‘Shape spaces’ alias ‘Moduli spaces in the differentiable category’

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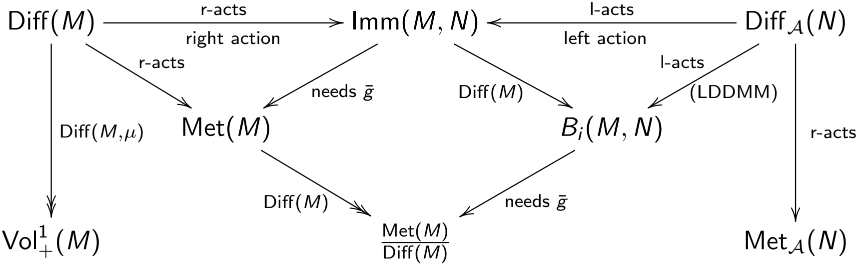
Dramatis personae: A diagram of actions of diffeomorphism groups.



M compact , N possibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar{g}$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M,N)$	mf of all immersions $M \to N$
$B_i(M,N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities

At the very birth of the notion of manifolds, in his Habilitationsschrift 1854, Riemann mentioned infinite dimensional manifolds explicitly:

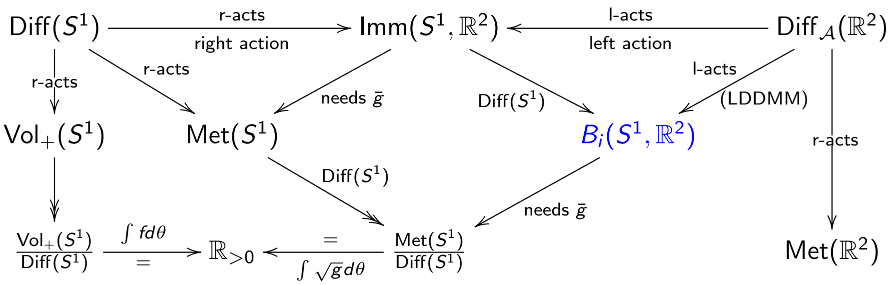
> *"Es giebt indess auch Mannigfaltigkeiten, in welchen die Ortsbestimmung nicht eine endliche Zahl, sondern entweder eine unendliche Reihe oder eine stetige Mannigfaltigkeit von Grössenbestimmungen erfordert. Solche Mannigfaltigkeiten bilden z.B. die möglichen Bestimmungen einer Function für ein gegebenes Gebiet, die möglichen Gestalten einer räumlichen Figur u.s.w."*

Translation into English by W.K. Clifford 1873:

> *"There are manifoldnesses in which the determination of position requires not a finite number, but either an endless series or a continuous manifoldness of determinations of quantity. Such manifoldnesses are, for example, the possible determinations of a function for a given region, the possible shapes of a solid figure, &c."*

If one reads this with a lot of good will one can interpret it as follows: When Riemann sketched the general notion of a manifold, he also forsaw the notion of an infinite dimensional manifold of mappings between manifolds, and of a manifold of shapes. He then went on to describe the notion of Riemannian metric and to talk about curvature.

The diagram in a simpler situation



$\mathrm{Diff}(S^1)$	Lie group of all diffeos on compact mf S^1
$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2)$, $\mathcal{A} \in \{\mathcal{B}, H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to $\mathrm{Id}_{\mathbb{R}^2}$
$\mathrm{Imm}(S^1, \mathbb{R}^2)$	mf of all immersions $S^1 \to \mathbb{R}^2$
$B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}/\mathrm{Diff}(S^1)$	shape space
$\mathrm{Vol}_+(S^1) = \left\{f\, d\theta : f \in C^\infty(S^1, \mathbb{R}_{>0})\right\}$	space of positive smooth probability densities
$\mathrm{Met}(S^1) = \left\{g\, d\theta^2 : g \in C^\infty(S^1, \mathbb{R}_{>0})\right\}$	space of metrics on S^1

Some words on smooth convenient calculus

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.
Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.

For more general locally convex spaces we sketch here the convenient approach as explained in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997].

I explain this to show how simple differential calculus can be!

