

Voronoi Choice Games

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Abstract. We study novel variations of Voronoi games (and associated random processes) that we call *Voronoi choice games*. These games provide a rich framework for studying questions regarding the power of small numbers of choices in multi-player, competitive scenarios in a geometric setting. Our work can be seen as a natural variation of the classical framework of Hotelling. While games based on Voronoi diagrams have appeared recently in the literature, our variations appear to be the first that take into account a notion of limited choices and the corresponding asymmetries in player strategies.

As a problem example, suppose a group of n miners (or players) are staking land claims through the following process: each miner has m associated points in the unit square (or, for symmetry, the unit torus), so the k th miner will have associated points $p_{k1}, p_{k2}, \dots, p_{km}$. We generally here think of m as being a small constant, such as 2. Each miner chooses one of these points as the base point for their claim. Generally the points will be distinct for each miner. Each miner obtains mining rights for the area of the square that is closest to their chosen base; that is, they obtain the Voronoi cell corresponding to their chosen point in the Voronoi diagram of the n chosen points. Each player's goal is simply to maximize the amount of land under their control. What can we say about the players' strategy and the equilibria of such games?

We provide several results on games in this setting. For example, we show that a correlated equilibrium can be found in polynomial time for the natural 1, 2, and 3-dimensional variations of the problem. For the one-dimensional version of the problem on the unit circle, we show that a pure Nash equilibrium exists when each player owns the part of the circle nearest to their point, but it is NP-hard to determine whether a pure Nash equilibrium exists when each player owns the arc starting from their point clockwise to the next point. We also consider a random version of the game, where players have m points chosen uniformly at random. We derive bounds on the expected number of pure Nash equilibria for a variation of the 1-dimensional game in this setting using interesting properties of random arc lengths on circles.

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1 Introduction

Consider the following prototypical problem: a group of miners are staking land claims. The k th miner—or player—has m associated points $p_{k1}, p_{k2}, \dots, p_{km}$ in the unit torus (which is the unit square with wraparound at the boundaries, providing symmetry). Each miner via some process will choose exactly one of their m points as the base for their claim. The resulting n points yield a Voronoi diagram, and each miner obtains their corresponding Voronoi cell. Each player’s goal is simply to maximize the amount of land under their control. We wish to study player behavior in this and similar games, focusing on equilibria.

As another application, political candidates can often be mapped according to their political views into a small-dimensional space; e.g., American candidates are often views as being points in a two-dimensional space, measuring how liberal/conservative they are on economic issues in one dimension and social issues on the other. Suppose parties must choose a candidate simultaneously, and their probability of winning is increasing in the area of the political space closest to their point. Again, the goal in this case is to maximize the corresponding area in a Voronoi diagram.

There are numerous variations one can construct from this setting. Most naturally, if the players are (lazy) security guards instead of miners, who have to patrol the area closest to their chosen base, their goal might be to minimize the area under their purview. Other alternatives stem from variations such as whether player choices are simultaneous or sequential, how the points for players are chosen, the underlying metric space, the type of equilibrium sought, and the utility function used to evaluate the final outcome.

However, the variations share the following fundamental features. There are n players, with the k th player having m_k associated points in some metric space. (We will focus on $m_k = m$ for a fixed m for all players.) Each player will have to choose to adopt one of their available points. A Voronoi diagram is then constructed, and each player is then associated with the corresponding area in the diagram. We refer to this general setting as *Voronoi choice games*. We discuss below how Voronoi choice games differ from similar recent work, but the key point is in the problems we study different players have different available choices; this asymmetry creates new problems and requires distinct methods.

We provide several results for this setting (as well as leave many challenging open questions). While details are given in the body of the paper, we briefly summarize our main results:

- We show that a correlated equilibrium can be found in polynomial time for several natural variations of the problem.
- For the one-dimensional version (on the unit circle), we prove that a pure Nash equilibrium exists when each player owns the arc of the circle nearest to their point, but it is NP-hard to determine whether a pure Nash equilibrium exists when a player owns the arc starting from their point clockwise to the next point for $m \geq 4$.

- For a version of the game where each player has m points chosen uniformly at random, in the latter one-dimensional case we derive bounds on the expected number of pure Nash equilibria.

The last result is the most technical, requiring a careful analysis based on interesting properties of distributions of random arcs on a circle. While we acknowledge that uniform random points are not motivated by practice, it leads to an interesting and, from the standpoint of probabilistic analysis and geometry, very natural class of games. Specifically, the uniform setting allows us to study the question of whether the game has pure Nash equilibria with high probability, a question for which we also offer conjectures for $k \geq 2$ dimensions based on simulations.

Our work focused on theoretical foundations for determining what kinds of results are possible in this problem space. However, we believe that the introduction of asymmetric “choice” into these types of Voronoi game models provides a rich framework with many interesting combinatorial, geometric, and game theoretic problems, as we describe throughout the paper. As such, we leave many natural open questions.

1.1 Related Work

The classical foundations for problems of this type can be found in the work by Hotelling [10], who studied the setting of two vendors who had to determine where to place their businesses along a line, corresponding to the main street in a town, with the assumption of uniformly distributed customers who would walk to the nearer vendor. Hotelling games have been considered for example in work on regret minimization and the price of anarchy, where the model studied players choosing points on a general graph instead of on the line as in the original model [4]. Recent work has also shown that for a Hotelling game on a given graph, once there are sufficiently many players a pure Nash equilibrium always exists [7]. A useful survey on economic location-based models is provided by Gabszewicz and Thisse [8].

Other variations of Voronoi games have appeared in the literature. More recent work refers to these generally as *competitive location games*; see for example [6,13,19,12], which discuss Voronoi games on graphs, for additional references.

Our setting appears different from previous work, in that it focuses on players with *limited* sets of choices that vary among the players. Our starting point was aiming to build connections between Voronoi games and random processes based on “the power of two choices” [3,14,1]. While in our games, each player has a limited (typically constant) set of distinct points to choose from, in previous work generally all players could choose from any point in the universe of possible choices. In economic terms, in relation to the Hotelling model, our work models that different businesses may have available a limited number of differing locations where they may establish their business. For example, businesses may have optioned the right to set up a franchise at specific locations in advance, and must then choose which location to actually build. While they could know the options

available to other competing franchises, they may have to decide where to build without knowing the choices made by competitors. In other situations, it may be possible for franchises to move (at some cost) to an alternative location.

1.2 Models

Before beginning, we explain our main problem variations. (We describe related sequential games, that we feel are worthy of further study, in Appendix A.)

We refer to the following as the *k-D Simultaneous Voronoi Game*.

- Each of the n players has m associated points from the k -dimensional unit torus $[0, 1]^k$. We assume that all players know about all of the possible points that can be chosen by every player (it is a game of complete information).
- The n players must simultaneously choose one of their m associated points.
- A Voronoi diagram is constructed for the n chosen points, and each player receives utility equal to the volume of its point's Voronoi cell in the maximization variation of the game. (In the minimization version, the utility could be the negation of the corresponding volume.)

The easiest version to think about is the 1-D version; each player chooses from m points on the unit circle, and after their choice they own an arc of the circle corresponding to all points closest to their chosen point. If each player tries to maximize their arc length, then the utility of a player is the length of their arc. (If each player tries to minimize their arc length, one can think of the negation of the arc length as the utility.) On the unit circle, there is another variant that we refer to as the *One Way 1-D Simultaneous Voronoi Game*, in which a player owns the arc starting from their point and continuing in a clockwise direction until the next chosen point. Our contributions include highlighting differences between the 1-D problem and the One Way 1-D problem, showing that in this case a small difference in the model subtlety leads to large differences in the behavior with respect to equilibria. Indeed, as we explain, we believe the One Way 1-D problem potentially offers more insight into the behavior of the k -D Simultaneous Voronoi Game for $k \geq 2$.

We focus on analyzing the equilibria of these games. The most common equilibrium to study is the Nash equilibrium [15], in which each player has a random distribution on strategies such that no player can improve their expected utility by changing their distribution. While Nash's results imply the Voronoi games above all have Nash equilibria, we do not determine the complexity of finding Nash equilibria for these games; this is left as an open question. We here focus on two variants of the Nash equilibrium, the correlated equilibrium and the pure Nash equilibrium. Whereas Nash equilibria have the players independently choosing their strategies, a correlated equilibrium allows the players' random distributions to be correlated (for example, by an external party). The stability requirement is then that given knowledge only of the overall distribution of outcomes and their own randomly chosen strategy, a player cannot improve their expected utility by deviating from their given strategy distribution [2].

Since a Nash equilibrium is a special case of a correlated equilibrium, a correlated equilibrium for the above games must exist. We discuss the computational complexity of finding a correlated equilibrium for the k -D Simultaneous Voronoi Game in Section 2.

A pure Nash equilibrium is a Nash equilibrium in which each player’s distribution has a support of size one. In other words, each player picks a single strategy to play and, given the other players’ strategies, no player can improve their utility by choosing a different strategy. Unlike the Nash and correlated equilibria, a pure Nash equilibrium is not guaranteed to exist.

Pure Nash equilibria can be viewed in the setting where each player can choose to switch to any of their adopted points at any time. In such a setting, the primary natural question is what are the stable states, where no player individually has the incentive to switch their adopted point. These stable states correspond to pure Nash equilibria, and hence may not even exist. A natural question is whether simple local dynamics—such as myopic best response, where at each time step some subset of players decides whether or not to switch the point it has adopted—reach a stable state quickly. In Section 3 we examine the computational complexity of determining the existence of stable states in the 1-D Simultaneous Voronoi Game and One Way 1-D Simultaneous Voronoi Game. For the former, we show (making use of known techniques) that a pure Nash equilibrium always exists; for the latter, we show that determining whether a pure Nash equilibrium exists is NP-complete.

We then consider the setting where players’ choices are made uniformly at random. We refer to these variations as “Randomized”; for example, given our NP-hardness result, it is natural to ask about the existence of a pure Nash equilibrium for the Randomized One Way 1-D Simultaneous Voronoi Game. Here we make significant progress on the issue by showing how to bound the expected number of pure Nash equilibria, through a careful analysis based on properties of distributions of random arcs on a circle.

