Beyond Standard Models and Grand Unifications: Anomalies, Topological Terms and Dynamical Constraints via Cobordisms

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Abstract

We classify and characterize fully all invertible anomalies and all allowed topological terms related to various Standard Models (SM), Grand Unified Theories (GUT), and Beyond Standard Model (BSM) physics. By all anomalies, we mean the inclusion of (1) perturbative/local anomalies captured by perturbative Feynman diagram loop calculations, classified by $\mathbb{Z}$ free classes, and (2) non-perturbative/global anomalies, classified by finite group $\mathbb{Z}_N$ torsion classes. Our theory built from [arXiv:1812.11967] fuses the math tools of Adams spectral sequence, Thom-Madsen-Tillmann spectra, and Freed-Hopkins theorem. For example, we compute bordism groups Ω_d^H and their invertible topological field theory invariants, which characterize dd topological terms and $(d-1)$d anomalies, protected by the following symmetry group H: $\mathrm{Spin} \times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\Gamma_q}$ for SM with $\Gamma_q := \mathbb{Z}_q$ as $q = 1, 2, 3, 6$; $\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}$ or $\mathrm{Spin} \times \mathrm{SO}(n)$ for SO(10) or SO(18) GUT as $n = 10, 18$; $\mathrm{Spin} \times \mathrm{SU}(n)$ for Georgi-Glashow SU(5) GUT as $n = 5$; $\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\Gamma_q}}{\mathbb{Z}_2^F}$ for Pati-Salam GUT as $q = 1, 2$; and others. Our approach offers an alternative view of all anomaly matching conditions built from the lower-energy (B)SM or GUT, in contrast to high-energy Quantum Gravity or String Theory Swampland program, as bottom-up/top-down complements. Symmetries and anomalies provide constraints of kinematics, we further suggest constraints of quantum gauge dynamics, and new predictions of possible extended defects/excitations plus hidden BSM non-perturbative topological sectors.

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1 Introduction

1.1 Physics Guide

The world where we reside, to our present knowledge, can be described by quantum theory, gravity theory, and the underlying long-range entanglement. Quantum field theory (QFT), specifically gauge field theory, under the name of *Gauge Principle* following Maxwell, Hilbert, Weyl [1], Pauli, and others, forms a cornerstone of the fundamental physics. Yang-Mills (YM) gauge theory [2], generalizing the U(1) abelian gauge group to a non-abelian Lie group, has been proven theoretically and experimentally essential to describe the Standard Model (SM) physics [3–5].

The SM of particle physics is a gauge theory encoding three of the four known fundamental forces or interactions (the electromagnetic, weak, and strong forces, but without gravity) in the universe. The SM also classifies all experimentally known elementary particles: fermions including three generations of quarks and leptons, and bosons including photon γ, gluon g, $W^{\pm}$ and Z gauge bosons and Higgs (while the graviton has not yet been detected). Physics experiments have confirmed that at a higher energy of SM, the electromagnetic and weak forces are unified into an electroweak interaction sector. Grand Unifications and Grand Unified Theories (GUT) predict that at further higher energy, the strong and the electroweak interactions will be unified into an electro-nuclear GUT interaction sector. The GUT interaction is characterized by one larger gauge group and its force carrier mediator gauge bosons with a single unified coupling constant.[1] Examples of GUT that we will encounter in this article includes Georgi-Glashow SU(5) GUT [6], Fritzsch-Minkowski SO(10) GUT [7] and Pati-Salam [8] and others.

In our present work, we aim to classify and characterize fully all anomalies and all allowed topological terms associated with various Standard Models (SM), Grand Unified Theories (GUT), and Beyond

[1]Unifying gravity with the GUT interaction gives rise to a Theory of Everything (TOE). However, in our present work, the gravity only plays the role of the background probed fields instead of dynamical gravity. As we will classify and characterize, the background probed gravity also gives new constraints, such as in the gravitational anomaly or the mixed gauge-gravitational anomaly. We however will comment the implications for dynamical gravity such as in Quantum Gravity in Sec. 7.

Standard Model physics (BSM) in 4d.[2]

By "anomalies" of a theory in physics terminology, physicists may mean one of the following:

(1): Classical global symmetry is violated in a quantum theory, such that the classical global symmetry fails to be a quantum global symmetry, e.g. Adler-Bell-Jackiw anomaly [9, 10].

