respect to $\mathbf{I}_b$ for $m \geq 2$. Then the set of all offspring of $\mathcal{I}^2_{m,\ell,1}$ is

$$\begin{aligned} O(\mathcal{I}^2_{m,\ell,1}) =& \{\mathcal{I}^1_{m+1,\ell,i} : i \geq 1\} \cup \{\mathcal{I}^2_{m+1,\ell,i} : i \geq 0\} \\ =& \{\mathcal{I}^j_{m+1,\ell,i} : i = 1, \dots, L-1, j = 1, 2\} \cup \{\mathcal{I}^2_{m+1,\ell,0}\} \cup \mathcal{T}_{m+1,\ell,1} \cup \mathcal{T}_{m+1,\ell,2}. \end{aligned}$$

Using Proposition 5.8(a), we have $\mathcal{I}^1_{m+1,\ell,i} = \{(S_{(2\ell,1^{m-1},2i)}, m+1), (S_{(2\ell,1^{m-1},2i+1)}, m+1)\}$. Similarly, Proposition 5.11 implies that $\mathcal{I}_{1,i} \approx_{\mu,S_{(2\ell,1^{m-1})},\widetilde{w}_1(m-1,\ell)} \mathcal{I}^1_{m+1,\ell,i}$ for $i \geq 1$, and $\mathcal{I}_{1,i} \approx_{\mu,S_{(2\ell+1,3^{m-1})},\widetilde{w}_2(m-1,\ell)} \mathcal{I}^2_{m+1,\ell,i}$ for $i \geq 2$ and $i = 0$; moreover, $[\mathcal{I}^2_{m+1,\ell,1}]_\mu \notin \mathbf{I}_{b,\mu}$. Thus $\mathcal{I}^2_{m,\ell,1}$ generates only one level-$(m+1)$ nonbasic island, namely, $\mathcal{I}^2_{m+1,\ell,1}$, with respect to $\mathbf{I}_b$. Moreover, $\mathcal{T}_{1,L} \approx_{\mu,S_{(2\ell,1^{m-1})},\widetilde{w}_1(m-1,\ell)} \mathcal{T}_{m,\ell,1}$, and $\mathcal{T}_{1,L} \approx_{\mu,S_{(2\ell+1,3^{m-1})},\widetilde{w}_2(m-1,\ell)} \mathcal{T}_{m,\ell,2}$. By induction and using Remark 5.5(b), conditions (2) and (3) of Definition 5.4 hold. $\square$

5.2. **Renewal equation for IIFSs.** We use the same method as in Section 3.2.

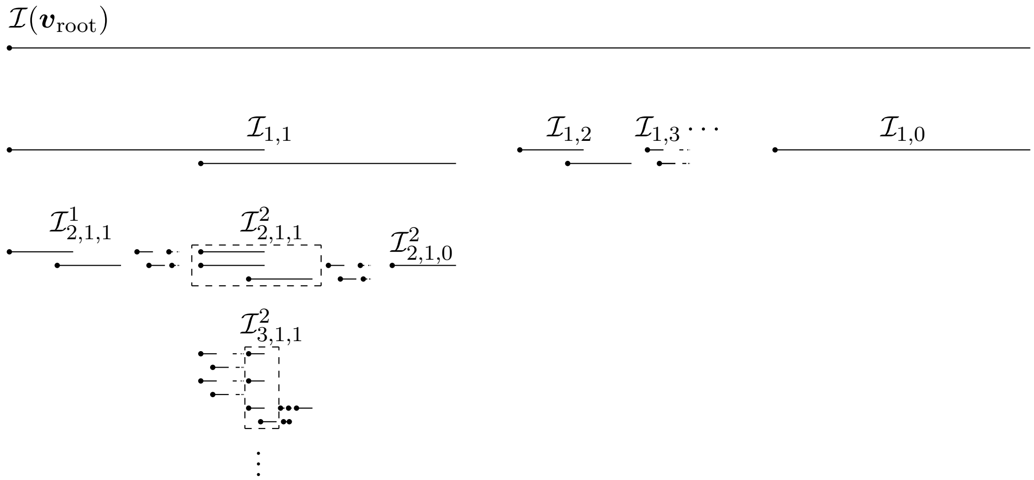

FIGURE 6. Figure for iterates of the IFS $\{S_i\}_{i=1}^{\infty}$ in (1.10) under the assumption that (1.11) holds with $L = 2$. Using the notation in the proof of Example 5.7, we have $\mathbf{I}_b := \{\mathcal{I}_{1,0}, \mathcal{I}_{1,1}\}$, $\mathbf{T}_b := \{\mathcal{T}_{1,2}\}$, and $\mathcal{I}^2_{k,1,1}$ is the only level-k nonbasic island with respect to $\mathbf{I}_b$. All nonbasic islands are boxed. The figure is drawn with $r = 1/4$ and $t = 2/3$.

Let $\{S_i\}_{i\in\Lambda}$ be an IIFS on a compact subset $X \subseteq \mathbb{R}$ satisfying the finite type condition with FTC set $\Omega \subseteq X$ and μ be the self-similar measure defined by $\{S_i\}_{i\in\Lambda}$. For $\mathcal{A} \subseteq \mathcal{V}_k$, $\mathcal{B} \subseteq \mathfrak{I}_k$ ($k \geq 0$), and $\mathcal{I} \in \mathfrak{I}$, we let $S_{\mathcal{A}}(\Omega)$, $S_{\mathcal{B}}(\Omega)$, $O(\mathcal{I})$ be defined as in (2.2), (2.3), and (2.4), respectively. We denote the contraction ratio of a contractive similitude τ by r_τ.

In the rest of this subsection, we assume that μ satisfies condition (B). Let $\mathbf{I}_b \subseteq \mathfrak{I}_{k_b}$ be the basic set of islands and $\mathbf{T}_b := \mathfrak{T}_{k_b}$ be the basic set of tails. Also, let $\mathbf{I}_{b,\mu} := \big\{[\mathcal{I}]_\mu : \mathcal{I} \in \mathbf{I}_b\big\}$ and $\mathbf{T}_{b,\mu} := \big\{[\mathcal{T}]_\mu : \mathcal{T} \in \mathbf{T}_b\big\}$. Let $\#\mathbf{I}_b + \#\mathbf{T}_b = \eta$. Since $\mathbf{I}_b \cup (\bigcup_{\mathcal{T}\in\mathbf{T}_b} \mathcal{T})$ is a disjoint union, using (3.2), we have

$$\begin{aligned}
&\sum_{\mathcal{I}\in\mathbf{I}_b} N(\lambda, -\Delta_{\mu|_{S_{\mathcal{I}}(\Omega)}}) + \sum_{\mathcal{T}\in\mathbf{T}_b} N(\lambda, -\Delta_{\mu|_T}) \leq N(\lambda, -\Delta_\mu) \\
&\leq \sum_{\mathcal{I}\in\mathbf{I}_b} N(\lambda, -\Delta_{\mu|_{S_{\mathcal{I}}(\Omega)}}) + \sum_{\mathcal{T}\in\mathbf{T}_b} N(\lambda, -\Delta_{\mu|_T}) + 2\eta - 2,
\end{aligned} \tag{5.11}$$

where T is the minimum open interval containing $S_{\mathcal{T}}(\Omega)$.

