HMS Seminar - Talk 2

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Version 1.0 : Morse homology on the loop space

Version 1.5 : the Cauchy-Riemann equation

Version 2.0 : Lagrangian Floer homology

Version 3.0 : Coherence

Insightful, but not rigorous

Slogan : Floer homology is Morse homology for the symplectic action on the loop space.

- Let (M, ω) be a symplectic manifold.
- Let $L_0, L_1 \subset M$ be two Lagrangians which intersect transverse.
- We define

$$\tilde{P}_c(L_0, L_1) = \{(\gamma, [u])\}$$

  to be the universal cover of the space of contractible paths $\gamma : [0,1] \to M$ with $y(0) \in L_0$ and $y(1) \in L_1$.

- The symplectic action is the functional

$$\mathcal{A}(\cdot) : (\gamma, [u]) \mapsto \int_{D^2} u^* \omega.$$

- Consider a family of paths γ_s with $s \in (-\epsilon, \epsilon)$, and $\gamma_0 = \gamma$.
- Get a vector field

$$X = \frac{d}{ds}\gamma_s$$

along γ.
- We compute

$$\frac{\delta \mathcal{A}}{\delta \gamma} = \int_0^1 \omega(X, \dot{\gamma}) dt.$$

- **Corollary:** Critical points of $\mathcal{A}$ are constant paths.

- Assume γ is constant.
- We compute

$$\frac{\delta^2 \mathcal{A}}{\delta^2 \gamma} = \int_0^1 \omega(X, \dot{X}) dt.$$

- By the polarization identity:

$$Hess\mathcal{A}(X, Y) = \int_0^1 \omega(X, \dot{Y})$$

which is symmetric.

- Using the complex structure and the symplectic form, we define a metric on M:

$$g_J(X,Y) = \omega(X, JY)$$

- This induces a metric on vector fields along γ by setting

$$\langle X, Y \rangle = \int_0^1 g_J(X,Y)dt$$

- The gradient equation is:

$$\int_0^1 \omega(X, \dot{\gamma})dt = d\mathcal{A}_\gamma(X) = \langle X, Y \rangle = \int_0^1 \omega(X, J\nabla\mathcal{A}_\gamma)dt$$

- Thus we got

$$\nabla\mathcal{A}_\gamma = -J\dot{\gamma}.$$

The gradient equation is

$$\frac{\partial \gamma_s}{\partial s} + J\dot{\gamma} = 0.$$

Setting $u(s,t) = \gamma_s(t)$, we get the Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0.$$

Writing $Hess\mathcal{A}(X,Y) = \langle X, DY \rangle$, we see that the self adjoint operator whch represents the Hessian with respect to the metric is:

$$DY = -J\dot{Y}$$

which is the 1-dimensional Dirac operator $-i\frac{d}{dt}$.

To understand index, we must know how many eigenvalues change sign.

When trying to understand the spectral theory, we search for solutions

$$Y(t) = e^{At}Y_0\ ,\ DY = \lambda Y$$

- Let $\Lambda(n)$ be the Lagrangian Grassmannian in $\mathbb{C}^n$.
- Then $\Lambda(n) \cong U(n)/O(n)$, and hence $H^1(\Lambda(n);\mathbb{Z}) \cong \mathbb{Z}$.
- The generator μ is called the **Maslov class**.
- We have $\pi_1(U(n)/O(n)) \cong \mathbb{Z}$, and the explicit classifying map of μ is

$$U(n)/O(n) \xrightarrow{det^2} S^1$$

  which is a π_1-isomorphism.
- Moreover, let $\gamma = \{L_t\}$. Then

$$\langle \mu, \gamma \rangle$$

  is the winding number of $det^2 \circ \gamma$.

- **Arnold.** The Maslov class is the signed intersection number of the loop γ with the Maslov cycle: the set of Lagrangian planes in $\mathbb{C}^n$ which are not transverse to $\mathbb{R}^n$.
- **Remark.** given a trivial $\mathbb{C}^n$-bundle $E \to \mathbb{D}^2$ and a Lagrangian sub-bundle $F \to S^1$, the Maslov index is the obstruction to trivializing it.
- Let

$$L_0, \{L_1(t)\}_{0 \leq t \leq 1}$$

be Lagrangian subspaces in $\Lambda(n)$ with $L_0 \pitchfork L_1(0)$ and $L_0 \pitchfork L_1(1)$.
- The Maslov index is the number of times $L_1(t)$ fails to be transverse to L_0, counted with sign and multiplicity.

Outline for section 2

Version 1.0 : Morse homology on the loop space

Version 1.5 : the Cauchy-Riemann equation

Version 2.0 : Lagrangian Floer homology

Version 3.0 : Coherence

Let (M, ω) be a symplectic manifold.

- An **almost complex structure** J is an automorphism $J : TM \to TM$ such that $J^2 = -Id$.
  1. J is ω-compatible if $g_J := \omega(\cdot, J\cdot)$ is a metric.
  2. J is ω-tame if $\omega(v, Jv) > 0$ for every nonzero vector $v \in T_pM$.
- Let (Σ, j) be a Riemann surface and (X, J) almost complex. A **pseudo-holomorphic curve** is a smooth map $u : \Sigma \to X$ such that

$$\overline{\partial}(u) = \frac{1}{2}(du + J \circ du \circ j) = 0.$$

- In conformal coordinates $s + it$ on the surface, this is equivalent to the usual CR-equation

$$\partial_t u = J(u)\partial_s u$$

but J is not constant.