(2): Quantum global symmetry is well-defined kinematically. However, there is an obstruction known as "'t Hooft anomaly [11]," to gauge the global symmetry, detectable via coupling the charge operator (i.e., symmetry generators or symmetry defects, which measures the global symmetry charge of charged objects) to background fields.[3] Specifically, we may detect an obstruction to even *weakly gauge* the symmetry or couple the symmetry to a *non-dynamical background probed field* (sometimes as background gauge field/connection). "'t Hooft anomaly [11]," is sometimes regarded as a "background gauged anomaly" in condensed matter. Namely, the path integral or partition function $\mathbf{Z}$ does not sum over background gauge fields. We only fix a background gauge field and the $\mathbf{Z}$ only depends on the background gauge connection as a classical field or as a classical coupling constant.

(3): Quantum global symmetry is well-defined kinematically. However, once we promote the global symmetry to a dynamical local gauge symmetry of the dynamical gauge theory, then the gauge theory becomes ill-defined. Some people call this as a "dynamical gauge anomaly" prohibiting a quantum theory to be well-defined. Namely, the path integral after summing over dynamical gauge fields becomes ill-defined. Therefore, the anomaly-free or anomaly-matching conditions are crucial to avoid the sickness and ill-defineness of quantum gauge theory.

In fact, it is obvious to observe that the anomalies from (3) are descendants of anomalies from (2). Anomalies from (2) can be related to anomalies from (3) via the gauging principle. Anomalies from (2) can be related to anomalies from (3) via the ungauging principle. Thus our key idea is that if we know the gauge group of a gauge theory (e.g., SM, GUT or BSM), we may identify its ungauged global symmetry group as an internal symmetry group, say $\mathbb{G}_{\text{internal}}$ via ungauging.[4]

To start, we should rewrite the global symmetries of an ungauging theory into the form of

$$G \equiv \left(\frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}}\right), \tag{1.1}$$

where the $G_{\text{spacetime}}$ is the spacetime symmetry, the $\mathbb{G}_{\text{internal}}$ is the internal symmetry,[5] the $\ltimes$ is a semi-direct product from a "twisted" extension,[6] and the N_{shared} is the shared common normal subgroup symmetry between $G_{\text{spacetime}}$ and $\mathbb{G}_{\text{internal}}$.

[2] We denote dd means the d spacetime dimensions. The $d+1$D means the d spatial and 1 time dimensions. The $\bar{D}$D means the $\bar{D}$ space dimensions.

[3] Throughout our article, we explicitly or implicitly use the modern language of symmetries and higher symmetries of QFTs, introduced in [12].

[4] By gauging or ungauging, also depending on the representation of the matter fields that couple to the gauge theory, we may gain or lose symmetries of higher symmetries [12]. It will become clear at this moment we only need to focus on the ordinary (0-form) internal global symmetries and their anomalies. See Sec. 7.

[5] Later we denote the probed background spacetime M connection over the spacetime tangent bundle TM, e.g. as $w_j(TM)$ where w_j is j-th Stiefel-Whitney (SW) class. We also denote the probed background internal-symmetry/gauge connection over the principal bundle E, e.g. as $w_j(E) = w_j(V_{\mathbb{G}_{\text{internal}}})$ where w_j is also j-th SW class.

[6] The "twisted" extension is due to the symmetry extension from $\mathbb{G}_{\text{internal}}$ by $G_{\text{spacetime}}$, for a trivial extension $\ltimes$ becomes a direct product $\times$.

In the later sections of our work, we will write down the ungauged global symmetry groups of SMs, GUTs and BSMs. Then we should determine, classify and characterize all of their associated anomalies and topological terms.

But by far there are some pertinent questions the readers may wonder in order to decide to keep reading our work (or decide to quit reading), which we should confront, comfort and answer immediately:

[I]. *What do we mean by all anomalies and all topological terms*?

By "all anomalies," we mean the inclusion of:

(i). **Perturbative/local anomalies** captured by perturbative Feynman diagram loop calculations, classified by the integer group $\mathbb{Z}$ classes, or the so-called free classes in mathematics. Some selective examples from QFT or gravity include:

- (1): Perturbative fermionic anomalies from chiral fermions with U(1) symmetry, originated from Adler-Bell-Jackiw (ABJ) anomalies [9, 10] with $\mathbb{Z}$ classes.
- (2): Perturbative bosonic anomalies from bosonic systems with U(1) symmetry with $\mathbb{Z}$ classes.
- (3): Perturbative gravitational anomalies [13].