We choose a subset $\mathbf{I} := \{\mathcal{I}_{1,\ell}\}_{\ell\in\Gamma_1} \subseteq \mathbf{I}_b$ such that for any $\mathcal{I} \in \mathbf{I}_b$, there exists a unique $\ell \in \Gamma_1$ satisfying $\mathcal{I} \in [\mathcal{I}_{1,\ell}]_\mu$. Similarly, choose a subset $\mathbf{T} := \{\mathcal{T}_{1,\ell}\}_{\ell\in\Gamma_2} \subseteq \mathbf{T}_b$ such that for any $\mathcal{T} \in \mathbf{T}_b$, there exists a unique $i \in \Gamma_2$ satisfying $[\mathcal{T}]_\mu = [\mathcal{T}_{1,\ell}]_\mu$. Denote $\Gamma := \Gamma_1 \cup \Gamma_2$. Define $I_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega)$ for $\ell \in \Gamma_1$, and for $\ell \in \Gamma_2$, let $T_{1,\ell}$ be the minimum open interval containing $S_{\mathcal{T}_{1,\ell}}(\Omega)$.

In view of Proposition 3.2 and (5.11), to study $N(\lambda, \Delta_\mu)$, it suffices to study $N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}})$ and $N(\lambda, -\Delta|_{\mu|_{T_{1,m}}})$ for $\ell \in \Gamma_1$ and $m \in \Gamma_2$.

Step 1. Derivation of a functional equation for $N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}})$ for $\ell \in \Gamma_1$. Similar to Step 1 in Section 3.2, for $\ell \in \Gamma_1$, we define

$$\mathbf{I}_{2,\ell} := \{\mathcal{I} : \mathcal{I} \in O(\mathcal{I}_{1,\ell}) \text{ and } [\mathcal{I}]_\mu \in \mathbf{I}_{b,\mu}\} \quad \text{and} \quad \mathbf{I}'_{2,\ell} := \{\mathcal{I} : \mathcal{I} \in O(\mathcal{I}_{1,\ell}) \text{ and } [\mathcal{I}]_\mu \notin \mathbf{I}_{b,\mu}\}.$$

For $k \geq 3$, if $\mathbf{I}'_{k-1,\ell} \neq \emptyset$, we define

$$\begin{aligned} \mathbf{I}_{k,\ell} &:= \{\mathcal{I} : \mathcal{I} \in O(\mathcal{J}) \text{ for some } \mathcal{J} \in \mathbf{I}'_{k-1,\ell} \text{ and } [\mathcal{I}]_\mu \in \mathbf{I}_{b,\mu}\}, \\ \mathbf{I}'_{k,\ell} &:= \{\mathcal{I} : \mathcal{I} \in O(\mathcal{J}) \text{ for some } \mathcal{J} \in \mathbf{I}'_{k-1,\ell} \text{ and } [\mathcal{I}]_\mu \notin \mathbf{I}_{b,\mu}\}. \end{aligned}$$

Without loss of generality, we assume that Γ_1 can be partitioned into two sub-collections, $\Gamma_{1,*}$ and $\Gamma'_{1,*}$, defined as follows. For $\ell \in \Gamma_1$, we say $\ell \in \Gamma_{1,*}$ if there exists a $\kappa_\ell \geq 2$, depending on ℓ, that is the smallest integer satisfying $\mathbf{I}'_{\kappa_\ell,\ell} = \emptyset$. Let $\Gamma'_{1,*} := \Gamma_1 \setminus \Gamma_{1,*}$ and define $\kappa_\ell := \infty$ for $\ell \in \Gamma'_{1,*}$.

Fix $\ell \in \Gamma_1$. Condition (B) implies that for any $2 \leq k \leq \kappa_\ell$, there exist finite subsets $G_{k,\ell}, G'_{k,\ell}, E_{k,\ell} \subseteq \mathbb{Z}$ such that

$$\begin{aligned} &\mathbf{I}_{k,\ell} = \{\mathcal{I}_{k,\ell,i} : i \in G_{k,\ell}\} \cup \Big(\bigcup_{i \in E_{k,\ell}} \mathcal{T}_{k,\ell,i}\Big), \qquad \mathbf{I}'_{k,\ell} = \{\mathcal{I}_{k,\ell,i} : i \in G'_{k,\ell}\}, \\ &0 \leq \# G'_{k,\ell} \leq M, \qquad G_{k,\ell} \cap G'_{k,\ell} = \emptyset, \qquad [\mathcal{T}_{k,\ell,i}]_\mu \in \mathbf{T}_{b,\mu} \qquad \text{for } i \in E_{k,\ell}, \end{aligned} \tag{5.12}$$

where $M > 0$ is a constant. Define

$$I_{k,\ell,i} := S_{\mathcal{I}_{k,\ell,i}}(\Omega) \qquad \text{for } 2 \leq k \leq \kappa_\ell \text{ and } i \in G_{k,\ell} \cup G'_{k,\ell}.$$

For $2 \leq k \leq \kappa_\ell$ and $i \in E_{k,\ell}$, let $T_{k,\ell,i}$ be the minimal open interval containing $S_{\mathcal{T}_{k,\ell,i}}(\Omega)$.

Proposition 5.13. *Assume that condition (B) holds.*

(a) *For $\ell \in \Gamma_1$, $2 \leq k \leq \kappa_\ell$, and $i \in G_{k,\ell}$, there exist unique $\xi_1(k,\ell,i) > 0$ and $c_1(k,\ell,i) \in \Gamma_1$ such that*

$$N(\lambda, -\Delta_{\mu|_{I_{k,\ell,i}}}) = N\big(\xi_1(k,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c_1(k,\ell,i)}}}\big).$$

(b) *For $\ell \in \Gamma_1$, $2 \leq k \leq \kappa_\ell$, and $i \in E_{k,\ell}$, there exist unique $\xi_2(k,\ell,i) > 0$ and $c_2(k,\ell,i) \in \Gamma_2$ such that*

$$N(\lambda, -\Delta_{\mu|_{T_{k,\ell,i}}}) = N\big(\xi_2(k,\ell,i)\lambda, -\Delta_{\mu|_{T_{1,c_2(k,\ell,i)}}}\big). \tag{5.13}$$

Proof. (a) The proof is similar to that of Proposition 3.3.