- For ω-compatible almost complex structures we define the energy

$$E(u) := \frac{1}{2}\int_S |du|^2 dvol_S$$

- The energy density depends on the metric, but the energy depends only on j, J, ω, since in local coordinates

$$|\eta|^2 dvol_S = \langle \eta \wedge *\eta \rangle_{g_J}.$$

- **The energy identity.**

$$E(u) = \int_S |\overline{\partial}_J u|^2 dvol_S + \int_S u^* \omega dvol_S$$

- Works with Lagrangian boundary conditions $u\big|_{\partial S} \in L_i$ as well.
- Thus,
  1. J-curves of fixed homology have fixed energy.
  2. Nullhomologous curves are constant.
  3. J-curves are harmonic maps.

(Sacks-Uhlenbeck-)Gromov compactness

Let (S, j) be a RS, (M, ω) compact symplectic manifold and $J_i \to J_\infty$ are domain-dependent almost complex structures.

- Any sequence of holomorphic curves $u_i : S_i \to M$ with bounded energy has a subsequence which "Gromov-converges" to some **stable** J_∞-holomorphic $u_\infty : S_\infty \to M$:

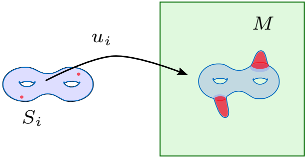


Figure: Energy is concentrating at two points

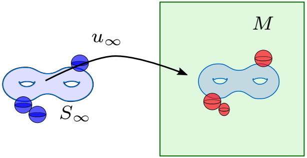


Figure: The limit curve recaptures lost energy

- All marked points and nodes are distinct on the domain.

Consider the sequence of holomorphic maps

$$u_n : \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$$
$$[x_0, x_1] \mapsto ([x_0, x_1], [nx_0, x_1])$$

- In affine charts, $u_n : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ given by

$$u_n(x) = (x, \frac{1}{nx}) \ , \ x = \frac{x_1}{x_0}.$$

- Away from zero, looks like uniform convergence to $\{(x, 0)\}$.
- But if we reparametrize $y = nx$, then we get

$$u_n(y) = (\frac{y}{n}, \frac{1}{y}).$$

- Away from $y = \infty$, this converges uniformly to $(0, \frac{1}{y})$ - the second coordinate axis.
- The Gromov limit is a map from a **nodal** sphere!

- Identify areas where the gradient blows-up $sup|du_i| \to \infty$. Away from these points, standard analytic estimates + elliptic bootstrapping force converge on compact subsets to a J_∞-holomorphic map.
- If we have a sequence of points $z_i \in S_i$ where the gradient blows-up, with limit z_∞ an interior point, then we can rescale $(\epsilon_i \to 0)$

$$v_i(z) := v_i(z_i + \epsilon_i z)$$

  so near this points so that the derivative does not blow-up anymore.
- A subsequence of v_i converges to a map $v_\infty : \mathbb{C} \to M$.
- By a **removal of singularity** for J-holomorphic maps, this extends to a map $\mathbb{CP}^1 \to M$ called a **sphere bubble**.

Idea of Proof (continued)

- The same logic applies to the case that z_∞ is a boundary point, gives a map $v_\infty : \mathbb{H} \to M$ and a **disc bubble**.
- Some work is needed to show this actually converges to a stable bubble tree and that homology and energy are all preserved in the limit.
- **Theorem.** If u is not constant and J-holomorphic then

$$E(u) \geq \hbar(M, \omega, J, L) \geq 0.$$

- The process is finite because of the apriori bound and quantization of energy!

Outline for section 3

Recall: The A-model (the symplectic side) in HMS should be the **Fukaya category** $Fuk(M, \omega)$.

- Objects in this new category are compact **Lagrangians** $+(...)$: half-dimensional submanifolds $L^n \subset M^{2n}$ on which $\omega\big|_L$ vanishes.
- It is defined over a Novikov field, whose elements are e.g. formal sums of the form $\sum_j c_j T^{r_j}$ with $r_j \to \infty$.
- When $L_0 \pitchfork L_1$, the morphism space $CF^\bullet(L_0, L_1)$ is freely generated over Λ by the intersection points in $L_0 \cap L_1$.

- There is a differential on the morphism spaces,

$$\mu^1 : CF(L_0, L_1) \to CF(L_0, L_1).$$

- For any $p, q \in L_0 \cap L_1$, the coefficient of q in $\mu^1(p)$ is defined by counting rigid J-holomorphic strips $u : \mathbb{R} \times [0, 1] \to M$ with boundary conditions like in the diagram:

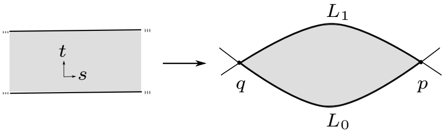


Figure: $\overline{\partial}_J(u) = \partial_s u + J(u)\partial_t u = 0$, $E(u) < \infty$

For every $d \geq 2$, there are multi-linear operation of degree $2-d$ on the morphism spaces,

$$\mu^d : CF(L_d, L_{d-1}) \otimes \ldots \otimes CF(L_1, L_0) \to CF(L_d, L_0).$$