(ii). **Non-perturbative/global anomalies**, classified by a product of finite groups such as $\mathbb{Z}_N$, or the so-called torsion classes in mathematics. Some selective examples from QFT or gravity include:

- (1): An SU(2) anomaly of Witten [14] with a $\mathbb{Z}_2$ class, which is a gauge anomaly.
- (2): A new SU(2) anomaly [15] with another $\mathbb{Z}_2$ class, which is a mixed gauge-gravity anomaly.
- (3): A higher ('t Hooft) anomaly for a pure SU(2) YM theory with a second-Chern-class topological term (or the so-called $\theta = \pi$ term) discovered in [16] and later refined via a mathematical well-defined topological term with four siblings of YM [17, 18] involving a discrete 0-form time-reversal symmetry and a 1-form center $\mathbb{Z}_2$-symmetry.
- (4): Global gravitational anomalies [19].

By "all topological terms," we mean the anomaly-inflow relation [20]: The $(d-1)$d anomalies can be systematically captured by a one higher dimensional dd topological invariants or dd topological terms. Recently Ref. [21] gives a non-perturbative description of anomaly inflow including local and global anomalies based on Dai-Freed theorem [22] and the Atiyah-Patodi-Singer η-Invariant [23].

Therefore, by determining all $(d-1)$d anomalies, we also determine all dd topological terms, and vice versa.[7]

[7]There is however a disclaimer. By all anomalies and all topological terms, their classifications and characterizations depend on the category of manifolds that can detect them. The categories of manifolds can be: TOP (topological manifolds), PL (piecewise linear), or DIFF (differentiable thus equivalently smooth), etc. These categories are different, and they are related by

$$\text{TOP} \supseteq \text{PL} \supseteq \text{DIFF}. \tag{1.2}$$

Since the SM, GUT and BSM are given by a continuum QFT data, in this work, we only focus on the DIFF manifolds and their associated all possible anomalies and topological terms. However, if we refine the data of QFT later in the future to include PL or TOP data from PL or TOP manifolds, we may also need to refine the corresponding SM, GUT and BSM. Thus, we will have a new set of so-called all anomalies and all topological terms. The tools we use in either case would be a certain version of cobordism theory suitable for a category of manifolds. See more in [24].
Another possible loop hole is that we do only focus on invertible anomalies captured by iTQFT, we do not study non-invertible anomalies. Different experts and different research fields may regard and define anomalies in different ways. After all, Laozi (B.C. 600) in Dao De Jing educated us that "The Way that can be told of is not an eternal way; The names that can be named are not eternal names."

[II]. *What tools are we using to classify and characterize all anomalies and all topological terms?*

The short answer is based on the Freed-Hopkin's theorem [25] and our prior work [26, 27].

The long answer is follows. Based on the Freed-Hopkin's theorem [25] and an extended generalization that we propose [26], there exists a one-to-one correspondence between "the invertible topological quantum field theories (iTQFTs) with symmetry (including higher symmetries or generalized global symmetries [12])" and "a cobordism group." In condensed matter physics, this means that there is a relation from iTQFT to "the symmetric invertible topological order (iTO, see a review [28]) with symmetry (including higher symmetries) or symmetry-protected topological state (SPTs, see [28–30]) that can be regularized on a lattice in its own dimensions."

More precisely, it is a one-to-one correspondence (isomorphism "≅") between the following two well-defined "mathematical objects" (which turn out to be abelian groups):

$$\left\{\begin{array}{c}\text{Deformation classes of the reflection positive}\\ \text{invertible } d\text{-dimensional extended}\\ \text{topological field theories (iTQFT) with}\\ \text{symmetry group } \frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}}\end{array}\right\}$$
$$\cong [MT(\frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}}), \Sigma^{d+1} I\mathbb{Z}]_{\text{tors}}. \tag{1.3}$$

We shall explain the notation above: MTG is the Madsen-Tillmann spectrum [31] of the group G, Σ is the suspension, $I\mathbb{Z}$ is the Anderson dual spectrum, and tors means the torsion group by taking only the finite group sector.