(b) (5.12) implies that for any $i \in E_{k,\ell}$, there exist unique $\tau_2(k,\ell,i) \in \mathscr{S}$, $w_2(k,\ell,i) > 0$, and $c_2(k,\ell,i) \in \Gamma_2$ such that $\mathcal{T}_{1,c_2(k,\ell,i)} \dot{\approx}_{\mu,\tau_2(k,\ell,i),w_2(k,\ell,i)} \mathcal{T}_{k,\ell,i}$. Combining this with Proposition 3.2(b), we have (5.13) holds with $\xi_2(k,\ell,i) := r_{\tau_2(k,\ell,i)} w_2(k,\ell,i)$. $\square$

For each integer $n \geq 1$ and $\ell \in \Gamma_1$, we define a partition $\mathcal{P}_{n,\ell}$ of $I_{1,\ell}$ as follows:

$$\begin{aligned}
&\mathcal{P}_{1,\ell} := \big\{x : x \text{ is an end-point of } I_{1,\ell}\big\};\\
&\mathcal{P}_{n,\ell} := \mathcal{P}_{n-1,\ell} \cup \big\{x : x \text{ is an end-point of some interval of the form}\\
&\qquad\qquad I_{n,\ell,i},\, T_{n,\ell,j},\, i \in G_{n,\ell} \cup G'_{n,\ell}, j \in E_{n,\ell}\big\} \quad \text{ for } 2 \leq n \leq \kappa_\ell.
\end{aligned}$$

Then $\mathcal{P}_{n,\ell}$, $\ell \in \Gamma_1$, $1 \leq n \leq \kappa_\ell$, are end-points of subintervals generated by a similar procedure as in Step 1 of Section 3.2. Thus we can now obtain the following analogues of (3.5) and (3.6):

$$\begin{aligned}
N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}}) = \sum_{j=2}^{\kappa_\ell} \Big(&\sum_{i \in G_{j,\ell}} N\big(\xi_1(j,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c_1(j,\ell,i)}}}\big)\\
&+ \sum_{i \in E_{j,\ell}} N(\xi_2(j,\ell,i)\lambda, -\Delta_{\mu|_{T_{1,c_2(j,\ell,i)}}})\Big) + \varepsilon(\kappa_\ell,\ell) \quad \text{for } \ell \in \Gamma_{1,*},
\end{aligned} \tag{5.14}$$

where $0 \leq \varepsilon(\kappa_\ell,\ell) \leq \#\mathcal{P}_{\kappa_\ell,\ell} - 2$, and

$$\begin{aligned}
N(\lambda, -\Delta_{\mu|_{I_{1,\ell}}}) = \sum_{j=1}^{n} \Big(&\sum_{i \in G_{j,\ell}} N\big(\xi_1(j,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c_1(j,\ell,i)}}}\big)\\
&+ \sum_{i \in E_{j,\ell}} N\big(\xi_2(j,\ell,i)\lambda, -\Delta_{\mu|_{T_{1,c_2(j,\ell,i)}}}\big)\Big)\\
&+ \sum_{i \in G'_{n,\ell}} N(\lambda, -\Delta_{\mu|_{I'_{n,\ell,i}}}) + \varepsilon(n,\ell) \qquad \text{for } n \geq 2, \ell \in \Gamma'_{1,*},
\end{aligned} \tag{5.15}$$

where $0 \leq \varepsilon(n,\ell) \leq \#\mathcal{P}_{n,\ell} - 2$.

Step 2. Derivation of a functional equation for $N(\lambda, -\Delta_{\mu|_{T_{1,\ell}}})$ for $\ell \in \Gamma_2$.

By the definition of a tail, for any $\ell \in \Gamma_2$, $\mathcal{T}_{1,\ell}$ can be uniquely expressed as $\mathcal{T}_{1,\ell} := \{\mathcal{I}_{1,\ell,i}\}_{i=a(\ell)}^{\infty}$ for some constant $a(\ell)$ such that, if let $\mathcal{R}_{1,\ell,j} := \{\mathcal{I}_{1,\ell,i}\}_{i=j}^{\infty}$ and $R_{1,\ell,j}$ be the minimal open interval containing $S_{\mathcal{R}_{1,\ell,j}}(\Omega)$ for any $j \geq a(\ell)+1$, then $S_{\mathcal{I}}(\Omega) \cap R_{1,\ell,j} \neq \emptyset$ if and only if $\mathcal{I} \in \mathcal{R}_{1,\ell,j}$.

Similarly, without loss of generality, we assume that Γ_2 can be partitioned into two subcollections, $\Gamma_{2,*}$ and $\Gamma'_{2,*}$, defined as follows: For $\ell \in \Gamma_2$, we say $\ell \in \Gamma_{2,*}$ if there exists an integer $\kappa_\ell \geq a(\ell)+1$, depending on ℓ, such that κ_ℓ is the minimum number satisfying $\mathcal{T}_{1,c'(\ell)} \dot{\approx}_\mu \mathcal{R}_{1,\ell,\kappa_\ell}$ for some $c'(\ell) \in \Gamma_2$; otherwise, $\ell \in \Gamma'_{2,*}$. Denote $\kappa_\ell := \infty$ for $\ell \in \Gamma'_{2,*}$. Define $I_{1,\ell,i} := S_{\mathcal{I}_{1,\ell,i}}(\Omega)$ for all $\ell \in \Gamma_2$ and $i \geq a(\ell)$.

Proposition 5.14. *Assume that condition (B) holds.*

(a) *For $\ell \in \Gamma_2$ and $1 \leq i \leq \kappa_\ell$, there exist unique $\xi_3(1,\ell,i) > 0$ and $c_3(1,\ell,i) \in \Gamma_1$ such that*

$$N(\lambda, -\Delta_{\mu|_{I_{1,\ell,i}}}) = N\big(\xi_3(1,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c_3(1,\ell,i)}}}\big). \tag{5.16}$$

(b) *For $\ell \in \Gamma_{2,*}$, there exist unique $\xi_4(1,\ell) > 0$ and $c_4(1,\ell) \in \Gamma_2$ such that*

$$N(\lambda, -\Delta_{\mu|_{R_{1,\ell,\kappa_\ell}}}) = N\big(\xi_4(1,\ell)\lambda, -\Delta_{\mu|_{T_{1,c_4(1,\ell)}}}\big). \tag{5.17}$$

Proof. The proof is similar to that of of Proposition 5.13; we omit the details. □

For each $\ell \in \Gamma_2$ and each integer $1 \le n \le \kappa_\ell$, we define a partition $\mathcal{P}_{n,\ell}$ of $T_{1,\ell}$ as follows:

$$\mathcal{P}_{1,\ell} := \big\{x : x \text{ is an end-point of } T_{1,\ell}\big\};$$
$$\mathcal{P}_{n,\ell} := \mathcal{P}_{n-1,\ell} \cup \big\{x : x \text{ is an end-point of } I_{1,\ell,n-1} \text{ or } R_{1,\ell,n}\big\} \qquad \text{for } 2 \le n \le \kappa_\ell.$$

Then $\mathcal{P}_{n,\ell}$ ($\ell \in \Gamma_2, 1 \le n \le \kappa_\ell$) are the end-points of subintervals generated by the following procedure. First, replace $T_{1,\ell}$ by the subintervals $I_{1,\ell,a(\ell)}$ and $R_{1,\ell,a(\ell)+1}$. If $\kappa_\ell = 2$, we stop this procedure; otherwise, keep $I_{1,\ell,a(\ell)}$ and replace $R_{1,\ell,a(\ell)+1}$ by the subintervals $I_{1,\ell,a(\ell)+1}$ and $R_{1,\ell,a(\ell)+2}$. Continue. We note that $\mathcal{P}_{n,\ell} \subseteq \operatorname{supp}(\mu)$ for all $\ell \in \Gamma_2$ and any $1 \le n \le \kappa_\ell$.