For any $p_d, \ldots, p_1$, the coefficient of $p_0 \cdot T^{\omega(\beta)}$ in $\mu^d(p_d, \ldots, d_1)$ is defined by counting rigid J-holomorphic polygons u with homotopy class $[u] = \beta$ and boundary conditions specified by the diagram:

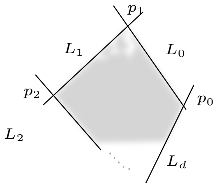


Figure: J-holomorphic polygon

Very schematically (see diagram on blackboard):

- Let $f_i : P_i \to X$ be "chains" of dimension $\dim(X) - d_i$.
- We consider the fiber product

$$R := \overline{\mathcal{M}}_k \times_{f_1,\ldots,f_k} (P_1 \times \ldots \times P_k)$$

  over X^k.
- $\pi_2 = ev_0 : \overline{\mathcal{M}}_k \to X$ induces a smooth map ev_0 from the manifold fiber product.
- We now put

$$\mu^k(P_k, \ldots, P_1) = (ev_0)_*(R)$$

  and use Poincaré duality to identify the resulting chain with a cochain.

Problems with the definition

- **Grading.** need to assign degrees so that the indices work out.
- **Compactness.** are the numbers we are counting finite?
- **Smooth structure.** how can we show that the manifold is smooth and of the expected dimension?
- What happens when L_0 and L_1 are NOT transverse?!

- Given a strip $u : S := Z \to (M, L_0, L_1)$, trivialize

$$u^*TM \cong \mathbb{Z} \times \mathbb{C}^n.$$

- Get u^*TL_0, u^*TL_1 paths of Lagrangians along $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$, respectively.
- We can further trivialize so that $u^*TL_0 \cong \mathbb{R} \times \{0\} \times \mathbb{R}^n$ is constant.
- Then $ind([u])$ is the Maslov index of the path TL_1 relative TL_0 as one goes from p to q.

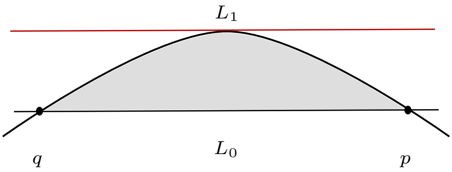


Figure: Fails to be transverse once $\Rightarrow$ the index $ind([u]) = 1$.

- Want: a way to assign to every $p \in L_0 \cap L_1$ a degree, $\deg(p)$ such that $\deg(q) - \deg(p) = ind([u])$ for every $u \in \pi_2(p, q)$ between them.
- That is impossible! Sources of ambiguity:
  1. **Connect sum with a disc** $v : (D^2, S^1) \to (M, L_i)$**.**

$$ind([u\#v]) = ind([u]) + \mu(v).$$

  2. **Connect sum with a sphere** $w : S^2 \to M$**.**

$$ind([u\#v]) = ind([u]) + 2\langle c_1(TM), [w]\rangle.$$

- Conclusion: we can only associate relative grading in $\mathbb{Z}/(N \cdot \mathbb{Z})$, where $N \cdot \mathbb{Z}$ is the group generated by the ambiguity terms.

Absolute grading (Kontsevich, Seidel)

Idea: resolve ambiguity by additional information. For simplicity, assume $c_1(M) = 0$ and there exists a global holomorphic volume form Ω.

- Let $\mathcal{L}$ be the bundle of linear Lagrangian subspaces of (TM, ω).
- As a consequence there exists a global **classical phase function**

$$\alpha : \mathcal{L} \to S^1 \times M$$
$$\alpha_x(\Lambda) := \frac{\Omega(e_1 \wedge \ldots \wedge e_n)^2}{|\Omega(e_1 \wedge \ldots \wedge e_n)^2|}.$$

where $e_1, \ldots, e_n$ is a choice of basis for $\Lambda \in \mathcal{L}_x$.

- We define a **grading** of L is defined as a choice of lift of the map

$$L \to \mathcal{L}|_{\mathcal{L}} \to S^1$$

to $L \to \mathbb{R}$.

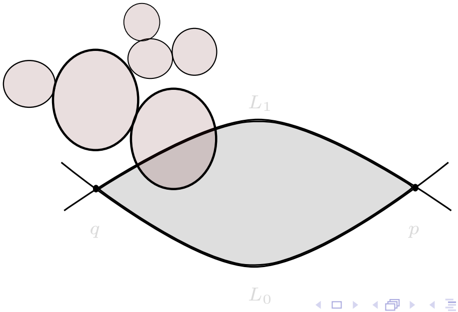
- We call the pair $\tilde{L} = (L, \tilde{\alpha}_L)$ a graded Lagrangian.
- **Remark.** There is a $\mathbb{Z}$-worth of choices of grading, which correspond in the Fukaya category to shifts, i.e.

$$\tilde{L}[k] = (L, \tilde{\alpha}_L + k).$$

- Given $p \in \tilde{L_0} \cap \tilde{L_1}$. In order to define the degree of p we:
  1. Choose a "canonical short path" (which means no crossing) in $\mathcal{L}_p$ between T_pL_0 and T_pL_1.
  2. Lift $\alpha(\Lambda_t)$ to $\tilde{\alpha}(\Lambda_t)$.
  3. The degree of the intersection point is the difference

$$\deg(p) := (\tilde{\alpha_1}(p) - \tilde{\alpha_1}) - (\tilde{\alpha_0}(p) - \tilde{\alpha_0}).$$

- Gromov compactness works with Lagrangian boundary conditions as well.
- We can define a topology on $\overline{\mathcal{M}}$ by knowing which sequences converge.
- **Fact.** Get compact, metrizable, Hausdorff topology on $\overline{\mathcal{M}}$!
- The structure of the bordification:
  1. **Sphere bubbling.** In good cases, happens in codimension two (or not at all). Image of multiple covers should cover via the simple ones.