In other words, we classify the deformation classes of symmetric iTQFTs and also symmetric invertible topological orders (iTOs), via this particular cobordism group defined as follows

$$\begin{aligned}\Omega^d_G &\equiv \Omega^d_{(\frac{G_{\text{spacetime}} \ltimes \mathbb{G}_{\text{internal}}}{N_{\text{shared}}})}\\ &\equiv \mathrm{TP}_d(G) \equiv [MTG, \Sigma^{d+1} I\mathbb{Z}].\end{aligned} \tag{1.4}$$

by classifying the cobordant relations of smooth, differentiable and triangulable manifolds with a stable G-structure, via associating them to the homotopy groups of Thom-Madsen-Tillmann spectra [31, 32], given by a theorem in Ref. [25]. Ref. [25] introduced TP which means the abbreviation of "Topological Phases" classifying the above symmetric iTQFT, where our notations follow [25] and [26]. (For an introduction of the mathematical background and mathematical notations explained for physicists, the readers can consult the Appendix A of [33] or [26].)

Now let us pause to trace back some recent history of relating these anomalies/topological terms to a cobordism theory. The d dimensional 't Hooft anomaly of ordinary 0-form global symmetries is known to be captured by a $(d+1)$ dimensional iTQFT. In the condensed matter literatures, these $(d+1)$d iTQFTs describe Symmetry-Protected Topological states (SPTs) or symmetric invertible topological orders (iTO)[8] [28–30]. SPTs and symmetric iTO are interacting systems (interacting systems of bosons and fermions at the lattice scale UV with a local Hilbert space) beyond the free fermion or K theory classification [34] for the (non-interacting) topological insulators/superconductors [35, 36]. The relations between the SPTs and the response probe field theory partition functions have been *systematically* studied, selectively, in [33, 37–40] (and References therein),[9] and climaxed to the evidence of cobordism

[8]We abbreviate both Symmetry-Protected Topological *state* and Symmetry-Protected Topological *states* as SPTs. We also abbreviate both Symmetry-Enriched Topologically ordered *state* and Symmetry-Enriched Topologically ordered *states* as SETs.

[9]For example, the interacting versions of 10 Cartan symmetry classes of fermionic superconductors/insulators classifications (e.g. [41] in 4d or 3+1D) from condensed matter can be captured precisely by bordism invariants as invertible TQFTs [33]

classification of SPTs [42, 43]. Recently, the iTQFTs and SPTs are found to be systematically classified by a powerful cobordism theory of Freed-Hopkins [25], following the earlier framework of Thom-Madsen-Tillmann spectra [31, 32].

A *new* ingredient in our work [26, 27] is a generalization of the calculations and the cobordism theory of Freed-Hopkins [25] involving higher symmetries: Instead of the ordinary group G or ordinary classifying space $\mathrm{B}G$, we consider a generalized cobordism theory studying spacetime manifolds endorsed with $G_{\text{spacetime}}$ structure, with an additional higher group $\mathbb{G}$ (i.e., generalized as principal-$\mathbb{G}$ bundles) and *higher classifying spaces* $\mathrm{B}\mathbb{G}$.[10]

[III]. *What do we mean by classifications and characterizations*?

- By *classification*, we mean that given certain physics theories or phenomena (here, higher-iTQFT and higher quantum anomalies), given a spacetime dimensions (here $d+1$d for higher-iTQFT or dd for higher quantum anomalies), and their spacetime $G_{\text{spacetime}}$-structure and the internal higher global symmetry $\mathbb{G}_{\text{internal}}$, we compute how *many classes* (a number to count them) there are? Also, we aim to determine the mathematical structures of classes (i.e. here group structure as for (co)bordism groups: would the classes be a finite group $\mathbb{Z}_N$ or an infinite group $\mathbb{Z}$ or others, etc.).
- By *characterization*, we mean that we formulate their mathematical invariants (here, we mean the bordism invariants) to fully describe or capture their mathematical essences and physics properties. Hopefully, one may further compute their physical observables from mathematical invariants.

Since some of readers are still with us reading this sentence (after we answer the three questions [I], [II] and [III]), we believe that these readers decide to be interested in understanding our results.