For $\ell \in \Gamma_2$, let $\mathcal{F}_{n,\ell} := \mathcal{F}(\mathcal{P}_{n,\ell})$ for $1 \le n \le \kappa_\ell$. We can now obtain analogues of (5.14) and (5.15) as follows. For $\ell \in \Gamma_{2,*}$,

$$\begin{aligned} N(\lambda, -\Delta_{\mu|_{T_{1,\ell}}}) = &\sum_{i=a(\ell)}^{\kappa_\ell - 1} N(\xi_3(1,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c_3(1,\ell,i)}}}) \\ &+ N(\xi_4(1,\ell)\lambda, -\Delta_{\mu|_{T_{1,c_4(1,\ell)}}}) + \varepsilon(\kappa_\ell,\ell), \end{aligned} \tag{5.18}$$

where $0 \le \varepsilon(\kappa_\ell,\ell) \le \#\mathcal{P}_{\kappa_\ell,\ell}$; while for $\ell \in \Gamma'_{2,*}$ and $n \ge 2$,

$$N(\lambda, -\Delta_{\mu|_{T_{1,\ell}}}) = \sum_{i=a(\ell)}^{n-1} N(\xi_3(1,\ell,i)\lambda, -\Delta_{\mu|_{I_{1,c_3(1,\ell,i)}}}) + N(\lambda, -\Delta_{\mu|_{R_{1,\ell,n}}}) + \varepsilon(n,\ell), \tag{5.19}$$

where $0 \le \varepsilon(n,\ell) \le \#\mathcal{P}_{n,\ell}$.

Step 3. Derivation of the vector-valued renewal equation. Define

$$\begin{aligned} f_\ell(t) = f_\ell^{(\alpha)}(t) &:= e^{-\alpha t} N(e^t, -\Delta_{\mu|_{I_{1,\ell}}}) \qquad \text{for } \ell \in \Gamma_1, \\ f_\ell(t) = f_\ell^{(\alpha)}(t) &:= e^{-\alpha t} N(e^t, -\Delta_{\mu|_{T_{1,\ell}}}) \qquad \text{for } \ell \in \Gamma_2. \end{aligned} \tag{5.20}$$

As in Step 2 of Section 3.2, there exists $t_0 \in \mathbb{R}$ such that $f_\ell(t) = 0$ for any $t < t_0$ and any $\ell \in \Gamma$. For each $t \in \mathbb{R}$, let n_t be the smallest integer such that

$$\begin{aligned} &t + \max\big\{\ln(\xi_1(n_t,\ell,i)) : \ell \in \Gamma'_{1,*},\ i \in G_{j,\ell}\big\} < t_0, \\ &t + \max\big\{\ln(\xi_2(n_t,\ell,i)) : \ell \in \Gamma'_{1,*},\ i \in E_{j,\ell}\big\} < t_0, \\ &t + \max\big\{\ln(\xi_3(1,\ell,n_t)) : \ell \in \Gamma'_{2,*}\big\} < t_0. \end{aligned} \tag{5.21}$$

We further simplify notation by letting

$$\begin{aligned} &\rho_s(j,\ell,i;\alpha,t) := \xi_s(j,\ell,i)^\alpha f_{c_s(j,\ell,i)}(t+\ln(\xi_s(j,\ell,i))), \quad s=1,2,3,\\ &\rho_4(1,\ell;\alpha,t) := \xi_4(1,\ell)^\alpha f_{c_4(1,\ell)}(t+\ln(\xi_4(1,\ell)), \end{aligned} \tag{5.22}$$

where the indices j,ℓ,i are to be specified. Then we obtain the following equations:

$$\begin{aligned} f_\ell(t) &= \sum_{j=2}^{k_\ell}\Big(\sum_{i\in D_{j,\ell}}\rho_1(j,\ell,i;\alpha,t)+\sum_{i\in E_{j,\ell}}\rho_2(j,\ell,i;\alpha,t)\Big)+z_\ell^{(\alpha)}(t), \quad \ell\in\Gamma_{1,*},\\ f_\ell(t) &= \Big(\sum_{i=1}^{\kappa_\ell-1}\rho_3(1,\ell,i;\alpha,t)\Big)+\rho_4(1,\ell;\alpha,t)+z_\ell^{(\alpha)}(t), \quad \ell\in\Gamma_{2,*},\\ f_\ell(t) &= \sum_{j=2}^{\infty}\Big(\sum_{i\in D_{j,\ell}}\rho_1(j,\ell,i;\alpha,t)+\sum_{i\in E_{j,\ell}}\rho_2(j,\ell,i;\alpha,t)\Big)+z_\ell^{(\alpha)}(t), \quad \ell\in\Gamma'_{1,*},\\ f_\ell(t) &= \sum_{i=a(\ell)}^{\infty}\rho_3(1,\ell,i;\alpha,t)+z_\ell^{(\alpha)}(t), \quad \ell\in\Gamma'_{2,*}. \end{aligned} \tag{5.23}$$

where $z_\ell^{(\alpha)}(t) := e^{-\alpha t}\varepsilon(\kappa_\ell,\ell)$ for $\ell\in\Gamma_{1,*}\cup\Gamma_{2,*}$ and $z_\ell^{(\alpha)}(t) := e^{-\alpha t}\big(N(e^t,-\Delta_{\mu|_{R_{1,\ell,n_t}}})+\varepsilon(n_t,\ell)\big)$ for $\ell\in\Gamma'_{1,*}\cup\Gamma'_{2,*}$.

We define $\mu_{\ell m}^{(\alpha)}$ as in Table I.

Table I. Definition of $\mu_{\ell m}^{(\alpha)}$, where $m\in\Gamma_1\cup\Gamma_2$

ℓ	j	i	$c_s(j,\ell,i)$	point	$\mu_{\ell m}^{(\alpha)}$
$\Gamma_{1,*}$	$2\le j\le\kappa_\ell$	$i\in G_{j,\ell}$	$c_1(j,\ell,i)=m$	$-\ln\big(\xi_1(j,\ell,i)\big)$	$\xi_1(j,\ell,i)^\alpha$
	$2\le j\le\kappa_\ell$	$i\in E_{j,\ell}$	$c_2(j,\ell,i)=m$	$-\ln\big(\xi_2(j,\ell,i)\big)$	$\xi_2(j,\ell,i)^\alpha$
$\Gamma'_{1,*}$	$j\ge 2$	$i\in G_{j,\ell}$	$c_1(j,\ell,i)=m$	$-\ln\big(\xi_1(j,\ell,i)\big)$	$\xi_1(j,\ell,i)^\alpha$
	$j\ge 2$	$i\in E_{j,\ell}$	$c_2(j,\ell,i)=m$	$-\ln\big(\xi_2(j,\ell,i)\big)$	$\xi_2(j,\ell,i)^\alpha$
$\Gamma_{2,*}$	$a(\ell)\le j\le\kappa_\ell-1$		$c_3(1,\ell,j)=m$	$-\ln\big(\xi_3(1,\ell,j)\big)$	$\xi_3(1,\ell,j)^\alpha$
			$c_4(1,\ell)=m$	$-\ln\big(\xi_4(1,\ell)\big)$	$\xi_4(1,\ell)^\alpha$
$\Gamma'_{2,*}$	$j\ge a(\ell)$		$c_3(1,\ell,j)=m$	$-\ln\big(\xi_3(1,\ell,j)\big)$	$\xi_3(1,\ell,j)^\alpha$