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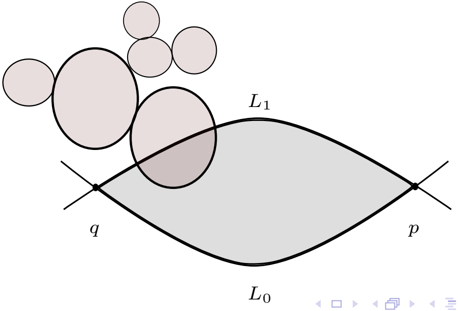
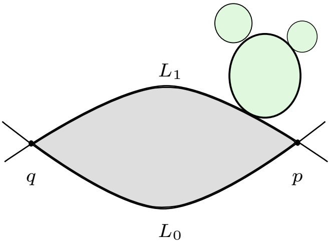
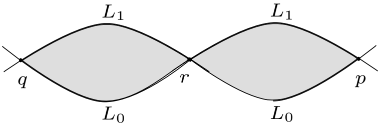


2 **Disc bubbling.** Serious issue! Happens in codimension one.



2 **Strip breaking.** Energy escapes to infinity (analogue: Morse theory)



Regularization: what does NOT work

Intuitively, we expect

- An implicit function theorem to equip $\overline{\mathcal{M}}$ with a smooth structure whenever the section σ is transverse to the zero section of $\mathcal{E}$;
- We hope to achieve this transversality by some dense set of perturbations of σ, with the resulting regularized moduli space essentially independent of this choice.
- **Finite dimensional Regularization.** Let $\mathcal{E} \to \mathfrak{B}$ be a smooth *finite dimensional* vector bundle, and let $s : \mathfrak{B} \to \mathcal{E}$ be a smooth section such that $\overline{\mathcal{M}} := s^{-1}(0)$ is compact. Then there exist arbitrarily small, compactly supported, smooth perturbation sections $p : \mathfrak{B} \to \mathcal{E}$ such that $s + p$ is transverse to the zero section, and hence $\overline{\mathcal{M}}_p := (s + p)^{-1}(0)$ is a smooth manifold. Moreover, the perturbed zero sets $\overline{\mathcal{M}}_{p_0}$ and $\overline{\mathcal{M}}_{p_1}$ of any two such perturbations p_0, p_1 are cobordant.

Geometric regularization paradigm (McDuff-Weirheim)

Step 1 (Banach setup). Set up the PDE (e.g. gradient flow equation) as smooth section σ : of a Banach space bundle $\mathcal{E} \to \mathfrak{B}$ over a Banach manifold $\mathfrak{B}$ of maps.

- The section should be **Fredholm** in the sense that the linearizations

$$D_b\sigma : T_b\mathfrak{B} \to \mathcal{E}_b$$

  at zeros $b \in \tilde{\mathcal{M}} := \sigma^{-1}(0)$ are Fredholm operators, and also equivariant under the action of Aut.
- The uncompactified moduli space is given as quotient of the zero set

$$\mathcal{M} = \tilde{\mathcal{M}}/Aut.$$

- Remark: should probably work in some "rougher space" that desired (like Sobolev spaces, Hölder spaces,...)

Toy model: Morse-Smale-Witten complex

- The base is

$$\mathfrak{B} = \Big\{\eta \in C^1(\mathbb{R}, \gamma^* TX) \mid \lim_{s\to\pm\infty} \gamma(s) \in \mathfrak{crit}(f)$$
$$\text{and } \lim_{s\to\pm\infty} |\dot{\gamma}(s)| = 0,$$

- The fiber of $\mathcal{E}$ are

$$\mathcal{E}_\gamma = \left\{\eta \in C^0(\mathbb{R}, \gamma^* TX) \mid \lim_{s\to\pm\infty} |\eta(s)| = 0\right\},$$

- The Banach section

$$\sigma(\gamma) = \dot{\gamma} + \nabla_f(\gamma).$$

- Even in this simple case, can check that re-parametrization action

$$\tau : \mathbb{R} \times C^1(\mathbb{R}, \gamma^* TX) \to C^1(\mathbb{R}, \gamma^* TX)\,,\ \tau(t, \gamma) := \gamma(t + \cdot).$$

is **not differentiable**!

- The base $\mathfrak{B}$ is taken to be the space of maps $W^{k,p}(S, M)$, with boundary conditions and asymptotics.
- Note that one can not work with maps from Deligne-Mumford space (there is NO such Banach manifold!)
- The fiber of $\mathcal{E}$ are

$$\mathcal{E}_u = \left\{ S, \Omega^{0,1} \otimes u^* TM \right\},$$

- The section σ is $\overline{\partial}_J(u)$, and is Fredholm when $L_0 \pitchfork L_1$ (Floer).

