HMS Seminar - Talk 1

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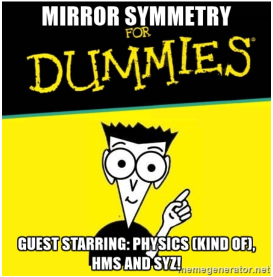

Fukaya categories : (naive) Lagrangian Floer homology, A_∞-structures

Introduction : what is mirror symmetry?

- The physical story
- Restatement 1: "Classical" MS
- Restatement 2: Homological Mirror Symmetry
- The SYZ conjecture

Fukaya categories : (naive) Lagrangian Floer homology, A_∞-structures

- A symplectic manifold (M^{2n}, ω) is a smooth manifold with a closed, non-degenerate 2-form. A symplectomorphism is a diffeomorphism $\phi : M \to M$ such that $\phi^*\omega = \omega$.
- Darboux theorem: there exists an open neighbourhood D of every point $p \in M$ that is symplectomorphic to

$$(\mathbb{R}^{2n}, \omega_{std} := \sum dx_i \wedge dy_i).$$

- A Lagrangian $L^n \subset M$ is a half dimensional submanifold with $\omega|_L = 0$.
- Weinstein tubular neighbourhood: Every Lagrangian $L \subset M$ has an open neighbourhood that is symplectomorphic to T^*L.
- "Everything is a Lagrangian!" (Weinstein's creed)
- Unfortunately, very hard to classify and understand directly:
  1. Possibly known for $\mathbb{R}^4$ (2016, Georgios D. Rizell).
  2. Not even a conjecture for $\mathbb{R}^6$!

15 minutes of symplectic topology

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- A Hamiltonian is just a function $H : M \to \mathbb{R}$.
- The Hamiltonian vector field X_H of H is defined by

$$\iota_{X_H}\omega = dH.$$

- Thus, every function generates a 1-parameter family of symplectomorphism $\{\phi_H^t\}$.
- The time-1 flow is called a Hamiltonian isotopy.
- **Corollary:** symplectic topology is very "flabby".
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Floer cohomology - formal properties

- In nice cases (not always!), we can associate to $L_0, L_1 \subset M$ a group $HF^*(L_0, L_1)$ which attempts to measure (in a Hamiltonian isotopy invariant way) how many times must L_0 and L_1 intersect.
- **Properties:**
  1. $\chi(HF^*(L_0, L_1)) = L_0 \cdot L_1$.
  2. $HF^*(\phi_H^1 L_0, L_1) \cong HF^*(L_0, L_1) \cong HF^*(L_0, \phi_H^1 L_1)$.
  3. $HF^*(L, L) \cong H^*_{sing}(L)$.
  4. If $L_0 L_1$, then $HF^*(L_0, L_1) = H^*(CF^*(L_0, L_1), d)$, where $CF^*(L_0, L_1)$ is freely generated by the intersection points.
- Thus, for example, we have a refined lower bound

$$|\phi_H L \cap L| =: rkCF^*(\phi_H L, L) \geq rkHF^*(L, L) = rkH^*(L).$$

- There are cases where we can define $CF^*(L_0, L_1)$ and d, but $d^2 \neq 0$. In that case, we say that (L_0, L_1) is **obstructed**.

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Floer cohomology - opening the blackbox (gently)

- Coefficients: a small field $\mathbf{k}$ ($= \mathbb{Z}_2$ for now, maybe $\mathbb{Q}$ or $\mathbb{C}$ later).
- Novikov field $\mathbb{K} = \Lambda_{\mathbf{k}}$ whose elements are formal sums

$$\mathbb{K} = \left\{ \sum_{i=0}^{\infty} c_i q^{\lambda_i} \mid c_i \in \mathbf{k}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = +\infty \right\}$$

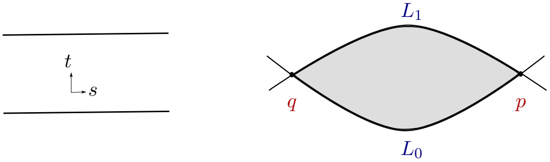
  Keeps track of areas of holomorphic curves.
- Morphism spaces are $\mathbb{K}$-vector spaces freely generated by intersection points:

$$CF(L_0, L_1) := \mathbb{K}\langle L_0 \cap L_1 \rangle.$$

- There is a differential on the morphism spaces,

$$d : CF^*(L_0, L_1) \to CF^*(L_0, L_1).$$

Given $p, q \in L_0 \cap L_1$, the coefficient of q in $d(p)$ is the number of J-holomorphic strips u like this:



weighted by $q^{\omega}(u)$.