Following from the above derivations, we have

Theorem 5.15. *Let μ be a self-similar measure defined by a finite type IIFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}$. Let $\mathbf{f},\mathbf{M}_\alpha$, and $\mathbf{z}$ be defined as in* (1.5)*. Assume that μ satisfies condition (B). Then $\mathbf{f}$ satisfies the vector-valued renewal equation $\mathbf{f}=\mathbf{f}*\mathbf{M}_\alpha+\mathbf{z}$.*

5.3. **Proof of Theorem 1.3 and a corollary.** The proof is similar to that for Theorem 1.1, except that $\sum_{k=0}^{L} m_{\ell k}^{(\alpha)}$ takes the following values:

$$\begin{aligned}
&\sum_{j=2}^{\kappa_\ell}\Big(\sum_{i\in G_{j,\ell}}\xi_1(j,\ell,i)^\alpha\Big|\ln(\xi_1(j,\ell,i))\Big|+\sum_{i\in E_{j,\ell}}\xi_2(j,\ell,i)^\alpha\Big|\ln(\xi_2(j,\ell,i))\Big|\Big), \quad \text{if } \ell\in\Gamma_1;\\
&\sum_{j=1}^{\kappa_\ell}\Big(\xi_3(1,\ell,j)^\alpha\Big|\ln(\xi_3(1,\ell,j))\Big|+\xi_4(1,\ell)^\alpha\Big|\ln(\xi_4(1,\ell))\Big|\Big), \quad \text{if } \ell\in\Gamma_{2,*};\\
&\sum_{j=1}^{\infty}\xi_3(1,\ell,j)^\alpha\Big|\ln(\xi_3(1,\ell,j))\Big|, \quad \text{if } \ell\in\Gamma_{2,*}'.
\end{aligned}$$

The derivation of renewal equation is given by (5.23), and the vector-valued renewal equation $\mathbf{f}=\mathbf{f}*\mathbf{M}_\alpha+\mathbf{z}$ is given by Theorem 5.15. $\square$

We state a corollory of Theorem 1.3 for IIFSs on $\mathbb{R}$ satisfying (OSC).

Corollary 5.16. *Let μ be a self-similar measure defined by an IIFS $\{S_i\}_{i\in\Lambda}$ satisfying (OSC) together with a probability vector $(p_i)_{i\in\Lambda}$. Assume that there are finitely many level-1 tails, and finitely many level-1 islands not contained in any level-1 tail. Let $\mathbf{M}_\alpha(\infty)$, $F_i(\alpha)$, and $\widetilde{\alpha}_i$ be defined as in (1.6) and (1.7). Assume that for each $i=0,\dots,L$, $\lim_{\alpha\to\widetilde{\alpha}_i^+}F_i(\alpha)>1$. Also, assume that for any $\mathcal{T}:=\{\mathcal{I}_i: i\ge 0\}\in\mathfrak{T}_1$, there exist constants $C,k_0>0$ such that $\mu|_{R_{1,k}}\le Cp_k\mu|_T\circ\tau_k^{-1}$ for any $k\ge k_0$, where I_k, $R_{1,k}$ and T are the open intervals corresponding to $\mathcal{I}_k$, $\mathcal{R}_{1,k}:=\{\mathcal{I}_i: i\ge k\}$ and $\mathcal{T}$, respectively, and τ_k is a similitude mapping I_0 onto I_k. Then conclusions (a) and (b) of Theorem 1.1 hold.*

Proof of Corollary 5.16. Example 5.6 implies that μ satisfies condition (B). The corollory follows from Theorem 1.1. $\square$

5.4. **A class of infinite IFSs with overlaps.** We first state a result concerning the existence of self-similar measure μ associated with an IIFS $\{S_i\}_{i\in\Lambda}$ and the proof is similar to that of [5, Theorem 2.8].

Proposition 5.17. *Let $\{S_i\}_{i\in\Lambda}$ be an IIFS on $\mathbb{R}$ and r_i be the contraction ratio of S_i. Assume there exists $c>0$ such that $r_i\le c<1$ for all $i\in\Lambda$. Then for any probability vector $(p_i)_{i\in\Lambda}$, there exists a unique probability measure μ satisfying the self-similar identity (2.1).*

In this subsection, we consider the family of IIFSs defined as in (1.10) and fix an FTC set $\Omega=(0,1)$. Proposition 5.17 implies the existence of a self-similar measure μ for any probability vector $(p_i)_{i=1}^\infty$. Assume that (1.11) holds in the rest of this section. Let $\mathcal{I}_{1,i}$, $i\ge 0$, and $\mathcal{T}_{1,L}$ be defined as in the proof of Examples 5.7. Thus Examples 5.7 implies that μ satisfies condition (B) with $\mathbf{I}_b:=\{\mathcal{I}_{1,i}: i=0,\dots,L-1\}$ being the basic set of islands and $\mathbf{T}_b:=\{\mathcal{T}_{1,L}\}$ being the basic set of tails. Let $\mathbf{I}_{b,\mu}:=\{[\mathcal{I}]_\mu:\mathcal{I}\in\mathbf{I}_b\}$. Without loss of generality, assume that $[\mathcal{I}_{1,i}]_\mu\ne[\mathcal{I}_{1,j}]$ for any distinct $i,j\in\{0,\dots,L-1\}$. Then $\mathbf{I}_b=\mathbf{I}$ and $\Gamma=\Gamma_1\cup\Gamma_2$, where $\Gamma_1=\{0,\dots,L-1\}$ and $\Gamma_2=\{L\}$. Moreover, $\Gamma_{1,*}=\{0\}$,

$\Gamma'_{1,*} = \{1, \dots, L-1\}$ and $\kappa_0 = 2$. Let $\mathbf{I}_{k,\ell}$, $\mathbf{I}'_{k,\ell}$, $G_{k,\ell}$, $G'_{k,\ell}$, and $E_{k,\ell}$ be defined as in the Section 5.2 for $2 \le k \le \kappa_\ell$ and $\ell \in \Gamma_1$.

Define

$$\mathcal{I}_{2,0,0} := \{(S_{(1,1)}, 2)\} \qquad \text{and} \qquad \mathcal{I}_{2,0,i} := \{(S_{(1,2i)}, 2), (S_{(1,2i+1)}, 2)\} \quad \text{for } i \ge 1.$$

Let $\mathcal{T}_{2,0,0} := \{\mathcal{I}_{2,0,i}\}_{i=L}^{\infty}$. Then $O(\mathcal{I}_{1,0}) = \{\mathcal{I}_{2,0,i} : 0 \le i \le L-1\} \cup \mathcal{T}_{2,0,0}$. Since $(id, 0) \sim_{\mu,S_1,p_1} (S_1, 1)$, $\mathcal{I}_{1,i} \approx_{\mu,S_1,p_1} \mathcal{I}_{2,0,i}$ for all $i \ge 0$, in particular, $\mathcal{T}_{1,L} \dot{\approx}_{\mu,S_1,p_1} \mathcal{T}_{2,0,0}$. It follows that $\mathbf{I}_{2,0} = \{\mathcal{I}_{2,0,i} : 0 \le i \le L-1\} \cup \mathcal{T}_{2,0,0}$ and $\mathbf{I}'_{2,0} = \emptyset$. Thus $G_{2,0} = \{0, \dots, L-1\}$, $G'_{2,0} = \emptyset$, and $E_{2,0} = \{0\}$. Moreover, the following Proposition holds.

