

Higher-Rank Tensor Field Theory of Non-Abelian Fracton and Embeddon

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Abstract

We introduce a new class of tensor gauge field theories in any dimension that is a hybrid class between symmetric higher-rank tensor gauge theory (i.e., higher-spin gauge theory) and anti-symmetric tensor topological field theory. Our theory describes a mixed unitary phase interplaying between gapless (which can live with or without Euclidean, Poincaré or anisotropic symmetry, at least in ultraviolet high energy) and gapped topological order phases. The “gauge structure” can be compact, continuous, and non-abelian. Our theory may be a new gem sitting in the virgin territory beyond the familiar gauge theories — outside the paradigm of Maxwell electromagnetic theory in 1865 and Yang-Mills isospin/color theory in 1954. We discuss its local gauge transformation in terms of the ungauged vector-like or tensor-like higher-moment global symmetry. Vector global symmetries along time direction may exhibit time crystals. We explore the relation of these long-range entangled matters to a non-abelian generalization of Fracton order in condensed matter, a field theory formulation of foliation, the spacetime embedding and Embeddon that we newly introduce, and fundamental physics applications to dark matter and dark energy.

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August 2019

Dedicated to
90 years of Gauge Principle since Hermann Weyl [Elektron und Gravitation, Zeit. für Physik 56, 330-352 (1929)]
and 65 years of Yang-Mills theory [Phys. Rev. 96, 191 (1954)] in 2019.

敬致楊振寧-密爾斯理論

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1 Introduction

Gauge theory is a powerful tool in quantum field theory. One of the earliest gauge theories is the renown *James Clerk Maxwell's dynamical U(1) gauge theory of electromagnetism* in 1865 [1]. *Gauge principle* is the underlying principle of gauge theory, proposed by Weyl [2] and pondered by Pauli and many others [3]. Chern introduced Chern characteristic classes laying the foundation of non-abelian gauge structure in terms of fiber bundles and connections in 1940s [4]. Earlier pioneer works do make an impact, leading to another famous gauge theory attempting for a physics realization: *Yang-Mills theory* in 1954 [5] with a non-abelian Lie gauge group, say SU(N). Yang-Mills theory incorporates the gauge principle over the isospin — by promoting the internal global symmetry of isospin (such as flavor symmetry) to a local symmetry. While its local symmetry transformation, say of matter fields, is compensated by that of the gauge field. Thus gauge theory is an ideal framework to mathematically formulate the interactions between *matters*, where the gauge field stands for the *force mediator*.

In this work, we develop a new type of gauge theory, obtained from a hybrid through the marriage between anti-symmetric tensor topological field theory (TQFT) and symmetric higher-rank tensor gauge theory. We find that the gauge structure ¹ can be non-abelian. Our work is inspired by the fracton order in condensed matter system (see a review article [6]). To the best of our understanding, our theory is a new example of *compact non-abelian higher-rank tensor gauge theory* previously unknown to the literature. It is also a first example (in fact, we construct a web of families of many examples) to have a continuous non-abelian gauge structure in any dimension in the fracton order literature.

Anti-symmetric tensor field theory has a long history as early as Kalb-Ramond work's on higher differential form gauge fields [7]. The later development of higher form continuum field theory can be found in [8, 9] and References therein, including higher-form gauge theory and higher-form global symmetries. We will focus on a particular *continuum* anti-symmetric tensor TQFT developed in Ref. [10–20] and references therein: A *continuum* gauge theory formulation [18, 19] of *group cohomology* or *higher group cohomology* type of TQFT such as Dijkgraaf-Witten theory [21], with a discrete finite gauge group. We term this finite gauge group for TQFT as G_{TQFT} .

Symmetric higher-rank tensor field theory also has a long history dating back to the study of arbitrary-

¹Here the gauge structure is similar to the usual concept of the gauge group, but the gauge structure is not precisely in a familiar framework as a Lie group of the well-known gauge theory. There is still a notion of commutative-ness or non-commutative-ness generator of the gauge structure, which we shall still call the commutative case as abelian, and the non-commutative case as non-abelian.

integer-spin bosonic fields, including massive and massless boson fields, see [22–25] and References therein. Singh-Hagen [24] studies the Fierz-Pauli theory [23] for massive arbitrary-integer-spin s boson fields. Fronsdal [25] studies their theory [23,24] in the massless limit, then finds that the massless arbitrary-integer-spin s boson fields can be understood in terms of *symmetric tensor bosonic field* of rank s , with certain constraints on the trace, double trace and divergence, etc. In this work, we will only focus on a certain *higher-spin theory* in terms of a class of specific *symmetric higher-rank tensor gauge theories*.

Our inspiration of *symmetric higher-rank tensor gauge theory* comes from the condensed matter systems studying the quantum spin liquids described by emergent higher-rank symmetric tensor gauge fields, say $A_{ijk\dots}$, where indices are symmetrized. If the symmetric tensor gauge field is in the U(1) value, say the closed integral of a cycle has

$$\oint A_{ijk\dots} \in 2\pi\mathbb{Z},$$

then it is commonly referred this compact abelian theory as the “higher-rank U(1) symmetric tensor gauge theory,” or “higher-rank U(1) spin liquids” in the condensed matter literature. However, we will refrain from using this terminology, due to the U(1) there may be an accidental misnomer. We will see that there is no precise occurrence of a U(1) gauge group in the theory, and the ungauged global symmetry is not simply a U(1) global symmetry group. (In facts, see the later Eq. (2.14), there is only a *group-analogous* structure that is abelian, instead of the familiar group structure.) Thus, we instead name this type theory as

$$a \text{ compact abelian higher-rank symmetric tensor gauge theory.} \quad (1.1)$$

Ref. [26] finds that the compact abelian *symmetric* tensor gauge theory is unstable in 2+1D, but it becomes stable to be gapless-ness with deconfinement in 3+1D.² In contrast, the *anti-symmetric* tensor gauge theory with a continuous gauge group is unstable and flows to a gapped phase with confinement in 2+1d and 3+1d [27, 28]. We will follow closely a version of compact abelian higher-rank symmetric tensor gauge theory developed and pioneered by Pretko and others, see for instance Ref. [29–33], where time and space indices are treated in an unequal and non-interchangeable footing. We name this type of theory as:

$$Anisotropic-type higher-rank symmetric tensor gauge theory. \quad (1.2)$$

We also develop another generalization of two kinds of higher-rank symmetric tensor gauge theory:

$$Euclidean-type higher-rank symmetric tensor gauge theory, \text{ or} \quad (1.3)$$

$$Lorentzian-type higher-rank symmetric tensor gauge theory. \quad (1.4)$$

Let us give a quick overview and outlines where we are heading in this work. It will become clear soon that we can formulate non-abelian gauge structures (for example $\left[\mathbb{Z}_2^C \times \left(\text{U}(1)_{x_{(d+1)}}\right)\right]$ introduced in Sec. 2) to a higher-rank tensor gauge theory (see later Eq. (2.27)) by gauging the higher-moment global symmetry³ and a discrete \mathbb{Z}_2^C charge conjugation symmetry together, relatively with a non-commutative semi-direct product (\ltimes) structure. This is the first ingredient to obtain our non-abelian tensor gauge theory. A second ingredient for our non-abelian gauge structure is by coupling several copies of $(\mathbb{Z}_2^C)^N$ to a twisted gauge theory of TQFT. The twisted- $(\mathbb{Z}_2^C)^N$ TQFT can be equivalent (strong and exactly dual) to a TQFT with a non-abelian gauge group with non-abelian anyonic excitations. For example, certain twisted- $(\mathbb{Z}_2^C)^3$ -TQFT in 3d (2+1D) can be dual to a non-abelian gauge theory of an order 8 dihedral D_8 or quaternion Q_8 gauge group [10, 34]. Similar non-abelian natures of field theories hold in 4d (3+1D) and higher dimensions [18, 19].

²We denote $n + 1d$ means the $n + 1$ spacetime dimensions, with n spatial and 1 time dimensions. We denote md means the m spacetime dimensions. We denote $n + 1D$ means the n spatial and 1 time dimensions.

³For example, a vector global symmetry $\text{U}(1)_{x_{(d+1)}}$ in spacetime or $\text{U}(1)_{x_{(d)}}$ in space.

In the remaining of Sec. 1, we will review the familiar gauge theories (U(1) Maxwell and non-abelian Yang-Mills theories), in contrast of the new non-abelian higher-rank tensor gauge theories we develop in Sec. 2. Our gauge field theories in Sec. 2 include the following properties:

- (1). Unitary (thus directly applicable to quantum matter and condensed matter systems),
- (2). Abelian (e.g. $\prod_{J=1}^N \text{U}(1)_{x_{(d+1)}}$) or non-abelian (e.g. $\prod_{J=1}^N \left[\mathbb{Z}_2^C \times \left(\text{U}(1)_{x_{(d+1)}} \right) \right]$),
- (3). Compact gauge structure (e.g. $\prod_{J=1}^N \left[\mathbb{Z}_2^C \times \left(\text{U}(1)_{x_{(d+1)}} \right) \right]$ is compact),
- (4). Continuous gauge structure (e.g. thanks to $\text{U}(1)_{x_{(d+1)}}$ or $\text{U}(1)_{x_{(d)}}$).

In Sec. 3, we examine the non-abelian tensor gauge theory interplayed between gapped and gapless phases, including the topological degeneracy from the zero modes, the gapless gapless degrees of freedom and dispersion relations. In Sec. 4, we discuss how our theories can be related to a non-abelian generalization of Fracton order in condensed matter, the spacetime embedding and the Embeddon that we newly introduce, and a field theory formulation of foliation. In Sec. 5, we conclude with discussions on the field theory quantization (either canonical quantization, or path integral), Feynman diagrams, and quantum Hamiltonian lattice models, and applications of our theories to time crystal and dark matter.

1.1 Overview of the familiar gauge theories

Gauge field allows a local transformation, depending on the spacetime coordinate. This local transformation is known as the gauge transformation. The whole partition function \mathbf{Z} or the path integral, defined through integrating out the gauge in-equivalent configuration phase space of gauge fields, is set to be invariant under the gauge transformation. We first review the familiar gauge theories and meanwhile set up our notations. For simplest, we focus on the 4 dimensions.

1.1.1 Abelian U(1) Maxwell gauge theory

In the modern formulation of Maxwell's U(1) gauge theory [1], we have a gauge field $A(x^\mu) = A_\mu(x^\mu) dx^\mu$ (locally as a 1-form as a differential form, but globally should be viewed as a U(1) gauge 1-connection), which under an infinitesimal gauge transformation becomes

$$A_\mu(r) \rightarrow A_\mu(r) + \delta A_\mu(r) = A_\mu(r) + \frac{1}{g} \partial_\mu \eta(r), \quad (1.5)$$

with the spacetime coordinate $r = (r_\mu)(dx^\mu)$ where $x^\mu := (x^0, x^j) := (x^0, \vec{x}) := (t, \vec{x})$. Locally dx^μ is a differential 1-form, the μ runs through the indices of coordinate of spacetime M . The coupling g can be related to electromagnetic coupling e as $g = -e$. The $\eta(r)$ is locally 0-form with a spacetime dependence. Just like the relation of a local curvature to the 1-connection, we have the field strength 2-form to the 1-form U(1) gauge field via

$$F = dA = \frac{1}{2} F_{\mu\nu} (dx^\mu \wedge dx^\nu) = \frac{1}{2} (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha) (dx^\mu \wedge dx^\nu), \quad (1.6)$$

with the exterior derivative d and the wedge product \wedge . The field strength is gauge invariant:

$$F_{\mu\nu}(r) = F_{\mu\nu}(r) + \delta F_{\mu\nu}(r) \rightarrow F_{\mu\nu}(r). \quad (1.7)$$

The whole gauge invariant partition function of U(1) gauge theory is meant to be written systematically as:

$$\mathbf{Z}_{\text{U(1),EM}} = \int [\mathcal{D}A] e^{i \int \frac{1}{2g^2} F \wedge \star F} = \int [\mathcal{D}A^\mu] e^{i \int d^4x \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu}}. \quad (1.8)$$

The $[\mathcal{D}A]$ is the path integral measure, for a configuration of the gauge field A . Here $\star F$ is F 's Hodge dual. We integrated over all allowed gauge inequivalent configurations $\int [\mathcal{D}A]$, while gauge redundancy is removed or mod out. The integration is under a weight factor $\exp(iS)$. e.g. In 4d (3+1D), we take the spacetime metric $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(+, -, -, -)$, and $\tilde{\epsilon}^{\mu_1\mu_2\mu_3\mu_4} = -\tilde{\epsilon}_{\mu_1\mu_2\mu_3\mu_4}$, while $\epsilon_{ijk} = \epsilon^{ijk}$. Generally we take μ, ν, μ_1, \dots are spacetime coordinates, while i, j, k are space coordinates only. This path integral may not be precisely mathematically well-defined, however it can be physically sensibly well-defined, for example being regularized via a higher energy cutoff such as a lattice cutoff scale. Our present work only focus on the physics side of rigor.

Indeed the above U(1) gauge theory follows the *gauge principle* [2] where the global U(1) symmetry of a point operator Φ say Φ is promoted to a local symmetry variation that can be absorbed by local symmetry variation of the 1-form gauge field. Then the 1-form gauge field plays the role of making the charged matter (namely the matter carrying gauge charge) interacting with each other. Thus the 1-form gauge field behaves as a *force mediator* between matter fields. The *force mediator* is known as a spin-1 gauge boson in physics.

1.1.2 Non-abelian SU(N) Yang-Mills gauge theories

For the non-abelian *Yang-Mills theory* (YM), the gauge field is locally a 1-form or a 1-connection

$$A = A_\mu dx^\mu = A_\nu^\alpha T^\alpha dx^\nu, \quad (1.9)$$

obtained from parallel transporting the SU(N) fiber of the principal SU(N) bundle over the spacetime of the base manifold M . Here T^α is the generator of Lie algebra \mathfrak{g} for the gauge group (say SU(N), with N is called the number of *color* for gauge theory, or the number of *isospin* for global symmetry in physics), with the commutator $[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma$, where $f^{\alpha\beta\gamma}$ is chosen to be a fully anti-symmetric structure constant. Then $A_\mu = A_\mu^\alpha T^\alpha$ is the Lie algebra valued gauge field, in the adjoint representation of the Lie algebra. In physics, A_μ can represent the gluon vector field of quantum chromodynamics. The non-abelian field strength is

$$F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu}^\alpha T^\alpha (dx^\mu \wedge dx^\nu) = \frac{1}{2} (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + i f^{\beta\gamma\alpha} A_\mu^\beta A_\nu^\gamma) T^\alpha (dx^\mu \wedge dx^\nu).$$

For the convenience to bridge to the physics convention, we rescale and redefine $A = A_\mu^\alpha \frac{T^\alpha}{i} g dx^\mu = A_\mu^\alpha (-ig T^\alpha) dx^\mu = A_\mu (-ig) dx^\mu$, thus we also redefine the field strength

$$F = \frac{1}{2} F_{\mu\nu}^\alpha (-ig T^\alpha) (dx^\mu \wedge dx^\nu) = \frac{1}{2} (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g f^{\beta\gamma\alpha} A_\mu^\beta A_\nu^\gamma) T^\alpha (dx^\mu \wedge dx^\nu). \quad (1.10)$$

The g is YM coupling constant. We also have the covariant derivative:

$$D_\mu = \partial_\mu - ig a_\mu^\alpha T^\alpha. \quad (1.11)$$

The field strength $F_{\mu\nu}^\alpha$ is not gauge invariant under the gauge transformation with $V = \exp(i\eta^\alpha T^\alpha)$:

$$A_\mu^\alpha T^\alpha \rightarrow V (A_\mu^\alpha T^\alpha + \frac{i}{g} \partial_\mu) V^\dagger \simeq (A_\mu^\alpha + \frac{1}{g} \partial_\mu \eta^\alpha - f^{\beta\gamma\alpha} \eta^\beta A_\mu^\gamma) T^\alpha + \dots, \quad (1.12)$$

since the field strength transforms to

$$F_{\mu\nu}^\alpha T^\alpha \rightarrow (F_{\mu\nu}^\alpha T^\alpha) - (f^{\beta\gamma\alpha}\eta^\beta F_{\mu\nu}^\gamma T^\alpha) + \dots \quad (1.13)$$