- Morally, $HF^*(L_0, L_1)$ is Morse homology of the "action functional" defined on the space of paths

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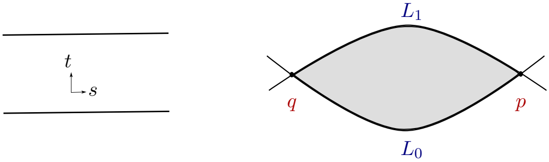

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Floer cohomology - opening the blackbox (gently)

- Analytically, it is very hard to actually define it in this way.
- Floer's Brilliant idea: replace an ODE on the path space by a PDE on the manifold!
- Solve the "gradient equation"

$$\frac{\partial u}{\partial s} + J(u(s,t))\frac{\partial u}{\partial t} = 0$$

for $u : \mathbb{R} \times [0,1] \to M$ subject to:

$$u(s,0) \in L_0,$$
$$u(s,1) \in L_1,$$
$$\lim_{s\to-\infty} u(s,t) = p, \lim_{s\to+\infty} u(s,t) = q.$$

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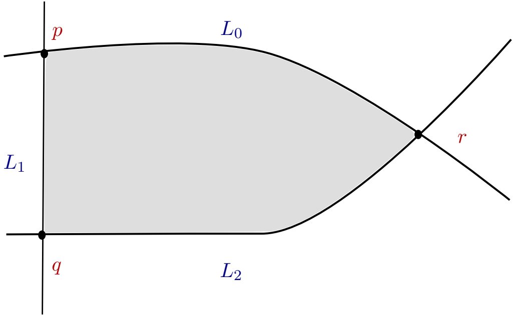
- Donaldson observed that there is a "composition"

$$[\mu^2] : HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2)$$

where the coefficient of r in $[\mu^2](p, q)$ is the number of holomorphic triangles u like this:



weighted by $q^{\omega}(u)$.

- As a result, the L_i are objects of the **Donaldson-Fukaya category**:

  Objects Unobstructed Lagrangians $L_i \subset M$.
  Morphism $Hom(L_i, L_j) = HF^*(L_i, L_j)$.

- Unfortunately, this does not capture the full story (e.g. insufficient for LES in Floer theory, functors,...)
- The solution are Fukaya categories - which keep track of the relations between the different $CF^*(L_0, L_1)$.
- There is a price to pay - Fukaya categories are **not**, well, categories,...
- The Donaldson-Fukaya category can be obtained from the Fukaya category by taking cohomology.

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Triangle product

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- We define the morphism spaces as the chain-complexes

$$Hom(L_i, L_j) := (CF^*(L_i, L_j), d)$$

We denote $\mu^1 = d$.

- Now, there is a bilinear map

$$\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$$

which is **not associative**.

- Instead, the associator satisfies

$$\mu^2(\cdot, \mu^2(\cdot,\cdot)) - \mu^2(\mu^2(\cdot,\cdot),\cdot) = \mu^3(\mu^1(\cdot),\cdot,\cdot) + \mu^3(\cdot,\mu^1(\cdot),\cdot) + \dots$$

, i.e. it is chain-homotopic to zero, and the homotopy defines another trilinear operation μ^3.

- **Q.** What does this hierarchy $\mu^1, \mu^2, \mu^3, \mu^4, ..., \mu^d, ...$ form??

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Let $\mathbb{K}$ be a field, and $\mathcal{A} = \mathcal{A}^\bullet$ a graded vector space.

- The *Hochschild cochains* of length $s \geq 0$ are

$$CC^{s+t}(\mathcal{A})^s := Hom^t(\mathcal{A}^{\otimes s}, \mathcal{A}),$$

and the Hochschild cochain complex is

$$CC^d(\mathcal{A}) := \prod_{d=s+t, s\geq 0} CC^{s+t}(\mathcal{A})^s.$$

- $\psi = \psi_0 + \psi_1 + \psi_2 + \ldots \in CC^d(\mathcal{A})$.
- It admits a *Gerstenhaber product* and *bracket*:

$$\phi \circ \psi(x_s, \ldots, x_1) := \sum_{i,j} \pm \phi(x_s, \ldots, \psi(x_{i+j}, \ldots, x_{i+1}), \ldots, x_1),$$

$$[\phi, \psi] := \phi \circ \psi \pm \psi \circ \phi.$$

An A_∞-**structure** is $\mu \in CC^2(\mathcal{A}), \mu^0 = 0$ such that $\frac{1}{2}[\mu, \mu] = 0$.

