Def An *anticanonical pair* (X, D) is a smooth proj. variety X w/ $D \in |-K_X|$ a reduced normal crossings divisor. Then $\exists\ \Omega \in H^0(K_X(\log D))$.

Analogous to the CY case. e.g. $\mathbb{H}^\bullet(\Omega^\bullet(\log D))$ hypercohomology of log complex computes $H^\bullet(X \setminus D, \mathbb{C})$. deRham

$h^{n,0}(X \setminus D) = 1$

As usual we also have a Kähler form so data we consider is (X, ω, J, Ω).

Def $L \subset X \setminus D$ is *special Lagrangian* if $\operatorname{Im} \Omega|_L = 0$, $\omega|_L = 0$.

Then $\operatorname{Re} \Omega|_L = \psi \operatorname{vol}_g$ where g is from the Kähler metric, $g(u,v) = \omega(u, Jv)$.

and $\psi \in C^\infty(L, \mathbb{R}^+)$.

Def $\alpha \in \Omega^1(L, \mathbb{R})$ is *ψ-harmonic* if $d\alpha = d^*(\psi\alpha) = 0$.

Lemma Each coho class in $H^1(L, \mathbb{R})$ has a unique ψ-harmonic representative.

Pf: If $\alpha = df$ is ψ-harmonic $0 = \psi^{-1} d^*(\psi f) = \Delta f - \psi^{-1}\langle d\psi, df\rangle$

so f satisfies maximum principle $\Rightarrow f$ constant

Existence: $D_\psi : \Omega^{odd}(L, \mathbb{R}) \to \Omega^{even}(L, \mathbb{R})$

$D_\psi(\alpha_1, \alpha_3, \dots) = (\psi^{-1} d^*(\psi\alpha_1), d\alpha_1 + d^*\alpha_3, \dots)$

ker D_ψ spanned by ψ-harmonic 1-forms + harmonic odd forms deg ≥ 3

coker D_ψ contains all even harmonic of deg ≥ 2 and ψ.

elliptic operator.

$\operatorname{index}(D_\psi) = \operatorname{index}(D_1) = \operatorname{index}(d + d^*) \Rightarrow \dim\{\psi\text{-harmonic forms}\} = H^1(L, \mathbb{R})$.

Prop. Infinitesimal special Lagr. deformations of L correspond to ψ-harmonic 1-forms.

Pf: Let $v \in C^\infty(NL)$ be a section of the normal bundle.

$$NL \xrightarrow{\sim} T^*L \qquad NL \xrightarrow{\sim} \Lambda^{n-1}T^*L$$
$$v \longmapsto -\iota_v\omega \qquad v \longmapsto \iota_v\Omega.$$

Can check by writing in local coordinates:

1) $\iota_v\Omega = \psi(*_g(-\iota_v\omega))$.

2) The deformation corresponding to v is special Lagrangian iff $-\iota_v\omega$ and $\iota_v\Omega$ are both closed, i.e. $\alpha := -\iota_v\omega$ is ψ-harmonic.

By McLean the deformations are unobstructed. Let ~~M = moduli space~~

$$T_L\underbrace{\{\text{moduli space of special Lagrangians}\}}_{B} \cong \{\psi\text{-harmonic 1-forms on } L\}$$

Furthermore B has an integral-affine structure (in fact two) defined by $T_{\mathbb{Z},L}B := H^1_\psi(L,\mathbb{Z})$ or $H^{n-1}_\psi(L,\mathbb{Z})$.

II. Complexified Moduli Space $\mathcal{M}$

Consider $\overset{\mathcal{M}}{\text{moduli of}}$ pairs (L,∇) with ∇ a flat $U(1)$-connection on the trivial cx. line bundle. Write $\nabla = d+iA$ A is a ψ-harmonic 1-form on L.

Note $\mathrm{hol}_\nabla \cong \mathrm{Hom}(H_1(L),U(1)) \cong H^1(L,\mathbb{R})/H^1(L,\mathbb{Z})$. (mod gauge equivalence) unique representative.

Then $T_{(L,\nabla)}\mathcal{M}$ is the set of pairs $(v,\alpha) \in C^\infty(NL) \oplus \Omega^1(L,\mathbb{R})$ s.t. $-\iota_v\omega$ is ψ-harmonic, as is α.