There are color-electric field and color-magnetic field related to the field strength

$$E_j^\alpha := E^{\alpha j} := F_{0j}^\alpha = \partial_0 A_j^\alpha - \partial_j A_0^\alpha + g f^{\beta\gamma\alpha} A_0^\beta A_j^\gamma = -\partial_t A^{\alpha j} - \nabla_j A^{\alpha 0} - g f^{\beta\gamma\alpha} A^{\beta 0} A^{\gamma j}. \quad (1.14)$$

$$\begin{aligned} B_j^\alpha := B^{\alpha j} &:= -\frac{1}{2}\varepsilon^{jkl} F_{kl}^\alpha = -\frac{1}{2}\varepsilon^{jkl} F^{\alpha kl} = +\frac{1}{2}\varepsilon_{jkl}(\partial_k A^{\alpha l} - \partial_l A^{\alpha k} - g f^{\beta\gamma\alpha} A^{\beta k} A^{\gamma l}) \\ &= (\vec{\nabla} \times \vec{A})^{\alpha j} - \frac{1}{2}g f^{\beta\gamma\alpha}(\vec{A}^\beta \times \vec{A}^\gamma)^j. \end{aligned} \quad (1.15)$$

Both color-electric field and color-magnetic field are *not* gauge-invariant

$$E_j^\alpha T^\alpha \rightarrow (E_j^\alpha T^\alpha) - (f^{\beta\gamma\alpha}\eta^\beta E_j^\gamma T^\alpha) + \dots, \quad B_j^\alpha T^\alpha \rightarrow (B_j^\alpha T^\alpha) - (f^{\beta\gamma\alpha}\eta^\beta B_j^\gamma T^\alpha) + \dots \quad (1.16)$$

But the Yang-Mills action and its path integral \mathbf{Z}_{YM} are gauge invariant (we set a normalization factor $\kappa_{\text{YM}} = \frac{1}{2}c(G)^{-1}$ for the convenience to match the standard convention):

$$\begin{aligned} \mathbf{Z}_{\text{YM}} &= \int [\mathcal{D}A] e^{i\frac{1}{g^2} \int -\text{Tr}(\kappa_{\text{YM}} F \wedge \star F)} = \int [\mathcal{D}A] e^{i\frac{1}{g^2} \int d^4x \text{Tr}((-\frac{1}{2})\kappa_{\text{YM}} F_{\mu\nu} F^{\mu\nu})} \\ &= \int [\mathcal{D}A] e^{i\frac{c(G)}{g^2} \int d^4x (-\frac{1}{2}\kappa_{\text{YM}} F_{\mu\nu}^\alpha F^{\alpha,\mu\nu})} = \int [\mathcal{D}A] e^{i \int d^4x \frac{1}{4} (F^{\alpha,\mu\nu} F_{\mu\nu}^\alpha)} \\ &= \int [\mathcal{D}A_\mu^\alpha] e^{i \int d^4x \frac{1}{4} ((F^{\alpha,0j} F_{0j}^\alpha) + (F^{\alpha,jk} F_{\alpha,jk}))} = \int [\mathcal{D}A_\mu^\alpha] e^{i \int d^4x (-\frac{1}{2}) ((E_j^\alpha)^2 - (B_j^\alpha)^2)}. \end{aligned} \quad (1.17)$$

In the next section, we move on to develop and summarize our new exotic higher-rank tensor gauge theory.

2 Class of New Non-Abelian Higher-Rank Tensor Gauge Theories

2.1 Euclidean or Lorentz Invariant Non-Abelian Higher-Rank Tensor Gauge Theory

Below we can formulate theories in both the Euclidean spacetime \mathbb{R}^{d+1} or in the Minkowski (or Lorentzian) spacetime $\mathbb{R}^{1,d}$. The two versions of theories have the same form of path integral and the action, thus we will present them together. We denote the space and time coordinates as $(t, \vec{x}_j) = (x_\mu)$. Our formulation below can be regarded as modifying Pretko's spacetime anisotropic theory [33] to a spacetime isotropic theory (up to the signature of Minkowski metric).

2.1.1 Matter field theory and higher-moment vector global symmetry

We can start from a matter field theory in $(d+1)d$ with the global symmetry including the spacetime symmetry and the internal symmetry. We focus on first the scalar charge matter theory with a vector global symmetry. Let the complex charge matter field called $\Phi(t, \vec{x}) = \Phi(x) \in \mathbb{C}$,

1. Internal symmetry:

The usual ordinary (0-form) U(1) global symmetry transformation acts on the charged matter as,

$$\Phi \rightarrow e^{i\eta}\Phi, \quad (2.1)$$

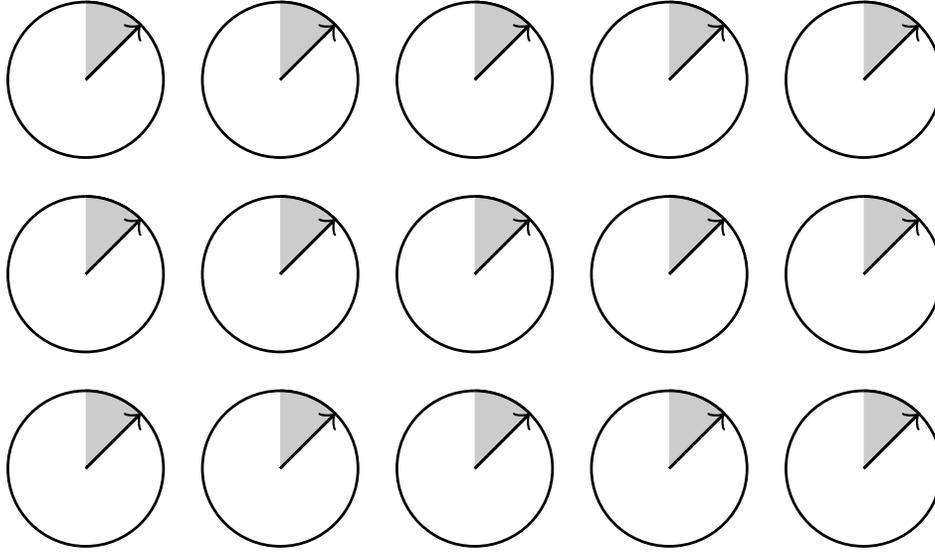


Figure 1: The ordinary (0-form) U(1) global symmetry transformation acts on the complex charged matter Φ , e.g. the rotor field. Thus we get Eq. (2.1): $\Phi \rightarrow e^{i\eta}\Phi$.

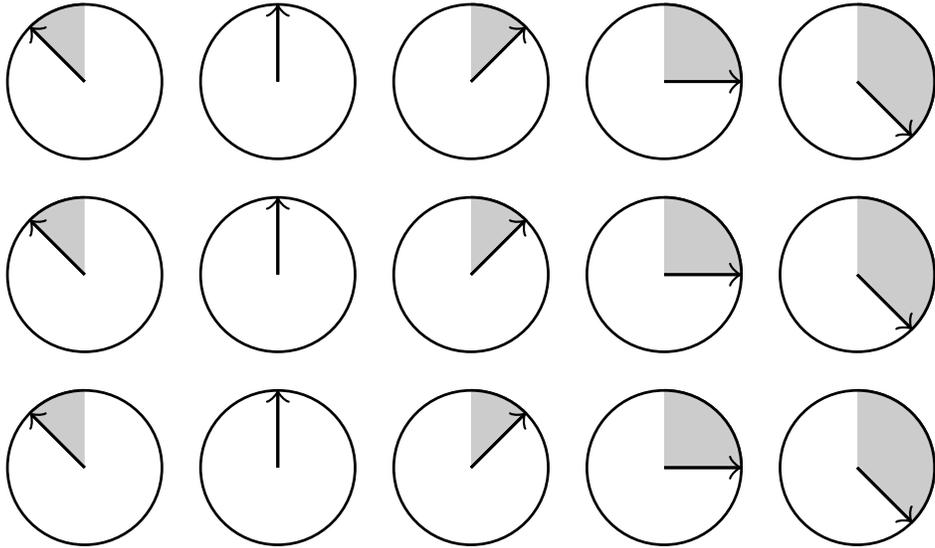


Figure 2: The vector global symmetry belongs to a generalized class of higher-moment symmetry. The vector U(1) global symmetry transformation acts on the complex charged matter Φ , e.g. the rotor field. Thus we get Eq. (2.2): $\Phi \rightarrow e^{i\eta_\nu(x)}\Phi := e^{i\Lambda \cdot x}\Phi$. The angle $\Lambda \cdot x$ depends on a reference point (say $x = 0$) and the distance x away from the reference point.

where η is a global parameter associated to the U(1) scalar charge, independent of spacetime (t, \vec{x}) . See Fig. 1.

In addition, we like to impose an additional vector global symmetry over the scalar charge matter

$$\Phi \rightarrow e^{i\eta_v(x)}\Phi := e^{i\Lambda \cdot x}\Phi. \quad (2.2)$$

Here Λ is a $(d+1)$ -vector on the spacetime, so the $\Lambda \cdot x$ takes the inner product with the spacetime coordinate. We term this

$$\text{U}(1)_{x_{(d+1)}} \text{ vector global symmetry} \quad (2.3)$$

due to the involvement of the $d+1$ d spacetime vectors, Λ and x . The vector global symmetry belongs to a generalized class of higher-moment symmetry. See Fig. 2.

In fact, the $\text{U}(1)_{x_{(d+1)}}$ involves several symmetry transformations under any $(d+1)$ linear-independent choice of $\Lambda = (\Lambda_0, \Lambda_1, \dots, \Lambda_d)$. So Eq. (2.2) can be chosen to be $e^{i\Lambda_0 \cdot x_0}, e^{i\Lambda_1 \cdot x_1}, \dots, e^{i\Lambda_d \cdot x_d}$. Thus Eq. (2.4) contains

$$\text{Eq. (2.4)} := \text{a set of } (d+1) \text{ independent } \text{U}(1)_{x_0}, \text{U}(1)_{x_1}, \dots, \text{U}(1)_{x_d}, \text{ vector global symmetries.} \quad (2.4)$$

Note that however $\text{U}(1)_{x_0}$ is not quite a standard global symmetry in terms of the Hamiltonian theory. Nonetheless we still abuse the name global symmetry for $\text{U}(1)_{x_0}$. The phenomenon here of $\text{U}(1)_{x_0}$ can be potentially related to a generalization of *time crystal* [35, 36], since

$$\Phi \rightarrow \Phi e^{i\Lambda_0 \cdot x_0} = \Phi e^{i\Lambda_0 \cdot t}, \quad (2.5)$$

while $x_0 = t$ is time coordinate. So the field configuration at a certain periodic time $t \simeq t + \frac{2\pi}{\Lambda_0} \mathbb{Z}$ is constrained. We will comment more in the Conclusion in Sec. 5.

Now let us write down the matter field theory Lagrangian. To recall, in order to have a Lagrangian kinetic term invariant under *only the ordinary global symmetry*, we have the familiar kinetic term $\partial^\mu \Phi^\dagger \partial_\mu \Phi$.

In order to have a Lagrangian kinetic term invariant under the vector global symmetry (alternatively, under both the ordinary and vector global symmetry), we should abandon the familiar kinetic term $\partial^\mu \Phi^\dagger \partial_\mu \Phi$ but design a new term:

$$(\Phi^\dagger \partial^\mu \partial^\nu \Phi^\dagger - \partial^\mu \Phi^\dagger \partial^\nu \Phi^\dagger)(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi). \quad (2.6)$$

It is easy to check that under Eq. (2.2) transformation,

$$(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi) \rightarrow e^{i2\eta_v(x)} (\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi + (i\partial_\mu \partial_\nu \eta_v) \Phi^2), \quad (2.7)$$

thus this term is covariant up to a factor, if η_v is in terms of a linear polynomial of x , namely here $\Lambda \cdot x$, rather than higher-order polynomial. Hence Eq. (2.6) is invariant under both the ordinary and vector global symmetry Eq. (2.1) and Eq. (2.2).

There is also a discrete \mathbb{Z}_2^C charge conjugation symmetry:

$$\Phi \rightarrow \Phi^\dagger \quad (2.8)$$

It is easy to see that the ordinary $\text{U}(1)$ and \mathbb{Z}_2^C forms a non-abelian $\text{U}(1) \rtimes \mathbb{Z}_2^C = \text{SO}(2) \rtimes \mathbb{Z}_2^C = \text{O}(2)$ -symmetry. The full internal symmetry in both Euclidean/Minkowski signature is

- Euclidean/Minkowski internal (including ordinary and higher-moment) global symmetry:

$$\mathbb{Z}_2^C \rtimes \left(\text{U}(1) \times \text{U}(1)_{x_{(d+1)}} \right)$$

2. Spacetime symmetry:

- Euclidean version's Poincaré symmetry: $\mathbb{R}^{d+1} \rtimes \text{O}(d+1)$, which includes Euclidean spacetime translation, rotation, and reflection (say R -symmetry, for Euclidean spacetime, this includes the parity P and time reversal T as the same symmetry). Note that the reflection R -symmetry is a \mathbb{Z}_2^R discrete symmetry, related to the 0-th homotopy group $\pi_0 \text{O}(d+1) = \mathbb{Z}_2$, so the reflection R flips between two disconnected components of $\text{O}(d+1)$.
- Minkowski version's Poincaré symmetry: $\mathbb{R}^{1,d} \rtimes \text{O}(1,d)$, which includes Minkowski spacetime translation, boost, rotation, and the parity P and time reversal T . Note that the P and T symmetry are both \mathbb{Z}_2 discrete symmetries, with $\mathbb{Z}_2^P: x_j \rightarrow -x_j$ for some j , and with $\mathbb{Z}_2^T: t \rightarrow -t$ over the spacetime coordinates. They are related to the $\pi_0 \text{O}(1,d) = (\mathbb{Z}_2)^2$, so the P and T flip between four disconnected components of $\text{O}(1,d)$.

$\text{U}(1)_{x_{(d+1)}}$ has a nontrivial action on the Poincaré symmetry, thus we can define a semi-direct product structure: $(\mathbb{R}^{d+1} \rtimes \text{O}(d+1)) \rtimes \text{U}(1)_{x_{(d+1)}}$. Combine the internal and spacetime symmetry together, we obtain the full global symmetry.

- Euclidean version's symmetry:

$$\left(\mathbb{R}^{d+1} \rtimes \text{O}(d+1) \right) \rtimes \left(\mathbb{Z}_2^C \rtimes \left(\text{U}(1) \times \text{U}(1)_{x_{(d+1)}} \right) \right). \quad (2.9)$$

- Minkowski version's symmetry is similar:

$$\left(\mathbb{R}^{1,d} \rtimes \text{O}(1,d) \right) \rtimes \left(\mathbb{Z}_2^C \rtimes \left(\text{U}(1) \times \text{U}(1)_{x_{(d+1)}} \right) \right). \quad (2.10)$$

Note that the 0-form symmetry $\mathbb{Z}_2^C \rtimes \text{U}(1)$ in fact commute with the Poincaré symmetry, thus we have the structure $(\mathbb{R}^{d+1} \rtimes \text{O}(d+1)) \times (\mathbb{Z}_2^C \rtimes \text{U}(1))$ if we disregard the vector global symmetry $\text{U}(1)_{x_{(d+1)}}$.

2.1.2 Symmetric higher-rank tensor gauge theory and the gauging procedure

Now we follow the gauge principle to gauge the higher-moment symmetry $\text{U}(1)_{x_{(d+1)}}$ in $(\text{U}(1) \times \text{U}(1)_{x_{(d+1)}})$. To recall the standard gauging procedure of the ordinary 0-form global symmetry, we promote the global symmetry transformation parameter η in Eq. (2.1) to a spacetime-dependent local transformation $\eta(x)$. Thus under $\Phi \rightarrow e^{i\eta(x)}\Phi$, the derivative term $\partial_\mu \Phi$ is not covariant unless we revise it to a covariant derivative term

$$(\partial_\mu - ig a_\mu)\Phi.$$

It is covariant under the familiar gauge transformation:

$$\Phi \rightarrow e^{i\eta(x)}\Phi, \quad a_\mu \rightarrow a_\mu + \frac{1}{g}\partial_\mu(\eta(x)). \quad (2.11)$$

See Fig. 3 for a demonstration of the gauge fluctuation away from the 0-form global symmetry.