The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth curves. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

The c^∞-topology. II

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.
In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since addition is no longer jointly continuous. Namely, $c^\infty(\mathcal{D} \times \mathcal{D})$ is strictly finer than $c^\infty\mathcal{D} \times c^\infty\mathcal{D}$.
The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following holds (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all cont. lin. funct. on E.
   - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
   - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.

   We call this *scalarwise* C^∞.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

Convenient vector spaces. II

5. If B is bounded closed absolutely convex, then E_B is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

Smooth mappings

Let E, and F be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called smooth or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.

If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C^∞_c.

Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection

$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R}, U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.

Main properties of smooth calculus, II

6. The exponential law holds: For c^∞-open $V \subset F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

   is a linear diffeomorphism of convenient vector spaces. **Note that this is the main assumption of variational calculus. Here it is a theorem.**

7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$. (*Smooth uniform boundedness theorem*, [KM97], theorem 5.26).
   A mapping $f : U \to L(F, G)$ is smooth iff $U \xrightarrow{f} L(F, G) \xrightarrow{\mathrm{ev}_x} G$ is smooth for all $x \in F$.

Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

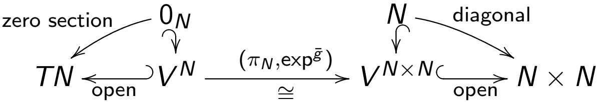
$$
\begin{aligned}
&\mathrm{ev} : C^\infty(E,F)\times E \to F, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : E \to C^\infty(F, E\times F), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E\times F, G)\\
&(\quad)^\vee : C^\infty(E\times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp} : C^\infty(F,G)\times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad) : C^\infty(F,F_1)\times C^\infty(E_1,E) \to\\
&\qquad\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g)\mapsto (h\mapsto f\circ h\circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}
$$

This ends our review of the standard results of convenient calculus. Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]; recent result: [Bochnak, Kucharz: Real analyticity is concentrated in dimension 2, 2018].
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33])
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]
- Affine schemes, resp., schemes. [Cherenack, P.: A Cartesian closed extension of a category of affine schemes. Cahiers Topologie Géom. Différentielle 23 (1982), no. 3, 291–316]. [Cherenack, P.: Extending schemes to a Cartesian closed category. Quaestiones Math. 9 (1986), no. 1-4, 95–133].

Manifolds of mappings

Let M be a compact (for simplicity's sake) fin. dim. manifold and N a manifold. We use an auxiliary Riemann metric $\bar{g}$ on N. Then



$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^*TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar{g}})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar{g}}_{f(x)}(s(x))$$

Manifolds of mappings II

Lemma: $C^\infty(\mathbb{R}, \Gamma(M; f^* TN)) = \Gamma(\mathbb{R} \times M; \mathrm{pr}_2{}^* f^* TN)$
By Cartesian Closedness (after handling local trivializations).
Lemma: Chart changes are smooth (C^∞)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.
Lemma: $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By Cartesian closedness.
Lemma: Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves
Corollary (of the chart structure):

$$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N).$$

Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [KM97], 38.4.

A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.

- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\mathrm{Evol}^X_t := g(t) = \mathrm{evol}_G(tX)$.

Diffeomorphism group of compact M

Theorem: *For each compact manifold M, the diffeomorphism group is a regular Lie group.*
Proof: $\mathrm{Diff}(M) \xrightarrow{open} C^\infty(M, M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\mathrm{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t, x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.
Let $X(t, x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\mathrm{Fl}_s^{\partial_t \times X}(t, x) = (t + s, \mathrm{Evol}^X(t, x))$ satisfies the ODE $\quad \partial_t \mathrm{Evol}(t, x) = X(t, \mathrm{Evol}(t, x))$. If $X(s, t, x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism. QED.

The principal bundle of embeddings: 'differentiable Chow variety'; 'nonlinear Grassmannian'

For finite dimensional manifolds M, N with M compact, $\mathrm{Emb}(M,N)$, the space of embeddings of M into N, is open in $C^\infty(M,N)$, so it is a smooth manifold. $\mathrm{Diff}(M)$ acts freely and smoothly from the right on $\mathrm{Emb}(M,N)$.