Then $(v,\alpha) \longmapsto -\iota_v\omega + i\alpha$ is an iso to cx. ψ-harmonic forms.

~~i.e. $J^\vee(v,\alpha) =$~~ $(\mathcal{M}, J^\vee, \omega^\vee, \Omega^\vee)\ldots$

Then let $A \in H_2(X, L)$ w/ $\partial A \neq 0 \in H_1(L, \mathbb{Z})$.

$z_A = \exp(-\int_A \omega)\, \mathrm{hol}_\nabla(\partial A) : M \to \mathbb{C}$ is a holomorphic function.

Pf $d \log z_A(v, \alpha) = \int_{\partial A} -\iota_v \omega + i\alpha$

Local coordinates on the mirror if

$H_2(X, L) \to H_1(L) \xrightarrow{0} H_1(X)$ otherwise, fin. reps of $H_1(X)$ in $H_1(L)$ and attach by a cylinder.

Def $\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2)) := \int_L \alpha_2 \wedge \iota_{v_1} \mathrm{Im}\, \Omega - \alpha_1 \wedge \iota_{v_2} \mathrm{Im}\, \Omega$

$(\alpha, \beta) \quad \beta \wedge \psi_{*g} \mathrm{Re}\, \alpha) - \mathrm{Im}\, \alpha \wedge \psi_{*g} \mathrm{Re}\, \beta$

Prop: $\omega^\vee$ is Kähler and compatible w/ $J^\vee$. $\boxed{\Xi}$

Rmk: $\pi : M \to B$ forget the connection is Lagr wrt $\omega^\vee$.

Def $\Omega^\vee((v_1, \alpha_1), \ldots, (v_n, \alpha_n)) := \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \cdots \wedge (-\iota_{v_n}\omega + i\alpha_n)$

clearly co linear b.c. $\Omega^\vee = d\log z_1 \wedge \cdots d\log z_n$.

Fibers $\pi : M \to B$ special Lagrangian. (X, ω, J, Ω).

If we assume ψ-harmonic forms on L have no zeros. Then the sp. Lag. tori fiber.

The superpotential

$CF^*(L, L) \xrightarrow{m_1} CF^*(L, L)[1]$.

$m_2($

FOOO $CF^*(L, L)$ has the structure of a "curved" or "obstructed" A_∞-algebra.

$m_1(m_1(x)) +$ ~~$m_2(m_0(x)) + \ldots$~~ $m_2(m_0, x) + m_2(x, m_0)$

Let $M(L, \beta) =$ J-holo. discs w/ ∂ on L rep. β.

Let $\beta \in \pi_2(X, L)$ virtual dimension $M(L, \beta) = n - 3 + \mu(\beta)$

$m_0 \in CF^*(L, L)$

$\mu(\beta) =$ Maslov index of β.

degree 0 part of the obstruction

Lemma $\mu(\beta) = 2\beta \cdot [D]$.

~~Pf: Defn of maslov index: volume determines a section of $\det(T\ldots)$. element of L~~

~~Failure to extend from~~

Pf L totally real $\Rightarrow$ volume element determines (Re Ω) a section $s \in \Lambda^n (T^{1,0}X|_L) = K^{-1}|_L$.

So s^2 defines a unique section of $S^1(K^{-2}|_L)$

The maslov index is by definition the (measure of) the failure of this to extend over the holo disc.

$\sigma = \Omega^{-1} \in H^0(K_X^{-1})$, $\sigma^2|_L = s^2$ coincide bc. special Lagrangian.

$\Rightarrow \mu(\beta) = 2\beta \cdot D$.

Assume
1) no discs index 0
2) holo discs index 2 regular
3) $\exists\, \mathbb{P}^1$ holo w/ $K \cdot [\mathbb{P}^1] \geq 0$.

Then $\forall \beta \in \pi_2(X, L)$ w/ $\mu(\beta) = 2$ $M(L, \beta)$ smooth compact of $\mathbb{R}$-dim n-1.

no bubbling b.c. 2 is minimal maslov index.