We provide additional results in the appendices, including an empirical investigation of the probability that myopic best response will find a stable state in the Randomized One Way 1-D Simultaneous Voronoi Game and Randomized 2-D Simultaneous Voronoi Game, and several related conjectures related to the Randomized One Way 1-D Simultaneous Voronoi Game.

2 Correlated Equilibria

Our goal in this section is to show that, for the k -D Simultaneous Voronoi Game, correlated equilibria can be found in polynomial time. We present the proof for $m = 2$, but our results easily extend for $m > 2$. We present the results for $k = 1, 2$, and 3. The results appear to extend to higher dimensions but the geometric details are technical. The results also apply to the One Way 1-D Simultaneous Voronoi Game.

Theorem 1. *For $k = 1, 2$, and 3 , and for a fixed m , there is a polynomial time algorithm for finding a correlated equilibrium in the k -D Simultaneous Voronoi Game.*

We appeal to [17] and [11], who present polynomial time algorithms for finding a correlated equilibrium of succinct games. (The running times for these algorithms are not specifically presented in the papers and appear rather large, but are still polynomial.) A succinct game is one that can be represented in polynomial space such that given each player’s strategy, their utilities can be computed in polynomial time. The k -D Simultaneous Voronoi Game is succinct because it can be represented in $O(nmk)$ space by a list of each players point choices and, given the players’ strategies, the utilities can be found by computing the Voronoi diagram of the chosen points.

Theorem 2 (Theorem 4.5, [11]). *Given a succinct game and a polynomial time algorithm for computing the expected utility of a player under any product distribution on strategies, there exists a polynomial time algorithm for finding a correlated equilibrium in that game.*

Jiang et al. proved Theorem 2 by constructing a linear program with a variable for each of the 2^n possible strategy profiles. The LP’s constraints are non-negativity, and the constraints requiring that the variables form a correlated equilibrium. They do not, however, enforce that the variables sum to one, or even at most one, and rather use the sum of these variables as the objective. Thus, since a correlated equilibrium is guaranteed to exist by Nash’s Theorem, this LP is unbounded and its dual is infeasible. They then run the ellipsoid algorithm for a polynomial number of steps on the dual LP (this takes polynomial time, since the dual LP has only polynomially many variables). They argue that the intermediate steps of the ellipsoid algorithm can be used to construct product distributions of which there is a convex combination that is a valid correlated equilibrium, and which can be found with a second linear program.

The second linear program’s separation oracle requires as a subroutine a polynomial time algorithm for computing the expected utility of a player given a product distribution over the strategies. (This requirement is referred to in [11] as the *polynomial expectation property*.) In our case, we represent this product distribution by letting $s(p_i)$ be the probability the point p_i is chosen, for $i \in \{1, \dots, 2n\}$. For a given point p , which is one of two choices for a player i , we use \bar{p} to refer to player i ’s other choice, so we must have $s(p) = 1 - s(\bar{p})$. Each player independently chooses a point according to the probabilities given by s . Our work is to demonstrate polynomial time algorithms for this subroutine, which we do in Section 2.1 and Section 2.2. We note that it is not immediate that such an algorithm should exist, even when each player has only $m = 2$ choices, as the number of possible configurations is m^n . Hence, we cannot simply sum over all configurations when calculating the expectation. In [17] it is noted that for certain congestion games, these expectations can be computed using dynamic programming, essentially adding one player in at a time and updating accordingly. Our approach is similar in spirit, but requires taking advantage of the underlying geometry.

2.1 The Algorithm for the 1-D Game

The 1-D version of the problem, while simpler, provides the intuition that helps us in higher dimensions.

Lemma 3: *Computing a player's expected utility under a product distribution on strategies in the 1-D Simultaneous Voronoi Game takes $O(n \log n)$ time.*

Proof. Computing the expected utility of a product distribution in the 1-D Simultaneous Voronoi Game is equivalent to finding the expected area of the Voronoi cell owned by a given point (conditioned on it being chosen) given the independent probabilities of each other player's points being chosen. In the 1-dimensional torus (a circle), for a given point p , let A_p be the area of p 's Voronoi cell (assuming it is chosen), $D_1(p)$ be the distance to the first chosen point clockwise from p , and $D_2(p)$ be the distance to the first point chosen counterclockwise from p . We have $E[A_p] = \frac{1}{2}(E[D_1(p)] + E[D_2(p)])$ by linearity of expectations.

To compute $E[D_1(p)]$, we start by sorting all the points. (We need only sort once for all points p .) We then loop over all points p' moving clockwise from p – excluding \bar{p} , which we know will not be chosen if p is chosen – and compute the probability that p' is the closest chosen point clockwise from p . The main insight is that, as we go through the points, the first time we have a point q such that we have already seen \bar{q} , then we can stop the computation, as one of these two points must be chosen.

Algorithm 1 describes how to compute $E[D_1(p)]$, assuming the points are given in sorted order (clockwise from p). We track D , the running value that will equal $E[D_1(p)]$ on return, and P , the probability that the closest point clockwise from P is further than those examined so far. As we visit a point p' , D increases by $s(p') \cdot P \cdot \text{dist}(p, p')$. The value of P decreases by a factor of $(1 - s(p'))$, unless \bar{p}' has already been visited; then P becomes 0 and we may terminate the calculation. Computing $E[D_2(p)]$ is entirely analogous. ■

Compute $E[D_1(p)]$

$D \leftarrow 0$ // value of $E[D_1(p)]$ computed so far

$P \leftarrow 1$ // probability of closest point clockwise from p further than seen so far

for each point q except p and \bar{p} , in ascending order of clockwise dist from p **do**

if \bar{q} has been previously visited **then**

return $D + P \cdot \text{dist}(p, q)$

end if

$D \leftarrow D + s(q) \cdot P \cdot \text{dist}(p, q)$

$P \leftarrow P \cdot (1 - s(q))$

end for

return D

Algorithm 1: Computing a player's expected utility in 1-D.

Observe that Lemma 3 makes no special use of the fact that the points lie in a torus, and in fact the Lemma also applies when they lie in a line segment. The generalization to the case where $m > 2$ is straightforward. One proceeds with the same computation to find $E[D_1(p)]$, except that one can stop and return the value for D only when one has reached all m points associated with a some player (in the case of a torus).

2.2 The Algorithm in Higher Dimensions

In this section we present a polynomial time algorithm for computing the expected utility of a product distribution over the chosen points in 2-D, and explain how it generalizes to three dimensions, and may generalize further to higher dimensions. Intuitively, the challenge in going beyond one dimension is that the interactions are more complicated; there are more than just two neighboring cells to consider. For convenience, the algorithm we present is for the unit square; it is readily modified for the unit torus.

Lemma 4: *Computing a player's expected utility under a product distribution on strategies in the 2-D Simultaneous Voronoi Game takes $O(n^3 \log n)$ time.*

Proof. We assume that the points are in general position (that is, no three are collinear); this assumption is not mandatory, but simplifies the proof. As in the one-dimensional case, we decompose the area of the cell into a sum and then exploit linearity of expectations. Given our initial point set, there are $O(n^2)$ possible vertices that can make up a vertex on p 's Voronoi cell, since each possible vertex is the circumcenter of p and two of the $2n - 2$ points that the other players can choose from. Choose one arbitrarily to be v_1 , and let $V(p) = \{v_1, \dots, v_t\}$ be the set of all possible vertices, in sorted order according to the angle between v_1, p , and v_i (measured in a counterclockwise direction from v_1). Let θ_i be the corresponding angle for v_i .

Let the random variable A_p be the area of p 's Voronoi cell. $A_p = \sum_{i=1}^t T_i$ where T_i is the area of p 's Voronoi cell between angles θ_i and θ_{i+1} . Thus, $E[A_p] = \sum_{i=1}^t E[T_i]$. Figure 1 provides an example of this decomposition for one possible realization of a point's Voronoi cell. Note that as depicted in this figure, a vertex may appear inside the Voronoi cell if neither of the points associated with that vertex are chosen by their corresponding players. Observe that region T_i is always a triangle since, by definition, no other vertex of the cell creates an angle in the range (θ_i, θ_{i+1}) . If we let D_i be the distance from p to the boundary of its Voronoi cell in the direction θ_i , then

$$E[T_i] = E \left[\frac{D_i D_{i+1} \sin(\theta_{i+1} - \theta_i)}{2} \right] = \frac{E[D_i D_{i+1}] \sin(\theta_{i+1} - \theta_i)}{2}.$$

We refer to the perpendicular bisector of the line (p, q) as the *boundary line* between p and q . If p and q own neighboring Voronoi cells, this line separates them. The point p has at most $2n$ possible boundary lines (one per possible

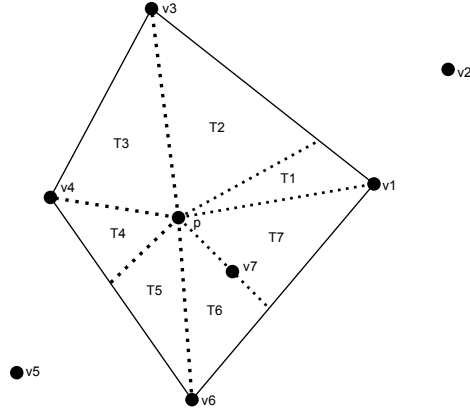


Fig. 1: A point p together with the possible vertices $V(p)$ and one possible Voronoi cell, decomposed into triangles T_i by dotted lines.

other point); we say one of p 's boundary lines is *chosen* in the event that the corresponding q is chosen by its player. Observe that the values of D_i and D_{i+1} are uniquely determined by the chosen boundary line closest to p within the (θ_i, θ_{i+1}) region. The notion of the closest boundary line is unambiguous within (θ_i, θ_{i+1}) since, by definition, none of the possible boundary lines intersect in (θ_i, θ_{i+1}) .