Theorem: *The quotient projection*

$$\mathrm{Emb}(M,N) \to \mathrm{Emb}(M,N)/\mathrm{Diff}(M) =: B(M,N)$$

is a principal fiber bundle with structure group $\mathrm{Diff}(M)$*.*

Proof: Auxiliary Riem. metric $\bar{g}$ on N. Given $f \in \mathrm{Emb}(M,N)$, view $f(M)$ as submanifold of N. $TN|_{f(M)} = \mathrm{Nor}(f(M)) \oplus Tf(M)$.

$\mathrm{Nor}(f(M)) : \xrightarrow[\cong]{\exp^{\bar{g}}} W_{f(M)} \xrightarrow{p_{f(M)}} f(M)$ tubular nbhd of $f(M)$.

If $g : M \to N$ is C^1-near to f, then
$\varphi(g) := f^{-1} \circ p_{f(M)} \circ g \in \mathrm{Diff}(M)$ and
$g \circ \varphi(g)^{-1} \in \Gamma(f^* W_{f(M)}) \subset \Gamma(f^* \mathrm{Nor}(f(M)))$.
This is the required local splitting. QED

The orbifold bundle of immersions: 'differentiable Chow variety including singularities'

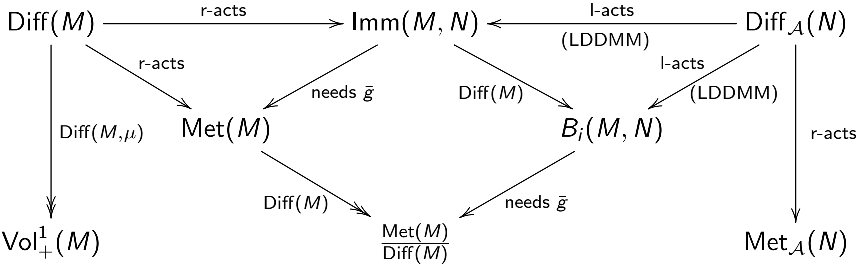
$\mathrm{Imm}(M, N)$, the space of immersions $M \to N$, is open in $C^\infty(M, N)$, and is thus a smooth manifold. The regular Lie group $\mathrm{Diff}(M)$ acts smoothly from the right, but no longer freely.

Theorem: [Cervera,Mascaro,M,1991] *For an immersion $f : M \to N$, the isotropy group $\mathrm{Diff}(M)_f = \{\varphi \in \mathrm{Diff}(M) : f \circ \varphi = f\}$ is always a finite group, acting freely on M; so $M \xrightarrow{p} M/\mathrm{Diff}(M)_f$ is a convering of manifold and f factors to $f = \bar{f} \circ p$. Thus*

$$\mathrm{Imm}(M, N) \to \mathrm{Imm}(M, N)/Diff(M) = B_i(M, N)$$

is a projection onto an honest infinite dimensional orbifold.

Recall the diagram of actions of diffeomorphism groups.



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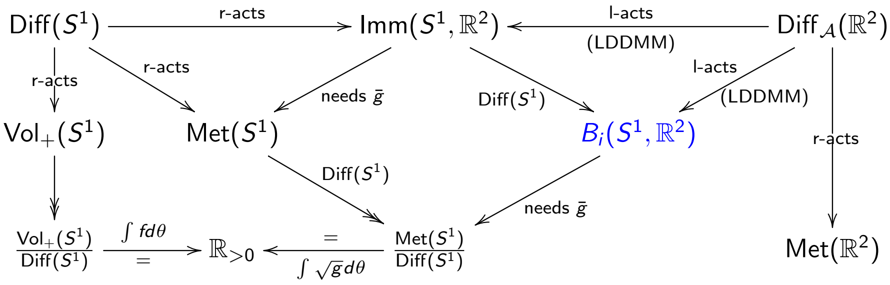

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Weak Riemannian metrics on Imm and B_i

$$\begin{array}{c} \operatorname{Imm}(M,N) \\ \big\downarrow{\scriptstyle \pi} \\ B_i(M,N) \end{array}$$

Let G be Diff(M)-invariant weak Riemannian metric G on Imm. The *vertical bundle* in TImm consists of the tangent spaces to the Diff(M)-orbits.