Proposition 5.18. *Let $\xi_i(\cdot,\cdot,\cdot)$ and $c_i(\cdot,\cdot,\cdot)$ be defined as in Proposition 5.13 for $i \in \{1,2\}$. Then $\xi_1(2,0,i) = \xi_2(2,0,0) = rp_1$, $c_1(2,0,i) = i$ for all $0 \le i \le L-1$, and $c_2(2,0,0) = L$.*

For $\ell \in \Gamma'_{1,*}$, $i \ge 1$, $j = 0$ or $j \ge 2$, and $k \ge 2$, let $\mathcal{I}^1_{k,\ell,i}$, $\mathcal{I}^2_{k,\ell,j}$, $\mathcal{T}_{k,\ell,1}$ and $\mathcal{T}_{k,\ell,2}$ be defined as in (5.9) and (5.10). Let $\widetilde{w}_1(\cdot,\cdot)$ and $\widetilde{w}_2(\cdot,\cdot)$ be defined as in (5.3). The proof of Example 5.7 shows that

$$\begin{aligned}
&\mathcal{I}_{1,i} \approx_{\mu,S_{(2\ell,1^{k-2})},\widetilde{w}_1(k-2,\ell)} \mathcal{I}^1_{k,\ell,i} && \text{for } i \ge 1,\\
&\mathcal{I}_{1,i} \approx_{\mu,S_{(2\ell+1,3^{k-2})},\widetilde{w}_2(k-2,\ell)} \mathcal{I}^2_{k,\ell,i} && \text{for } i = 0 \text{ and } i \ge 2,\\
&\mathcal{I}^2_{k,\ell,1} \notin \mathbf{I}_{b,\mu}.
\end{aligned} \tag{5.24}$$

Moreover,

$$\mathcal{T}_{1,L} \dot{\approx}_{\mu,S_{(2\ell,1^{k-2})},\widetilde{w}_1(k-2,\ell)} \mathcal{T}_{k,\ell,1} \quad \text{and} \quad \mathcal{T}_{1,L} \dot{\approx}_{\mu,S_{(2\ell+1,3^{k-2})},\widetilde{w}_2(k-2,\ell)} \mathcal{T}_{k,\ell,2}. \tag{5.25}$$

Thus $\mathbf{I}_{k,\ell} = \{\mathcal{I}^j_{k,\ell,i} : i = 1, \dots, L-1, j = 0, 1\} \cup \{\mathcal{I}^2_{k,\ell,0}\} \cup \mathcal{T}_{k,\ell,1} \cup \mathcal{T}_{k,\ell,2}$ and $\mathbf{I}'_{k,\ell} = \{\mathcal{I}^2_{k,\ell,1}\}$. For $k \ge 2$ and $\ell \in \Gamma'_{1,*}$, define

$$\begin{aligned}
\mathcal{I}_{k,\ell,i} &:= \mathcal{I}^1_{k,\ell,i} \quad \text{for } i = 1, \dots, L-1, \qquad \mathcal{I}_{k,\ell,0} := \mathcal{I}^2_{k,\ell,0},\\
\mathcal{I}_{k,\ell,i+L-2} &:= \mathcal{I}^2_{k,\ell,i} \quad \text{for } i = 2, \dots, L-1, \quad \text{and} \quad \mathcal{I}_{k,\ell,2L-2} := \mathcal{I}^2_{k,\ell,1}.
\end{aligned}$$

It follows that $G_{k,\ell} = \{0, \dots, 2L-3\}$, $G'_{k,\ell} = \{2L-2\}$, and $E_{k,\ell} = \{1,2\}$. Combining this with (5.24) and (5.25), we have the following result.

Proposition 5.19. *Assume the hypothese of Proposition 5.18. Let $\xi_i(\cdot,\cdot,\cdot)$ and $c_i(\cdot,\cdot,\cdot)$ be defined as in Proposition 5.13 for $i \in \{1,2\}$. Then for any $k \ge 2$ and all $\ell \in \Gamma'_{1,*}$,*

$$\begin{aligned}
&\xi_1(k,\ell,0) = r^{k+\ell-2}\widetilde{w}_2(k-2,\ell) \quad \textit{and} \quad c_1(k,\ell,0) = 0;\\
&\xi_1(k,\ell,i) = r^{k+\ell-2}\widetilde{w}_1(k-2,\ell) \quad \textit{and} \quad c_1(k,\ell,i) = i \quad \textit{for all } 1 \le i \le L-1;\\
&\xi_1(k,\ell,i) = r^{k+\ell-2}\widetilde{w}_2(k-2,\ell) \quad \textit{and} \quad c_1(k,\ell,i) = i-L+2 \quad \textit{for all } L \le i \le 2L-3;\\
&\xi_2(k,\ell,0) = r^{k+\ell-2}\widetilde{w}_1(k-2,\ell) \quad \textit{and} \quad c_2(k,\ell,0) = L;\\
&\xi_2(k,\ell,1) = r^{k+\ell-2}\widetilde{w}_2(k-2,\ell) \quad \textit{and} \quad c_2(k,\ell,1) = L.
\end{aligned}$$

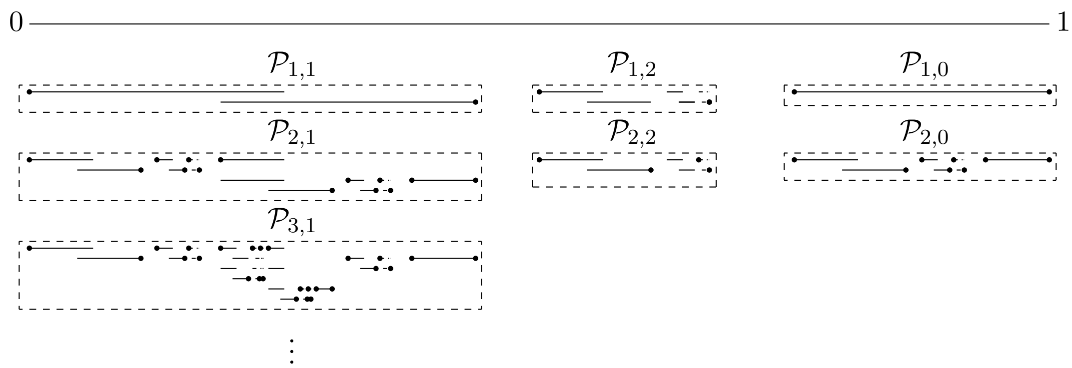

FIGURE 7. Partitions $\mathcal{P}_{n,\ell}$ defined in Section 3 for an IIFS $\{S_i\}_{i=1}^{\infty}$ in (1.10), assuming that (1.11) holds with $L = 2 \in \Gamma_{2,*}$ and $\kappa_L = 2$. The figure is drawn with $r = 1/4$, $t = 2/3$.