Follow [33], the way we do is to promote the global parameters η and $\eta_v(x) = \Lambda \cdot x$ to depend on the spacetime: $\eta \rightarrow \eta(x)$ and $\eta_v \rightarrow \eta_v(x) = \Lambda(x) \cdot x$. In this case, in order to make Eq. (2.7) covariant, we revise it to a new term:

$$(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi - ig A_{\mu\nu} \Phi^2) \quad (2.12)$$

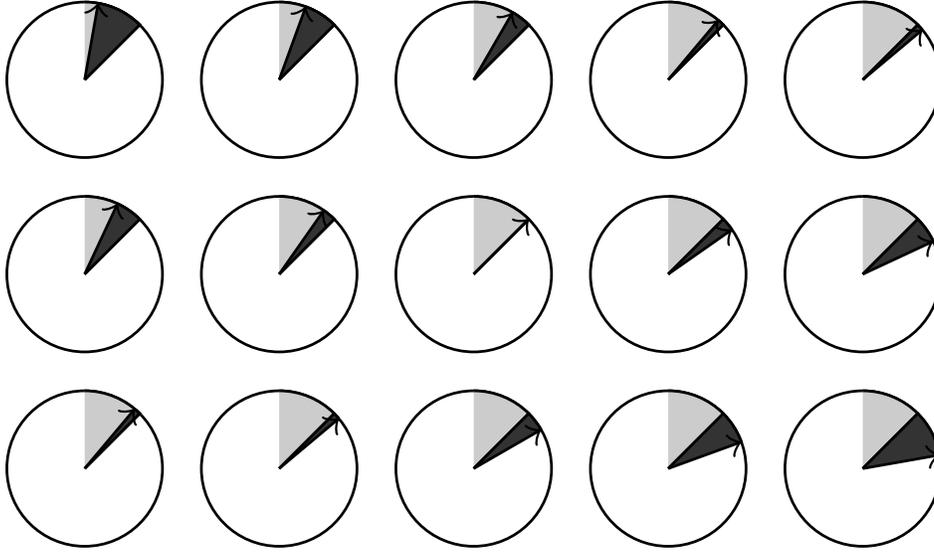


Figure 3: A demonstration of the gauge fluctuation (the dark gray color) deviates away from the 0-form global symmetry transformation of Fig. 1 (the light gray color). See Eq. (2.11).

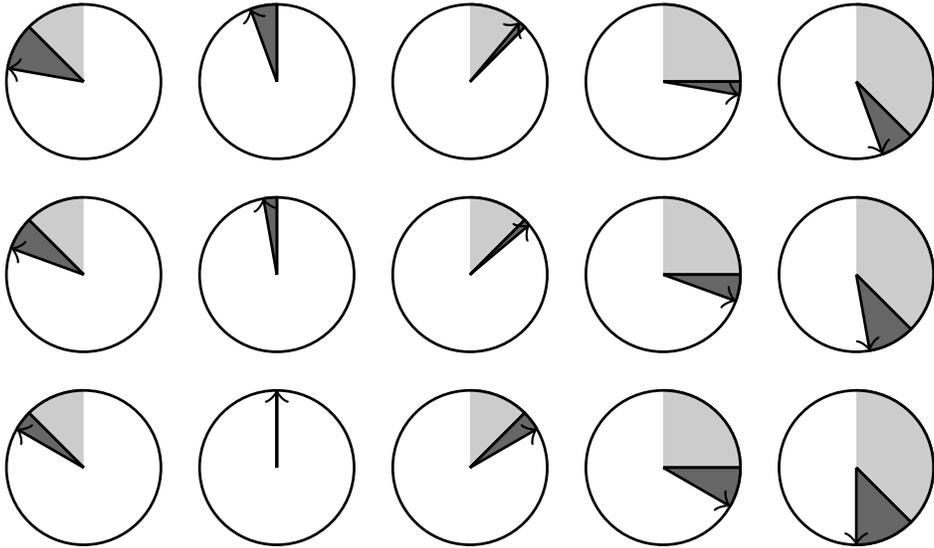


Figure 4: A demonstration of the gauge fluctuation (the dark gray color) deviates away from the vector global symmetry of Fig. 2 (the light gray color, which belongs to a generalized class of higher-moment symmetry). See Eq. (2.13).

such that it is covariant under the new gauge transformation

$$\Phi \rightarrow e^{i\eta_v(x)}\Phi, \quad A_{\mu\nu} \rightarrow A_{\mu\nu} + \frac{1}{g}\partial_\mu\partial_\nu(\eta_v(x)). \quad (2.13)$$

The $A_{\mu\nu}$ is a symmetric rank-2 tensor of $(d+1) \times (d+1)$ components in a $(d+1)d$ spacetime.

A major difference of ours apart from previous work is that we treat spacetime indices isotropically and equally (up to the metric signature) instead of an-isotropically as in [33]. After gauge $U(1)_{x_{(d+1)}}$ higher-moment symmetry, we have a ‘‘gauge group-analogous structure’’

$$\left[U(1)_{x_{(d+1)}} \right], \quad (2.14)$$

where the big bracket $[\dots]$ stands for being dynamically gauged.⁴ We can define the rank-3 field strength $F_{\mu\nu\xi}$ of the symmetric rank-2 tensor $A_{\nu\xi}$ as:

$$F_{\mu\nu\xi} = \partial_\mu A_{\nu\xi} - \partial_\nu A_{\mu\xi}. \quad (2.15)$$

$F_{\mu\nu\xi}$ is anti-symmetric with respect to the first two indices $\mu \leftrightarrow \nu$. And we can define the kinetic Lagrangian term for gauge fields as:

$$\mathcal{L}_{A,\text{kinetic}} := \frac{1}{g^2} |F_{\mu\nu\xi}|^2 := \frac{1}{g^2} F_{\mu\nu\xi} F^{\mu\nu\xi} = \frac{1}{g^2} (\partial_\mu A_{\nu\xi} - \partial_\nu A_{\mu\xi})(\partial^\mu A^{\nu\xi} - \partial^\nu A^{\mu\xi}). \quad (2.16)$$

The equation of motion (EOM) for a pure gauge theory is

$$\partial^\mu F_{\mu\nu\xi} = \partial^\mu (\partial_\mu A_{\nu\xi} - \partial_\nu A_{\mu\xi}) = 0. \quad (2.17)$$

We certainly can write down the classical field theory by giving the action, with or without matter field (which we can set $\Phi = 0$):

$$\int_{M^{d+1}} d^{d+1}x \left(\frac{1}{g^2} |F_{\mu\nu\xi}|^2 + |(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi - ig A_{\mu\nu} \Phi^2)|^2 + V(|\Phi|^2) \right). \quad (2.18)$$

Here $V(|\Phi|^2)$ is a potential term. The M^{d+1} is a $d + 1$ d spacetime manifold. Ideally, we hope to discuss a quantum theory. Formally, we propose a schematic path integral:

$$\mathbf{Z}_{\text{rk-2-sym-}\Phi} = \int [\mathcal{D}A_{\mu\nu}] [\mathcal{D}\Phi] \exp(i \int_{M^{d+1}} d^{d+1}x \left(\frac{1}{g^2} |F_{\mu\nu\xi}|^2 + |(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi - ig A_{\mu\nu} \Phi^2)|^2 + V(|\Phi|^2) \right)). \quad (2.19)$$

We shorthand ‘‘rk-2-sym- Φ ’’ for the rank-2 symmetric tensor gauge theory coupled to matters. Unfortunately, this field theory has no free nor quadratic term that we can start with to do a perturbative theory to extract the higher-order term effects. In fact, due to the leading order term is already highly nontrivial interacting quartic interactions, a fully fledged quantum field theory by a field quantization is still an open question. We comment more the quantum aspects of the theory in Conclusion Sec. 5.

2.1.3 Field strength, electric and magnetic tensors: Independent components and representations

The rank-3 field strength $F_{\mu\nu\xi}$ can also be interpreted as the higher-rank generalized electric tensor field $\bar{\mathbf{E}}$ and magnetic tensor field $\bar{\mathbf{B}}$. To do so, we take those $F_{\mu\nu\xi}$ have an *odd number of time indices* (0 or t) as the electric tensor field $\bar{\mathbf{E}}$, while those $F_{\mu\nu\xi}$ have an *even number of time indices* (0 or t) as the magnetic tensor field $\bar{\mathbf{B}}$. Above in this section, we have a full general discussion in any dimension. Below, we focus

⁴Again, it is only ‘‘group-analogous structure’’ but not quite a group, because the 0-form symmetry (\mathbb{Z}_2^C) and higher moment symmetry ($U(1)_{x_{(d+1)}}$) are fundamentally different. Note that we only gauge $U(1)_{x_{(d+1)}}$ but we do not gauge $U(1)$ in ($U(1) \times U(1)_{x_{(d+1)}}$), but the $U(1)$ global symmetry would be lost altogether after gauging $U(1)_{x_{(d+1)}}$. Importantly, we do not introduce the usual 1-form $U(1)$ gauge field, because we do not gauge 0-form $U(1)$ symmetry. We introduce the symmetric rank-2 tensor $A_{\nu\xi}$ to only gauge the $U(1)_{x_{(d+1)}}$ symmetry.

on the 4d (3+1D) spacetime, where $i, j, k, \ell \dots \in \{1, 2, 3\}$ are space coordinates only without time. Let us count independent (\equiv indpt) components. We define:

$$\begin{aligned} \bar{\mathbf{E}} &:= \begin{cases} \bar{\mathbf{E}}_{ij} := F_{0ij} = -F_{i0j} = \partial_0 A_{ij} - \partial_i A_{0j}, \text{ with } 3^2 = 9\text{-indpt components} \\ \bar{\mathbf{E}}_\ell := \frac{1}{2} \epsilon^{\ell ij} F_{ij0}, \text{ with } i, j \text{ summed.} \\ \text{where } F_{ij0} = -F_{j i 0} = \partial_i A_{j0} - \partial_j A_{i0}, \text{ thus } i \neq j \text{ for 3-indpt components} \end{cases} \quad (2.20) \\ \bar{\mathbf{B}} &:= \begin{cases} \bar{\mathbf{B}}_{\ell k} := \frac{1}{2} \epsilon^{\ell ij} F_{ijk} = \frac{1}{2} \epsilon^{\ell ij} (\partial_i A_{jk} - \partial_j A_{ik}), \text{ with } i, j \text{ summed.} \\ \text{where } F_{ijk} = -F_{jik} = \partial_i A_{jk} - \partial_j A_{ik}, \text{ thus } i \neq j \text{ with } \frac{3 \cdot 2}{2} 3 = 9\text{-indpt components.} \\ \bar{\mathbf{B}}_j := F_{0j0} = -F_{j00} = \partial_0 A_{j0} - \partial_j A_{00}, \text{ with 3-indpt components.} \end{cases} \quad (2.21) \end{aligned}$$

Thus both $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ tensor contains 12 components. Here the 9 means the spatial rank-2 tensor (say $i, j \in \{1, 2, 3\}$), and the 3 means spatial vector (say $j \in \{1, 2, 3\}$). We can understand the above in terms of representation of spatial rotation symmetry group $\text{SO}(3)$. The $\text{SO}(3)$ has a $(2n+1)$ -dimensional representation, we need that $n=0$ gives the trivial representation $\mathbf{1}$, $n=1$ which gives the vector representation $\mathbf{3}$, and the $n=2$ gives the 5-dimensional representation $\mathbf{5}$. The $\bar{\mathbf{E}}_\ell$ and $\bar{\mathbf{B}}_j$ give the the vector representation $\mathbf{3}$ with 3 components. The $\bar{\mathbf{E}}_{ij}$ and $\bar{\mathbf{B}}_{\ell k}$ give the decomposition of representations $(\mathbf{1} + \mathbf{3} + \mathbf{5})$ with 9 components. Thus, we have both tensor fields decomposed in terms of spatial rotational $\text{SO}(3)$ representations:

$$\begin{aligned} \bar{\mathbf{E}} &= \mathbf{3} + (\mathbf{1} + \mathbf{3} + \mathbf{5}), \\ \bar{\mathbf{B}} &= \mathbf{3} + (\mathbf{1} + \mathbf{3} + \mathbf{5}). \end{aligned} \quad (2.22)$$

2.1.4 Gauge the \mathbb{Z}_2^C -charge conjugation symmetry: Non-abelian higher-moment continuous gauge-structure

There is a discrete charge conjugation \mathbb{Z}_2^C ordinary unitary 0-form symmetry, which flips the complex scalar Φ , the real gauge field $A_{\mu\nu}$, and the vector ‘‘gauge symmetry transformation’’ parameter $\eta_v(x)$ as follows

$$\Phi \rightarrow \Phi^\dagger, \quad A_{\mu\nu} \rightarrow -A_{\mu\nu}, \quad \eta_v(x) \rightarrow -\eta_v(x). \quad (2.23)$$

Thus, the Eq. (2.12) under \mathbb{Z}_2^C -symmetry transformation becomes its complex conjugation:

$$(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi + ig A_{\mu\nu} \Phi^2) \rightarrow (\Phi^\dagger \partial_\mu \partial_\nu \Phi^\dagger - \partial_\mu \Phi^\dagger \partial_\nu \Phi^\dagger - ig A_{\mu\nu} (\Phi^\dagger)^2), \quad (2.24)$$

such that the action Eq. (2.18) is \mathbb{Z}_2^C -invariant. It is easy to see that the ordinary $\text{U}(1)$ global symmetry and \mathbb{Z}_2^C of Eq. (2.9) and Eq. (2.10) does not commute. Take the symmetry transformation operator as $U_{\text{U}(1)}$ and $U_{\mathbb{Z}_2^C}$ respectively, we see that the transformations act on $A_{\mu\nu}$ and Φ as:

$$\begin{aligned} U_{\mathbb{Z}_2^C} U_{\text{U}(1)} A_{\mu\nu} &= U_{\mathbb{Z}_2^C} (A_{\mu\nu} + \frac{1}{g} \partial_\mu \partial_\nu \eta_v) = -A_{\mu\nu} + \frac{1}{g} \partial_\mu \partial_\nu \eta_v. \\ U_{\text{U}(1)} U_{\mathbb{Z}_2^C} A_{\mu\nu} &= U_{\mathbb{Z}_2^C} (-A_{\mu\nu}) = -A_{\mu\nu} - \frac{1}{g} \partial_\mu \partial_\nu \eta_v. \end{aligned} \quad (2.25)$$

$$\begin{aligned} U_{\mathbb{Z}_2^C} U_{\text{U}(1)} \Phi &= U_{\mathbb{Z}_2^C} (e^{i\eta_v} \Phi) = e^{i\eta_v} \Phi^\dagger. \\ U_{\text{U}(1)} U_{\mathbb{Z}_2^C} \Phi &= U_{\mathbb{Z}_2^C} (\Phi^\dagger) = e^{-i\eta_v} \Phi^\dagger. \end{aligned} \quad (2.26)$$

Thus $U_{\mathbb{Z}_2^C} U_{\text{U}(1)} \neq U_{\text{U}(1)} U_{\mathbb{Z}_2^C}$, we have a non-abelian symmetry group structure the $\mathbb{Z}_2^C \times \text{U}(1) = \text{O}(2)$.

Below we work out gauging \mathbb{Z}_2^C after gauging $\text{U}(1)_{x_{(d+1)}}$

Thus, it prompts us to also gauge the \mathbb{Z}_2^C symmetry. After gauge $U(1)_{x_{(d+1)}}$ higher-moment symmetry in Eq. (2.13), and then we can further gauge \mathbb{Z}_2^C ordinary 0-form symmetry where the big bracket [...]. So among the global symmetry of Eq. (2.9) and Eq. (2.10), after we denote such a gauge theory with the new “gauge group-analogous structure” as:

$$\left[\mathbb{Z}_2^C \times \left(U(1)_{x_{(d+1)}} \right) \right], \quad (2.27)$$

importantly the ordinary $U(1)$ -global symmetry of Eq. (2.9) and Eq. (2.10) is not a global symmetry anymore after gauging $U(1)_{x_{(d+1)}}$ the way we did in Eq. (2.13). Again $\left[\mathbb{Z}_2^C \times \left(U(1)_{x_{(d+1)}} \right) \right]$ is not quite a group, which is definitely not just $O(2)$, but only as what we call a “gauge group-analogous structure.”