Step 2 (Space of perturbations). Find a family of smooth sections $(p : \mathfrak{B} \to \mathcal{E})_{p \in \mathcal{P}}$ parametrized by a Banach manifold $\mathcal{P}$. For each $p \in \mathcal{P}$, we denote $\sigma_p := \sigma + p$. Then we need the following properties to hold:

1. the perturbed solution space $(\sigma_p)^{-1}(0)$ is invariant under the action of Aut.
2. For each $p \in \mathcal{P}$ the perturbed solution space $(\sigma_p)^{-1}(0)$ has the same compactification properties as the unperturbed space $\sigma^{-1}(0)$.
3. The "universal moduli space"

$$\tilde{\mathcal{M}}_{univ} := \{(b,p) \in \mathfrak{B} \times \mathcal{P} \,|\sigma_p(b) = 0\,\}$$

is cut out transversely and has the structure of a Banach manifold.

Example: For Morse theory, the perturbations could be

$$p(\gamma) = \nabla f(\gamma) - \nabla' f(\gamma)$$

where ∇' is the gradient with respect to another metric g'.

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In Floer theory

The first and third requirement combined with some linear algebra$+$ unique continuation $\Rightarrow$ suffice to have injectivity of u and nonvanishing of du somewhere along almost every Aut-orbit in Σ.

- **Floer.** under nice conditions (i.e., no bubbles and homology class $\beta \neq 0$) the third requirement always holds for Floer trajectories.
- In contrast, for J-holomorphic curves, we must restrict to $\mathfrak{B}^*$ which consists of simple J-holomorphic curves (i.e., not multiply covered).
- Similarly for discs, but now nowhere injectivity does not imply multiply covered!

Step 3 (Sard-Smale theorem). This is automatic.

- Given a family of perturbations $\mathcal{P}$ as described, the Sard-Smale theorem guarantees a comeagre set $\mathcal{P}_{reg} \subset \mathcal{P}$ of regular values of the canonical projection

$$pr : \tilde{\mathcal{M}}_{univ} \to \mathcal{P}.$$

- Moreover, for $p \in \mathcal{P}_{reg}$ the perturbed section σ_p is transverse to the zero section, yet it is still Aut-equivariant.
- Hence, by the implicit function theorem, $\sigma_p^{-1}(0)$ is a smooth submanifold of finite dimension on which Aut acts, and the dimension is given by the Fredholm index.
- **Example:** For Morse theory, this would pick out metrics that satisfy the Morse-Smale condition.

Step 4 (Quotient). Check that the action of Aut on

$$\tilde{\mathcal{M}}_p := \sigma_p^{-1}(0)$$

is smooth, free, and properly discontinuous.

- Then the moduli space

$$\mathcal{M}_p := \tilde{\mathcal{M}}_p / Aut$$

is a smooth manifold.

Step 5 (gluing). Construct a gluing map

$$\oplus : (R_0, \infty) \times \sigma_p^{-1}(0) \tilde{\times} \sigma_p^{-1}(0) \hookrightarrow \sigma_p^{-1}(0)$$

that is an embedding.

- The construction of $\oplus$ involves a **pre-gluing map**

$$\tilde{\oplus} : (R_0, \infty) \times \sigma_p^{-1}(0) \tilde{\times} \sigma_p^{-1}(0)$$

  and an implicit function theorem to determine exact solutions.
- Need ambient metric topology on the compactified moduli space...
- This technique is usually **only** applied to glue 0-dimensional components or compact subsets of the fiber product! Otherwise, need coherence ...

The remaining steps (we skipped **Step 6** ... we'll get back to it later):

- **Step 7 (compactness).** Check that the complement of the gluing image, is compact.
- **Step 8 (invariance).** Prove that the algebraic structures (e.g. the Morse chain complex) arising from different choices in the previous steps, in particular the choice of perturbation, are equivalent in an appropriate sense.
- This usually involves constructing a cobordism and a homotopy-of-choices argument.

Pointed-Boundary Riemann surface

To make things more rigorous, we define the domain of our operations to be a **Pointed-Boundary Riemann surface**:

- A Riemann surface $\hat{S}$ with a nonempty boundary ∂S, with a collection of marked points

$$\Sigma = \Sigma^- \cup \Sigma^+ \subset \partial \hat{S}$$

  divided into **incoming** and **outgoing**.
- We denote $S = \hat{S} \backslash \Sigma$.
- Examples include $D \subset \mathbb{C}$ the closed unit disc, and
  1. $Z \subset \mathbb{R} \times [0, 1]$ with coordinates (s, t), an incoming point $s = -\infty$ and an outgoing point $s = +\infty$.
  2. $Z^{\pm} = \mathbb{R}^{\pm} \times [0, 1]$ the incoming and outgoing half-infinite strips.
- A set of **Lagrangian labels** for S is a family of Lagrangian submanifolds $\{L_C\} \subset M$, indexed by arcs $C \in \pi_0(\partial S)$.

This is nice, but we need to rigidify it a bit.