Since $\mathcal{T}_{1,L} := \{\mathcal{I}_{1,i} : i \geq L\}$, $a(L) = L$. Let $\mathcal{R}_{1,L,j} := \{\mathcal{I}_{1,L,i} : i \geq j\}$ for any $j \geq L+1$.

Proposition 5.20. *Assume the hypothese of Proposition 5.18. Let $\xi_3(1,\cdot,\cdot)$, $c_3(1,\cdot,\cdot)$ $\xi_4(1,\cdot)$, and $c_4(1,\cdot)$ be defined as in Proposition 5.14. Then the following hold.*

(a) $1 \leq c_3(1,L,i) \leq L-1$ *and* $\xi_3(1,L,i) = r^{i-c_3(1,L,i)} p_{2i}/p_{2c_3(1,L,i)}$ *for all* $L \leq i \leq \kappa_L$.
(b) *If* $L \in \Gamma_{2,*}$, *then* $\xi_4(1,L) = r^{\kappa_L - L} p_{2\kappa_L}/p_{2L}$, *and* $c_4(1,L) = L$.

Proof. The results follows from Propositions 5.10 and 5.14. □

Let $I_{1,i}$ be defined as (5.5) for $\ell \geq 0$ and $T_{1,L} := (S_{2L}(0), t)$. Define

$$I_{k,\ell,2L-2} := S_{\mathcal{I}_{k,\ell,2L-2}}(\Omega) = S_{(2\ell+1,3^{k-2})}(I_{1,1}) \qquad \text{for } k \geq 2 \text{ and } \ell \in \Gamma'_{1,*}.$$

Let $\mathcal{P}_{n,\ell}$, $\ell \in \{0,\dots,L-1\}$ and $\mathcal{P}_{n,L}$ be the partitions of $I_{1,\ell}$ and $T_{1,L}$, respectively, defined as in Section 5.2 (see Figure 7). Then $\#\mathcal{P}_{\kappa_0,0} = 2L$, $\#(\mathcal{P}_{n,\ell}) \leq 2Ln$ for $n \geq 1$ and $\ell \in \Gamma'_{1,*}$, and $\#\mathcal{P}_{n,L} \leq 2n$ for $1 \leq n \leq \kappa_L$.

Let $f_\ell(t)$ be defined as in (5.20) for $\ell \in \Gamma$. Applying Propositions 5.18, 5.19, and 5.20 to (5.23), we have

$$\begin{aligned}
f_0(t) &= \big(rp_1\big)^\alpha \sum_{i=0}^{L} f_i\big(t+\ln(rp_1)\big) + e^{-\alpha t}\varepsilon(2,0),\\
f_\ell(t) &= \sum_{k=0}^{\infty} \big(r^{\ell+k}\widetilde{w}_1(k,\ell)\big)^\alpha \sum_{i=1}^{L} f_i\big(t+\ln(r^{\ell+k}\widetilde{w}_1(k,\ell))\big)\\
&\quad + \sum_{k=0}^{\infty} \big(r^{\ell+k}\widetilde{w}_2(k,\ell)\big)^\alpha \Big(\sum_{j=2}^{L} f_j\big(t+\ln(r^{\ell+k}\widetilde{w}_2(k,\ell))\big) + f_0\big(t+\ln(r^{\ell+k}\widetilde{w}_2(k,\ell))\big)\Big)\\
&\quad + z_\ell^{(\alpha)}(t), \qquad \text{for } \ell \in \Gamma'_{1,*},
\end{aligned}$$

and if $L \in \Gamma_{2,*}$,

$$f_L(t) = \Big(\sum_{i=1}^{\kappa_L} \rho_3(1,L,i;\alpha,t)\Big) + \rho_4(1,L;\alpha,t) + z_L^{(\alpha)}(t);$$

otherwise, i.e., $L \in \Gamma'_{2,*}$,

$$f_L(t) = \Big(\sum_{i=1}^{\infty} \rho_3(1,L,i;\alpha,t)\Big) + z_L^{(\alpha)}(t),$$

where ρ_3 and ρ_4 are defined in (5.22). Let $\mu_{\ell m}^{(\alpha)}$, $\ell, m \in \Gamma_1 \cup \Gamma_2$, be the discrete measure defined as in Table I.

Proposition 5.21. *Let $F_\ell(\alpha)$ and $\tilde{\alpha}_l$ be defined as in (1.7) for $\ell \in \Gamma$. Then $\widetilde{\alpha}_\ell = 0$ and $F_\ell(0) > 1$ for all $\ell \in \Gamma$.*

Proof. Similar to that of Proposition 4.2. □

Proposition 5.22. *For $\ell \in \Gamma'_{1,*}$, there exists a constant $C > 0$ such that $N\big(\lambda, -\Delta_{\mu|_{I_{n_t,\ell,2L-2}}}\big) \le C$.*

Proof. We note that $\mu|_{I_{n_t,\ell,2L-2}} \le (\widetilde{w}_1(n_t-2,\ell)/p_2)\mu \circ S^{-1}_{(2\ell+1,3^{n_t-2})}$ on $I_{n_t,\ell,2L-2}$. The proof is similar to that of Proposition 4.3. □

Proposition 5.23. *Assume that (1.12) holds. If $L \in \Gamma'_{2,*}$, then there exists a constant $C > 0$ such that $N\big(\lambda, -\Delta_{\mu|_{R_{1,L,n_t}}}\big) \le C$.*

Proof. Let $\tau(x) = r^{n_t-L}x + t(1-r^{n_t-L})$. Using (1.11), (1.12), and the proof of Proposition 5.10, we have $\mathcal{T}_{1,L} \dot{\approx}_\tau R_{1,L,n_t}$ and $\mu|_{I_{1,n_t+i}} \le C(p_{2n_t}/p_{2L})\mu|_{I_{1,L+i}} \circ \tau^{-1}$ on I_{1,n_t+i} for any $i \ge 0$. Thus $\mu|_{R_{1,L,n_t}} \le (p_{2n_t}/p_{2L})\mu \circ \tau^{-1}$ on R_{1,L,n_t}. The result can now be deduced by using the method in Proposition 4.3. □

Proposition 5.24. *Assume that* (1.12) *holds. Let α be the unique number such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to* 1. *Then there exists some $\sigma > 0$ such that for all $\ell \in \Gamma$, $z_\ell^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$.*

Proof. The proof is similar to that of Proposition 4.4. □

Proof of Theorem 1.4. Combine Propositions 5.21, and 5.24, and Theorem 1.3. □

Example 5.25. *Let μ be the self-similar measure defined by an IIFS as in* (1.10) *with probability vector $(p_i)_{i=1}^\infty$. Assume that $r = 1/4$, $t = 2/3$, $p_1 = 1/3$, and $p_{2k} = p_{2k+1} = 1/4^k$ for all $k \geq 1$. Then $\dim_s(\mu) \approx 0.93168$.*