Let us temporarily turn off the matter Φ and focus on promoting the global \mathbb{Z}_2^C to a local symmetry that can be gauged in a pure tensor gauge theory:

$$A_{\mu\nu} \rightarrow e^{i\gamma_c(x)} A_{\mu\nu}, \quad C_\nu \rightarrow C_\nu + \frac{1}{g_c} \partial_\nu \gamma_c(x) \quad (2.28)$$

with a new g_c coupling and the new 1-form \mathbb{Z}_2^C -gauge field C coupling to the 0-form symmetry \mathbb{Z}_2^C -charged object $A_{\mu\nu}$. Note $A_{\mu\nu}$ is real-valued, so a generic $e^{i\gamma_c(x)}$ complexified the $A_{\mu\nu}$. However, what we really mean is to restrict gauge transformation so it is only \mathbb{Z}_2^C -gauged (not $U(1)^C$ -gauged

$$e^{i\gamma_c(x)} := (-1)^{\gamma'_c(x)} \in \{\pm 1\} \quad (2.29)$$

so $\gamma'_c(x)$ is an integer and $A_{\mu\nu}$ stays in real. Thus $\gamma'_c(x)$ can jumps between even or odd integers, while the \mathbb{Z}_2^C -gauge transformation is better formulated on a lattice. We can directly rewrite the above Eq. (2.28) on a simplicial complex and a triangulable manifold.

We define a new covariant derivative with respect to \mathbb{Z}_2^C :

$$D_\mu^c := (\partial_\mu - i g_c C_\mu). \quad (2.30)$$

We need to modify $U(1)_{x_{(d+1)}}$ -gauge transformation Eq. (2.13) involving the \mathbb{Z}_2^C -covariant derivative and combine with \mathbb{Z}_2^C -gauge transformation Eq. (2.28):

$$\begin{aligned} A_{\mu\nu} &\rightarrow e^{i\gamma_c(x)} A_{\mu\nu} + \frac{1}{g} D_\mu^c D_\nu^c (\eta_v(x)) \\ &= e^{i\gamma_c(x)} A_{\mu\nu} + \frac{1}{g} \left(\partial_\mu \partial_\nu (\eta_v) - i g_c \partial_\mu (C_\nu \eta_v) - i g_c C_\mu (\partial_\nu \eta_v) - (g_c)^2 C_\mu C_\nu \eta_v \right). \\ C_\nu &\rightarrow C_\nu + \frac{1}{g_c} \partial_\nu \gamma_c(x). \end{aligned} \quad (2.31)$$

We re-define Eq. (2.15)'s $F_{\mu\nu\xi}$ into the new gauge covariant field strength $\hat{F}_{\mu\nu\xi}^c$:

$$\boxed{\hat{F}_{\mu\nu\xi}^c := D_\mu^c A_{\nu\xi} - D_\nu^c A_{\mu\xi} := (\partial_\mu - i g_c C_\mu) A_{\nu\xi} - (\partial_\nu - i g_c C_\nu) A_{\mu\xi}}, \quad (2.32)$$

which is covariant under both the modify $U(1)_{x_{(d+1)}}$ gauge transformation and \mathbb{Z}_2^C gauge transformation Eq. (2.31):

$$\begin{aligned} \hat{F}_{\mu\nu\xi}^c &\rightarrow e^{i\gamma_c(x)} \hat{F}_{\mu\nu\xi}^c - i \frac{g_c}{g} (C_\mu \partial_\nu \partial_\xi \eta_v - C_\nu \partial_\mu \partial_\xi \eta_v) + \frac{(-i g_c)^2}{g} (C_\mu \partial_\nu (C_\xi \eta_v) - C_\nu \partial_\mu (C_\xi \eta_v)) \\ &\quad - i \frac{g_c}{g} (\partial_\mu (C_\nu (\partial_\xi \eta_v)) - \partial_\nu (C_\mu (\partial_\xi \eta_v))) - \frac{g_c^2}{g} (\partial_\mu (C_\nu C_\xi \eta_v) - \partial_\nu (C_\mu C_\xi \eta_v)) \\ &= e^{i\gamma_c(x)} \hat{F}_{\mu\nu\xi}^c - i \frac{g_c}{g} (\partial_\mu C_\nu - \partial_\nu C_\mu) D_\xi^c \eta_v \\ &= e^{i\gamma_c(x)} \hat{F}_{\mu\nu\xi}^c \end{aligned} \quad (2.33)$$

invariant up to a \pm phase Eq. (2.29) $e^{i\gamma_c(x)} := (-1)^{\gamma_c(x)} \in \{\pm 1\}$. Here we can eliminate $(\partial_\mu C_\nu - \partial_\nu C_\mu)D_\xi^c \eta_\nu$ via $(\partial_\mu C_\nu - \partial_\nu C_\mu) = 0$ because $dC = 0$ is locally flat via the local constraint for \mathbb{Z}_2 -gauge field C . Let us put things altogether in the next paragraph.

Since C is a 1-form \mathbb{Z}_2 -gauge field, we can either (1) treat C as a \mathbb{Z}_2 -valued 1-cochain, or (2) treat it as a $U(1)$ gauge field but impose a locally flat condition $dC = 0$ and $\oint C = m\pi \pmod{2\pi}$, with $m \in \mathbb{Z}_2$. For the later purpose for (2), this can be done by introducing a Lagrange multiplier $(d-1)$ -form B field (in $3+1d$, we have $d = 3$). We introduce the famous level-2 anti-symmetric (\equiv asym) tensor BF theory,

$$\mathbf{Z}_{\text{level-2-BF}} = \int [\mathcal{D}B][\mathcal{D}C] \exp(i \int_{M^{d+1}} i \frac{2}{2\pi} B dC). \quad (2.34)$$

So far we have a $(d-1)$ -form \mathbb{Z}_2 -gauge B field and 1-form \mathbb{Z}_2 -gauge field C , obviously $BdC := B \wedge dC$. Note it is also a standard convention to omit the wedge product \wedge when we are taking the wedge product between the differential forms. Furthermore, ideally and for simplicity, by leaving out the matter Φ , we can aim for a full quantum theory for a gauge theory of a gauge-analogous structure Eq. (2.27). Formally, we propose a schematic path integral:

$$\mathbf{Z}_{\substack{\text{rk-2-sym} \\ \text{asym-BF}}} := \int [\mathcal{D}A_{\mu\nu}][\mathcal{D}B][\mathcal{D}C] \exp(i \int_{M^{d+1}} d^{d+1}x \left(\frac{1}{g^2} |\hat{F}_{\mu\nu\xi}^c|^2(d^{d+1}x) + i \frac{2}{2\pi} B dC \right)), \quad (2.35)$$

non-trivially fully gauge-invariant under Eq. (2.31), the gauge transformation of B field via a local $(d-2)$ -form η_B :

$$B \rightarrow B + d\eta_B \quad (2.36)$$

and Eq. (2.33). We shorthand ‘‘rk-2-sym’’ for the rank-2 symmetric tensor gauge theory. We shorthand ‘‘asym-BF’’ for the anti-symmetric tensor BF theory. Our theory is unitary. At this stage, we only deal with the kinematics of the theory, we discuss its possible quantum dynamics in Conclusion Sec. 5.

2.1.5 Coupling to anti-symmetric tensor topological field theories

Another new ingredient of our approach is that we can introduce \mathbb{Z}_2^C -gauge theory Eq. (2.35) which is an anti-symmetric tensor topological field theory. A formal way to introduce this \mathbb{Z}_2^C -gauge theory in $(d+1)d$ is that it is a group cocycle element ω_{d+1} of the topological gauge theory specified by the group cohomology H^{d+1} of the gauge group G_g :

$$\omega_{d+1} \in H^{d+1}(G_g, \mathbb{R}/\mathbb{Z}) := \mathcal{H}^{d+1}(BG_g, \mathbb{R}/\mathbb{Z}), \quad (2.37)$$

or the topological cohomology \mathcal{H}^{d+1} of the classifying space of BG_g in the second expression [21]. The cohomology group $H^{d+1}(G_g, \mathbb{R}/\mathbb{Z})$ always forms an abelian group.

If we only have a single copy of an abelian symmetric higher-rank tensor gauge theory with a single gauge structure $\left[\mathbb{Z}_2^C \times \left(U(1)_{x_{(d+1)}} \right) \right]$, then we can determine possible \mathbb{Z}_2^C -gauge theory via $H^{d+1}(\mathbb{Z}_2^C, \mathbb{R}/\mathbb{Z})$. More generally, we can consider N copies of such abelian symmetric higher-rank tensor gauge theory with N copies of gauge structure

$$\prod_{J=1}^N \left[\mathbb{Z}_2^C \times \left(U(1)_{x_{(d+1)}} \right) \right]. \quad (2.38)$$

So we have the ordinary gauge group

$$G_g = [(\mathbb{Z}_2^C)^N] \quad (2.39)$$

thus a $[(\mathbb{Z}_2^C)^N]$ -gauge theory in addition to some higher-moment gauge structure (formally not a group), then we can specify a group cocycle [21], which is computed systematically in Ref. [10, 11]

$$\omega_{d+1} \in \mathbf{H}^{d+1}(G_g, \mathbb{R}/\mathbb{Z}) = \mathbf{H}^{d+1}((\mathbb{Z}_2^C)^N, \mathbb{R}/\mathbb{Z}). \quad (2.40)$$

See Table 1 for explicit results for a generic cohomology group.

- If the ω_{d+1} is a trivial cocycle, namely the relation

$$\omega_{d+1} = \delta\beta_d,$$

which is a coboundary term under the coboundary operator δ and a lower dimensional d -cochain $\beta_d \in C^d(G_g, \mathbb{R}/\mathbb{Z})$. This means that $\omega_{d+1} \simeq 1$ is identically to the identity in the cohomology group.

- If the ω_{d+1} is a nontrivial cocycle, namely $\omega_{d+1} \neq \delta\beta_d$ is not exact for any cochain β_d , the gauge theory is commonly called as the *twisted group cohomology gauge theory*, or *Dijkgraaf-Witten gauge theory* (\equiv DW) of gauge group G_g [21]. Ref. [10, 11, 15, 18–20] had systematically studied these topological gauge theories as (i) discrete cocycle partition functions, and, (ii) continuum TQFTs. See Table 2 and Sec. 3.1 for the overview of their continuum TQFT formulations Ref. [11, 15, 18, 20].

Cohomology group	Type I	Type II	Type III	Type IV	Type V
	\mathbb{Z}_{N_i}	$\mathbb{Z}_{N_{ij}}$	$\mathbb{Z}_{N_{ijl}}$	$\mathbb{Z}_{N_{ijlm}}$	$\mathbb{Z}_{\text{gcd}\otimes_i^5(N^{(i)})}$	$\mathbb{Z}_{\text{gcd}\otimes_i^m(N_i)}$	$\mathbb{Z}_{\text{gcd}\otimes_i^d N^{(i)}}$
$\mathbf{H}^1(G, \mathbb{R}/\mathbb{Z})$	1						
$\mathbf{H}^2(G, \mathbb{R}/\mathbb{Z})$	0	1					
$\mathbf{H}^3(G, \mathbb{R}/\mathbb{Z})$	1	1	1				
$\mathbf{H}^4(G, \mathbb{R}/\mathbb{Z})$	0	2	2	1			
$\mathbf{H}^5(G, \mathbb{R}/\mathbb{Z})$	1	2	4	3	1		
$\mathbf{H}^6(G, \mathbb{R}/\mathbb{Z})$	0	3	6	7	4		
$\mathbf{H}^d(G, \mathbb{R}/\mathbb{Z})$	$\frac{1-(-1)^d}{2}$	$\frac{d}{2} - \frac{1-(-1)^d}{4}$	1

Table 1: Cohomology group: Table from Ref. [11] exhibits the exponent of the $\mathbb{Z}_{\text{gcd}\otimes_i^m(N_i)}$ class in the cohomology group $\mathbf{H}^d(G_g, \mathbb{R}/\mathbb{Z})$ for a finite Abelian group $G_g = \prod^k \mathbb{Z}_{N_u}$. We denote the greatest common divisor \equiv gcd. We define a shorthand of $\mathbb{Z}_{\text{gcd}(N_{i_1}, N_{i_2}, \dots, N_{i_m})} \equiv \mathbb{Z}_{N_{i_1, \dots, i_m}} \equiv \mathbb{Z}_{\text{gcd}\otimes_i^m(N_i)}$, etc also for other higher gcd. The definition of the Type m in the top row is from its number (m) of cyclic gauge groups in the gcd class $\mathbb{Z}_{\text{gcd}\otimes_i^m(N_i)}$. The number of exponents can be systematically obtained by Künneth formula and Universal Coefficient Theorem. This table can also be independently derived by gathering the data of from field theory approach. For example, we obtain $\mathbf{H}^5(G_g, \mathbb{R}/\mathbb{Z}) = \prod_{1 \leq i < j < l < m < n \leq k} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{ijl}})^4 \times (\mathbb{Z}_{N_{ijlm}})^3 \times \mathbb{Z}_{N_{ijlmn}}$, etc. We can derive the continuum field theory from the group cohomology result, or the other way around [11, 15, 18]. For the purpose of present work, we simply take Eq. (2.39): $G_g = [(\mathbb{Z}_2^C)^N]$.

Now that $\omega_{d+1}(\{C_I\}) \in \mathbf{H}^{d+1}(G_g, \mathbb{R}/\mathbb{Z})$ maps to the $\mathbb{R}/\mathbb{Z} = \text{U}(1)$ coefficient, a complex $\text{U}(1)$ phase, physically $\omega_{d+1}(\{C_I\})$ is the weight of the quantum amplitude in the path integral. Thus, the sum over such a $\omega_{d+1}(\{C_I\})$ is also related to the orbifold construction.

$$\mathbf{Z}_{\text{rk-2-sym}}^{\text{asym-tw}} := \int \left(\prod_{I=1}^N [\mathcal{D}A_{I, \mu\nu}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp \left(i \int_{M^{d+1}} \frac{1}{g^2} |\hat{F}_{\mu\nu}^c|^2 (d^{d+1}x) + i \frac{2}{2\pi} \sum_{I=1}^N B_I dC_I \right) \cdot \omega_{d+1}(\{C_I\}) \quad (2.41)$$

where $\omega_{d+1} \in \mathbf{H}^{d+1}((\mathbb{Z}_2^C)^N, \mathbb{R}/\mathbb{Z})$. We shorthand “asym-DW” for the anti-symmetric tensor twisted (tw) Dijkgraaf-Witten (DW) gauge theory. Adding the DW topological term, Eq. (2.36) needs to be modified

to $B \rightarrow B + d\eta_B + \dots$, some examples of the additional \dots terms are shown in Table 2. We can apply the variation principle on the A, B and C fields to gain their EOM (i.e., Euler-Lagrange equation):

$$\frac{\delta}{\delta A^{\nu\xi}}(\dots) \Rightarrow D^{c\mu} \hat{F}_{\mu\nu\xi}^c = (\partial^\mu - ig_c C^\mu) \left((\partial_\nu - ig_c C_\nu) A_{\mu\xi} - (\partial_\nu - ig_c C_\nu) A_{\mu\xi} \right) = 0. \quad (2.42)$$

$$\frac{\delta}{\delta B}(\dots) \Rightarrow \frac{2}{2\pi} dC = 0, \quad (2.43)$$

$$\frac{\delta}{\delta C}(\dots) \Rightarrow \frac{2}{2\pi} dB = \# C_I \wedge C_J \wedge \dots \wedge dC_K - i4g_c \# \star (A^{\nu\xi} \hat{F}_{\mu\nu\xi}^c dx^\mu) \quad (2.44)$$

Here $\#$ are some factors depending on the data ω_{d+1} in terms of the continuum TQFTs given in Table 2. In Eq. (2.42), Eq. (2.43) and Eq. (2.44), we only list down the schematic result since we do not yet precisely provide the data of the group cohomology cocycle ω_{d+1} , which will be given later explicitly in Sec. 3.1. The allowed *topological term* (or the twisted term) $C_I \wedge C_J \wedge \dots \wedge dC_K$ depends on the spacetime dimensions and the copies of G_g , see Table 1 and Table 2. Here are some comments about EOM:

- ⎧ In Eq. (2.42), the dependence of A and C reveals a first ingredient of the *non-abelian* gauge structure.
- ⎧ In Eq. (2.43), in any case, $dC = 0$ is always a locally flat \mathbb{Z}_2 gauge field.
- ⎧ In Eq. (2.44), the dependence of B to A and C reveals a second ingredient for the possible *non-abelian* gauge structure. See more details in Sec. 3.1.2

Of course, the discussions in this whole subsection have two versions, simple by changing the spacetime metric signatures: *Euclidean-type higher-rank symmetric tensor gauge theory* Eq. (1.3) and *Lorentzian-type higher-rank symmetric tensor gauge theory* Eq. (1.4). Their physics interpretations would be different.