Algorithm 2 provides an algorithm for computing $E[D_i D_{i+1}]$ given a product distribution on the points. This algorithm first sorts the possible boundary lines in increasing order of distance from p within (θ_i, θ_{i+1}) . This can be done by sorting them by their position along the ray $(p, (\theta_i + \theta_{i+1})/2)$. The algorithm then iterates through the boundary lines computing the probability that each one is the closest chosen boundary line to p . The value $D(p, q, i)$ in the algorithm is the distance from p to the boundary line between p and q along angle θ_i . The running time is dominated by sorting the boundary lines which takes $O(n \log n)$ time. Thus, the total running time to compute $E[A_p]$ is $O(n^3 \log n)$. ■

Corollary 1. *Computing a player's expected utility under a product distribution on strategies in the 3-D Simultaneous Voronoi Game takes $O(n^5 \log n)$ time.*

Proof. The approach of Lemma 4 generalizes naturally to three dimensions. The idea is to partition the space into pyramids radiating out from p , in which none of the $O(n)$ possible boundary planes intersect. Given this partitioning, we can again use linearity of expectations by computing the expected volume of p 's Voronoi cell in each pyramid separately. Within a given pyramid, we sort the boundary planes by their distance to p , and then iterate through them computing the probability that each one is the closest chosen boundary plane. We then multiply these probabilities by the volumes of the polyhedra induced by the corresponding boundary planes, and sum the results to get the expected volume of the pyramid.

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Compute  $E[D_i D_{i+1}]$ 
   $D \leftarrow 0$  // value of  $E[D_i D_{i+1}]$  computed so far
   $P \leftarrow 1$  // probability of closest boundary line to  $p$  in this arc being further
  than seen so far
  for each point  $q$  except  $p$  and  $\bar{p}$  whose boundary line with  $p$  intersects
   $(\theta_i, \theta_{i+1})$ , in increasing order of the boundary line's distance to  $p$  do
    if  $\bar{q}$  has been previously visited then
      return  $D + P \cdot D(p, q, i) \cdot D(p, q, i + 1)$ 
    end if
     $D \leftarrow D + s(q) \cdot P \cdot D(p, q, i) \cdot D(p, q, i + 1)$ 
     $P \leftarrow P \cdot (1 - s(q))$ 
  end for
return  $D$ 

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Algorithm 2: Computing part of a player's expected utility in 2-D.

Identifying the space partition in three dimensions is trickier than it is in two. Here, the pyramids can be found by computing the lines of intersection between all of the possible boundary planes, projecting them onto a sphere centered at p , and then finding all of the faces induced by these lines on the sphere. Each face forms the base of one pyramid. Identifying these faces can be done by constructing the graph formed by the lines and their intersections and then proceeding analogously to finding the faces of a planar graph as in [16]. This takes time linear in the number of edges, faces and vertices.

With $O(n^2)$ lines of intersection, there will be at most $O(n^4)$ vertices and faces. For each pyramid, we must sort the $O(n)$ possible boundary planes, thus the running time for finding the expected utility in three dimensions is $O(n^5 \log n)$. ■

This same strategy generalizes further to $k > 3$ dimensions, where we are now projecting $(k - 2)$ -dimensional hyperplanes formed by the intersection of the $(k - 1)$ -dimensional boundary hyperplanes onto a k -dimensional hypersphere. However, determining methods for algorithmically identifying the “faces” on the hypersphere is outside the scope of this work.

3 Pure Nash Equilibria

In this section, we show a fundamental difference between the One Way 1-D Simultaneous Voronoi Game and the 1-D Simultaneous Voronoi Game. Recall that for these problems each of the n players has a choice of m points on the unit circle; all players simultaneously choose one of their m points. The utility for the One Way variation of given player is equal to the distance to the nearest chosen point clockwise from its chosen point, while for the standard variation the utility is the size of the Voronoi cell (in this case, an arc).

We show the standard Voronoi variation always has at least one pure Nash equilibrium (for any number of choices per player), while it is NP-hard to de-

termine if the maximization version of the One Way variation has a pure Nash equilibrium. We also suggest the implications of these results for the higher dimensional setting.

3.1 Existence of Pure Nash Equilibria in the 1-D Simultaneous Voronoi Game

In the argument that follows we assume the choices of points are distinct. The analysis can be easily modified for the case where multiple players can choose the same point if ownership of that point is determined by a fixed preference order (and other players have zero utility). However, if players choosing the same point share utility, then the theorem does not hold, as shown in [13].

Theorem 5. *A pure Nash equilibrium for the maximization and minimization versions of the 1-D Simultaneous Voronoi Game exists for any set of points.*

Proof. We follow an approach utilized in [6, Lemma 4]. We define a natural total ordering on multi-sets of numbers A and B . For any two such multi-sets A and B , if $|A| < |B|$ we have $A \succ B$. When $|A| = |B|$, we have $A \succ B$ if $\max A > \max B$. If $\max A = \max B$, then let A' be A with one copy of the value $\max A$ removed, and similarly for B' ; then $A \succ B$ also when $\max A = \max B$ and $A' \succ B'$.

Now consider a collection of choices in the 1-D Simultaneous Voronoi Game, and let $A = \{a_0, a_1, \dots, a_{n-1}\}$ be the corresponding arc lengths, starting from some chosen point and then in clockwise order around the circle, induced by the choice of points. Each player's payoff is given by a value of the form $(a_i + a_{i+1})/2$ for a suitable value of i . (One player's payoff is $(a_{n-1} + a_0)/2$.) Consider the maximization version of the game. If some player has a move that improves their utility, let them make that move. Without loss of generality, suppose this player's payoff was given by $(a_i + a_{i+1})/2$, and it moves somewhere on the arc with length a_j . Note this means $a_j > a_i + a_{i+1}$. We see that the arc lengths A' are those of A but with a_i, a_{i+1} , and a_j replaced by $a_i + a_{i+1}, x$, and y where $x + y = a_j$. Hence $A \succ A'$, so after a finite number of moves, this version of myopic best response converges to a pure Nash equilibrium.

The argument for the minimization variation is analogous. ■

We note that we leave as an open question to determine a bound on the number of steps a myopic best response approach would take to reach a pure Nash equilibrium; in particular, we do not yet know if the approach of Theorem 5 yields a pure Nash equilibrium in a polynomial number of steps.

We might have hoped that the above technique could allow us to show that for the 2-D Simultaneous Voronoi Game (and higher dimensions) that a pure Nash equilibrium exists. Unfortunately, that is not the case. One can readily find choices of 2 points for each of 3 players where no pure Nash equilibrium exists for both the maximization and minimization version of the problem. We have generated many such examples randomly, computing the Voronoi diagrams for

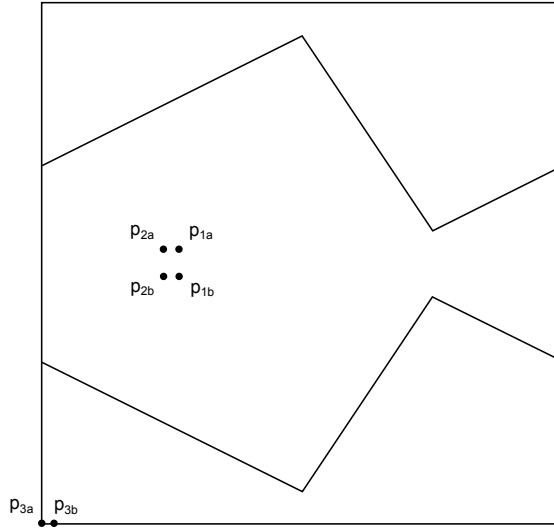


Fig. 2: This is an example of a 2-D Simultaneous Voronoi Game in which no pure Nash equilibria exists. Player 1 has points $(p_{1a}, p_{1b}) = ((1/4 + \varepsilon^2, 1/2 + \varepsilon), (1/4 + \varepsilon^2, 1/2 - \varepsilon))$, player 2 has points $(p_{2a}, p_{2b}) = ((1/4 - \varepsilon^2, 1/2 + \varepsilon), (1/4 - \varepsilon^2, 1/2 - \varepsilon))$, and player 3 has points $(p_{3a}, p_{3b}) = ((0, 0), (\varepsilon, 0))$. The Voronoi diagram shown is constructed from the points $(1/4, 1/2)$ and $(0, 0)$.

the eight resulting configurations. One example in two dimensions is depicted in Figure 2. In this example, player 1 has choices $((1/4 + \varepsilon^2, 1/2 + \varepsilon), (1/4 + \varepsilon^2, 1/2 - \varepsilon))$, player 2 has choices $((1/4 - \varepsilon^2, 1/2 + \varepsilon), (1/4 - \varepsilon^2, 1/2 - \varepsilon))$, and player 3 has choices $((0, 0), (\varepsilon, 0))$ for sufficiently small $\varepsilon > 0$. The idea here is that player 3's choice is irrelevant, and players 1 and 2 are dividing up the Voronoi cell owned by point $(1/4, 1/2)$ in the Voronoi diagram of the points $(1/4, 1/2)$ and $(0, 0)$. If both player 1 and 2 choose their first point, or both choose their second point, then they divide the cell with a vertical line, which due to the geometry of the cell favors player 1. However, if one of them chooses their first point and the other chooses their second point then they divide the cell with a roughly horizontal line, which gives each of them roughly half. Thus, there is no choice of points for which neither wants to deviate. This example applies equally in higher dimensions (by making higher dimensional coordinates all zero).

This example can be extended to show that for any number of players n , there are settings of points for the players so no pure Nash equilibrium exists, showing that this setting differs from previous work on symmetric Hotelling games on graphs, where it has been shown that when there are sufficiently many players a pure Nash equilibrium always exists [7]. Specifically, use the same example but for players $4, \dots, n$ both of their choices will be in an epsilon-small neighborhood of $(0, 0)$. We note that such examples do not disprove the possibility that a

pure Nash equilibrium exists with probability $1 - o(1)$ if the points are chosen randomly.