Explicitly, the first few equations $\mu = (0, \mu^1, \mu^2, \mu^3, \ldots)$ has to satisfy are exactly $d^2 = 0$, the Leibniz rule, and an associator relation:

$$\mu^1(\mu^1(x_1)) = 0$$
$$\pm \mu^2(x_2, \mu^1(x_1)) \pm \mu^2(\mu^1(x_2), x_1) = \mu^1(\mu^2(x_2, x_1))$$
$$\pm \mu^2(x_2, \mu^2(x_1, x_0)) \pm \mu^2(\mu^2(x_2, x_1), x_0) = \pm\mu^3(\mu^1(x_2), x_1, x_0) \pm \ldots$$

so we can define an associative cohomology algebra $H(\mathcal{A}, \mu)$.

Finally, we note that given any such μ, we can define a differential $\partial_\mu(\phi) = [\mu, \phi]$ on $CC^\bullet(\mathcal{A})$. Then $HH^\bullet(\mathcal{A}) := H(CC^\bullet(\mathcal{A}), \partial_\mu)$ is called **Hochschild cohomology**.

- **Meta-theorem (Fukaya):** making some choices, we can define an A_∞-structure μ^d on $Fuk(M)$ extending the Floer differential and triangle product.
- The category $(Fuk(M), \mu^d)$ is independent of all choices up to an A_∞-quasi-isomorphism.
- $Fuk(M)$ is the right setting to discuss various relations between Lagrangians, or rather, the split-closure derived category $D^\pi Fuk(M)$ which is a formal enlargement with objects

$$\{L_0 \to L_1 \to \ldots \to L_k\}.$$

- This is *much more* then a book-keeping device ...

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Supersymmetric string theory (very roughly...)

- Strings propagate in space-time $\Rightarrow$ span a world-sheet (=surface Σ).
- Come in two flavors: open string ($\partial\Sigma \neq \phi$) and closed string.
- Get a 2D QFT on Σ, with some fields taking value in space-time (=manifolds).
- The theory is required to be **supersymmetric** and **conformal**.
- Most common example is a nonlinear σ-model - where the input data is a **Calabi-Yau manifold**.
  1. (X, J) is a smooth, complex manifold.
  2. $\mathcal{K}_X := \bigwedge^n T^*X \cong_{holo} \mathcal{O}_X$, and thus there exists a nonvanishing holomorphic volume form Ω.
  3. $\omega^{\mathbb{C}} = B + i\omega \in \Omega^{1,1}(X)$ is a complexified Kähler form.

- To every such SCFT one can associate two $\mathcal{N} = 1$-supersymmetric TFT's: the A- and B-model.
- There exists super-symmetric operators $(Q, Q^\dagger)$ whose simultanuous eigenspaces are $H^q(\bigwedge^p TX)$ and $H^q(X, \Omega^p_X)$.
- If there exists another CY $(\hat{X}, \hat{J}, \hat{\omega}^{\mathbb{C}})$ such that

$$H^q(\bigwedge^p TX) \to H^q(X, \Omega^p_X)$$

and the TFT's are equivalent:

$$A(X) \cong B(\hat{X})\,,\ A(\hat{X}) \cong B(X).$$

We will say that X and $\hat{X}$ are **mirror symmetric**.
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- First thing we expect from CY mirrors:

$$H^q(X, \bigwedge^p T_X) = H^q(\hat{X}, \Omega^p_{\hat{X}}).$$

- We can always define a map

$$v_1 \wedge \ldots \wedge v_p \mapsto \iota_{v_1 \wedge \ldots \wedge v_p} \Omega.$$

- Because of the CY assumption this is an isomorphism

$$\bigwedge^p T_X \cong \Omega^{n-p}_X.$$

- So there is an equality

$$H^{n-p,q}(X) \cong H^{p,q}(\hat{X}).$$

Let (M, ω, J) be a Kähler manifold.