- A set of **strip-like ends** for S consists of proper holomorphic embeddings

$$\epsilon_\zeta : Z^\pm \to S$$

one for each $\zeta \in$, satisfying

$$\epsilon_\zeta^{-1}(\partial S) = \mathbb{R}^\pm \times \{0,1\},$$
$$\lim_{s \in \pm\infty} \epsilon_\zeta(\cdot, s) = \zeta.$$

- Needed for:
  1. Analytic framework, since the $W^{1,p}$-norms ($p > 2$) are not conformally invariant.
  2. Standard coordinates on the ends appear when we define the inhomogeneous term in the Floer equation.
  3. Makes gluing more precise.

Equip the surface with a choice of Lagrangian labels.

- We are simply going to solve the transversality problem by moving the Lagrangians around.
- For any pair of Lagrangians $L_0, L_1 \subset M$, a choice of **Floer datum** consists of $J \in C^\infty([0,1], \mathcal{J})$, $H \in C^\infty([0,1], \mathcal{H})$with the following property: if X is the time-dependent Hamiltonian vector field of H and ϕ^1 its "flow", then $\phi^1(L_0)$ intersects L_1 transversally.
- Define $\mathcal{C}(L_0, L_1)$ to be the set of **Hamiltonian chords**: maps $y : [0,1] \to M$ such that $y(0) \in L_0, y(1) \in L_1$ and

$$\frac{dy}{dt} = X(t, y(t)).$$

- Chords correspond bijectively to points in $\phi^1(L_0) \cap L_1$, hence there are only finitely many of them!

Denote $\mathcal{J}$ the space of ω-compatible almost complex structures, and $\mathcal{H}$ the space of Hamiltonians $H : M \to \mathbb{R}$.

- We choose:
  1. 1-form α that coincides with dt on the strip-like ends.
  2. An S-dependent almost complex structure, J,
  3. An S-dependent Hamiltonian function $H : S \times M \to \mathbb{R}$.

  Both (J, H) are required to coincides with the Floer data on the strip-like ends. That is, for every puncture ζ:

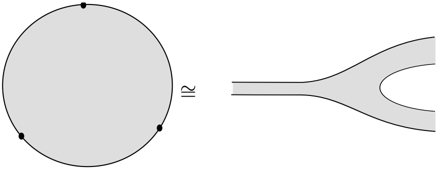
$$\epsilon_i^* J(s,t) = J_{\zeta_i}(t)$$
$$\epsilon_i^* H(s,t) = H_{\zeta_i}(t)$$

For illustration, let's start by looking at μ^2.

- Assume we have fixed choices and defined

$$CF^*(L_0, L_1) = CF^*(L_0, L_1; J_{L_0,L_1}, H_{L_0,L_1})$$
$$CF^*(L_1, L_2) = CF^*(L_1, L_2; J_{L_0,L_1}, H_{L_0,L_1})$$
$$CF^*(L_0, L_2) = CF^*(L_0, L_2; J_{L_0,L_1}, H_{L_0,L_1})$$

- The product should be defined by solving a PDE with domain

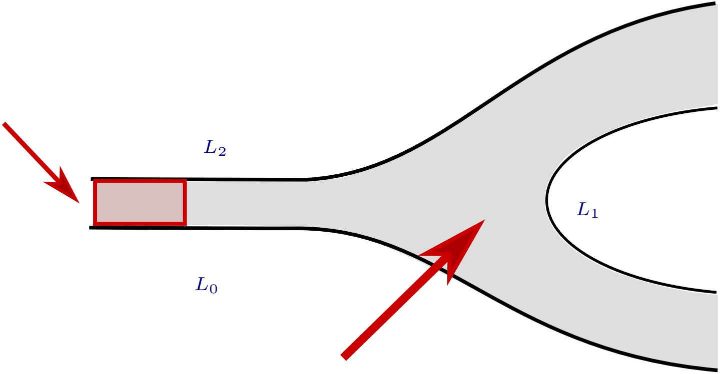


Inhomogeneous Floer equation (picture)

$dt = \epsilon_0^* \alpha$

$J_{L_0 L_2 = \epsilon_0^* J}$

$H_{L_0 L_2 = \epsilon_0^* H}$



(α, J, H)

Inhomogeneous Floer equation

Consider the moduli space

$$\mathcal{M}_{\dots}(\zeta_0; \zeta_1, \zeta_2)$$

consisting of solutions $u : S \to M$ to the elliptic PDE with boundary conditions:

$$\begin{aligned} (du - X_H \otimes \alpha)^{0,1} &= 0 \\ u\big|_{\partial^i S} &\in L_i \\ \lim_{s \to +\infty} (\epsilon_i^+)^* u(s,t) &= \zeta_i \\ \lim_{s \to -\infty} (\epsilon_i^-)^* u(s,t) &= \zeta_0 \end{aligned}$$

Remark: the anti-holomorphic part is taken with respect to J.

Properties: smooth structure

In good cases (e.g. no sphere or disc bubbling), and for generic choice of perturbation data:

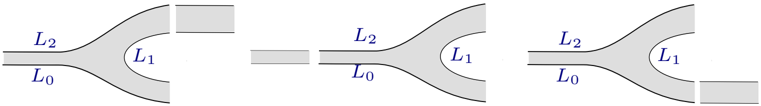
- $\mathcal{M}(\zeta_0; \zeta_1, \zeta_2)$ is a manifold of dimension

$$\deg(\zeta_0) - \deg(\zeta_1) - \deg(\zeta_2).$$

- Follows from transversality + index calculation.