2.2 Anisotropic Non-Abelian Higher-Rank Tensor Gauge Theory for Space and Time

Let us consider the *anisotropic-type higher-rank symmetric tensor gauge theory* Eq. (1.2). Let us list down the modification for this new case Eq. (1.2) from the previous subsection.

2.2.1 Electric and magnetic tensors: Independent components and SO(3) representations

First, we need to introduce a symmetric rank-2 tensor A_{ij} of $d \times d$ components in a $(d+1)d$ spacetime. We also need a scalar potential A_0 . Instead of the $F_{\mu\nu\xi}$ in Eq. (2.15), or the electromagnetic tensor \bar{E} in Eq. (2.20) and \bar{B} in Eq. (2.21), we redefine the electric field and magnetic field tensors similar to Pretko's Ref. [29–33]:

$$\tilde{E}_{ij} := -\partial_0 A_{ij} + \partial_i \partial_j A_0 = -\partial_t A_{ij} + \partial_i \partial_j A_0. \quad (2.45)$$

$$\tilde{B}_{ij} := \varepsilon_{i\ell m} \partial_\ell A_{mj} \quad (2.46)$$

(Naively we can also write $\tilde{B}_{ijk} \propto \partial_j A_{ik} - \partial_k A_{ij}$ as a rank-3 tensor, but it is not necessary.) Now that \tilde{E}_{ij} and \tilde{B}_{ij} are rank-2 tensors. In $(3+1)d$ spacetime, we have $3 \times 3 = 9$ components for each. Let us count the independent \tilde{E}_{ij} and \tilde{B}_{ij} in terms of the spatial rotational SO(3) representations as we did in Eq. (2.22). Note that \tilde{E}_{ij} is symmetric, thus we have \tilde{E}_{ii} 3 components and $\tilde{E}_{i \neq j}$ another 3 components. Note that \tilde{B}_{ij} is neither symmetric nor anti-symmetric, thus 9 components. We have both tensor fields decomposed in terms of SO(3) representations:

$$\begin{aligned} \tilde{E} &= (\mathbf{1} + \mathbf{5}), \\ \tilde{B} &= (\mathbf{1} + \mathbf{3} + \mathbf{5}). \end{aligned} \quad (2.47)$$

The $\mathbf{1}$ in \tilde{E} is from the identity component $\frac{1}{3} \sum_{i=1}^3 \tilde{E}_{ii} (\mathbb{1}_3)$. The $\mathbf{5}$ in \tilde{E} are from other components.

2.2.2 Matter field theory and symmetric higher-rank tensor gauge theory

We restrict our global symmetry transformation to the space in contrast to Eq. (2.2):

$$\Phi \rightarrow e^{i\eta_v(x)}\Phi := e^{i\vec{\Lambda}\cdot\vec{x}}\Phi. \quad (2.48)$$

Here $\vec{\Lambda}$ is a (d) -vector on the space, so the $\vec{\Lambda}\cdot\vec{x}$ takes the inner product with the space coordinate. In contrast to Eq. (2.4), we term this

$$U(1)_{x(d)} \text{ vector global symmetry.} \quad (2.49)$$

Promoting the vector global symmetry transformation to a gauge transformation, follow Sec. 2.1.1, this yields Pretko's [31] with some couplings g_0 and λ :

$$\mathbf{Z}'_{\text{rk-2-sym-}\Phi} := \int [\mathcal{D}A_{I,0}][\mathcal{D}A_{I,ij}][\mathcal{D}\Phi] \exp(i \int_{M^{d+1}} d^{d+1}x \left(\frac{1}{g^2} (|\tilde{\mathbf{E}}_{ij}|^2 - |\tilde{\mathbf{B}}_{ij}|^2) + |(\partial_t - ig_0 A_0)\Phi|^2 - \lambda |(\Phi \partial_i \partial_j \Phi - \partial_i \Phi \partial_j \Phi - ig A_{ij} \Phi^2)|^2 + V(|\Phi|^2) + \dots \right)). \quad (2.50)$$

gauge invariant under:

$$\Phi \rightarrow e^{i\eta_v(x)}\Phi, \quad A_0 \rightarrow A_0 + \frac{1}{g}\partial_0(\eta_v(x)) = A_0 + \frac{1}{g}\partial_t(\eta_v(x)), \quad A_{ij} \rightarrow A_{ij} + \frac{1}{g}\partial_i\partial_j(\eta_v(x)). \quad (2.51)$$

There are other possible terms listed in \dots , see, for instance, Ref. [31].

2.2.3 Gauge \mathbb{Z}_2^C and another non-abelian higher-moment-gauged tensor theory

Similar to Sec. 2.1.4, we aim to obtain a non-abelian tensor gauge theory by gauging the ordinary 0-form \mathbb{Z}_2^C charge conjugation symmetry:

$$\Phi \rightarrow \Phi^\dagger, \quad A_0 \rightarrow -A_0, \quad A_{ij} \rightarrow -A_{ij}, \quad \eta_v(x) \rightarrow -\eta_v(x). \quad (2.52)$$

Under the same \mathbb{Z}_2^C -covariant derivative Eq. (2.30)'s $D_\mu^c := (\partial_\mu - ig_c C_\mu)$, similar to Eq. (2.31) and Eq. (2.29), and we couple the gauged theory to DW topological terms as in Sec. 2.1.5, we have the combined gauge transformation

$$\begin{aligned} A_0 &\rightarrow e^{i\gamma_c(x)}A_0 + \frac{1}{g}D_0^c(\eta_v(x)) = e^{i\gamma_c(x)}A_0 + \frac{1}{g}D_t^c(\eta_v(x)). \\ A_{ij} &\rightarrow e^{i\gamma_c(x)}A_{ij} + \frac{1}{g}D_\mu^c D_\nu^c(\eta_v(x)) \\ &= e^{i\gamma_c(x)}A_{ij} + \frac{1}{g}\left(\partial_i\partial_j(\eta_v) - ig_c\partial_i(C_j\eta_v) - ig_c C_i(\partial_j\eta_v) - (g_c)^2 C_i C_j \eta_v\right). \\ C_j &\rightarrow C_j + \frac{1}{g_c}\partial_j\gamma_c(x). \\ e^{i\gamma_c(x)} &:= (-1)^{\gamma_c(x)} \in \{\pm 1\}. \\ B &\rightarrow B + d\eta_B + \dots \end{aligned} \quad (2.53)$$

A \mathbb{Z}_2^C -gauge transformation can be much easier defined in a discretized spacetime. We also refine the $\tilde{E}_{ij}, \tilde{B}_{ij}$ to the covariant derivative Eq. (2.30) version's $\tilde{E}_{ij}^c, \tilde{B}_{ij}^c$:

$$\tilde{E}_{ij}^c := -D_0^c A_{ij} + D_i^c D_j^c A_0 = -D_t^c A_{ij} + D_i^c D_j^c A_0. \quad (2.54)$$

$$\tilde{B}_{ij}^c := \varepsilon_{ilm} D_\ell^c A_{mj} \quad (2.55)$$

Examples of the additional ... terms are shown in Table 2. Again we couple a symmetric tensor gauge theory to a DW term $\omega_{d+1}(\{C_I\}) \in H^{d+1}(G_g, \mathbb{R}/\mathbb{Z})$ shown in Ref. [11, 15, 18–20] to gain another unitary field theory with anisotropic space time symmetry:⁵

$$\mathbf{z}'_{\text{rk-2-sym}} := \int \left(\prod_{I=1}^N [\mathcal{D}A_{I,0}] [\mathcal{D}A_{I,ij}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp\left(i \int_{M^{d+1}} \frac{1}{g^2} (|\tilde{E}_{ij}^c|^2 - |\tilde{B}_{ij}^c|^2) (d^{d+1}x) \right. \\ \left. + i \frac{2}{2\pi} \sum_{I=1}^N B_I dC_I \right) \cdot \omega_{d+1}(\{C_I\}) \quad (2.56)$$

3 Non-Abelian Tensor Gauge Theory Interplayed Between Gapped And Gapless Phases

Now we provide evidence that our theories in Eq. (2.41) and Eq. (2.56) are in the interplay between gapped phases (see Sec. 3.1) and gapless phases (see Sec. 3.2). Let us organize our statements line by line.

1. The *phases* mean that *the states of matter in the phase diagram* of the quantum matter or condensed matter sense.
2. The *gapped* phases mean that the excitations in the phases are massive and highly energetic expansive. In fact, in the TQFT limit, it costs infinite energy to break up the extend operators (such as line and surface operators) with open ends — only the open ends can host massive particles or heavy objects.
3. The *gapless* phases mean that the dispersion of excitations is continuous and massless in the large system size limit.
4. The anti-symmetric tensor TQFTs describe gapped phases shown in Sec. 3.1. The *gapped excitations* are the anyonic particles (described by the worldline of C field), or the anyonic strings/ $(d-3)$ -branes (described by the worldsheet/volume of B field).
5. The symmetric tensor gauge theories describe gapless phases shown in Sec. 3.2.

3.1 Anti-Symmetric Tensor Topological Field Theory and Gapped Phase

Shown in Table 2, here we quickly overview the continuum field theory formulation [15, 18, 19] of topological gauge theory from a cohomology group [21]: $H^d(G_g, \mathbb{R}/\mathbb{Z})$ for a finite Abelian group $G_g = \prod_{u=1}^k \mathbb{Z}_{N_u}$, or

⁵Readers should beware that the definitions of B and \tilde{B} are fundamentally different. We use B for the any-symmetric tensor field (here $(d-2)$ -form gauge field), here we use \tilde{B} to represent the magnetic tensor field in its anisotropic version.

specifically a gauge group $(\mathbb{Z}_2^C)^m$ of gauged charge conjugation symmetry.

The first column of Table 2 shows the continuum anti-symmetric tensor TQFT actions and their *gauge transformations to all order in analytic exact forms*. Namely, Table 2 contains a finite gauge transformations, instead of only infinite gauge transformations. There is no further higher order term we can add to the *gauge transformations* beyond Table 2.

In fact, the term in the path integral of Eq. (2.41) and Eq. (2.56) $\int([\mathcal{D}B_I][\mathcal{D}C_I]) \exp(i \int_{M^{d+1}} i \frac{2}{2\pi} \sum_{I=1}^N B_I dC_I)$. $\omega_{d+1}(\{C_I\})$ can be replaced to this continuum theory

$$\int([\mathcal{D}B_I][\mathcal{D}C_I]) \exp(i \int_{M^{d+1}} i \frac{2}{2\pi} \sum_{I=1}^N B_I \wedge dC_I) + \#C_I \wedge C_J \wedge \cdots \wedge dC_K). \quad (3.1)$$

with some coefficients $\#$, shown in Table 2.

The EOMs follow Eq. (2.42), Eq. (2.43), Eq. (2.44). The first two do not depend on the topological term ω_{d+1} while the third EOM does:

$$\frac{\delta}{\delta C} = 0 \implies \frac{2}{2\pi} dB = +\xi_\omega - 4ig_c \star (A^{\nu\xi} F_{\mu\nu}^c dx^\mu) = 0. \quad (3.2)$$

The ξ_ω depends on ω_{d+1} and is proportional to:

- (1). $\frac{p_{IJ}}{2\pi} dC^J$
- (2). $c_{123} C^2 \wedge C^3 + \text{permutations } (1 \leftrightarrow 2 \leftrightarrow 3)$
- (3). 0
- (4). $\frac{N_I N_J p_{IJK}}{(2\pi)^2 N_{IJ}} C^J \wedge dC^K$
- (5). $c_{1234} C^2 \wedge C^3 \wedge C^4 + \text{permutations } (1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4)$

corresponding to the five cases listed in Table 2.

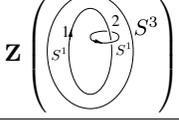
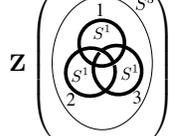
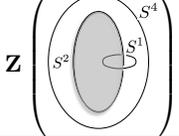
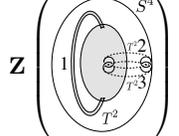
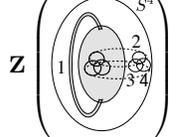
(i). TQFT actions	(ii). Group-cohomology cocycles associated to (i)	(iii). Path-integral link invariants; Quantum statistic spacetime-braiding data characterized by (i)
3d (2+1D) spacetime		
$\int \sum_I \frac{N_I}{2\pi} B^I \wedge dC^I + \frac{p_{IJ}}{2\pi} C^I \wedge dC^J$ $C^I \rightarrow C^I + \frac{d\gamma_c^I}{g_c},$ $B^I \rightarrow B^I + d\eta_B^I.$	$\omega_3 = \exp\left(\frac{2\pi i p_{IJ}}{N_I N_J} a_I (b_J + c_J - [b_J + c_J]_{N_J})\right)$ $\in \mathbb{H}^3((\mathbb{Z}_2^C)^2, \mathbb{R}/\mathbb{Z})$	
$\int \sum_I \frac{N_I}{2\pi} B^I \wedge dC^I + c_{123} C^1 \wedge C^2 \wedge C^3$ $C^I \rightarrow C^I + \frac{d\gamma_c^I}{g_c},$ $B^I \rightarrow B^I + d\eta_B^I + 2\pi \frac{\tilde{c}_{IJK}}{N_I} C^J \frac{\gamma_c^K}{g_c}$ $- \pi \frac{\tilde{c}_{IJK}}{N_I} \frac{\gamma_c^J}{g_c} \frac{d\gamma_c^K}{g_c}.$	$\omega_3^{\text{Top}} = \exp\left(\frac{2\pi i p_{123}}{N_{123}} a_1 b_2 c_3\right)$ $\in \mathbb{H}^3((\mathbb{Z}_2^C)^3, \mathbb{R}/\mathbb{Z})$	
4d (3+1D) spacetime		
$\int \frac{N_I}{2\pi} B^I \wedge dC^I$ $C^I \rightarrow C^I + \frac{d\gamma_c^I}{g_c},$ $B^I \rightarrow B^I + d\eta_B^I.$	$\omega_4 = 1$	
$\int \sum_I \frac{N_I}{2\pi} B^I \wedge dC^I + \sum_{I,J} \frac{N_I N_J}{(2\pi)^2} \frac{p_{IJK}}{N_{IJ}} C^I \wedge C^J \wedge dC^K$ $C^I \rightarrow C^I + \frac{d\gamma_c^I}{g_c},$ $B^I \rightarrow B^I + d\eta_B^I + \varepsilon_{IJ} \frac{N_J}{2\pi N_{IJ}} \frac{p_{IJK}}{N_{IJ}} \frac{d\gamma_c^J}{g_c} \wedge C^K,$ <p style="text-align: center;">here K is fixed.</p>	$\omega_4 = \exp\left(\frac{2\pi i p_{IJK}}{(N_{IJ} \cdot N_K)} (a_I b_J)(c_K + d_K - [c_K + d_K]_{N_K})\right)$ $\in \mathbb{H}^4((\mathbb{Z}_2^C)^3, \mathbb{R}/\mathbb{Z})$	
$\int \sum_I \frac{N_I}{2\pi} B^I \wedge dC^I + c_{1234} C^1 \wedge C^2 \wedge C^3 \wedge C^4$ $C^I \rightarrow C^I + \frac{d\gamma_c^I}{g_c},$ $B^I \rightarrow B^I + d\eta_B^I - \pi \frac{\tilde{c}_{IJKL}}{N_I} C^J C^K \frac{\gamma_c^L}{g_c}$ $+ \pi \frac{\tilde{c}_{IJKL}}{N_I} C^J \frac{\gamma_c^K}{g_c} \frac{d\gamma_c^L}{g_c} - \frac{\pi}{3} \frac{\tilde{c}_{IJKL}}{N_I} \frac{\gamma_c^J}{g_c} \frac{d\gamma_c^K}{g_c} \frac{d\gamma_c^L}{g_c}.$	$\omega_4^{\text{Top}} = \exp\left(\frac{2\pi i p_{1234}}{N_{1234}} a_1 b_2 c_3 d_4\right)$ $\in \mathbb{H}^4((\mathbb{Z}_2^C)^4, \mathbb{R}/\mathbb{Z})$	