If the G-orthogonal complement to the vertical bundle (called the *horizontal bundle*) is a complement, then $T\pi$ restricted to the horizontal space yields an isomorphism

$$(\ker T_f \pi)^{\perp, G} \cong T_{\pi(f)} B_i.$$

Otherwise one has to induce the quotient metric, or use the completion. One needs concepts like *robust Riemannian manifolds* to make this work. For Imm it mostly works very well.
This gives a metric on B_i such that $\pi : \operatorname{Imm} \to B_i$ is a *Riemannian submersion*.

[Micheli, M, Mumford, Izvestija 2013]

Riemannian submersions

$$\operatorname{Imm}(M,N) \xrightarrow{\pi} B_i := \operatorname{Imm}(M,N)/\operatorname{Diff}(M)$$

- Horizontal geodesics on $\operatorname{Imm}(M,N)$ project down to geodesics in shape space.
- O'Neill's formula connects sectional curvature on $\operatorname{Imm}(M,N)$ and on B_i.
- A formula for sectional curvature using the inverse metric G^{-1} on the cotangent bundle as much as possible thies nicely with O'Neill's formula and allows curvature computations in some highly complex shape space situations.

[Micheli, M, Mumford, Izvestija 2013]

The simplest (L^2-) metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$

$$G^0_c(h,k) = \int_M \langle h(\theta), k(\theta)\rangle ds \quad \text{where } ds = |c'(\theta)| d\theta.$$

Problem: The induced geodesic distance vanishes.
Movies about vanishing: $\mathrm{Diff}(S^1)$ $\mathrm{Imm}(S^1, \mathbb{R}^2)$
Geodesic equation is a relative of Burger's equation:

$$c_{tt} = -\frac{1}{2|c_\theta|}\partial_\theta\left(\frac{|c_t|^2\, c_\theta}{|c_\theta|}\right) - \frac{1}{|c_\theta|^2}\langle c_{t\theta}, c_\theta\rangle c_t.$$

Conserved momenta for G^0 along any geodesic $t \mapsto c(\quad, t)$:

$\langle v, c_t\rangle \lvert c_\theta\rvert^2 \in \mathfrak{X}(S^1)$	reparam. mom.
$\int_{S^1} c_t ds \in \mathbb{R}^2$	linear moment.
$\int_{S^1} \langle Jc, c_t\rangle ds \in \mathbb{R}$	angular moment.

Weak Riem. metrics on $\mathrm{Emb}(M,N) \subset \mathrm{Imm}(M,N)$.

$$G_f^P(h,k) = \int_M \bar{g}(P^f h, k)\,\mathrm{vol}(f^*\bar{g})$$

where $\bar{g}$ is some fixed metric on N, $g = f^*\bar{g}$ is the induced metric on M, $h,k \in \Gamma(f^*TN)$ are tangent vectors at f to $\mathrm{Imm}(M,N)$, and P^f is a positive, selfadjoint, bijective (scalar) pseudo differential operator of order $2p$ depending smoothly on f.
Also P has to be $\mathrm{Diff}(M)$-invariant: $\varphi^* \circ P_f = P_{f\circ\varphi} \circ \varphi^*$.
Good example: Sobolev type metrics $P^f = (1 + \Delta^{f^*\bar{g}})^p$, where $\Delta^{f^*\bar{g}}$ is the Bochner-Laplacian on M induced by the metric $f^*\bar{g}$. p can be real.
Geodesic equation is well-posed. For that $f \mapsto P^f$ has to have a smooth extension to Sobolev completions $\mathrm{Imm}_{H^s}(M,N)$ for all $s > \frac{\dim(M)}{2} + 1$.
Curvature formulas are available even for general P.