Assume L is spin. Then $M(L,\beta)$ is oriented.

$M(L,\beta)$ covers L n_β times. i.e.
$M(L, \beta, pt) \to L$ degree n_β.

Def: $m_0(L, \nabla) := \sum_{\substack{\beta \\ \mu(\beta)=2}} n_\beta(L) \exp\left(-\int_\beta \omega\right) \mathrm{hol}_\nabla(\partial\beta)$

~~In this situation n_β are locally constant.~~ $W = m_0 : M \to \mathbb{C}$ is holomorphic

This is the "superpotential" on the mirror.

Assumption 3 not always necessary especially if a $\mathbb{P}^1$ bubbles off

its a codim 2 condition

2 neither but then must on $M(L,\beta)$.

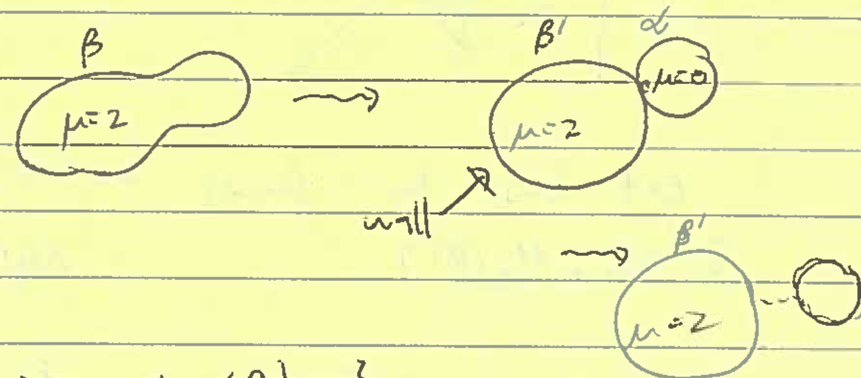
do virtual count (virtual fund cycle on $M(L,\beta)$).

$[M(L,\beta)]^{vir}$.

n_β is still constant, Lagr. isotopies induce a cobordism between virtual fund. cycles

Assumption 1: Problem

Never holds except toric case.



But $\{$pts of L lying on $\partial\beta$ w/ $\mu(\beta)=0\}$ is codim 2.

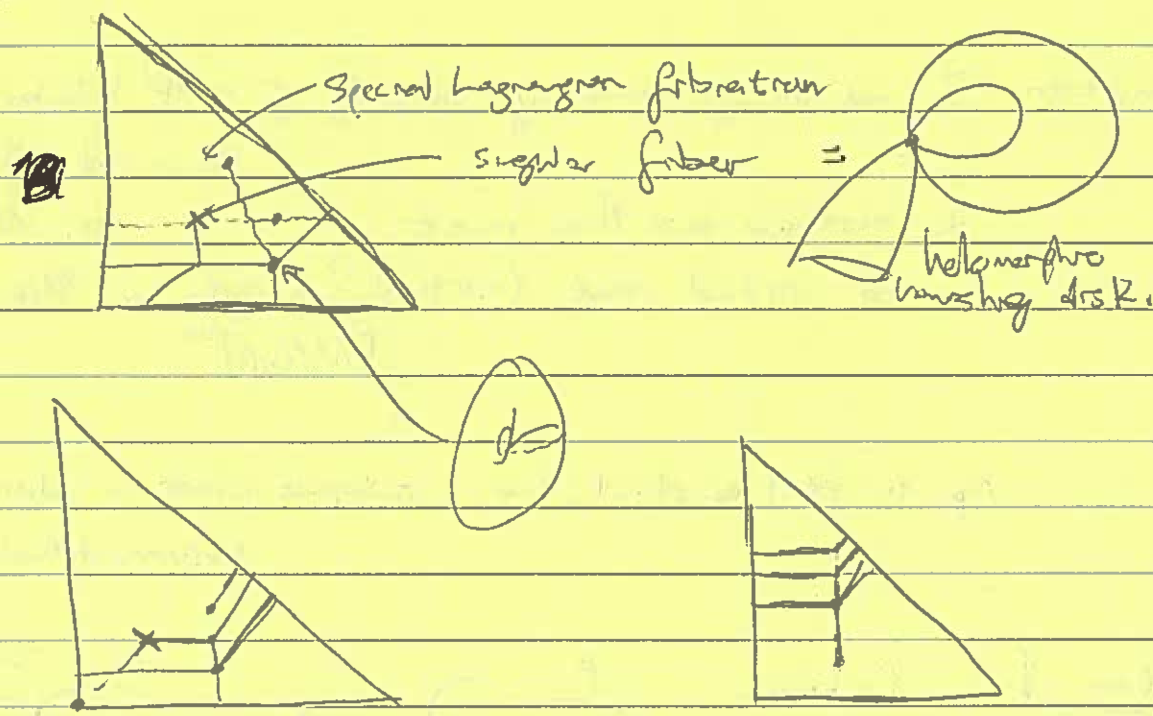
Thus, for a generic $p\in L$, the discs β s.t. $p\in\partial\beta$ $\mu(\beta)=2$ is finite and so don't intersect