Properties: bordification

- When the expected dimension is zero, $\mathcal{M}$ is a finite collection of points.
- When it is one, $\mathcal{M}$ is a 1-manifold with ends which correspond bijectively to 3 types of codim-1 boundaries:



- Gives the graded Leibniz relation:

$$\mu^2(\mu^1(\cdot),\cdot) \pm \mu^2(\cdot,\mu^1(\cdot)) \pm \mu^1(\mu^2(\cdot,\cdot)) = 0$$

which is part of the A_∞-relations.

Properties: canonical orientation

- After fixing spin structures, $\mathcal{M}$ is canonically oriented "relative the orientation of the ends". That is, there exists a canonical isomorphism

$$\lambda(T\mathcal{M}) \cong \mathfrak{o}_{\zeta_0} \otimes \mathfrak{o}_{\zeta_1}^{\wedge} \otimes \mathfrak{o}_{\zeta_2}^{\wedge}$$

- Thus, every rigid element ϕ defines a map of orientation lines

$$\mu_\phi : \mathfrak{o}_{\zeta_2} \otimes \mathfrak{o}_{\zeta_1} \to \mathfrak{o}_{\zeta_0}.$$

- Thus, we can (finally!) define

$$\mu^2([\zeta_2],[\zeta_1]) = \sum_{\zeta_0,\beta} \#\mu_\phi([\zeta_2],[\zeta_1]) \cdot q^{\omega(\beta)} \cdot (-1)^{\deg(\zeta_1)}.$$

where we sum over all classes such that the expected dimension is zero (hence all elements in $\mathcal{M}$ are rigid), and $[\zeta_i]$ are the generators of the orientation lines.

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This is a convenient formalism to discuss operations arising from moduli spaces.

- A (non-unital, non-symmetric) **operad** $\mathcal{P}$ of vector spaces consists of a collection of vector spaces

$$\{\mathcal{P}(n)\}_{n\geq 0}$$

as well as composition maps

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(n+m-1)\,,\; i = 1, \ldots, m$$

satisfying certain axioms.
- Think of $\mathcal{P}(n)$ as the factory which contains all the operations with m inputs and one output... this is just associtivity.

- An **algebra** V over an operad $\mathcal{P}$ is a vector space V with maps

$$\Phi_m : \mathcal{P}(m) \to Hom(V^{\otimes m}, V)$$

compatible with composition such that for all $\phi \in \mathcal{P}(m), \psi \in \mathcal{P}(n)$ we have

$$\Phi_{n+m+1}(\phi \circ_i \psi)(v_1, \ldots, v_{n+m-1}) =$$
$$\Phi_m(\phi)(v_1, \ldots, v_{i-1}, \Phi_n(v_i, \ldots, v_{i+n-1}), \ldots, v_{n+m-1}).$$

- Analogy: Operad $\Rightarrow$ Algebra $\Rightarrow$ Module.

Example: $\mathcal{P} = Ass$

- Let **k** denote the ground field.
- Consider the operad $\mathcal{P}$ defined by: $\mathcal{P}(0) = \mathcal{P}(1) = \{0\}$, and $\mathcal{P}(m) = \mathbf{k}$ when $m \geq 2$; and $\circ_i$ are always the identity for all values of i, m, n.
- What is an algebra over $\mathcal{P}$?
- **Claim.** This is just the usual notion of an associative algebra over the field **k** with bilinear multiplication

$$\Phi_2(1) \in Hom(V^{\otimes 2}, V)$$

- **Proof.** Follows easily from

$$\Phi_2(1)(\Phi_2(1)(a, b), c) = \Phi_3(1)(a, b, c) = \Phi_2(1)(a, \Phi_2(1)(b, c)).$$

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- There is of course no reason to restrict ourselves to vector spaces ...
- A **topological operad** is the same definition as before, except V is a "CW-complex" and not a "vector space" and the maps are required to be "cellular" instead of "linear".
- Operads that are valued in chain complexes are called **DG-operads**. The structure maps are required to be chain maps etc.
- The same generalizes easily to algebras over such operads.
- Given any topological operad $\mathcal{O}$, we can take cellular chains to obtain a DG-operad

$$\mathcal{P}(m) := C_*^{cell}(\mathcal{O}(m)).$$

- $\mathcal{K}_m$ is the polytope with one vertex for every way of parenthesizing the m elements $\{a_1, \ldots, a_m\}$. Thus:
  1. There is an edge between two vertices if one expression can be reached from the other by changing one pair of parentheses,
  2. There is a 2-cell relating expressions that can be reached from each other by two re-parenthesizings, etc...
- $\mathcal{K}_\bullet$ is called the topological A_∞-operad. An algebra over it is an A_∞-space.
- When connected, any A_∞-space Y is homotopy equivalent to a loop space ΩX (Stasheff).

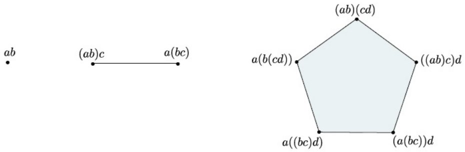

Figure: $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$

- **Claim.** DG-algebras over the DG-operad

$$\mathcal{P}(m) := C_*^{cell}(\mathcal{K}_m)$$

are exactly A_∞-algebras.