[M Mumford, 2005a, 2005b], [Bauer, Bruveris, Harms, M, 2011, 2012], O.Müller

Elastic metrics on plane curves

Here $M = S^1$ or $[0, 2\pi]$.The elastic metrics on $\operatorname{Imm}(M, \mathbb{R}^2)/\text{transl.}$ is (where $ds = |c'(\theta)| d\theta$, $D_s = \frac{\partial_\theta}{|c'(\theta)|}$, $v = D_s c = \frac{c'}{|c'|}$)

$$G_c^{a,b}(h, k) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle \langle D_s k, n\rangle + b^2 \langle D_s h, v\rangle \langle D_s k, v\rangle \, ds\,.$$

Theorem. *The metric $G^{a,b}$ is the pullback of the flat L^2 metric via:*

$$R^{a,b} : \operatorname{Imm}([0, 2\pi], \mathbb{R}^2) \to C^\infty([0, 2\pi], \mathbb{R}^3)$$
$$R^{a,b}(c) = |c'|^{1/2} \left(a \begin{pmatrix} v \\ 0 \end{pmatrix} + \sqrt{4b^2 - a^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \quad .$$

The image takes values in a cone $C^{a,b}$ in $\mathbb{R}^3$. The metric $G^{a,b}$ is flat on open curves, geodesics are the preimages under the R-transform of geodesics on the flat space $\operatorname{im} R$ *and the geodesic distance between* $c, \overline{c} \in \operatorname{Imm}([0, 2\pi], \mathbb{R}^2)/\operatorname{trans}$ *is given by the integral over the pointwise distance in the image* $\operatorname{Im}(R)$*. The curvature on $B([0, 2\pi], \mathbb{R}^2)$ is non-negative.*

[BauerBruverisMarsland M 2014]

Right invariant weak Riemannian metrics on Diffeomorphism groups.

For $M = N$ (compact connected) the space $\mathrm{Emb}(M, M)$ equals the *diffeomorphism group* $\mathrm{Diff}(M)$ *of* M. Reparameterization invariant metrics are right invariant metrics G on $\mathrm{Diff}(M)$, as explained by Arnold 1966. G is uniquely determined by its restriction to the Lie algebra $T_{\mathrm{Id}}\mathrm{Diff}(M) = \mathfrak{X}(M)$ (with the negative of the usual Lie bracket), viewed as $G_{\mathrm{Id}} = G : \mathfrak{X}(M) \to \mathfrak{X}(M)'$. The following works for regular convenient Lie groups.

A smooth curve $t \mapsto \varphi(t) \in \mathrm{Diff}(M)$ *is a geodesic iff the right logarithmic derivative* $u(t) := (\partial_f \varphi(t)) \circ \varphi(t)^{-1}$

$$\boxed{\partial_t u = -G_{\mathrm{Id}}^{-1}\,\mathrm{ad}(u)^*\; G_{Id}(u) = -G_{\mathrm{Id}}^{-1}\mathcal{L}_u G_{\mathrm{Id}}(u)}$$

Thus the geodesic equation exists in general $\Leftrightarrow$

$$\Leftrightarrow \quad \mathrm{ad}(X)^* G_{\mathrm{Id}}(X) = \mathcal{L}_X G_{\mathrm{Id}} X \in G(\mathfrak{X}(M)) \subset \mathfrak{X}(M)' \quad \forall X \in \mathfrak{X}(M)$$

$\Leftrightarrow$ Existence of the Christoffel symbols.

[Arnold 1966] has the stronger condition $\mathrm{ad}(X)^* G(Y) \in G(\mathfrak{X}(M))$.

$(\mathrm{Diff}_{\mathcal{N}}, G)$ induces metrics on differentiable Chow varieties

Left action of $\mathrm{Diff}_{\mathcal{A}}(N)$ on $B(M,N)$. M can be a finite set: *Landmark space*. Given a right invariant metric G of high enough Sobolev order we can induce a metric on each orbit through a submanifold $S \subset N$ of type M such that $\mathrm{ev}_S : \mathrm{Diff}_{\mathcal{A}}(N) \to B(M,N)$ becomes a Riemannian submersion. Subtle, since here the G-orthogonal space to the fiber is NOT a complement as a rule. Needs the notion of robust weak Riemannian manifolds: Horizontally lifted tangent vectors from TB are vector fields in the G_{Id}-completion of $\mathfrak{X}_{\mathcal{A}}(M)$.
Nevertheless geodesics, curvature and efficient numerical procedures are available:
LDDMM=large deformation diffeomorphic metric matching (Beg, Miller, Trouve, Younes).