Thus, we have $n_\beta(L,p)$ and $\forall p\in L$. $\rightsquigarrow m_0(L,\nabla,p)$

Wall-chamber structure on L. Really just a cochain rather than a cocycle in L. Enlarge moduli space to get a cycle, but,

Restrict to surface case: Here there are walls in another sense most Lagr. don't bound a $\mu(\beta)=0$ disc.

Typical situation



Let L_t be a family $t \in (-\varepsilon, \varepsilon)$ L_0 bounds $\overset{\text{unique}}{\mu\text{-I}}$ ⓪ Disk d_0.

$\partial(ev_{1*} M_{\varepsilon}(\beta))$ $\qquad\qquad$ $M_{\varepsilon}(\beta) := \coprod_t M_{\varepsilon}(L_t, \beta)$

$$= \sum_{\substack{m \geq 1 \\ \mu(\alpha)=0}} ev_{1*}\left[\tilde{M}_2(\beta - m\alpha) \times_{ev_2} \tilde{M}_1(m\alpha)\right]$$

~~Typical~~ $\beta = \beta_0 + m\alpha$ deform to holo. disks for $t<0$ or $t>0$
but not both

Lack of transversality perturb J s.t. each disk

$$F_t(q) := \sum_{m \in \mathbb{Z}} n_{\beta_0 + m\alpha}(L_t)\, q^m \qquad F_\varepsilon(q) = F_{-\varepsilon}(q)\left[1 + \tilde{n}_{\alpha} q + \tilde{n}_{2\alpha} q^2 + \cdots\right]^{(\partial\beta_0 \cdot \partial\alpha)}$$

Then ~~contrib~~ note $z_{\beta_0} F_t(z_\alpha)$ = contrib. $\beta_0 + m\alpha$ to $m_0(L, \nabla)$.

$\cancel{m_0(L_\varepsilon, \nabla)_\varepsilon =}$

~~Upon~~ ~~Thus, $m_0(L_t$~~

~~Prop~~ $\cancel{m_0(L, \nabla)}$

Prop Upon crossing a wall bounding a unique maslov index 0 disc representing $\alpha \in H_2(X, L)$

$m_0(L, \nabla)$ changes by the change of variables as a Laurent series in z_β

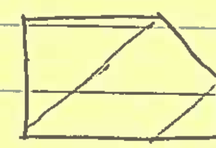
$$z_\beta \longmapsto z_\beta\, h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]} \qquad h(z_\alpha) = 1 + O(z_\alpha)$$

indep of β.

Examples Toric Varieties

No index 0 discs ☺.

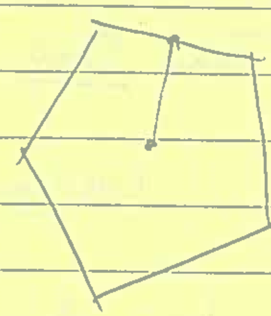
Let $\mu : (X, D) \to \Delta$ be the moment map.



$X \setminus D \cong (\mathbb{C}^*)^n \qquad \Omega = d\log x_1 \wedge \dots \wedge d\log x_n.$

and fibers of μ are $\left.\begin{matrix} |x_1| = r_1 \\ \vdots \\ |x_n| = r_n \end{matrix}\right\}$ special Lagrangian.