Given that the 1-D Simultaneous Voronoi Game appears to have a very special structure (in terms of the existence of pure Nash equilibrium) that differs from k -D Simultaneous Voronoi Game for $k \geq 2$, it is natural to seek a 1-D variant that might shed more insight into the behavior in higher dimensions. This motivates us to look at the One Way 1-D Simultaneous Voronoi Game.

3.2 NP-Hardness of the One Way 1-D Simultaneous Voronoi Game

In contrast to the result of the previous subsection, we prove the NP-hardness of the maximization version of the One Way 1-D Simultaneous Voronoi Game whenever each player has $m \geq 4$ choices. We leave as an open question to find a reduction for $m = 2$ or 3 , as well as for the minimization version.

In what follows we denote a player r 's i th point by $Z(r, i)$, as we have multiple different players we describe.

Theorem 6. *The problem of determining if a pure Nash equilibrium (PNE) exists in the maximization version of a One Way 1-D Simultaneous Voronoi Game is NP-hard for $m \geq 4$.*

Our proof will ultimately only use at most 4 choices per player, however observe that we can essentially reduce the number of choices available to a player by placing leftover choices arbitrarily close in front of one of the desired choices, thus making the desired choices dominate the leftover choices.

Before we prove Theorem 6, we first introduce a gadget that will be useful in our proof. This is the Utility Enforcement Gadget (UEG), depicted in Figure 3. For now, in what follows d and ε are suitable constants (we specify their values further in the proof of the theorem). The purpose of the UEG is to prevent a PNE in which a given player r has utility less than a given value d . The UEG consists of a region of length $d + 6\varepsilon$, where ε is a value small enough that $d - \varepsilon$ is greater than any other utility r can achieve less than d . We also introduce 5 new players – b_1, b_2, x, y and z – and one new choice for r : $Z(r, k + 1)$. Players b_1 and b_2 each only have one choice; these form the boundaries of the UEG's region. $Z(r, k + 1)$ is $d - \varepsilon$ before $Z(b_2, 1)$ with nothing in between them, so r will always prefer $Z(r, k + 1)$ to any of its other choices with utility less than d . Players x and y each have two choices and z has one choice. x, y and z 's choices are arranged in the UEG's region such that they have a stable configuration if and only if r does not choose $Z(r, k + 1)$.

Let G be an instance of the One Way 1-D Simultaneous Voronoi Game, and let G' be that instance modified to add a UEG (scaling as necessary) for a given player r and utility d .

Lemma 7: *G' has a PNE if and only if G has a PNE in which r has utility at least d .*

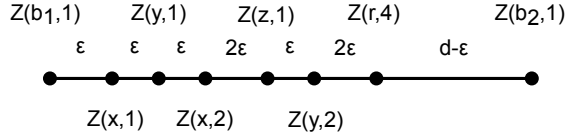


Fig. 3: A Utility Enforcement Gadget for player r and utility d . The arrows point to the stable configuration for b_1, b_2, x, y and z when r does not choose $Z(r, k+1)$.

Proof. If G has a PNE in which r has utility at least d , then G' has a PNE in which all of the players from G make the same choices and x chooses $Z(x, 1)$ and y chooses $Z(y, 2)$. Thus r is the only player in G whose preference is affected by the UEG, and if his utility is at least d , then he will not wish to deviate to $Z(r, k+1)$ which only yields a utility of $d - \varepsilon$.

If G has only PNEs in which r has utility less than d , then G' will not have a PNE, since the preference of every player from in G is unaffected by the addition of the UEG except for r . In G' , r will wish to deviate from any configuration in which r has utility less than d by choosing $Z(r, k+1)$. However, there can be no PNE in which r chooses $Z(r, k+1)$, because that will cause y to prefer $Z(y, 1)$ to $Z(y, 2)$ when x chooses $Z(x, 1)$ and prefer $Z(y, 2)$ to $Z(y, 1)$ when x chooses $Z(x, 2)$, while x prefers $Z(x, 2)$ to $Z(x, 1)$ when y chooses $Z(y, 1)$ and prefers $Z(x, 1)$ to $Z(x, 2)$ when y chooses $Z(y, 2)$. ■

Now that we have defined the UEG, we can move on to the main proof.

Proof (of Theorem 6).

We reduce from the NP-complete Monotone 1-in-3 SAT problem [18]. In this problem, we are given a set of k variables v_1, v_2, \dots, v_k and l clauses c_1, c_2, \dots, c_l . Each clause is a set of three variables, and the problem is to determine whether there is an assignment of boolean values to the variables such that in each clause, exactly one of its values is true.

Our reduction has $n = 2k + 6l$ players. The circle is partitioned into $k + l$ equal sized regions using $k + l$ boundary players (denoted $r_{b_1}, \dots, r_{b_{k+l}}$) who each have only one choice as shown in Figure 4. Within one of these regions, we will sometimes use “left” to refer to counterclockwise and “right” to refer to clockwise. We think of l of the players as corresponding to clauses (denoted r_{c_1}, \dots, r_{c_l}), and refer to them as clause players. We think of k of the regions as corresponding to variables. Each clause player will have one choice in each region that corresponds to one of that clause’s variables. Which variable region the clause player chooses will correspond to which of the clause’s variables is set to true. A variable is true if every clause player corresponding to a clause containing the variable chooses its point falling in the variable’s region. A variable is false if none of the clause players corresponding to clauses containing the variable choose points in the variable’s region. Therefore, any intermediate configuration – some but not all of the relevant clause players choose points in a variable’s region – is invalid. To prevent such an invalid configuration from yielding a PNE,

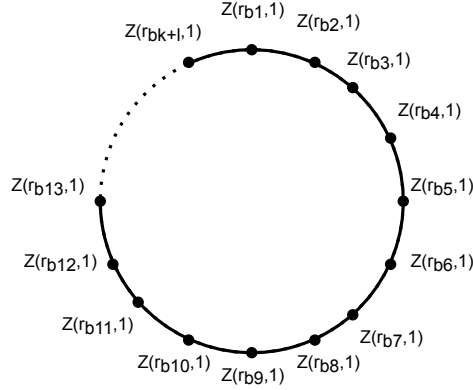


Fig. 4: The partitioning of the circle using $k + l$ boundary players that each have only one point choice.

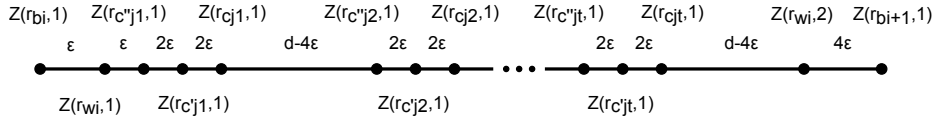


Fig. 5: A variable region where for a variable contained in clauses c_{j_1}, \dots, c_{j_t} . ϵ is much smaller than d .

we introduce an additional two players per clause, that we will refer to as the *shadow clause players*, denoted by $r_{c'_i}$ and $r_{c''_i}$. Each variable region will also contain a *wrap player* (denoted r_{w_1}, \dots, r_{w_k}) who has one point choice on the left of the region and one choice on the right side. We place a wrap player so that it prefers its left point unless one of the left most shadow clause players' points are chosen, in which case it prefers its right point.

In what follows, $d = 1$ and ϵ is a suitably small constant much smaller than d . To enforce a unit length for the circle, we simply scale d and ϵ appropriately. A variable region will be arranged as in Figure 5. The clause players within a variable region appear in the order of the clauses. The clause players are spaced d apart, and each of the two shadow clause players appear 2ϵ and 4ϵ before their corresponding clause player. The wrap player has one point ϵ before the first shadow clause player, and one 4ϵ before the terminating boundary point. This ensures that if neither of the left most shadow clause players is chosen, then the wrap player will choose its left point, and not affect any clause players. However if one of the left most shadow clause players is chosen, then the wrap player will choose its right point thereby reducing the utility of the rightmost clause player by 4ϵ .

Finally, we will have one UEG per clause player to enforce that its utility is at least d . These contribute the final $3l$ players (the boundaries for the UEGs were included in the boundary players).

To complete the proof, we argue that any PNE in the game corresponds to a valid assignment of the variables and any valid assignment to the variables of corresponds to a PNE in the game.

Suppose that there exists a PNE such that there is a variable region with at least one clause player and at least one shadow clause player. Then we must have that either there is a clause point followed immediately by a shadow clause point, or the first point is a shadow clause point and the last point is a clause point. In the former case, we immediately have that the utility of the clause player is less than d , so such a PNE cannot exist by Lemma 7. In the latter case, by the construction of the variable region, the wrap player in the region will prefer the point on the right of the region, where it will cause the utility of the clause player to once again be less than d .

Thus, any PNE has the property that each clause player is assigned to a variable region, and each variable region either has all clause players, or all shadow clause players. This immediately gives us an assignment of variables that satisfies the 1-in-3 SAT formula, by setting each variable to be true if and only if there are clause players in the variable's region.

Suppose we have a variable assignment which satisfies the formula. This corresponds to a PNE in the game. Each clause player will choose the point in the region corresponding to the true variable in its clause. So as long as all of the shadow clause players are assigned to other variable regions, none of the clause players will want to deviate since they will all achieve the maximum possible utility of d . It remains to show that there is assignment of shadow clause players to remaining regions such that none of the shadow clause players want to deviate. This can be achieved algorithmically. First, pick an arbitrary assignment of the shadow clause players to the variable regions that are false. Next, go through the shadow clause players in descending order of their corresponding clauses, and choose the best response for each one. Because we ordered the clause players (and shadow clause players) in the regions by the order of the clauses, and the utility of a player is solely determined by the player to the right, no choice made by a player later in this procedure affects the utility of a player who chose earlier in this procedure. Thus, at the end of the procedure all of the shadow clause players will still be in their best response. ■

We conjecture that determining if a pure Nash equilibrium exists in the k -D Simultaneous Voronoi Game is NP-hard for $k \geq 2$; however, we suspect that building that corresponding gadgets may prove technically challenging.