- **Gromov-Witten invariants** count the number of isolated rational curves in a given homology class, subject to some generic point constraints.
- **Examples:**
  1. How many lines $(d = 1)$ pass through two generic points? 1.
  2. How many conics $(d = 2)$ pass through five generic points? 1.
  3. Number of lines $(d = 1)$ on a quintic three-fold: 2875.
  4. Number of conics $(d = 2)$ on a quintic three-fold: 609250.
- **Gromov (1985)**. Gromov-Witten invariants are symplectic invariants! We can actually define GW invariants for any symplectic manifold...
- But as before, we must count **J-holomorphic curves**: solutions to a *perturbed* Cauchy-Riemann PDE (with a possibly non-integrable J!)

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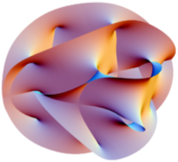
- Idea: we instead look at the "correlation functions" (=expectation values for observables). These will have a mathematical formulation. This defines an additional structure called **Yukawa coupling** $\langle \cdot, \cdot, \cdot \rangle$:
- On $H^1(X, T_X) \cong \mathrm{H}^{2,1}(X)$ this is connected to periods and the Gauss-Manin connection.
- On $H^{1,1}(X)$ this is the GW invariant!
- **Classical MS:** if X and $\hat{X}$ are mirrors then

$$(H^{1,1}(X), \langle \cdot, \cdot, \cdot \rangle_A) \cong (H^{2,1}(X), \langle \cdot, \cdot, \cdot \rangle_B)$$

Early success

- **Idea** (Candelas, de la Ossa, Green and Parkes, 1991): We can use the periods of the holomorphic volume form on the mirror to the to quintic three-fold predict Gromov-Witten invariants! (agreed with known results for $d = 1, 2, 3$).
- Proved mathematically by Givental and Lian-Li-Yau in many improtant cases (including all CY and Fano complete intersections in toric varieties)
- ... and so a new field in mathematics was born!



- Open strings propagate $\Rightarrow$ the worldsheet is a surface with boundary.
- Constraints on the values of the fields at the boundary are called "D-branes".
- Field theory axioms $\Rightarrow$ there is a category of D-branes.

  A-model branes are Lagrangian manifolds + flat bundles.
  B-model branes are complex analytic manifolds + holomorphic bundles.
- **Kontsevich (1994):** If X and $\hat{X}$ are mirrors, then

$$D^{\pi}Fuk(X) \cong D^{b}Coh(\hat{X}),$$
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- Morphism in $D^{b}Coh(X)$ are $\mathcal{H}om$'s and $\mathcal{E}xt$'s (mirror symmetry related the intersection theories!)

Why is that AWESOME

- Can now understand the conjecture is simple cases like surfaces (CY 3-folds are complicated!)
- Can (finally!) compute things on the A-side, find auto-equivalences on the B-side,...
- Open-string mirror symmetry implies closed-string mirror symmetry by taking Hochschild cohomology! (GPS,2016):

$$QH^*(X) \cong HH^*(D^\pi Fuk(X))$$
$$H^*(X, \wedge T_X) \cong_{HKR} HH^*(D^b Coh(X))$$

- $Fuk(X)$ and $Coh(X)$ turns out to be very important mathematical objects (Heegaard Floer theory, Seidel-Smith Khovanov homology, representation theory, birational geometry,...)

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Storminger-Yau-Zaslow conjecture (1996)

- The crucial missing bit: what does it mean for two manifolds to be mirror to each other geometrically? and when does it happen??
- A Lagrangian is **special** if $Im(\Omega|_L) = 0$.
- SYZ (very roughly): $X, \hat{X}$ are mirrors $\Rightarrow$ they carry mutually dual special Lagrangian torus fibrations

$$\mathbb{T}^n \to X \to B \, , \, \hat{\mathbb{T}}^n \to \hat{X} \to B,$$
$$\hat{\mathbb{T}}^n = Hom(\pi_1(T), U(1)).$$

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$\hat{X}$ is the moduli space of points of $\hat{X}$.

- Every $p \in \hat{X}$ yields a skyscraper sheaf $\mathcal{O}_p \in D^b Coh(\hat{X})$.
- By HMS, this corresponds to a Lagrangian L_p, such that

$$HF^*(L_p, L_p) \cong Ext^*(\mathcal{O}_p, \mathcal{O}_p) = \bigwedge \mathbb{C}^n$$

- Thus, as graded vector spaces,

$$H^*(L_p) \cong H^*(\mathbb{T}^n).$$

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