[Micheli, M, Mumford, 2013]

Advantages of Sobolev type metrics

- Positive geodesic distance if $p \geq 1$.
- Geodesic equations are well posed, usually for $p \geq 1$.
- Spaces are geodesically complete for $p > \frac{\dim(M)}{2} + 1$.

[Bruveris, M, Mumford, 2013], [Bruveris 2015] for plane curves. Higher dim. for Imm proof still lacking.
A remark in [Ebin, Marsden, 1970], and [Bruveris, Vialard, J.EMS 2017] for diffeomorphism groups.

- Geodesic equations for Sobolev metrics on $\mathrm{Diff}(S^1)$, $\mathrm{Diff}(\mathbb{R})$, and the Virosoro groups of order $-\frac{1}{2}, 0, \frac{1}{2}, 1$ lead to named equations: Burgers, Korteveg-de Vrieß, Camassa Holm, Hunter-Saxton, quasi geostrophic, De Gasperi-Procesi.
- L^2-metric on $\mathrm{Diff}(M, \mu)$ and $\int_{\mathbb{R}^n} g((1 + \frac{1}{\varepsilon^2}\operatorname{grad}\operatorname{div})X, Y)\mu$ on $\mathrm{Diff}(\mathbb{R}^n)$ describe incompressible Euler flow of fluid mechanics.
- High order Sobolev metrics on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ induce metrics on $B(M, \mathbb{R}^n)$ which have found convincing applications in computational anatomy etc. M may be a finite set: *Landmark space*. See lectures by M.Miller and L.Younes on Friday. LDDMM=large deformation diffeomorphic metric matching

A sidestep to Invariant theory

Theorem [BBM 2016] *Let M be a compact connected smooth compact manifold, of dimension ≥ 2. Let G be a smooth (equivalently, bounded) bilinear form on the space* $\mathrm{Dens}_+(M)$ *of smooth positive densities, which is invariant under the action of* $\mathrm{Diff}(M)$. *Then*

$$G_\mu(\alpha, \beta) = C_1(\mu(M)) \int_M \frac{\alpha}{\mu} \frac{\beta}{\mu} \, \mu + C_2(\mu(M)) \int_M \alpha \cdot \int_M \beta$$

for smooth functions C_1, C_2 of the total volume $\mu(M)$.

This is the Fisher-Rao metric on the space of smooth positive probability measures on M, used in **Information Geometry**

Conjecture *Let M be a compact connected smooth oriented manifold with corners, of dimension $m = \dim(M) \geq 2$. Then the associative algebra of $\mathrm{Diff}_0(M)$-invariant covariant tensor fields on $\mathrm{Dens}_+(M)$ is generated by:*

$$\mu \mapsto F(\mu(M)), \qquad F(\mu(x)) \quad \text{for } x \in \partial^m M, \quad F \in C^\infty(\mathbb{R})$$

$$(\mu, \alpha) \mapsto \int_M \alpha$$

$$(\mu, \alpha_1, \dots, \alpha_p) \mapsto \int_C \frac{\alpha_1}{\mu} \dots \frac{\alpha_k}{\mu} d\Big(\frac{\alpha_{k+1}}{\mu_2}\Big) \wedge \dots \wedge d\Big(\frac{\alpha_p}{\mu_2}\Big),$$