4 Random Voronoi Games

We now consider random variations of the Voronoi games we have considered, where each player's choices are chosen uniformly at random. While it is not clear such a model corresponds to any specific real-world scenario, such random problems are intrinsically interesting combinatorially and in relation to other similar studied problems. For example, in the context of load-balancing in distributed

peer-to-peer systems, the authors of [5] study a model where one begins with a Voronoi diagram on N points (chosen uniformly at random from the universe, say the unit torus) corresponding to N servers. Then M agents sequentially enter; each is assigned k random points from the universe; and each agent chooses the one of its k points that lies in the Voronoi cell with the smallest number of agents, or *load* for that server. Other “power-of-choice” problems, such as the Achlioptas process [1], have spurred new understanding of phenomena such as explosive percolation.

Given our hardness results, a natural question is whether the One Way 1-D Simultaneous Voronoi Game has (or does not have) a pure Nash equilibrium with high probability in the random variation. While we have not proven this result, we have proven bounds on the expected number of pure Nash equilibria in the random setting that are interesting in their own right and nearly answer this question. In particular, our careful analysis builds on the interesting relationship between random arcs on a circle and weighted sums of exponentially distributed random variables.

The following is our main result.

Theorem 8. *The expected number of PNE for the maximization version of the One Way 1-D Simultaneous Voronoi Game is at most m , and at least $0.19^{m-1}m$.*

Interestingly, our bounds depend on the number of choices m , not the number of players. Unfortunately, this means that we cannot use these expectation bounds directly to show that, for example, the probability of a PNE existing is exponentially small in n . But the bounds provide insight by showing that pure Nash equilibria are typically few in number in the random case.

In proving Theorem 8, we need the following well-known property of exponential random variables. Let X_1, \dots, X_n be i.i.d. exponential random variables. Let $S_n = \sum_{j=1}^n X_j$. Recall that S_n is said to have a *gamma distribution*, and we use facts about the gamma distribution later in our analysis. Similarly, S_{n-1}/S_n is said to have a *beta distribution*, and we use facts about beta distributions as well. For example,

Lemma 9: $S_n, \frac{S_1}{S_2}, \frac{S_2}{S_3}, \dots, \frac{S_{n-1}}{S_n}$ are all mutually independent.

See e.g. [9] for its discussion of exponential random variables. In particular, the X_i can be viewed as the gaps in the arrivals of a Poisson process; the $n - 1$ arrivals before the last are uniformly distributed on the interval $[0, S_n]$, from which one can derive the lemma above.

Another important fact we use is the following:

Fact 10: We have $\mathbb{E} \left[(S_j/S_{j+1})^t \right] = \prod_{i=0}^{t-1} \frac{j+i}{j+1+i} = \frac{j}{j+t}$, since $S_j/S_{j+1} \sim \text{Beta}(j, 1)$.

We note that in some of our arguments, the ordering of the variables becomes reversed, and we consider sums of the form $\sum_{j=n-i+1}^n X_j$. Of course this has the same distribution as $\sum_{j=1}^i X_i$, and the corresponding version of Lemma 9 holds.

Where convenient, we therefore refer to $\sum_{j=n-i+1}^n X_j = S_i$ where there is no ambiguity as to the desired meaning.

Proof (of Theorem 8).

We begin by using linearity of expectations to write the expected number of PNE in terms of the probability that each player choosing their first choice will yield a stable configuration.

$$\begin{aligned} \mathbb{E}[\# \text{ PNE}] &= m^n \cdot \Pr[\text{first choices are stable}] \\ &= m^n \cdot \mathbb{E}[\Pr[\text{first choices are stable} \mid \text{position of first choices}]]. \end{aligned}$$

Partition the circle into arcs according to the players' first choice points. Let A_i be the length of the i th smallest arc. As shown in [9], the A_i are distributed jointly as

$$A_i \sim \frac{1}{S_n} \sum_{j=n-i+1}^n \frac{X_j}{j}$$

where again the X_j are i.i.d. exponential random variables of mean 1 and $S_n = \sum_{j=1}^n X_j$. We say that an arc is stable if the player whose point starts the arc (going clockwise) does not wish to deviate to any of their other points. Given the position of the first choices, the probability each arc is stable depends only on the other choices available to the player that owns the arc, and hence the stability of the arcs are independent. Therefore

$$\begin{aligned} &\Pr[\text{first choices are stable} \mid \text{position of first choices}] \\ &= \prod_{i=1}^n \Pr[i\text{th smallest arc is stable} \mid \text{position of first choices}]. \end{aligned}$$

If the arc is the i th smallest, then it will be stable if the other choices fall in the same arc, one of the $i - 1$ smaller arcs, or the front A_i -length portion of the $(n - i)$ larger arcs – except in the latter two cases, we must take into account that if a choice falls immediately backward into the arc directly counterclockwise of the current arc, then the arc is not stable. We therefore have the following calculation:

$$\begin{aligned} &\Pr[i\text{th smallest arc is stable} \mid \text{position of first choices}] \\ &= \left((n - i)A_i + \sum_{j=1}^i A_j - \min(A_i, \text{length of arc before } i) \right)^{m-1} \\ &\leq \left((n - i)A_i + \sum_{j=1}^i A_j \right)^{m-1} \\ &= S_n^{-(m-1)} \left((n - i) \sum_{j=n-i+1}^n \frac{X_j}{j} + \sum_{j=1}^i \sum_{k=n-j+1}^n \frac{X_k}{k} \right)^{m-1} \end{aligned}$$

$$\begin{aligned}
&= S_n^{-(m-1)} \left((n-i) \sum_{j=n-i+1}^n \frac{X_j}{j} + \sum_{k=n-i+1}^n (k-n+i) \frac{X_k}{k} \right)^{m-1} \\
&= S_n^{-(m-1)} \left(\sum_{j=n-i+1}^n (n-i+j-n+i) \frac{X_j}{j} \right)^{m-1} \\
&= S_n^{-(m-1)} \left(\sum_{j=n-i+1}^n X_j \right)^{m-1} \\
&= \left(\frac{S_i}{S_n} \right)^{m-1}.
\end{aligned}$$

Note we have used $\sum_{j=n-i+1}^n X_j = S_i$ for convenience. Our resulting bound has a surprisingly clean form in terms of the S_i .

Thus, by Lemma 9 and Fact 10

$$\begin{aligned}
\Pr[\text{first choices are stable}] &\leq \mathbb{E} \left[\prod_{i=1}^n \left(\frac{S_i}{S_n} \right)^{m-1} \right] \\
&= \mathbb{E} \left[\prod_{i=1}^{n-1} \left(\frac{S_i}{S_{i+1}} \cdot \frac{S_{i+1}}{S_{i+2}} \cdots \frac{S_{n-1}}{S_n} \right)^{m-1} \right] \\
&= \mathbb{E} \left[\prod_{i=1}^{n-1} \left(\frac{S_i}{S_{i+1}} \right)^{i(m-1)} \right] = \prod_{i=1}^{n-1} \mathbb{E} \left[\left(\frac{S_i}{S_{i+1}} \right)^{i(m-1)} \right] \\
&= \prod_{i=1}^{n-1} \frac{i}{i+i(m-1)} = \frac{1}{m^{n-1}}.
\end{aligned}$$

It then follows that

$$\mathbb{E}[\#\text{PNE}] \leq m^n \cdot \frac{1}{m^{n-1}} = m.$$

We can similarly find a lower bound, although some additional technical work is required.

$$\begin{aligned}
&\Pr[i\text{th smallest arc is stable} \mid \text{position of first choices}] \\
&= \left((n-i)A_i + \sum_{j=1}^i A_j - \min(A_i, \text{length of arc before } i) \right)^{m-1} \\
&\geq \left((n-i)A_i + \sum_{j=1}^i A_j - A_i \right)^{m-1}
\end{aligned}$$

$$\begin{aligned}
&= \left((n-i-1)A_i + \sum_{j=1}^i A_j \right)^{m-1} \\
&= S_n^{-(m-1)} \left(\sum_{j=n-i+1}^n (n-i-1+j-n+i) \frac{X_j}{j} \right)^{m-1} \\
&= S_n^{-(m-1)} \left(\sum_{j=n-i+1}^n \frac{(j-1)X_j}{j} \right)^{m-1}
\end{aligned}$$

Hence

$$\Pr[\text{first choices are stable}] \geq \mathbb{E} \left[S_n^{-(m-1)n} \prod_{i=1}^n \left(\sum_{j=n-i+1}^n \frac{(j-1)X_j}{j} \right)^{m-1} \right].$$

A simple stochastic domination argument (provided in Appendix B) shows that the expectation on the right side decreases if, in each term in the product, we equalize the coefficient, so that instead of terms of the form $\frac{(j-1)X_j}{j}$, the coefficient for all terms of the sum in the i th term of the product is the average $c_i = \frac{1}{i} \sum_{j=n-i+1}^n \frac{j-1}{j}$. This gives

$$\begin{aligned}
\Pr[\text{first choices are stable}] &\geq \mathbb{E} \left[S_n^{-(m-1)n} \prod_{i=1}^n \left(\sum_{j=n-i+1}^n c_i X_j \right)^{m-1} \right] \\
&= \mathbb{E} \left[S_n^{-(m-1)n} \prod_{i=1}^n (c_i S_i)^{m-1} \right] \\
&= \left(\prod_{i=1}^n c_i^{m-1} \right) \mathbb{E} \left[\prod_{i=1}^n \left(\frac{S_i}{S_n} \right)^{m-1} \right].
\end{aligned}$$

Observe that this expectation is the same as the one we computed in the upper bound. Therefore

$$\begin{aligned}
\mathbb{E}[\# \text{ PNE}] &\geq \left(\prod_{i=1}^n c_i^{m-1} \right) m \\
&\geq 0.19^{m-1} m.
\end{aligned}$$

The proof of the final inequality is presented in Appendix B. ■

We note that similar calculations can be done for the minimization version, although we have not yet found a clean form for the upper bound. We can, however, state the following lower bound, showing the expected number of pure Nash equilibria is at least inverse polynomial in n for a fixed number of choices m .