Prop μ biholo. to $\mathrm{Log}^{-1}(\mathrm{int}\,\Delta) \subset (\mathbb{C}^*)^n$



$\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$

$(z_1, \dots, z_n) \mapsto \frac{-1}{2\pi}(\log|z_1|, \dots, \log|z_n|).$

Take $z_1, \dots, z_n$ to be coordinates defined before and $\checkmark$ image.

Prop $W = m_0(L, \nabla) = \sum_{F \text{ facet}} e^{-2\pi\alpha(F)} z^{v(F)}$ $\quad v$ = normal to F outgoing vector.

$\alpha(F)$ s.t. $\langle v(F), \phi \rangle + \alpha(F) = 0$ "eqn of F.

Claim Unique disc β_i intersecting $D_i \subset D$

s.t. $\partial\beta_i$ passes through $p \in L$, $p = (x_1, \dots, x_n)$

$(\omega^{v_1} x_1, \dots, \omega^{v_n} x_n)$ $\omega \in \mathbb{D}$. ($\omega = 0$ corresp

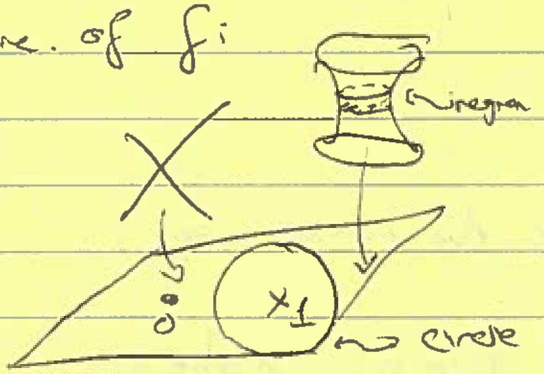
$X = \mathbb{CP}^2 \setminus \{\ell \cup C\}$ $\quad X = \mathbb{C}^2 \setminus V(xy-1)$.

$\Omega = \dfrac{dx \wedge dy}{xy-1}$, $\quad$ Admits an S^1 action $\quad (x,y) \mapsto (e^{i\theta}x, e^{-i\theta}y)$.

$$f : X \to \mathbb{C}$$
$$(x,y) \mapsto xy$$

ω = FS form on $\mathbb{P}^2$, (normalized) s.t. $\int_{\text{line}} \omega = 1$.

Then pic. of f:



Horizontal distrib. sympl. orthog to fibers

$T_{r,0} := \{(x,y) \mid |x| = |y|,\ f(x,y) \in C_r(1)\}$.

$T_{r,\lambda} := \{(x,y) \mid |x|/|y| = \text{constant},\ f(x,y) \in C_r(1), \text{ and } \int_{\text{region}} \omega = \lambda\}$.

Then Claim: $T_{r,\lambda}$ special Lagrangian

Lagrangian ✓ (transport of S^1-orbit)

Let $H = |xy-1|^2$, $\xi = (ix, -iy)$

X_H, ξ span tangent space of $T_{r,\lambda}$ $\forall r, \lambda$.

$i_\xi \Omega = \dfrac{ix\,dy + iy\,dx}{xy-1} = i\, d\log(xy-1)$ Thus, $\operatorname{Im} \Omega(\xi, X_H) = d\log |xy-1| (X_H) = 0$.

For any $\beta \in \pi_2(\mathbb{CP}^2, T_{r,\lambda})$

$$\mu(\beta) = 2\,\beta \cdot \text{fiber} + \beta \cdot \mathbb{CP}^1_\infty$$

on interior of loop. line at ∞.