Table 2: Continuum field theory formulation [15, 18, 19] of topological gauge theory from cohomology group [21] of Table 1. Here we list down a gauge group $G_g = (\mathbb{Z}_2^C)^m$ because of the limitation of our model's charge conjugation symmetry, but we can replace it to a more generic gauge group $(\mathbb{Z}_{N_I})^m$, see Ref. [11, 15, 18–20]. The first column shows the proposed continuum anti-symmetric tensor TQFT actions for these theories and their *gauge transformations to all order in analytic exact forms*. The second column shows the group-cohomology cocycle data ω as a certain partition-function solution of Dijkgraaf-Witten theory, where ω belongs to the group-cohomology group, $\omega \in \mathbb{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}) = \mathbb{H}^{d+1}(G, \mathbb{U}(1))$. The notations such as the mod N_J summation for $[\dots]_{N_J}$ are introduced in Ref. [11]. The third column shows the path integral form which encodes the braiding process of particles (1-worldline) and strings (2-worldsheet) in the spacetime. In 2+1D (3d), C and B are 1-forms, while γ_c and η_B are 0-forms. In 3+1D (4d), B is a 2-form, C and η_B are 1-forms, while γ_c is a 0-form. The $I, J, K \in \{1, 2, 3, \dots\}$ belongs to the gauge subgroup indices, $N_{12\dots u} \equiv \text{gcd}(N_1, N_2, \dots, N_u)$ is defined as the greatest common divisor (gcd) of N_1, N_2, \dots, N_u . Here $p_{IJ} \in \mathbb{Z}_{N_{IJ}}, p_{123} \in \mathbb{Z}_{N_{123}}, p_{IJK} \in \mathbb{Z}_{N_{IJK}}, p_{1234} \in \mathbb{Z}_{N_{1234}}$ are integer coefficients. The $c_{IJ}, c_{123}, c_{IJK}, c_{1234}$ are quantized coefficients labeling distinct topological gauge theories, where $c_{12} = \frac{1}{(2\pi)} \frac{N_1 N_2 p_{12}}{N_{12}}, c_{123} = \frac{1}{(2\pi)^2} \frac{N_1 N_2 N_3 p_{123}}{N_{123}}, c_{1234} = \frac{1}{(2\pi)^3} \frac{N_1 N_2 N_3 N_4 p_{1234}}{N_{1234}}$. The quantization labelings are described and derived in [10, 11]. Be aware that we define both $p_{IJ\dots}$ and $c_{IJ\dots}$ as constants with *fixed-indices* I, J, \dots without summing over those indices. We additionally define $\tilde{c}_{IJ\dots} \equiv \epsilon_{IJ\dots} c_{12\dots}$ with the $\epsilon_{IJ\dots} = \pm 1$ as an anti-symmetric Levi-Civita alternating tensor where I, J, \dots are *free indices* needed to be Einstein-summed over. The $c_{12\dots}$ is fixed.

Readers can find the explicit derivation of:

- { *link invariants and braiding statistics* with extended operators from a continuum TQFT in [18].
- { $\dim(\mathcal{H}_d) = \text{GSD}$ associated to d -dimensional space (counting zero energy modes) in [19].

3.1.1 Abelian TQFTs v.s. Top-Type Non-Abelian TQFTs

In fact, when we have the number of copies of charge conjugation (\mathbb{Z}_2^C) -symmetry as the same as $N = d + 1$ of total spacetime dimensions, then we are allowed to enjoy an exotic cocycle $\omega_{d+1}^{\text{Top}} \in \mathbb{H}^{d+1}((\mathbb{Z}_2^C)^{N=d+1}, \mathbb{R}/\mathbb{Z})$, the path integral in the form of continuum gauge theory in Eq. (3.1) can be written as

$$\int ([\mathcal{D}B_I][\mathcal{D}C_I]) \exp(i \int_{M^{d+1}} \sum_{I=1}^N \frac{N_I}{2\pi} B^I \wedge dC^I + p \frac{N_1 N_2 \dots N_{d+1}}{(2\pi)^d N_{1\dots d, d+1}} C^1 \wedge \dots \wedge C^{d+1}) \cdot \omega_{d+1}^{\text{Lower}} \quad (3.3)$$

with $p \in \mathbb{Z}_{N_{1\dots d, d+1}} = \mathbb{Z}_2$, with all of $N_I = 2$ thanks to (\mathbb{Z}_2^C) , such that the cocycle $\omega_{d+1}^{\text{Top}}$ is known as the Top Type (For this Top Type, see References therein [34] and [10, 11, 15, 18, 19]). These Top Type TQFTs are *non-abelian* topological orders in nature. While other Lower types $\omega_{d+1}^{\text{Lower}}$ of cocycle twists *alone* produce *abelian* topological orders. Combine both the Lower and Top Types of cocycle twists as in Eq. (3.3) *also* produce *additional non-abelian* topological orders.

By non-Abelian topological orders, we mean that some of the following properties are matched:

- The $\dim \mathcal{H}_{S^d; \sigma_1, \sigma_2, \sigma_3, \dots} = \mathbf{Z}(S^d \times S^1; \sigma_1, \sigma_2, \sigma_3, \dots) = \text{GSD}$ on a sphere S^d with operator insertions (or the insertions of anyonic particle/string excitations on S^d) show the following behavior:
 - (1) $\mathbf{Z}(S^d \times S^1; \sigma_1, \sigma_2, \sigma_3, \dots) = \dim \mathcal{H}_{S^d; \sigma_1, \sigma_2, \sigma_3, \dots}$ will grow exponentially as k^n for a certain set of large n number of insertions, for some number k . The anyonic particle causes this behavior is called non-Abelian anyon or non-Abelian particle. The anyonic string causes this behavior can be called non-Abelian string [10].
 - (2) $\mathbf{Z}(S^d \times S^1; \sigma_1, \sigma_2, \sigma_3) = \dim \mathcal{H}_{S^d; \sigma_1, \sigma_2, \sigma_3} > 1$ for a certain set of three insertions.
- Prepare a certain configuration (with a degenerate Hilbert space $\dim \mathcal{H}_{T^d} > 1$, e.g. as the previous remark), the adiabatic braiding process of gapped excitations will rotate the ground state-vector $|\Psi_{\text{initial}}\rangle$ in the Hilbert space into a new state-vector $|\Psi_{\text{final}}\rangle$ not parallel nor up to a phase. Namely, $|\Psi_{\text{final}}\rangle = U_{\text{braid}} |\Psi_{\text{initial}}\rangle \not\propto e^{i\theta_{\text{abelian}}} |\Psi_{\text{initial}}\rangle$. This means the U_{braid} matrix is generically non-commutative thus non-abelian. In Table 3, we give these examples of abelian TQFTs v.s. non-abelian TQFTs in 2+1D and in 3+1D.
- The $\dim \mathcal{H}_{T^d} = \mathbf{Z}(T^d \times S^1) = \text{GSD}$ for a discrete gauge theory of a gauge group G on T^d spatial torus, behaves as $\text{GSD} < |G_g|^d$, i.e. reduced to a smaller number than Abelian GSD. This criterion however works only for 1-form gauge theory (here the 1-form C field). See the detail explanation in the footnote 7 of [19].

$\dim \mathcal{H}_{T^d} = \text{GSD}_{T^d},$ $G_g = (\mathbb{Z}_2^C)^{N=d+1}$ in $d + 1\text{D}$	Abelian TQFT $\text{GSD}_{T^d} = G_g ^d$	Non-abelian TQFT $\text{GSD}_{T^d} < G_g ^d$
2+1D	$(2^3)^d = 64$	22
3+1D	$(2^4)^d = 4096$	1576

Table 3: Data of zero energy modes counting below the TQFT energy gap, as $\dim \mathcal{H}_{T^d} = \text{GSD}_{T^d}$. Data is taken from [10, 19].

3.1.2 Non-abelian tensor gauge theory with gapless modes, massive particles and gapped non-abelian anyonic strings in 3+1D (4d), and gapped non-abelian anyonic branes in 4+1D (5d)

Below let us write down the 3+1D (4d) and 4+1D (5d) models explicitly that have non-abelian anyonic excitations. Both are non-abelian tensor gauge theory with gapless modes. Both have massive particles (from the end of worldline of C field). The 3+1D (4d) model has gapped anyonic strings (the end of worldsheet of B field). The 4+1D (5d) model has gapped anyonic branes (the end of worldvolume of B field). These models may have applications to dark matter and dark energy, see Conclusion Sec. 5.

Here are the 3+1D (4d) and 4+1D (5d) models with schematic path integrals written in generality:⁶

$$\begin{aligned} \mathbf{z}_{\text{rk-2-sym}}^{\text{3+1D}} := & \int \left(\prod_{I=1}^N [\mathcal{D}A_{I,\mu\nu}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp\left(i \int_{M^{d+1}} \frac{1}{g^2} (|\hat{\mathbf{E}}^c|^2 - |\hat{\mathbf{B}}^c|^2) (d^{d+1}x) \right. \\ & \left. + \sum_{I=1}^N \frac{N_I}{2\pi} B^I \wedge dC^I + p \frac{N_1 N_2 N_3 N_4}{(2\pi)^3 N_{1234}} C^1 \wedge C^2 \wedge C^3 \wedge C^4 \right) \cdot \omega_4^{\text{Lower}}(\{C_I\}). \quad (3.4) \end{aligned}$$

$$\begin{aligned} \mathbf{z}_{\text{rk-2-sym}}^{\text{4+1D}} := & \int \left(\prod_{I=1}^N [\mathcal{D}A_{I,\mu\nu}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp\left(i \int_{M^{d+1}} \frac{1}{g^2} (|\hat{\mathbf{E}}^c|^2 - |\hat{\mathbf{B}}^c|^2) (d^{d+1}x) \right. \\ & \left. + \sum_{I=1}^N \frac{N_I}{2\pi} B^I \wedge dC^I + p \frac{N_1 N_2 N_3 N_4 N_5}{(2\pi)^4 N_{12345}} C^1 \wedge C^2 \wedge C^3 \wedge C^4 \wedge C^5 \right) \cdot \omega_5^{\text{Lower}}(\{C_I\}). \quad (3.5) \end{aligned}$$

For models of Euclidean/Lorentz version in Sec. 2.1, we have

$$(|\hat{\mathbf{E}}^c|^2 - |\hat{\mathbf{B}}^c|^2) := |\hat{F}_{\mu\nu\xi}^c|^2 = \left((\partial_\mu - ig_c C_\mu) A_{\nu\xi} - (\partial_\nu - ig_c C_\nu) A_{\mu\xi} \right) \left((\partial^\mu - ig_c C^\mu) A^{\nu\xi} - (\partial^\nu - ig_c C^\nu) A^{\mu\xi} \right),$$

where $\hat{\mathbf{E}}^c, \hat{\mathbf{B}}^c$ should be the covariant version of $\bar{\mathbf{E}}, \bar{\mathbf{B}}$ in Eq. (2.20) and Eq. (2.21).

For models of anisotropic version in Sec. 2.2, we have $(|\hat{\mathbf{E}}^c|^2 - |\hat{\mathbf{B}}^c|^2) := (|\tilde{\mathbf{E}}_{ij}^c|^2 - |\tilde{\mathbf{B}}_{ij}^c|^2)$ given by Eq. (2.54) and Eq. (2.55), and $[\mathcal{D}A_{I,\mu\nu}]$ should be replaced by $[\mathcal{D}A_{I,0}][\mathcal{D}A_{I,ij}]$.

3.2 Symmetric Higher-Rank Tensor Gauge Theory and Gapless Phase

We discuss that the symmetric higher-rank tensor gauge theory with an action alone $\int_{M^{d+1}} \frac{1}{g^2} |\hat{F}_{\mu\nu\xi}^c|^2 (d^{d+1}x)$ in $d+1$ dimensions (at least $d+1 \geq 3+1$) can contain gapless phase with massless modes.

3.2.1 Degrees of freedom for gapless modes

Degrees of freedom (\equiv dof) counting for gapless modes:

⁶Readers should beware that the definitions of B and \mathbf{B} are fundamentally different. We use B for the any-symmetric tensor field (here $(d-2)$ -form gauge field), while we use $\mathbf{B}, \bar{\mathbf{B}}, \tilde{\mathbf{B}}, \hat{\mathbf{B}}$, etc., to represent the magnetic tensor field from the field strength $F_{\mu\nu\xi}$ or its anisotropic version. Similarly, we use $\hat{\mathbf{B}}^c$, etc., to represent the magnetic tensor field from the field strength $\hat{F}_{\mu\nu\xi}^c$

1. U(1) gauge theory of Sec. 1.1.1:

Let us recall the standard dof counting for gapless (or massless) modes, e.g., the photons, in the spin-1 Maxwell theory in 3+1D (4d) in Sec. 1.1.1. Let a single photon moves in the speed of light say with a momentum \vec{p}_z along an arbitrary z direction.

Let us first count the photon dof physically. Naively the photon can either oscillate along the longitudinal direction (1 dof along \vec{p}_c) and the transverse direction (2 dof perpendicular to \vec{p}_z). However, the longitudinal mode does not make sense because the photon cannot oscillate forward faster than the speed of light. Thus there are only 2 transverse modes: say along the p_x and p_y direction. Then their linear combinations $p_x + ip_y$ and $p_x - ip_y$ are effectively the two eigenvectors for the spin angular momentum operator \hat{S}_z of the photon. Namely, there is a map of 2 dof:

$$p_x \pm ip_y \implies \langle \hat{S}_z \rangle = \pm 1 \implies |S, S_z\rangle = |1, \pm 1\rangle.$$

These are in fact also the 2 dof from the 2 helicity from the 2 transverse modes

$$\hat{S}_z \cdot \hat{P}_z = \pm 1.$$

Let us also count the dof mathematically. We see that a photon field is a 4-vector A_μ with $\mu = 0, 1, 2, 3$. We can choose the gauge on η such that Eq. (1.5) $A_0 \rightarrow A_0 + \frac{1}{g}\partial_0\eta = 0$, which gives 1 dof out of 4 dof. Furthermore, the gauge transformation $A_0 \rightarrow A_0 + \frac{1}{g}\partial_0\eta$ can still hold if $\eta \rightarrow \eta + \eta'(x_j)$ where $\eta'(x_j)$ is a pure function of space indices x_j . Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most $4 - 1 - 1 = 2$ dof, which agrees with the physics story given earlier.

2. SU(N) YM gauge theory of Sec. 1.1.2:

Similarly, the dof counting of gluon field A_μ^α is similar to the previous counting. We have 2 dof for each A_μ^α and there are $N^2 - 1$ independent of gluons with index α in the adjoint representation. However, as we know due to the color confinement [37], the gapless dof of gluon field A_μ^α do not appear at the low energy, but only above the confinement energy gap scale $\geq \Lambda_{\text{YM}}$. At low energy $< \Lambda_{\text{YM}}$, we only observe a confinement energy gap without massless modes.

3. Euclidean or Lorentz invariant non-abelian higher-rank tensor gauge theory of Sec. 2.1:

Ref. [26] finds that the compact abelian *symmetric* tensor gauge theory is unstable to confinement in 2+1D, but it becomes stable to be gapless-ness with deconfinement in 3+1D. We thus focus on 3+1D (or above) for its richer physics in a gapless phase.

Let us count the dof in 3+1D. We see that the rank-2 symmetric tensor field is $A_{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3$ naively with 10 dof. We can choose the gauge on η^v such that Eq. (2.13) $A_{00} \rightarrow A_{00} + \frac{1}{g}\partial_0\partial_0\eta^v = 0$, which kills 1 dof out of 10 dof. Furthermore, the gauge transformation $A_{00} \rightarrow A_{00} + \frac{1}{g}\partial_0\partial_0\eta^v = 0$ can still hold if $\eta^v \rightarrow \eta^v + \eta'(x_j) + t \cdot \eta''(x_j)$ where the two additional redundant dof $\eta'(x_j)$ and $\eta''(x_j)$ are pure functions of space indices x_j . Since the shifted function has a linear t dependence, we have $\partial_0\partial_0\eta^v \rightarrow \partial_0\partial_0\eta^v$. Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most $10 - 1 - 2 = 7$ dof.