Theorem 11. *The expected number of PNE for the minimization version of the One Way 1-D Simultaneous Voronoi Game is at least $\frac{m(m-1)}{(mn-1)n^{m-1}}$.*

Proof. This proof will proceed similarly to that of Theorem 8. We use the same notation, except that here $S_i = \sum_{j=1}^i X_j$.

An arc is stable if the player whose point starts the arc does not wish to deviate to any of their other points. If the arc is the i th smallest, then it will be stable if the other choices fall in the preceding arc, or in a portion of length $A_j - A_i$ at the start of the j th smallest arc for $j > i$. Therefore

$$\begin{aligned}
& \Pr[i\text{th smallest arc is stable} \mid \text{position of first choices}] \\
&= \left(\sum_{j=i+1}^n (A_j - A_i) + \min(A_i, \text{length of arc before } i) \right)^{m-1} \\
&\geq \left(\sum_{j=i}^n (A_j - A_i) + A_1 \right)^{m-1} \\
&= \left(\sum_{j=i}^n A_j - (n-i+1)A_i + A_1 \right)^{m-1} \\
&= S_n^{-(m-1)} \left(\sum_{j=i}^n \sum_{k=n-j+1}^n \frac{X_k}{k} - (n-i+1) \sum_{k=n-i+1}^n \frac{X_k}{k} + \frac{X_n}{n} \right)^{m-1} \\
&= S_n^{-(m-1)} \left(\sum_{k=1}^{n-i} k \frac{X_k}{k} + \sum_{k=1}^n (n-i+1) \frac{X_k}{k} - (n-i+1) \sum_{k=n-i+1}^n \frac{X_k}{k} + \frac{X_n}{n} \right)^{m-1} \\
&= S_n^{-(m-1)} \left(\sum_{k=1}^{n-i} X_k + \frac{X_n}{n} \right)^{m-1} \\
&= S_n^{-(m-1)} \left(S_{n-i} + \frac{X_n}{n} \right)^{m-1}.
\end{aligned}$$

Using the fact that probabilities of each arc being stable are independent when conditioning on their positions, we have

$$\begin{aligned}
\Pr[\text{first choices are stable}] &\geq \mathbb{E} \left[S_n^{-(m-1)n} \prod_{i=1}^n \left(S_{n-i} + \frac{X_n}{n} \right)^{m-1} \right] \\
&\geq \mathbb{E} \left[S_n^{-(m-1)n} \left(\frac{X_n}{n} \right)^{m-1} \prod_{i=1}^{n-1} S_i^{m-1} \right] \\
&= \mathbb{E} \left[S_n^{-(m-1)n} X_n^{m-1} \prod_{i=1}^{n-1} S_i^{m-1} \right] / n^{m-1}
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[S_n^{-(m-1)n} \left(\left(1 - \frac{S_{n-1}}{S_n} \right) S_n \right)^{m-1} \prod_{i=1}^{n-1} S_i^{m-1} \right] / n^{m-1} \\
&= \mathbb{E} \left[S_n^{-(m-1)(n-1)} \left(1 - \frac{S_{n-1}}{S_n} \right)^{m-1} \prod_{i=1}^{n-1} S_i^{m-1} \right] / n^{m-1} \\
&= \left(\mathbb{E} \left[\prod_{i=1}^{n-1} \left(\frac{S_i}{S_n} \right)^{m-1} \right] - \mathbb{E} \left[\left(\frac{S_{n-1}}{S_n} \right)^{m-1} \prod_{i=1}^{n-1} \left(\frac{S_i}{S_n} \right)^{m-1} \right] \right) / n^{m-1}.
\end{aligned}$$

Observe here the occurrence of

$$\mathbb{E} \left[\prod_{i=1}^{n-1} \left(\frac{S_i}{S_n} \right)^{m-1} \right] = \mathbb{E} \left[\prod_{i=1}^n \left(\frac{S_i}{S_n} \right)^{m-1} \right]$$

which we found to equal $m^{-(n-1)}$ in the proof of Theorem 8. Evaluating the other expectation using Lemma 9 and Fact 10 we have

$$\begin{aligned}
&\mathbb{E} \left[\left(\frac{S_{n-1}}{S_n} \right)^{m-1} \prod_{i=1}^{n-1} \left(\frac{S_i}{S_n} \right)^{m-1} \right] \\
&= \mathbb{E} \left[\left(\frac{S_{n-1}}{S_n} \right)^{m-1} \prod_{i=1}^{n-1} \left(\frac{S_i}{S_{i+1}} \cdot \frac{S_{i+1}}{S_{i+2}} \cdots \frac{S_{n-1}}{S_n} \right)^{m-1} \right] \\
&= \mathbb{E} \left[\left(\frac{S_{n-1}}{S_n} \right)^{n(m-1)} \prod_{i=1}^{n-2} \left(\frac{S_i}{S_{i+1}} \right)^{i(m-1)} \right] \\
&= \mathbb{E} \left[\left(\frac{S_{n-1}}{S_n} \right)^{n(m-1)} \right] \prod_{i=1}^{n-2} \mathbb{E} \left[\left(\frac{S_i}{S_{i+1}} \right)^{i(m-1)} \right] \\
&= \frac{n-1}{n-1+n(m-1)} \prod_{i=1}^{n-2} \frac{i}{i+i(m-1)} \\
&= \frac{n-1}{(mn-1)m^{n-2}}.
\end{aligned}$$

Putting it all together we have

$$\begin{aligned}
\mathbb{E}[\# \text{ PNE}] &= m^n \Pr[\text{first choices are stable}] \\
&\geq m^n \left(m^{-(n-1)} - \frac{n-1}{(mn-1)m^{n-2}} \right) / n^{m-1} \\
&= m \left(1 - \frac{m(n-1)}{mn-1} \right) / n^{m-1} \\
&= m \left(\frac{m-1}{mn-1} \right) / n^{m-1}
\end{aligned}$$

$$= \frac{m(m-1)}{(mn-1)n^{m-1}}.$$

■

4.1 Empirical Results

We have done several experiments regarding Random Voronoi games, which prove consistent with our theoretical results and suggest some interesting conjectures, particularly for the 2-D Simultaneous Voronoi Game. Our discussion of these results can be found in Appendix C.

5 Conclusion

We have introduced a new but we believe important set of variants on Voronoi games, where each player has a limited number of points to choose from. We believe these variations are motivated both by natural economic settings, and because of the possible connections to other “power-of-choice” processes in which participants choose from a limited set of random options.

In particular, we note that the Voronoi choice games we propose offer the chance to consider randomized versions of the problem, where the set of possible choices for each player is chosen uniformly over of the space. We have conjectured that the Random k -D Simultaneous Voronoi Game has a pure Nash equilibrium with probability $1 - o(1)$, based on a simulation study. While this is perhaps the most natural open question in this setting, there remain several other questions for both the simultaneous and sequential versions of Voronoi choice problems, in the worst case and with random point sets.

References

1. Dimitris Achlioptas, Raissa M D’Souza, and Joel Spencer. Explosive percolation in random networks. *Science*, 323(5920):1453–1455, 2009.
2. Robert J Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1(1):67–96, 1974.
3. Yossi Azar, Andrei Z Broder, Anna R Karlin, and Eli Upfal. Balanced allocations. *SIAM journal on computing*, 29(1):180–200, 1999.
4. Avrim Blum, MohammadTaghi Hajiaghayi, Katrina Ligett, and Aaron Roth. Regret minimization and the price of total anarchy. In *Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing*, pages 373–382. ACM, 2008.
5. John Byers, Jeffrey Considine, and Michael Mitzenmacher. Simple load balancing for distributed hash tables. In *Peer-to-peer Systems II*, pages 80–87. Springer, 2003.
6. Christoph Dürr and Nguyen Kim Thang. Nash equilibria in voronoi games on graphs. In *Proceedings of the 5th Annual European Symposium on Algorithms*, pages 17–28. Springer, 2007.

7. Gaëtan Fournier and Marco Scarsini. Hotelling games on networks: efficiency of equilibria. *Available at SSRN 2423345*, 2014.
8. J. J. Gabszewicz and J.-F. Thisse. Location. In *Handbook of Game Theory with Economic Applications*. Volume 1, Chapter 9. R. Aumann and S. Hart, editors. Elsevier Science Publishers, 1992.
9. Lars Holst. On the lengths of the pieces of a stick broken at random. *Journal of Applied Probability*, pages 623–634, 1980.
10. Harold Hotelling. Stability in competition. *The Economic Journal*, 39(153):41–57, 1929.
11. Albert Xin Jiang and Kevin Leyton-Brown. Polynomial-time computation of exact correlated equilibrium in compact games. *Games and Economic Behavior*, 21(1-2):183–202, 2013.
12. Masashi Kiyomi, Toshiki Saitoh, and Ryuhei Uehara. Voronoi game on a path. *IEICE TRANSACTIONS on Information and Systems*, 94(6):1185–1189, 2011.
13. Marios Mavronicolas, Burkhard Monien, Vicky G Papadopoulou, and Florian Schoppmann. Voronoi games on cycle graphs. In *Proceedings of Mathematical Foundations of Computer Science*, pages 503–514. Springer, 2008.
14. M Mitzenmacher, Andrea W Richa, and R Sitaraman. The power of two random choices: A survey of techniques and results. In *Handbook of Randomized Computing*, pages 255–312, 2000.
15. John Nash. Non-cooperative games. *The Annals of Mathematics*, 54(2):286–295, 1951.
16. Takao Nishizeki and Norishige Chiba. *Planar graphs: Theory and algorithms*. Elsevier, 1988.
17. Christos H Papadimitriou and Tim Roughgarden. Computing correlated equilibria in multi-player games. *Journal of the ACM*, 55(3):14, 2008.
18. Thomas J Schaefer. The complexity of satisfiability problems. In *Proceedings of the tenth annual ACM symposium on Theory of computing*, pages 216–226. ACM, 1978.
19. Sachio Teramoto, Erik D Demaine, and Ryuhei Uehara. Voronoi game on graphs and its complexity. In *IEEE Symposium on Computational Intelligence and Games*, pages 265–271. IEEE, 2006.