- **Proof.** Given such an algebra (A, d), the differential is denoted μ^1, and the action of the fundemental class $[\mathcal{K}_m]$ is denoted μ^m. But

$$d[\mathcal{K}_s] = \sum_{i,j\geq 2,k\geq 2} [\mathcal{K}_j] \circ_i [\mathcal{K}_k]$$

and this translates that to the A_∞-equations:

$$\sum_{j \text{or} k=1} \mu^j(..., \mu^k(...), ...) = \sum_{j,k\geq 2} \mu^j(..., \mu^k(...), ...)$$

Operadic actions from geometry

- Basic situation: a family of diagrams

$$
\begin{array}{ccc}
 & \mathcal{O}_m & \\
 & \uparrow & \\
X \longleftarrow & \overline{\mathcal{M}}_m & \longrightarrow X^m
\end{array}
$$

with a family of composition operations

$$\circ_i : \overline{\mathcal{M}}_k \times_{\dots} \overline{\mathcal{M}}_l \to \overline{\mathcal{M}}_{k+l-1}$$

Operadic actions from geometry (continued)

- These are required to satisfiy the following relations
  1. **Operadic action axiom.** This is a pullback diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_k \times \overline{\mathcal{M}}_l & \longrightarrow & \overline{\mathcal{M}}_{k+l-1} \\ \downarrow & & \downarrow \\ \mathcal{O}_k \times \mathcal{O}_l & \longrightarrow & \mathcal{O}_{k+l-1} \end{array}$$

  2. **Maurer-Cartan axiom.** The boundary $\partial\overline{\mathcal{M}}_k$ is covered by the images of the lower moduli spaces.
  3. ...
- The same integration process as before defines multi-linear operations.
- The axioms ensure that they assemble to an operad action.
- Problem: in order to regulate the moduli space, we had to PERTURB.

- A **pointed-boundary disc**, $d \geq 0$, is a pointed-boundary Riemann surface S whose compactification $\hat{S}$ is a disc, and which has one incoming point (ζ_0) at infinity and d outgoing ones $(\zeta_1, \ldots, \zeta_d)$.
- Let

$$Conf_{d+1}(\partial \mathbb{D}^2)$$

  be the configuration space of $d+1$-tuples of points on the circle whose numbering is compatible with their cyclic order.
- $Aut(\mathbb{D}) \cong PSL_2(\mathbb{R})$ acts freely and properly on this configuration space. space.
- In the stable range $d \geq 2$, d-pointed discs have a trivial stablizer, and there is a universal family of them, denoted by

$$\mathcal{S}_d \to \mathcal{R}_d := Conf_{d+1}(\partial \mathbb{D}^2)/Aut(\mathbb{D})$$

- The moduli space $\mathcal{R}_d$ of conformal structures on the disc admits a natural compactification to a $(d-2)$-dimensional polytope $\overline{\mathcal{R}}_d$, whose top-dimensional facets correspond to nodal degenerations of D to a pair of discs $D_1 \cup D_2$, with each component carrying at least two of the marked points.

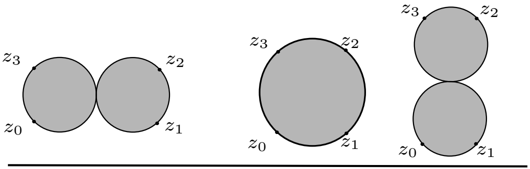

Figure: The 1-dimensional associahedron $\mathcal{R}_4$

- This is yet another incarnation of the A_∞-operad $\mathcal{K}_\bullet$.

There is an underlying combinatorical structure of a tree that gives "gluing instructions".

- Given two marked discs S_1 and S_2, and assuming we have made a choice of strip-like ends, and given ζ_1^+, ζ_2^- and a gluing length $l > 0$, set

$$S_1' = S_1 \setminus \epsilon^+_{\zeta_1}((l, +\infty) \times [0,1]),$$
$$S_2' = S_2 \setminus \epsilon^-_{\zeta_2}((-\infty, l) \times [0,1]),$$
$$S := S_1' \cup S_2' / \sim .$$

- The compactification correspondes to some $l = 0$.
- Can use that to give $\overline{\mathcal{R}}_d$ the structure of a manifold with boundary and corners (associtive gluing maps!)
- Every nodal disc inherits a "thick-thin" decomposition (has nothing to do with the one from hyperbolic geometry).

Universal and consistent choices

- We make a universal and consistent choice strip-like ends (can't do it for interior marked points!)
- We make a universal choice of perturbation data (α, J, H), inductive in d, and for every family of Lagrangian labels $(L_0, \dots, L_d)$.
- It should be smoothly varying in $\mathcal{S}$ and **consistent**: which means the restriction to corner strata agrees with the pre-assigned choice, and smoothly varying with respect to the corner charts.
- **Proposition**. Such choices exist because of the contractibility of the associahedron.
- Construct Floer moduli space for entire families

$$\mathcal{M}_{\mathcal{S}\to\mathcal{R},\dots}$$

  simultaneously.
- Modify an existing choice to achieve transversality in the same way as before.