$$1 \leq k < p, 1 \leq p - k \leq m$$

$$C \text{ a component of } \partial^{m-p+k} M$$

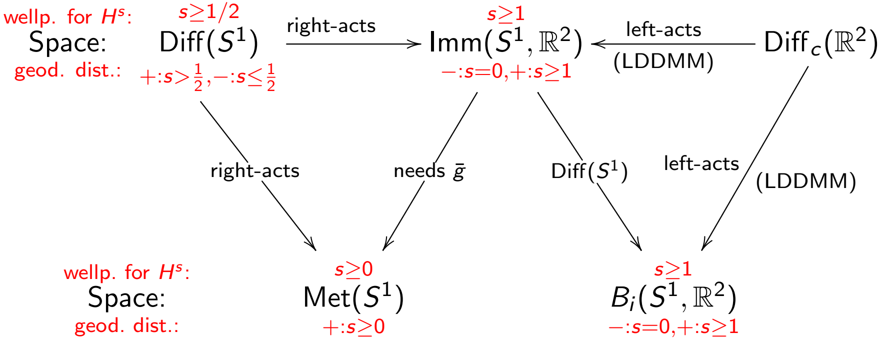
$$(\mu, \alpha_1, \dots, \alpha_p) \mapsto \int_M \frac{\alpha_1}{\mu} \dots \frac{\alpha_p}{\mu} \cdot \mu, \quad 2 \leq p,$$

Problems of Sobolev type metrics

• Analytic solutions to the geodesic equation are complicated. Numerics are in general computationally expensive. Although the methods sketched in this conference seem to be geometrically correct way for shape analysis, many applications use much simpler and more error prone methods. Danger of too much reliance on the miracle like deep learning methods using big data.

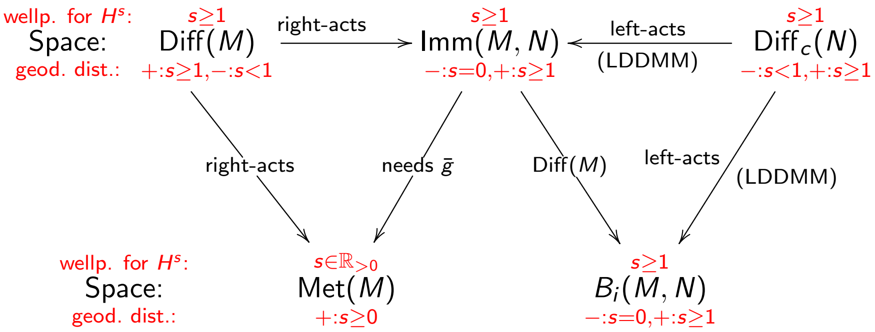
• Curvature of shape space with respect to these metrics is complicated. Too much negative curv. components make geodesic shooting inaccurate. Too much positive components mixes up statistics: Need for better statistics (PCA) on curves spaces!

Well-posedness and geodesic distance in dim. 1



[M. Mumford, 2005],
[Bauer, Bruveris, Harms, M, 2012] for KdV on Virasoro $\mathbb{R} \times \mathrm{Diff}_c(\mathbb{R})$
[Bauer, Bruveris, Harms, M, 2013, 2013],

Well-posedness and geodesic distance in $\dim(M) > 0$



[M. Mumford, 2005],
[Bauer, Bruveris, Harms, M, 2012] for KdV (L^2) on Virasoro $\mathbb{R} \times \mathrm{Diff}_c(\mathbb{R})$
[Bauer, Bruveris, Harms, M, 2013, 2013],
[Bauer, Harms, Preston, 2018] for $H^{-1/2}$ on $\mathrm{Diff}(M^2, \mu)$
[Jerrard, Maor, 2018] $W^{s,p}$ has vanishing Geod. distance on $\mathrm{Diff}_c(M)$ if $s < \min\{\frac{dim(M)}{p}, 1\}$, allways positive if $s > \frac{\dim(M)}{p}$ or $s \geq 1$.

Universal Teichmüller space

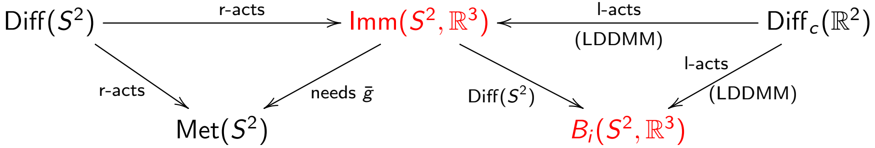
The quotient $\mathcal{T} := \mathrm{Diff}(S^1)/PSL(2,\mathbb{R})$, also known as universal Teichmüller space, is naturally a coadjoint orbit of the Virasoro group, and as such it carries a natural invariant Kähler structure;
The corresponding Riemann metric is called Weil-Petersen metric:
For $u \in \mathfrak{X}(S^1) \cong C^\infty(S^1)$, Fourier series $u(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ with $\overline{a}_n = a_{-n}$, let