$x = y = \pm\sqrt{1+c}$. $y^2 = 1+c$. $|c| < 1$. $xy - 1 = c$. $y = x$. $x = y$ is a section of f? $xy-1$. $x=y$: $|x| = |y|$, $|xy - 1| = r$, ~~$|x|/|y| = \text{const}$~~ ~~$y=x$~~

Lemma:

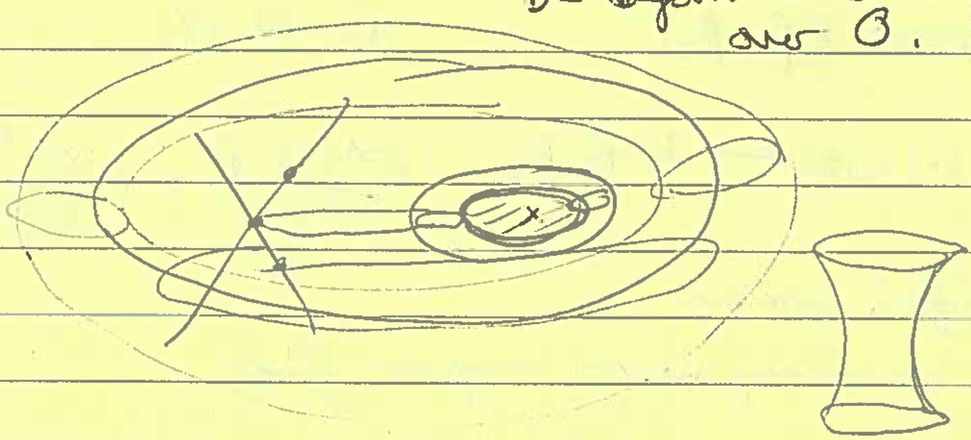
$T_{r,\lambda}$ bounds maslov index 0 disc iff $r = 1$.

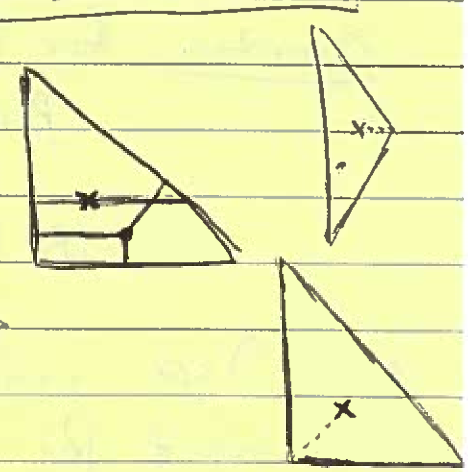
Pf: compose w/ f. f must be constant.

Then topologically β must be in the nodal fiber.

Prop for $r > 1$ $W = z_1 + z_2 + \dfrac{e^{-\Lambda}}{z_1 z_2}$ same as toric case.

β_c ~~deform to circle~~ over 0.





Prop For $r < 1$: Basis of $\pi_2(\mathbb{CP}^2, T_{r,0})$:

α, β, ~~H hyperplane class~~

		μ	x-axis	y-axis	$f^{-1}(\varepsilon)$
α	vanishing disc of node of $xy = 0$.	0	-1	-1	0
β	$y = x \mid$ disc bounded by	2	0	0	1
H	hyperplane class	6	1	1	2

$\mu = 2 \Rightarrow \beta$ or $H - 2\beta + k\alpha$, $\alpha \in \{-1, 0, 1\}$.
all bound.

$$\rightsquigarrow W = u + \frac{e^{-\Lambda}(1+w)^2}{u^2 w}$$

$u = z_1 + z_2$ $w = z_1/z_2$ change variables
so that w is globally defined.

$r \ll 1 \rightsquigarrow r > 1$

$\alpha \longleftrightarrow \beta_1 - \beta_2$ $w \longleftrightarrow z_1/z_2$

$\beta \longleftrightarrow \beta_2$ $u \longleftrightarrow z_2$

$H - 2\beta - \alpha \longleftrightarrow H - \beta_1 - \beta_2$ $e^{-\Lambda}/u^2 w \longleftrightarrow e^{-\Lambda}/z_1 z_2$

relative homotopy classes:

incorrect!

Correction For $\lambda > 0$

$\beta \longleftrightarrow \{\beta_1, \beta_2\}$ $u \longleftrightarrow z_1 + z_2$

$H - 2\beta + \{-1, 0, 1\}\alpha \longleftrightarrow H - \beta_1 - \beta_2$ $\dfrac{e^{-\Lambda}(1+w)^2}{u^2 w} \longleftrightarrow \dfrac{e^{-\Lambda}}{z_1 z_2}$

For $\lambda < 0$ modified correction
to account for monodromy around singular fiber.