Let us also count the dof in $d + 1$ D. The rank-2 symmetric tensor field is $A_{\mu\nu}$ naively with $\frac{(d+2)(d+1)}{2}$ dof. We can choose the gauge on η^v such that Eq. (2.13) $A_{00} \rightarrow A_{00} + \frac{1}{g}\partial_0\partial_0\eta^v = 0$, which reduces 1 dof. Furthermore, the gauge transformation $A_{00} \rightarrow A_{00} + \frac{1}{g}\partial_0\partial_0\eta^v = 0$ can still hold if $\eta^v \rightarrow \eta^v + \eta'(x_j) + t \cdot \eta''(x_j)$ since the shifted function (with 2 dof) has a linear t dependence, we have $\partial_0\partial_0\eta^v \rightarrow \partial_0\partial_0\eta^v$. Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most $\frac{(d+2)(d+1)}{2} - 1 - 2 = \left(\frac{(d+2)(d+1)}{2} - 3\right)$ dof.

4. Anisotropic non-abelian higher-rank tensor gauge theory for space and time of Sec. 2.2.

Again due to 3+1D may becomes stable to be gapless-ness with deconfinement Ref. [26], let us count the dof in 3+1D or in $d + 1$ D. The rank-2 symmetric tensor field is A_{ij} naively with $\frac{(d+1)d}{2}$ dof and A_0 for 1 dof. We can choose the gauge on η^v such that Eq. (2.51) $A_0 \rightarrow A_0 + \frac{1}{g}\partial_0\eta^v = 0$, which reduces 1 dof. Furthermore, the gauge transformation $A_0 \rightarrow A_0 + \frac{1}{g}\partial_0\eta^v = 0$ can still hold if $\eta^v \rightarrow \eta^v + \eta^v(x_j)$ since the shifted function (with 1 dof) has a linear t dependence, we have $\partial_0\eta^v \rightarrow \partial_0\eta^v$. Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most $\frac{(d+1)d}{2} + 1 - 1 - 1 = \left(\frac{(d+1)d}{2} - 1\right)$ dof. In 3+1D, we have 5 dof.

See a summary in Table 4.

3.2.2 Dispersions of gapless modes

Now we determine the dispersions of gapless modes, as the scaling of energy $\omega_E(k) \propto k^z$ and the momentum k to the dispersion's dynamical exponent z . For the dimensional analysis, below we denote the scaling of the spatial dimensions as $[L]$.

gapless dof	Euclidean/Lorentz invariant of Sec. 2.1	Anisotropic for space and time of Sec. 2.2
3+1D	7	5
$d+1$ D	$\left(\frac{(d+2)(d+1)}{2} - 3\right)$	$\left(\frac{(d+1)d}{2} - 1\right)$
dispersion 3+1D	$\omega_E(k) \propto c_v k$	$\omega_E(k) \propto c_v k$
E and B tensor SO(3) Rep	$\begin{aligned} \bar{\mathbf{E}} &= \mathbf{3} + (\mathbf{1} + \mathbf{3} + \mathbf{5}), \\ \bar{\mathbf{B}} &= \mathbf{3} + (\mathbf{1} + \mathbf{3} + \mathbf{5}). \end{aligned}$ in Eq. (2.22)	$\begin{aligned} \tilde{\mathbf{E}} &= (\mathbf{1} + \mathbf{5}), \\ \tilde{\mathbf{B}} &= (\mathbf{1} + \mathbf{3} + \mathbf{5}). \end{aligned}$ in Eq. (2.47)

Table 4: We collect the data of the degrees of freedom (dof) and dispersion relation for gapless modes, and the electric E and magnetic B tensors in terms of the spatial rotational SO(3) representation.

1. Obviously both Maxwell and Yang-Mills theories have photons and gluons with linear dispersion relation $\omega_E(k) \propto ck$, where c is known as the speed of light constant. (Alternatively, the effective speed of light in the quantum matter material.) This is easy to derive based on the scaling between $E^2 = c^2 B^2$, where the ratio gives the (square of) the speed of light.
2. Euclidean or Lorentz invariant non-abelian higher-rank tensor gauge theory of Sec. 2.1:
We see that from Eq. (2.13), Eq. (2.20) and Eq. (2.21),

$$\begin{cases} [A_{\mu\nu}] & \sim \frac{1}{g}[\partial_\mu\partial_\nu(\eta_v(x))] \sim [L]^{-2}, \\ [\bar{\mathbf{E}}_{ij}] & \sim [\partial_0 A_{ij} - \partial_i A_{0j}] \sim [L]^{-3}, \\ [\bar{\mathbf{B}}_{\ell k}] & \sim [\frac{1}{2}\epsilon^{\ell ij}(\partial_i A_{jk} - \partial_j A_{ik})] \sim [L]^{-3}. \\ [\bar{\mathbf{E}}_{ij}]^2 & \sim [\bar{\mathbf{B}}_{\ell k}]^2, \text{ as } [\partial_0]^2 \sim (v_c)^2[\partial_i]^2. \end{cases} \quad (3.6)$$

So $[t] \sim [x] \sim [L]$, thus it is a linear dispersion relation $\omega_E(k) \propto v_c k$ with $z = 1$.

3. Anisotropic non-abelian higher-rank tensor gauge theory for space and time of Sec. 2.2:
From Eq. (2.51) and Eq. (2.45),

$$\begin{cases} [A_0] & \sim \frac{1}{g}[\partial_0(\eta_v(x))] \sim \frac{1}{g}[\partial_t(\eta_v(x))] \sim [L]^{-1}, \\ [A_{ij}] & \sim \frac{1}{g}[\partial_i\partial_j(\eta_v(x))] \sim [L]^{-2}, \\ [\tilde{\mathbf{E}}_{ij}] & \sim [-\partial_0 A_{ij} + \partial_i\partial_j A_0] \sim [L]^{-3}, \\ [\tilde{\mathbf{B}}_{ij}] & \sim [\epsilon_{ilm}\partial_\ell A_{mj}] \sim [L]^{-3}. \\ [\tilde{\mathbf{E}}_{ij}]^2 & \sim [\tilde{\mathbf{B}}_{ij}]^2, \text{ as } [\partial_0]^2 \sim (v_c)^2[\partial_i]^2. \end{cases} \quad (3.7)$$

So $[t] \sim [x] \sim [L]$, it is still a linear dispersion relation $\omega_E(k) \propto v_c k$ with $z = 1$.

See a summary in Table 4. In fact, for other tensor gauge models, there are other possibilities of gapless modes with different dispersion of $\omega_E(k) \propto k^z$ with $z = 1, 2, 3$, etc., e.g., see Ref. [26, 30].

4 Discussions: Fracton, Embeddon and Foliation

4.1 Fracton

The name Fracton is introduced in [38, 39]. Famous examples include the Chamon quantum glass model [40, 41], the Haah's cubic code [42] and others. The characterizations and physics definitions of fracton orders are that the excitations (including gapless or gapped excitations) have restricted mobility when acted by local operators [6]. The fractonic excitations have either of the following:

- (1). The excitations cannot move without creating additional excitations (other fractonic excitations).
- (2). The excitations can move only in sub-dimensions or certain directions. These excitations are also known as sub-dimensional particles.

It is easy to see that the theory in Eq. (2.19) and Eq. (2.50), indeed have the property (1). By looking at one term, for instance, we obtain

$$\begin{aligned}
& (\Phi \partial_x \partial_y \Phi - \partial_x \Phi \partial_y \Phi - i g A_{xy} \Phi^2) \tag{4.1} \\
& \simeq \Phi(x, y) ((\Phi(x + \Delta x, y + \Delta y) - \Phi(x, y + \Delta y) - \Phi(x + \Delta x, y) + \Phi(x, y)) \\
& - (\Phi(x + \Delta x, y) - \Phi(x, y)) (\Phi(x, y + \Delta y) - \Phi(x, y)) - i g A_{xy}(x, y) \Phi^2(x, y)) \\
& = \Phi(x, y) \Phi(x + \Delta x, y + \Delta y) - \Phi(x + \Delta x, y) \Phi(x, y + \Delta y) - i g A_{xy}(x, y) \Phi^2(x, y) \tag{4.2}
\end{aligned}$$

in a discretized manner. To minimize the action by the variational principle, say this $\Phi(x, y) \Phi(x + \Delta x, y + \Delta y) - \Phi(x + \Delta x, y) \Phi(x, y + \Delta y)$ term, it means that the $U(1)_{x(d+1)}$ or $U(1)_{x(d)}$ vector global symmetry charge should be arranged in a higher-moment quadrupole manner. Indeed, the property that the *dipole* should not be created but the *quadrupole* can be created, is noticed and analyzed in various Pretko's work [33].

We hope to explore (2) in a new setting in terms of the spacetime embedding in the next subsection.

4.2 Embedding to Embeddon

Let us construct another infinite series of new theories similar to Eq. (2.41) and Eq. (2.56), by the idea of embedding. We device new kinds of path integral as:

$$\int \left(\prod_{I=1}^N [\mathcal{D}A_{I,\mu\nu}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp \left(i \int_{M^{d+1}} \frac{1}{g^2} |\hat{F}_{\mu\nu\xi}^c|^2 + \int_{M_{\text{sub-M}}^n} i \left(\frac{2}{2\pi} \sum_{I=1}^N B_I dC_I + \# C_I \wedge C_J \wedge \cdots \wedge dC_K \right) \right). \tag{4.3}$$


 NO ambient space

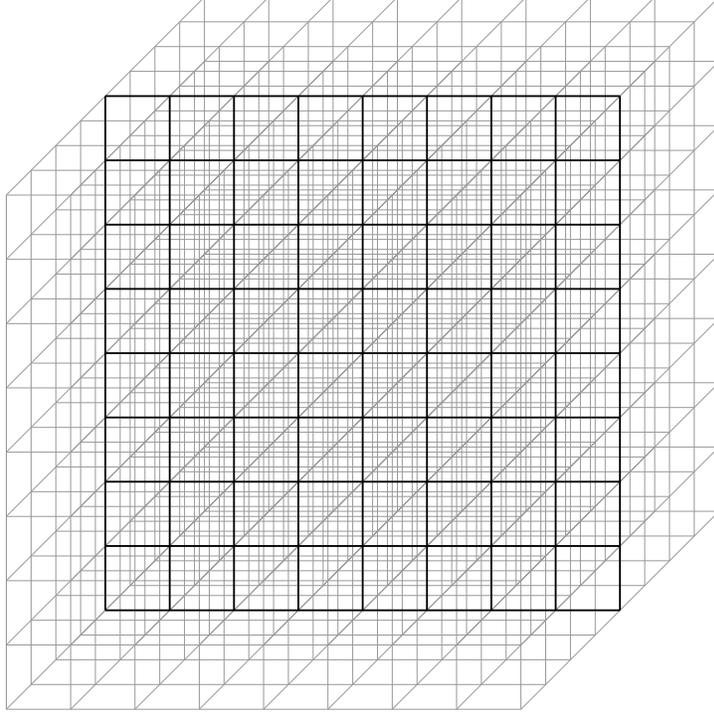


Figure 5: Interpretations of the embeddon: (1) Quantization of spacetime embedding [43]. (2) The anyonic objects (e.g. particles/strings/branes) live in the embedded manifold $M^n_{\text{sub-}M} \subset M^{d+1}$ in nd , thus the anyonic objects are embedded inside the sub-dimensions.

Here we introduce a notation $\underbrace{\quad\quad\quad}_{\text{NO ambient space}}$ means that there is no volume form of the ambient space.

We introduce the lower dimensional nd anti-symmetric TQFT on the submanifold $M^n_{\text{sub-}M} \subset M^{d+1}$. For example, the submanifold $M^n_{\text{sub-}M} \subset M^{d+1}$ is a darkened slice in Fig. 5. We thus have a constraint $N \leq n \leq d + 1$ for the twisted topological term $\omega_n(\{C_I\})$ that we can add from Table 1 and Table 2, here $\omega_n \in \mathbb{H}^n((\mathbb{Z}_2^C)^N, \mathbb{R}/\mathbb{Z})$ for a lower-dim twisted cohomology group.

Physically, what we are doing is to embed an anti-symmetric TQFT in a lower dimensional space, into a higher dimensional space where the symmetric tensor gauge theory resides. Mathematically, this can be done by gauging the sub-sector of the Eq. (2.39)'s $G_g = [(\mathbb{Z}_2^C)^N]$. The anyonic objects (e.g. particles/strings/branes) described by ω_n live in nd , thus the anyonic objects are embedded inside the sub-dimensions. We shall name those embedded objects in the quantum theory as *embeddon*.

4.3 Foliation

We devise new kinds of path integral as another infinite series of new theories similar to Eq. (2.41) and Eq. (2.56), similar to Eq. (4.3) but different in spirit:

$$\int \left(\prod_{I=1}^N [\mathcal{D}A_{I,\mu\nu}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp \left(i \int_{M^{d+1}} \frac{1}{g^2} |\hat{F}_{\mu\nu\xi}^c|^2 + \right. \\ \left. i \left(\frac{2}{2\pi} \sum_{I=1}^N B_I dC_I + \# C_I \wedge C_J \wedge \cdots \wedge dC_K \right) \underbrace{(\wedge \cdots \wedge dx')}_{\text{ambient space}} \right) \quad (4.4)$$

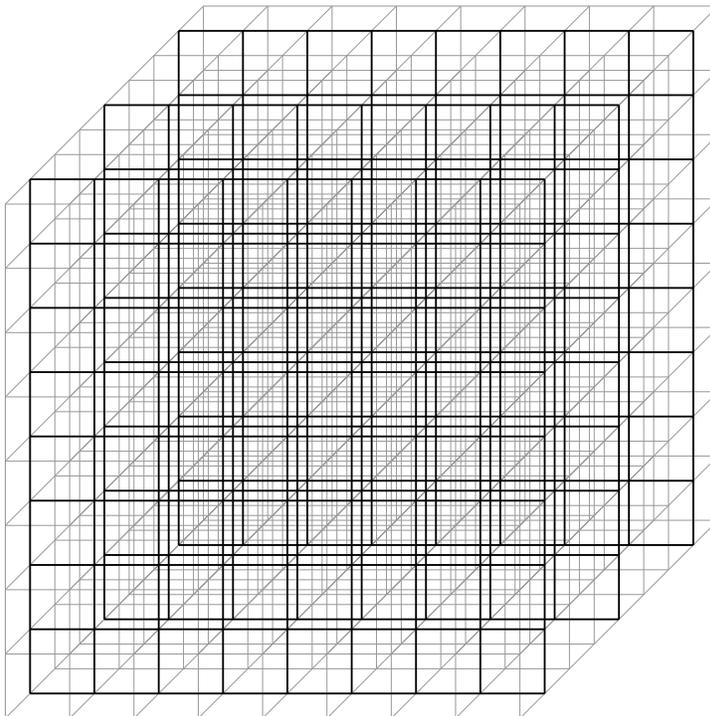


Figure 6: From the sequences of embedding to foliation in the spacetime.

Here $\underbrace{(\wedge \cdots \wedge dx^i)}_{\text{ambient space}}$ is the volume of the ambient space. We gain another theory by a sequence of embedding of submanifolds into a new picture of foliation. By this stacking of embedded TQFT picture along any codimension direction (in the dual $d-n$ -dimensions in space), we can also easily reproduce the exponential growth of ground state degeneracy (GSD) associated to d -dimensional space (counting zero energy modes) and thus exponential growth of the ground state subspace $\dim(\mathcal{H}_d)$.

Our theory for this physics picture thus potentially correspond to the GSD counting in [44] and the *foliation picture* [45] on a wider class of manifolds. We may also construct the sub-dimensional sectors with BF theory or Chern-Simons theory in nd with $n \leq d + 1$, either via the embedding in Sec. 4.2 or the foliation in Sec. 4.3, potentially bridge to another recent work [46]. Our theory can also combine the *foliation* (in this subsection) and the *anisotropic properties* (Sec. 2.2), likely reproduce a field theory interpretation for a lattice construction similar to [47].

5 Conclusions: Quantization, Feynman diagram, Lattice Model, and Dark Matter

1. We have constructed a new family of hybrid classes between symmetric higher-rank tensor gauge theories and anti-symmetric tensor topological field theories (TQFTs) in *any dimension*. In condensed matter community, it is believed that the compact abelian symmetric rank-2 tensor field theory in Eq. (2.19) without matter field in 3+1D (4d) or above describes a gapless deconfined phase [26]. Thus our theory in 3+1D (4d) could describe a unitary mixture phase including the. gapless sectors (which can live with or without Euclidean, Poincaré or anisotropic symmetry, from compact abelian symmetric tensor gauge theories, thus possibly can be relativistic or non-relativistic), and the gapped

topological order phases (from anti-symmetric tensor TQFTs).