$$\|u\|_{\mathrm{WP}}^2 = \sum_{n\in\mathbb{Z}} |n^3 - n||a_n|^2 = \int_{S^1} L(u).u\, d\theta$$

where $L = \mathcal{H}(\partial_\theta^3 + \partial_\theta)$ and $\mathcal{H}$ is the periodic Hilbert transform, given by convolution with $\frac{1}{2\pi}\,\mathrm{cotan}(\frac{\theta}{2})$.
The kernel $\ker(\|\cdot\|_{\mathrm{WP}}) = \mathfrak{sl}(2,\mathbb{R})$.
This metric admits soliton like solutions, called Teichons.

[Mumford, Sharon, 2004, 2006], [Kushnarev, 2009]

A shape space with strongly negatively curved parts



$$G_f^{\Phi}(h,k) = \int_M \Phi(f)\overline{g}(h,k)\,\mathrm{vol}(g)$$

The pathlength metric induced by G^{Φ} separates points on B_i if either:

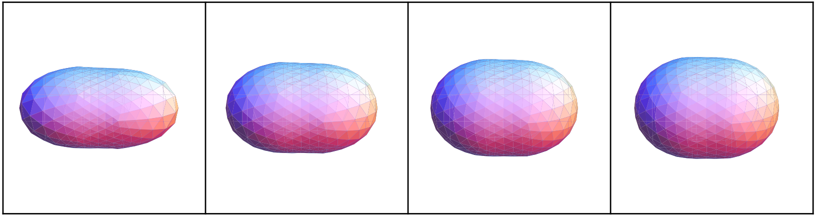
- $\Phi \geq C_1 + C_2 \|\mathrm{Tr}^g(S)\|^2$ with $C_1, C_2 > 0$ or
- $\Phi \geq C_3\,\mathrm{Vol}$

This leads us to consider $\Phi = \Phi(\mathrm{Vol}, \|\mathrm{Tr}^g(S)\|^2)$. Special cases:

- G^A-metric: $\Phi = 1 + A\|\mathrm{Tr}^g(S)\|^2$
- Conformal metrics: $\Phi = \Phi(\mathrm{Vol})$

[BauerHarms M 2012]

Negative Curvature: Movies



Movies: Ex1: $\Phi = 1 + .4\,\mathrm{Tr}(L)^2$ Ex2: $\Phi = e^{Vol}$ Ex3: $\Phi = e^{Vol}$

Final mention: Weak Riemann metrics on Met(M)

All of them are Diff(M)-invariant; natural, tautological.

$$
\begin{aligned}
G_g(h,k) &= \int_M g_2^0(h,k)\,\mathrm{vol}(g) = \int \mathrm{Tr}(g^{-1}hg^{-1}k)\,\mathrm{vol}(g), \quad L^2\text{-metr.}\\
\text{or } &= \Phi(\mathrm{Vol(g)}) \int_M g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{conformal}\\
\text{or } &= \int_M \Phi(\mathrm{Scal}^g).g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{curvature modified}\\
\text{or } &= \int_M g_2^0((1+\Delta^g)^s h,k)\,\mathrm{vol}(g) \qquad \text{Sobolev order } s \text{ real}
\end{aligned}
$$

where Φ is a suitable real-valued function, $\mathrm{Vol} = \int_M \mathrm{vol}(g)$ is the total volume of (M,g), Scal is the scalar curvature of (M,g), and where g_2^0 is the induced metric on $\binom{0}{2}$-tensors.

[DeWitt, 1967], [Ebin, 1970], [Freed, Groisser, 1989], [Gil-Medrano, M, 1991], [Bauer, Harms, M, 2013], [Bauer, Bruveris, Harms, M, 2018].

Movie using soliton solution for the $\dot{H}^1$-metric on space of plane curves

All these marvellous applications and uses of convenient calculus are due to the inspiration by David.

Thank you, David, for this, and for collaborating with me!

Thank you, audience, for your attention!

Thank you, organizers, for this great conference!