Proof. We note that (1.11) and (1.12) hold with $L = 2$; moreover $L \in \Gamma_{2,*}$ and $\kappa_L = 2$. Theorem 1.4 implies that α is the unique positive number satisfying $|I_3 - \mathbf{M}_\alpha(\infty)| = 0$, where I_3 is the 3×3 identity matrix. Since

$$\mathbf{M}_\alpha(\infty) = \begin{pmatrix} (rp_1)^\alpha & (rp_1)^\alpha & (rp_1)^\alpha \\ a & b & a+b \\ 0 & (rq)^\alpha & (rq)^\alpha \end{pmatrix},$$

where $q = 1/4$, $a := \sum_{i=0}^\infty (r^{1+i}\widetilde{w}_2(i,1))^\alpha$ and $b := \sum_{i=0}^\infty (r^{1+i}\widetilde{w}_1(i,1))^\alpha$, α is the unique positive number satisfying $(1-(rp_1)^\alpha)((rq)^\alpha(a+1)+b-1)+a(rp_1)^\alpha = 0$. Hence, $\dim_s(\mu) = 2\alpha$, which can be easily approximated. □

6. Comments and questions

We do not know whether the assumption concerning the error estimates in Theorem 1.1(b) can be removed.

It is of interest to express the eigenvalue counting function in terms of the properties of the measure and the domain, as in the original Weyl law. Also, in view of Weyl's conjecture stated in the introduction, it is of interest to study the second order term in the asymptotic expansion of the eigenvalue counting function.

It is interesting to extend our results to higher dimensions. It is expected that additional efforts are needed to estimate the error terms in the renewal equation.

Appendix. Vector-valued renewal theorem

For convenience, we state the vector-valued renewal theorem by Lau *et al.* [16], which is used in the proof of Theorem 1.1. We first introduce some terminology and refer the reader to [23, 16] for any unexplained term. Let $\mathbf{F}$ be a matrix-valued Radon measure that vanishes on $(-\infty, 0)$, i.e.,

$$\mathbf{F} = \begin{bmatrix} F_{11} & \dots & F_{1n} \\ \vdots & & \vdots \\ F_{n1} & \dots & F_{nn} \end{bmatrix},$$

where $F_{ij}(x) = \mu_{ij}(-\infty, x]$ and each μ_{ij} is a Radon measure (i.e., positive Borel regular measure) on $\mathbb{R}$ that vanishes on $(-\infty, 0)$. Define $\mathbf{F}(\infty) := [F_{ij}(\infty)]$ and let $\mathbf{m} = [m_{ij}] = [\int_0^\infty x\, dF_{ij}]$ be the *moment matrix.* If

$$\sum_{i=1}^{n} F_{ij}(0) < \sum_{i=1}^{n} F_{ij}(\infty) \quad \text{for } 1 \le j \le n,$$

each column of $\mathbf{F}$ is said to be *nondegenerate* at 0. For any *path* $\gamma = (i_1, \dots, i_k)$ with $i_j \in \{1, \dots, n\}$ for any $j = 1, \dots, k$, we define

$$\mu_\gamma := \mu_{i_1 i_2} * \mu_{i_2 i_3} * \cdots * \mu_{i_{k-1} i_k}.$$

In this case, γ is called a *cycle* if $i_1 = i_k$; in particular, if it is a cycle and $i_j, \dots, i_{k-1}$ are distinct, then γ is said to be a *simple cycle.* We denote by $\mathbb{R}_\mathbf{F}$ the closed subgroup of $(\mathbb{R}, +)$ generated by

$$\bigcup \bigl\{\mathrm{supp}(\mu_\gamma) : \ \gamma \text{ is a simple cycle on } \{1, \dots, n\}\bigr\}.$$

The following theorem, stated in [23], is modified from [16, Theorem 4.3].

Theorem A.1. (Lau *et al.* [16]) *Let $\mathbf{F}$ be an $n \times n$ matrix-valued Radon measure defined on $\mathbb{R}$ that vanishes on $(-\infty, 0)$ and assume that each column of $\mathbf{F}$ is nondegenerate at 0. Suppose $\mathbf{F}(\infty)$ is irreducible and has maximal eigenvalue 1. Let $\mathbf{U} = \sum_{k=0}^\infty \mathbf{F}^{*k}$ and let $\mathbf{z}$ be a directly Riemann integrable function on $\mathbb{R}$ that vanishes on $(-\infty, x_o)$ for some $x_o \in \mathbb{R}$. Then $\mathbf{f} = \mathbf{z} * \mathbf{U}$ is a bounded Borel measurable solution of*

$$\mathbf{f}(x) = (\mathbf{f} * \mathbf{F})(x) + \mathbf{z}(x), \qquad x \in \mathbb{R}, \tag{A.1}$$

and it is unique in the class of Borel measurable solutions that vanish on $(-\infty, x_o)$. Furthermore, the following hold:

(a) *If $\mathbb{R}_\mathbf{F} = \mathbb{R}$, then*

$$\lim_{x\to\infty} \mathbf{f}(x) = \left(\int_{-\infty}^{\infty} \mathbf{z}(t)\, dt\right) \mathbf{A},$$

where

$$\mathbf{A} = \frac{1}{\gamma}\begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ \vdots & & \vdots \\ u_nv_1 & \dots & u_nv_n \end{bmatrix}, \quad \gamma = [v_1, \dots, v_n]\begin{bmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{u} = [u_1, \dots, u_n], \mathbf{v} = [v_1, \dots, v_n]$ *are the unique normalized positive right and left 1-eigenvectors of* $\mathbf{F}(\infty)$*, respectively.* ($\mathbf{A} = \mathbf{0}$ *if one of the* m_{ij} *is* ∞.)

(b) *If* $\mathbb{R}_{\mathbf{F}} = \langle\lambda\rangle$ *for some* $\lambda > 0$*, then for each* $x > 0$*,*

$$\lim_{k\to\infty}\big[f_1(x + a_{11} + k\lambda), \dots, f_n(x + a_{1n} + k\lambda)\big] = \left(\sum_{k=-\infty}^{\infty}\mathbf{z}(x + k\lambda)\right)\mathbf{A},$$

where $a_{1j} \in \operatorname{supp}(\mu_{\gamma(1,j)})$ *and* $\gamma(1, j)$ *is any path from* 1 *to* j *such that* $\mu_{\gamma(1,j)} \neq 0$.

Acknowledgements. Part of this work was carried out when the first author was visiting the Center of Mathematical Sciences and Applications of Harvard University. He is indebted to Professor Shing-Tung Yau for the opportunity and thanks the center for its hospitality and support.

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College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China, and Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, USA.

E-mail address: smngai@georgiasouthern.edu

Key Laboratory of High Performance Computing and Stochastic Information Processing (HPCSIP) (Ministry of Education of China), College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, P. R. China.

E-mail address: twmath2016@163.com

Key Laboratory of High Performance Computing and Stochastic Information Processing (HPCSIP) (Ministry of Education of China), College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, P. R. China.

E-mail address: xieyuanyuan198767@163.com