However, the mixture of such gapless phase and gapped phase is only the effective field theory description, at least in the ultraviolet high energy (UV).

What are the infrared low energy (IR) fates and quantum dynamics of our theories? First of all, the continuous non-abelian gauge structure of our theories is $\left[\mathbb{Z}_2^C \times \left(\mathrm{U}(1)_{x_{(d+1)}}\right)\right]$ in Eq. (2.27), a higher-moment continuous $\mathrm{U}(1)_{x_{(d+1)}}$ twisted by the discrete gauged \mathbb{Z}_2^C . So one may expect that the *gapless* and *deconfinement* of compact abelian symmetric higher-rank tensor gauge theories should be maintained. We may speculate the gapped anti-symmetric tensor TQFT sector would not severely interfere the gapless sector, but only introduce additional massive degrees of freedom of anyonic particles (in 2+1D)/strings (in 3+1D)/branes (in 4+1D), etc. as massive high-energetic excitations above certainly TQFT energy gap Δ_E .

To confirm the speculation, we need to know the quantum dynamics of our theories. It may be challenging. To make an analogy, it is like writing down a Yang-Mills theory [5] is still quite far from to understanding its quantum dynamics and confinement mechanism [37]. Let us make a few remarks before coming back to the quantization issue in the Remark 5.

2. *Causes of non-abelian higher-moment global symmetry:*

It is easy to see that the non-abelian global symmetry is due to the non-commutative nature of $\left(\mathbb{Z}_2^C \times \left(\mathrm{U}(1) \times \mathrm{U}(1)_{x_{(d+1)}}\right)\right)$ displayed in Eq. (2.9) and Eq. (2.10).

3. *Causes of non-abelian gauge structure:*

There are two ingredients to cause the non-abelian gauge structure.

- The first ingredient is due to the “gauge group-analogous structure” is Eq. (2.27) $\left[\mathbb{Z}_2^C \times \left(\mathrm{U}(1)_{x_{(d+1)}}\right)\right]$ once we gauge both $\mathrm{U}(1)_{x_{(d+1)}}$ and \mathbb{Z}_2^C . Even without any topological twisted term ω_{d+1} , our theory is still a non-abelian tensor gauge theory (say in Eq. (2.41) and others).
- The second ingredient is due to the topological twisted term ω_{d+1} involving the Top Type explained in Sec. 3.1.1.

4. *Fracton, Embeddon, Foliation:* We explain how our theories can incorporate the phenomenon of Fracton in Sec. 4.1, almost directly given from Ref. [33]. We introduce the spacetime embedding in terms of field theory setup and the Embeddon in Sec. 4.2, and how it bridges to build up the foliation in Sec. 4.3.

5. *Feynman diagrams (Dyson graphs), path integral and (canonical) quantization:*

A fully fledged quantum field theory, even of Eq. (2.19) or (2.50), with matter and with or without gauge fields, by a field quantization is still an open question. The usual canonical quantization by identifying the field variable Φ and the conjugate momentum variable may not be transparent for this quartic higher-order intrinsic interacting field theory.⁷ On the other hand, the path integrals we formulated are so far schematic.

However, we can at least attempt to identify the interaction terms and write down the Feynman diagrams in the perturbative sense. We do *not* include the Feynman rules, as it depends on whether we can use the propagators of free fields as the starting point to compute the scattering amplitudes. As we mentioned in Sec. 4.1 that the quadrupole (instead of dipole) of interacting matters at four different locations is the better configuration to think about the low energy physics of the theory; one can be afraid that the free field description of the theory (Eq. (2.19)) may not be ideal. In any case, we list the Feynman diagrams for $|\left(\Phi\partial_\mu\partial_\nu\Phi - \partial_\mu\Phi\partial_\nu\Phi - igA_{\mu\nu}\Phi^2\right)|^2$ in Figure 7.

⁷Note that there is a very recent work by Seiberg [48] commenting about the quantization of the quartic matter field theory and its spectrum, however without the gauge fields.

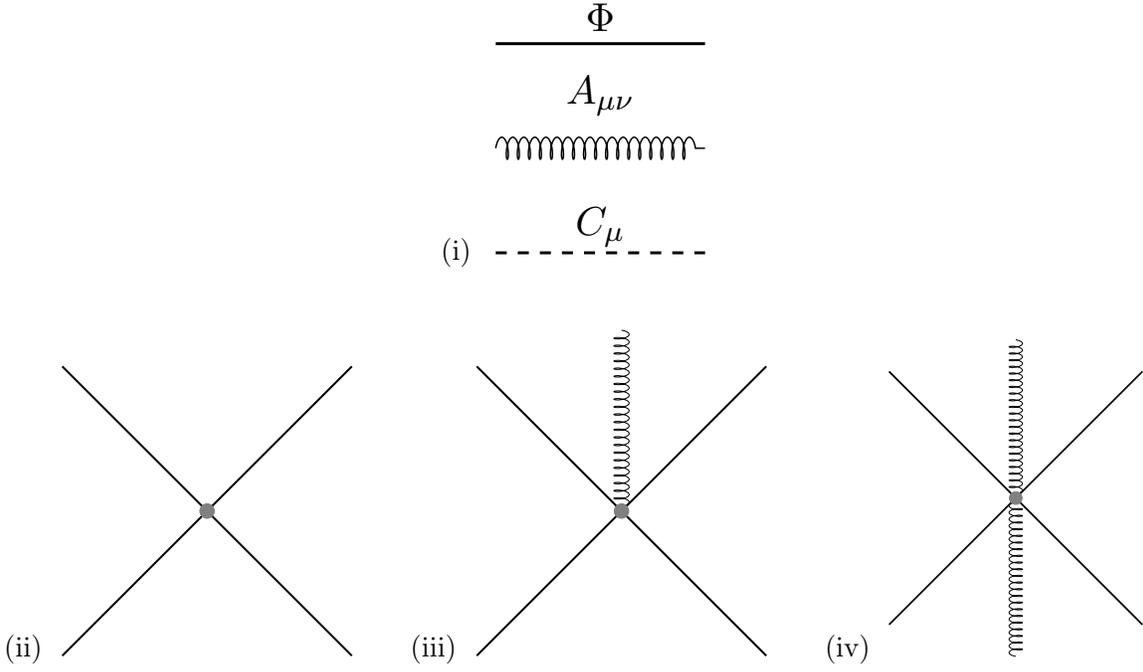


Figure 7: Feynman diagrams relevant for the tree-level interactions of $|(\Phi\partial_\mu\partial_\nu\Phi - \partial_\mu\Phi\partial_\nu\Phi - igA_{\mu\nu}\Phi^2)|^2$. (i) The solid line, the curly spring, and the dashed line are for the propagators for the complex scalar Φ , the real rank-2 tensor A , and the 1-form C field respectively. (ii) The $|(\Phi\partial_\mu\partial_\nu\Phi - \partial_\mu\Phi\partial_\nu\Phi)|^2$ in Eq. (2.19) contains multiple terms involving the 4 points of Φ field interactions (subfigure (ii)), involving their momentums. (iii) The $ig(\Phi\partial_\mu\partial_\nu\Phi - \partial_\mu\Phi\partial_\nu\Phi)A_{\mu\nu}\Phi^2$ and its complex conjugation in Eq. (2.19) contains the 5 points of four Φ field and one $A_{\mu\nu}$ interactions (subfigure (iii)). (iv) The $g^2A_{\mu\nu}^2|\Phi|^4$ in Eq. (2.19) contains the 6 points of four Φ field and two $A_{\mu\nu}$ interactions (subfigure (iv)).

On the other hand, there are free field descriptions for rank-2 symmetric tensor A in Eq. (2.41). Thus the propagator of A read from $|\hat{F}_{\mu\nu\xi}^c|^2 = \left((\partial_\mu - ig_c C_\mu)A_{\nu\xi} - (\partial_\nu - ig_c C_\nu)A_{\mu\xi}\right)\left((\partial^\mu - ig_c C^\mu)A^{\nu\xi} - (\partial^\nu - ig_c C^\nu)A^{\mu\xi}\right)$ and its Feynman diagrams may be a more sensible to do than that of Eq. (2.19). We list the Feynman diagrams for $|\hat{F}_{\mu\nu\xi}^c|^2$ in Figure 8.

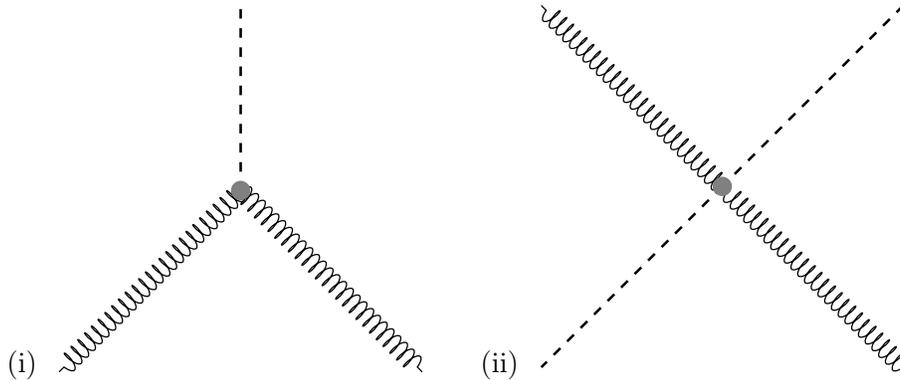


Figure 8: Feynman diagrams relevant for tree-level interactions of $|\hat{F}_{\mu\nu\xi}^c|^2$. The conventions follow Fig. 7. (i) The interaction term from ∂ACA in Eq. (2.41). (ii) The interaction term from $CACA$ in Eq. (2.41).

In Figure 8 and in $|\hat{F}_{\mu\nu\xi}^c|^2$, it is important to notice that there is a kinetic term for A in the Lagrangian $\partial A\partial A$ term. But there is no kinetic term for C . In fact, as we already knew the B and C are dual gauge fields of the topological BF theory sector for the TQFT. Thus we expect the physical

observables of C fields captured by the topological links of extend operators listed in Table 2.

6. *Universality of field theory*: We have attempted to develop our theories based on field theories in any dimension. We hope that the universality of our field theories can capture the wider universality of a large class of lattice models. Although in the fracton literature [6], many known exact solvable lattice models are powerful, it is still important for us to re-collect the universal data out of arbitrary lattice models.
7. *Lattice models*: An open question is that whether our field theories can be regularized on the lattice as a Hamiltonian quantum mechanical model? (This question addresses a different pursuit: Instead of attempting to define our field theory as a quantum field theory, we may define our theory as a unitary many-body quantum mechanic model.) Several previous works can help, for example,
 - By generalizing the Kitaev quantum double model [49], exactly solvable quantum Hamiltonian models of twisted topological gauge theories and Dijkgraaf-Witten theories are known and constructed in 2+1D [50] and in 3+1D [51].
 - Exactly solvable quantum Hamiltonian models of twisted fracton models, analogous to twisted topological gauge theories [50, 51], are constructed in 3+1D [52, 53], or in any dimension [54].

Since our theory is a hybrid class between symmetric higher-rank tensor gauge theory (i.e., higher-spin gauge theory) and anti-symmetric tensor TQFT of Dijkgraaf-Witten/group-cohomology types [50, 51], combining the above two versions of Hamiltonian models potentially can bridge to the quantum Hamiltonians for Eq. (2.41) and Eq. (2.56).

8. *Time crystal*: We mentioned in Sec. 2.1.1 and in Eq. (2.5) about a potential application of our theories to time crystal [35, 36]. Since Eq. (2.5) shows

$$\Phi \rightarrow \Phi e^{i\Lambda_0 \cdot x_0} = \Phi e^{i\Lambda_0 \cdot t},$$

while $x_0 = t$ is time coordinate. The field configuration can respect a certain periodic time constraint $t \simeq t + \frac{2\pi}{\Lambda_0} \mathbb{Z}$. Of course, what we said is only for the possible kinematics from the vector-like global symmetry (say at a UV effective field theory). In order to know whether there is any time crystal order formed in the ground state (namely in the quantum vacua at low energy), we need to know the quantum dynamics (say as the IR fate). This is again another hard question, related to the perturbative and quantization issue in Remark 5 and the further much challenging non-perturbative dynamics [37].

9. *Dark matter*:

There are several possibilities for our theories to play a fundamental physics role for the dark matter.:

- (1). Eq. (2.19) or (2.50), with matter and with or without gauge fields
- (2). Eq. (2.41) and Eq. (2.56) with gauge fields and TQFT sectors, and their higher-moment tensor generalization.

In cosmology, the present research suggests that the total mass-energy of the Universe contains:

- 5% of ordinary matter (e.g. baryonic and leptonic) and energy,
- 27% dark matter,
- 68% of dark energy as an unknown form of energy.

Thus, dark matter constitutes 85% of total mass, while dark energy plus dark matter constitute 95% of total mass-energy content.

Hypothetically, we can interpret our model (1) with fractonic matter Φ of complex scalar bosons as a source of dark matter decoupling from the Standard Model particle physics. We may also interpret

our model (2) objects of anyonic particles or anyonic extended objects (including anyonic strings and anyonic branes, etc.), that can be embedded or foliated in the spacetime. In either case, the candidate dark matters from our model have the properties:

1. Non-baryonic and non-leptonic.
2. Non-supersymmetry (or no need to have supersymmetry).
3. Weakly interacting massive particles (WIMPs) or extended objects as hypothetical particles for dark matter. It can be massive if we interpret dark matters as anyonic particles. But our model also provide non-particles but extended objects as dark matter candidate: Anyonic extended objects (anyonic strings/branes, etc.) in Eq. (2.41) and Eq. (2.56), e.g. in Table 2. They can be highly energetic gapped excitations that we can not probe easily at low energy, but they affect the topological degenerate zero energy subspace, see Table 3.
4. Weakly interacting (nearly) massless or gapless matters: Models of (1) (Eq. (2.19) or (2.50)) can have massless or gapless (or nearly depending on our input on the mass) matters, decoupled (or nearly decoupled) from Standard Model sectors.
5. Cold dark matter: Our dark matter candidates are highly quantum and highly entangled states of matter (both for particle-like or extended string/brane-like objects). They exist and are robust at the absolute zero temperature without thermal effect. Although our candidates can still play a role for warm and hot environment in the Universe.
10. *Dark energy*: Dark energy is an unknown energy form hypothetically permeating all of space and time. It tends to accelerate the expansion of the Universe. Dark energy is the accepted hypothesis to explain that the Universe is expanding at an accelerating rate especially after 1990s observations. We do *not* yet have a concrete model for producing such an energy to expand the space from our model. However, we should note that there are an enormous amount of static and dynamical energy from the gauge structure of our models (in both Eq. ((1)) and Eq. ((2))). For a simpler compact abelian tensor gauge model, the huge amount of static energy is emphasized already in [29]. Related issues on how matters interact and their relations through gravity via a Mach’s principle is revisited in [55] for the fracton context. We expect that more immense energy stored in our non-abelian models with both gapless and gapped TQFTs sectors.

6 Acknowledgements

The idea of this family of theories were pursued since 2016 by JW. This theory is found in July 2019. JW wishes to thank Edward Witten for a conversation in 2016, and thank Stephen L. Adler for a conversation in 2019. JW thanks Trithep Devakul for a collaboration on the upcoming work. JW thanks the participants of “Fracton Phases of Matter and Topological Crystalline Order on December 3-5, 2018” in PCTS Princeton [43] and “Simons Collaboration on Ultra-Quantum Matter meeting on September 12-13, 2019” at Harvard, also at a CMSA seminar, for helpful feedbacks [56]. JW thanks Shing-Tung Yau for informing the indispensable impact of Chern’s work in 1940s [4] prior to the later formulation of Yang-Mills theory [5]. JW has been benefited by presenting this model informally at these conferences, special thanks the valuable feedback of Michael Hermele and Michael Pretko. JW was supported by NSF Grant PHY-1606531 and Institute for Advanced Study. This work is also supported by NSF Grant DMS-1607871 “Analysis, Geometry and Mathematical Physics” and Center for Mathematical Sciences and Applications at Harvard University.

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