A Some Words on Models

In this paper we focus on games where players choose their strategy simultaneously. While possibly less interesting game theoretically, sequential variations where players arrive one at a time and decide their choice with knowledge of the preceding choices of previous players are interesting. For example, we might consider the *Random 2-D Sequential Voronoi Problem*:

- The n players arrive sequentially. Each player has m associated points that are chosen independently and uniformly at random from the unit torus $[0, 1]^2$.
- On arrival, each player chooses a point. The player’s final payoff will be the area of their cell in the Voronoi diagram. The player’s goal is to choose the point that maximizes (or, alternatively, minimizes) their final area.

In this variation, the primary natural question would be to determine how each successive player calculates its optimal strategy, and in particular under what conditions a player should not greedily choose the point that yields the maximum area at the point of time of their arrival. Other natural questions would involve the distribution of the sizes of the resulting cells (under the greedy strategy or some other strategy), and how they differ from that of a Voronoi diagram of n uniform random points. The “unfairness” in this setting might be measured by the difference in the minimum and maximum cell areas, or the “second moment” $\sum_i X_i^2$ where X_i is the area obtained by the i th player; one could consider the unfairness compared to the number of choices.

B Random Voronoi Games Proofs

We here show a statement made in proving the second part of Theorem 8, that

$$E \left[\prod_{i=1}^n \left(\sum_{j=n-i+1}^n \frac{(j-1)X_j}{j} \right)^{m-1} \right] \geq E \left[\prod_{i=1}^n \left(c_i \sum_{j=n-i+1}^n X_j \right)^{m-1} \right],$$

where $c_i = \frac{1}{i} \sum_{j=n-i+1}^n \frac{j-1}{j}$. We actually prove a more general statement. Denote the product of the components of a vector $\mathbf{x} \in \mathbb{R}^m$ by $\Pi(\mathbf{x}) = \prod_{j=1}^m x_j$.

Lemma 12: *Let A be an $n \times m$ matrix with non-negative entries, let $s, t \in [m]$ such that $s \neq t$, and let $\varepsilon > 0$. Let $X_j \sim \text{Exp}(1)$ i.i.d. for $j \in [m]$ and let $\mathbf{X} = (X_1, \dots, X_m)$. If $a_{is} \leq a_{it}$ for all $i \in [n]$ then*

$$E[\Pi(A\mathbf{X})] \geq E[\Pi(B\mathbf{X})],$$

where the perturbation B is defined by

$$b_{ij} = \begin{cases} a_{ij} + \varepsilon, & i = 1 \text{ and } j = s \\ a_{ij} - \varepsilon, & i = 1 \text{ and } j = t \\ a_{ij}, & \text{otherwise.} \end{cases}$$

Repeatedly applying this allows us to average all non-zero entries in each row.

The intuition of the proof is rather simple; if we expand the products out and use linearity of expectation, we can match up the terms in a way that coordinates the powers of X_s and X_t , controlling their moments. Because of this, on a term by term basis, shifting weight from a_{1t} to a_{1s} can only reduce the value.

The following mechanics are somewhat unwieldy but formalize this idea.

Proof. We can write $\Pi(A\mathbf{X})$ as a sum of m^n terms as follows:

$$\Pi(A\mathbf{X}) = \sum_{f:[n] \rightarrow [m]} \prod_{i=1}^n a_{if(i)} X_{f(i)}$$

Denote the number of preimages of $j \in [m]$ in a function $f : [n] \rightarrow [m]$ by $\sigma_f(j) = |f^{-1}(j)|$. By linearity of expectation and independence we get that

$$\begin{aligned} \mathbb{E}[\Pi(A\mathbf{X})] &= \sum_{f:[n] \rightarrow [m]} \mathbb{E} \left[\prod_{i=1}^n a_{if(i)} X_{f(i)} \right] \\ &= \sum_{f:[n] \rightarrow [m]} \prod_{i=1}^n a_{if(i)} \prod_{j=1}^m \mathbb{E} \left[X_j^{\sigma_f(j)} \right] \\ &= \sum_{f:[n] \rightarrow [m]} \left(\prod_{i=1}^n a_{if(i)} \prod_{j=1}^m \sigma_f(j)! \right). \end{aligned}$$

For notational convenience write

$$P_f = \prod_{i=1}^n a_{if(i)} \prod_{j=1}^m \sigma_f(j)!$$

and then we can express the expectation of $\Pi(B\mathbf{X})$ as:

$$\mathbb{E}[\Pi(B\mathbf{X})] = \sum_{f:[n] \rightarrow [m]} \left(\prod_{i=1}^n b_{if(i)} \prod_{j=1}^m \sigma_f(j)! \right) = \sum_{f:[n] \rightarrow [m]} b_{1f(1)} P_f$$

whereas their difference is:

$$\begin{aligned} \mathbb{E}[\Pi(A\mathbf{X}) - \Pi(B\mathbf{X})] &= \sum_{f:[n] \rightarrow [m]} (a_{1f(1)} - b_{1f(1)}) P_f \\ &= \varepsilon \left(\sum_{f(1)=t} P_f - \sum_{f(1)=s} P_f \right), \end{aligned}$$

so it suffices to show that $\sum_{f(1)=t} P_f \geq \sum_{f(1)=s} P_f$.

We now partition the sum to groups $\{f, f', g, g'\}$ of size four as follows. Take $f : [n] \rightarrow [m]$ such that $f(1) = t$ and $\sigma_f(t) \leq \sigma_f(s) + 1$. Let $S = f^{-1}(s)$ and

$T = f^{-1}(t) \setminus \{1\}$. Use the bijection from Lemma 13 to map $T \subset S \cup T$ to some $T' = \phi(T) \subset S \cup T$ of size $\sigma_f(s)$ such that $T \subset T'$. Define $f', g, g' : [n] \rightarrow [m]$ by

$$f'(i) = \begin{cases} t, & i \in T'; \\ f(i), & \text{otherwise} \end{cases}$$

$$g(i) = \begin{cases} s, & i = 1; \\ f(i), & \text{otherwise} \end{cases}$$

$$g'(i) = \begin{cases} s, & i = 1; \\ g(i), & \text{otherwise.} \end{cases}$$

Observe that $\sigma_g(s) = \sigma_{f'}(t) = \sigma_f(s) + 1$, $\sigma_g(t) = \sigma_{f'}(s) = \sigma_f(t) - 1$, $\sigma_{g'}(s) = \sigma_{f'}(s) + 1 = \sigma_f(t)$ and $\sigma_{g'}(t) = \sigma_{f'}(t) + 1 = \sigma_f(s)$.

We claim that $P_f + P_{f'} \geq P_g + P_{g'}$. To see that, set aside the common non-negative factor

$$C = \prod_{i \in [n] \setminus (S \cup T \cup \{1\})} a_{if(i)} \prod_{j=1}^m \sigma_f(j)!$$

and observe that

$$P_f = C \cdot \prod_{i \in S} a_{is} \prod_{i \in T} a_{it}$$

$$P_{g'} = C \cdot \prod_{i \in S \setminus T'} a_{is} \prod_{i \in T'} a_{it}$$

$$P_g = C \cdot \frac{\sigma_f(s) + 1}{\sigma_f(t)} \cdot \prod_{i \in S} a_{is} \prod_{i \in T} a_{it}$$

$$P_{f'} = C \cdot \frac{\sigma_f(s) + 1}{\sigma_f(t)} \cdot \prod_{i \in S \setminus T'} a_{is} \prod_{i \in T'} a_{it}.$$

Therefore

$$\begin{aligned} & (P_{f'} - P_g) - (P_{g'} - P_f) \\ &= C \left(\frac{\sigma_f(s) + 1}{\sigma_f(t)} - 1 \right) \left(\prod_{i \in S \setminus T'} a_{is} \prod_{i \in T'} a_{it} - \prod_{i \in S} a_{is} \prod_{i \in T} a_{it} \right) \\ &\geq 0 \end{aligned}$$

since $\sigma_f(t) \leq \sigma_f(s) + 1$ and the s th column of A is dominated by the t th column.

Note that when $\sigma_f(t) = \sigma_f(s) + 1$ we get $T' = T$ and thus $f' = f, g' = g$; in this case $P_f = P_g$ so no harm is done by counting it twice. \blacksquare

Lemma 13: Let n and k be integers such that $0 < 2k \leq n$. Then there is a monotone bijection $\phi : \binom{[n]}{k} \rightarrow \binom{[n]}{n-k}$; that is, ϕ maps k -subsets of $[n]$ to $(n-k)$ -subsets of $[n]$ and satisfies $A \subset \phi(A)$.

Proof. Build a bipartite graph whose sides are $L = \binom{[n]}{k}$ and $R = \binom{[n]}{n-k}$. Connect $A \in L$ to $B \in R$ iff $A \subset B$. This is a $\binom{n-k}{k}$ -regular bipartite graph, and as such it has a perfect matching (e.g., by Hall's theorem). ■

We now turn to the actual calculation of the product of the terms $c_i = \frac{1}{i} \sum_{j=n-i+1}^n \frac{j-1}{j}$. Denote the i th Harmonic and the i th 2-Harmonic numbers, respectively, by $H_i = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i}$ and $H_i^{(2)} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{i^2}$. Recall that the Harmonic series diverges but the 2-Harmonic series converges to $\pi^2/6$.

Now write $c_i = 1 - \frac{1}{i} \left(\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{n-i+1} \right) = 1 + (H_{n-i} - H_n)/i$.

Lemma 14: $c_1 + c_2 + \dots + c_n = n - H_n^{(2)}$ for all $n \in \mathbb{N}$.

Proof. First we claim that $H_n^2 + H_n^{(2)} = 2 \sum_{k=1}^n H_k/k$. Indeed,

$$\begin{aligned} H_n^2 &= \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right)^2 = \sum_{j,k=1}^n \frac{1}{jk} = 2 \sum_{1 \leq j \leq k \leq n} \frac{1}{jk} - \sum_{1 \leq j=k \leq n} \frac{1}{jk} \\ &= 2 \sum_{k=1}^n \frac{1}{k} \sum_{j=1}^k \frac{1}{j} - \sum_{k=1}^n \frac{1}{k^2} = 2 \sum_{k=1}^n \frac{H_k}{k} - H_n^{(2)}. \end{aligned}$$

Now for any $i \in [n]$ we have

$$\frac{H_{n-i} - (H_n - H_i)}{i} = \frac{1}{i} \sum_{k=1}^{n-i} \left(\frac{1}{k} - \frac{1}{i+k} \right) = \frac{1}{i} \sum_{k=1}^{n-i} \frac{(i+k) - k}{k(i+k)} = \sum_{k=1}^{n-i} \frac{1}{k(i+k)},$$

so

$$\begin{aligned} \sum_{i=1}^n \frac{H_{n-i} - (H_n - H_i)}{i} &= \sum_{i=1}^n \sum_{k=1}^{n-i} \frac{1}{k(i+k)} = \sum_{k=1}^n \sum_{i=1}^{n-k} \frac{1}{k(i+k)} \\ &= \sum_{k=1}^n \sum_{j=k+1}^n \frac{1}{kj} = \sum_{k=1}^n \frac{H_n - H_k}{k} \\ &= H_n^2 - \sum_{k=1}^n \frac{H_k}{k}, \end{aligned}$$

and thus

$$\sum_{i=1}^n c_i = \sum_{i=1}^n \left(1 + \frac{H_{n-i} - H_n}{i} \right) = n + \sum_{i=1}^n \frac{H_{n-i} - (H_n - H_i) - H_i}{i}$$

$$= n + H_n^2 - 2 \sum_{i=1}^n \frac{H_i}{i} = n - H_n^{(2)}.$$

■

Corollary 2. $\prod_{i=1}^n c_i \geq \exp(-(\pi^2/6 + o(1)))$; in particular, $\prod_{i=1}^n c_i \geq 0.19$.

Proof. Use $\ln(x) \geq (x-1)/x$ for $0 < x < 1$ along with the fact that

$$1 - \frac{1}{n} = c_1 \geq c_2 \geq \dots \geq c_n = 1 - \frac{H_n}{n}$$

to establish that

$$\begin{aligned} \ln \left(\prod_{i=1}^n c_i \right) &= \sum_{i=1}^n \ln(c_i) \geq \sum_{i=1}^n \frac{c_i - 1}{c_i} \geq \sum_{i=1}^n \frac{c_i - 1}{c_n} = \frac{1}{c_n} \left(-n + \sum_{i=1}^n c_i \right) \\ &= \frac{-H_n^{(2)}}{1 - H_n/n} \geq - \left(\frac{\pi^2}{6} + O\left(\frac{1}{n}\right) \right). \end{aligned}$$

■

C Empirical Results

We describe our empirical results for finding PNE in random instances of Voronoi choice games. Specifically, we investigated the likelihood that a myopic best response process would converge to a PNE on random instances of the One Way 1-D Simultaneous Voronoi Game, the 1-D Simultaneous Voronoi Game and the 2-D Simultaneous Voronoi Game. We investigated both the maximization and minimization variants of each of these games with $n \in \{10, 100, 1000, 10000\}$ and $m \in \{2, 3, 4\}$. (For the 2-D case we present results only for n up to 1000.) We note that we present limited results for the 1-D Simultaneous Voronoi Game, as we were always able to find a PNE. This is not surprising, given Theorem 5. However, it offers some weight suggesting that the worst-case time for myopic best response to converge to an equilibrium for this problem is polynomial in n ; this remains an interesting open question.

Our experimental procedure is as follows. For each type of game, for each value of n and m , we generated 1000 random instances of the game by choosing each player's points uniformly at random from the unit torus. For each instance, we first randomly assign each player to one of its points. We then repeatedly iterate through all of the players, moving them to their best response to the current set of points. If in a pass through the players, none of them change strategies, we have found a PNE. Otherwise, we give up after 1000 such passes. (The number 1000 appears suitable; in our experiments, no instances successfully converged after more than 764 passes). If no PNE was found, using the same set of random points, we try a new random starting assignment of points to players, and then try another 1000 passes through the players. If after 100 such random

initial configurations no PNE has been found, we declare that we have failed to find a PNE for that problem instance. (For the 2-D Simultaneous Voronoi Game, our results are only up to 10 initial configurations; as we shall see this appears sufficient for our exploration.)

The program for the 2-D Simultaneous Voronoi problem was built on top of the Delaunay triangulation program in the Computational Geometry Algorithms Library,¹ which supports fast dynamic updates of a Delaunay triangulation. By maintaining this underlying structure, our program was able to efficiently update the Voronoi cell areas during the myopic best response process.

Our results are presented in Table 1 through Table 6. Table 1 through Table 4 show in how many instances out of 1000 random instances our algorithm found a PNE, for various values of n , m and numbers of random initial configurations attempted. Table 5 and Table 6 give the maximum number of passes of myopic best response that our algorithm took before successfully finding a PNE for the variants of the 1-D Simultaneous Voronoi Game and 2-D Simultaneous Voronoi Game respectively.

Based on this data, we propose a few conjectures. Looking at Table 2, it seems possible that the probability that a PNE exists for a random instance of the maximization variant of the One Way 1-D Simultaneous Voronoi Game converges as n grows to a constant in $(0,1)$ for $m = 2$. For $m \geq 3$ for the maximization variant, and for $m \geq 2$ for the minimization variant, Table 1 and Table 2 suggest that the probability of a PNE exists may converge to 0, or may be difficult to find through myopic best response. Both of these possibilities are consistent with our results on the expected number of solutions.

For the 2-D Simultaneous Voronoi Game, for both the maximization and the minimization version, Table 3 and Table 4 suggest that a PNE exists for a random instance with high probability, and further can be found by myopic best response.

Interestingly, the expected number of passes before finding a PNE in our experiments seems to grow like $\Theta(\log n)$ for the maximization variant of both the 1-D Simultaneous Voronoi Game and the 2-D Simultaneous Voronoi Game, but seems to possibly grow like $\Theta(n)$ (or possibly some other superlogarithmic function) for the minimization variants. Again, we are left with many interesting open questions.

¹ <http://www.cgal.org/>

m	n	1 attempt	2 attempts	5 attempts	10 attempts	100 attempts
2	10	216	222	231	239	242
2	100	17	17	18	21	24
2	1000	1	1	1	1	1
2	10000	0	0	0	0	0
3	10	38	40	45	46	46
3	100	0	0	0	0	0
3	1000	0	0	0	0	0
3	10000	0	0	0	0	0
4	10	6	6	6	6	6
4	100	0	0	0	0	0
4	1000	0	0	0	0	0
4	10000	0	0	0	0	0

Table 1: The number of successes over 1000 iterations of finding a PNE for a random instance of the minimization variant of the One Way 1-D Simultaneous Voronoi Game using a given number of attempts at myopic best response with random initial strategy profiles.

m	n	1 attempt	2 attempts	5 attempts	10 attempts	100 attempts
2	10	616	633	652	663	668
2	100	531	556	578	588	609
2	1000	497	525	547	556	589
2	10000	497	518	548	563	592
3	10	175	197	223	240	285
3	100	9	13	19	26	51
3	1000	0	0	0	0	0
3	10000	0	0	0	0	0
4	10	35	44	56	65	112
4	100	0	0	0	0	0
4	1000	0	0	0	0	0
4	10000	0	0	0	0	0

Table 2: The number of successes over 1000 iterations of finding a PNE for a random instance of the maximization variant of the One Way 1-D Simultaneous Voronoi Game using a given number of attempts at myopic best response with random initial strategy profiles.

m	n	1 attempt	2 attempts	5 attempts	10 attempts
2	10	987	995	998	999
2	100	982	994	997	999
2	1000	978	999	1000	1000
3	10	986	988	995	996
3	100	983	995	1000	1000
3	1000	967	1000	1000	1000
4	10	970	993	998	1000
4	100	969	1000	1000	1000
4	1000	948	998	1000	1000

Table 3: The number of successes over 1000 iterations of finding a PNE for a random instance of the minimization variant of the 2-D Simultaneous Voronoi Game using a given number of attempts at myopic best response with random initial strategy profiles.

m	n	1 attempt	2 attempts	5 attempts	10 attempts
2	10	992	997	999	999
2	100	994	999	1000	1000
2	1000	991	1000	1000	1000
3	10	987	992	997	998
3	100	990	999	1000	1000
3	1000	993	999	1000	1000
4	10	992	1000	1000	1000
4	100	990	1000	1000	1000
4	1000	993	1000	1000	1000

Table 4: The number of successes over 1000 iterations of finding a PNE for a random instance of the maximization variant of the 2-D Simultaneous Voronoi Game using a given number of attempts at myopic best response with random initial strategy profiles.

n	min/ $m = 2$	min/ $m = 3$	min/ $m = 4$	max/ $m = 2$	max/ $m = 3$	max/ $m = 4$
10	6	8	11	5	6	8
100	12	22	37	8	9	12
1000	42	87	173	11	12	16
10000	117	361	764	12	15	19

Table 5: The maximum number of passes of myopic best response before an instance converged to a PNE for various 1-D Simultaneous Voronoi Games.

n	$\min/m = 2$	$\min/m = 3$	$\min/m = 4$	$\max/m = 2$	$\max/m = 3$	$\max/m = 4$
10	7	8	13	7	8	10
100	15	14	19	13	13	16
1000	29	30	57	13	14	19

Table 6: The maximum number of passes of myopic best response before an instance successfully converged to a PNE for various 2-D Simultaneous Voronoi Games.