Here we concern theories of 4d SMs, GUTs and BSMs and their anomalies and topological terms. Their 4d 't Hooft anomalies captured by 5d iTQFTs. These 5d iTQFTs or bordism invariants are defined on the dd manifolds ($d=5$). In fact, in our work, we present all

$$(d-1)\text{d 't Hooft anomalies captured by } d\text{d iTQFTs, for } d=1,2,3,4,5,$$

associated with various SMs, GUTs and BSM (ungauged) symmetries. The manifold generators for the bordism groups are actually the closed dd manifolds. We should clarify that although there are 't Hooft anomalies for $(d-1)$d QFTs (so $\mathbb{G}_{\text{internal}}$ may not be gauge-able on the boundary), the SPTs/topological invariants defined in the closed dd actually have $\mathbb{G}_{\text{internal}}$ always gauge-able in that dd. This is related to the fact that the *bulk* dd SPTs in condensed matter physics has an onsite local internal $\mathbb{G}_{\text{internal}}$-symmetry (or on-$n$-simplex-symmetry as a generalization for higher-SPTs), thus this $\mathbb{G}_{\text{internal}}$ must be gauge-able. By gauging the topological terms, this idea has been used to study the vacua of YM gauge theories coupling to gauged SPT terms (like the orbifold techniques, but in any dimenion), for example, in [33] and references therein. There are other uses and interpretations of our cobordism theory data that we will explain in Sec. 7.

We should emphasize that several recent pursuits are also along the fusions between the non-perturbative physics of SMs, GUTs, BSMs via a cobordism theory:

- Garca-Etxebarria-Montero [44] studies global anomalies of some SMs and GUTs model via a Dai-Freed theorem and Atiyah-Hirzebruch spectral sequence (AHSS) [22].

[10] Although most of our results in this article focus on the ordinary symmetry group, this framework do allow us to consider possible higher symmetries and higher anomalies [26, 27].

- Wang-Wen [45], independently, studies the non-perturbative definitions (e.g. on a lattice) and the global anomalies of SO(10) GUTs or SO(18) GUTs via $\Omega_5^{\frac{\text{Spin}(5)\times\text{Spin}(n)}{\mathbb{Z}_2^F}}$ with $n = 10, 18$ and SU(5) GUTs via $\Omega_5^{\text{Spin}}(\text{BSU}(5))$. They ask what are the possible anomalies for a fermionic theory with $\frac{\text{Spin}(5)\times\text{Spin}(10)}{\mathbb{Z}_2^F}$-symmetry.[11] Under the interaction effects, the answer turns out to be a $\mathbb{Z}_2$ class (or a mod 2 class) global anomaly captured by the 5d iTQFT:

$$e^{\text{i}\pi\int_{M^5} w_2(TM)w_3(TM)}, \tag{1.5}$$

where $w_n(TM)$ is the n-th-Stiefel-Whitney class for the tangent bundle of 5D spacetime M^5. We note that on a M^5, we have a $\frac{\text{Spin}(d=5)\times\text{Spin}(n)}{\mathbb{Z}_2^F}$ connection — a mixed gravity-gauge connection, rather than the pure gravitational Spin($d = 5$) connection, such that $w_2(TM) = w_2(V_{\text{SO}(N)})$ and $w_3(TM) = w_3(V_{\text{SO}(N)})$, where $w_n(V_{\text{SO}(N)})$ is the n-th-Stiefel-Whitney class for an SO(N) gauge bundle. The M^5 can be a non-spin manifold. This is the same global anomaly known as "a new SU(2) anomaly" studied in Ref. [15]. But Ref. [45] and [15] show explicitly, since

$$\text{Spin}(10) \supset \text{Spin}(3) = \text{SU}(2),$$

if the SO(10) GUT chiral fermion theory is free from "the new SU(2) anomaly [15]" (which indeed is true), then the SO(10) GUT chiral fermion theory contains *no* anomaly at all. Thus this SO(10) GUT is all anomaly-free [15,45]." This leads to a possible non-perturbative construction of SO(10) GUTs on the lattice proposed in [46,47], rooted in the idea of gapping the mirror chiral fermion of Eichten-Preskill [48]. However, we will not pursue this idea of [45] nor [46,47] (such as the lattice regularization) further in this work, but leave this for a future exploration [49].

- McNamara-Vafa [50] the cobordism classes and the constraints on the Quantum Gravity or String Theory Swampland.

- Davighi-Gripaios-Lohitsiri [51] studies also the global anomalies in various SMs and BSMs, also based on Atiyah-Hirzebruch spectral sequence.

- Freed-Hopkins [52] studies the global anomaly relevant for M theory.

The outline of our article is the following. We consider the following models/theories, their co/bordism groups, TP groups, topological terms and anomalies, written in terms of iTQFTs.

1. Standard Models in Sec. 2 and Sec. 3:

   - Sec. 2: Spin $\times\, \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma_n}$ model for $\Gamma_n \coloneqq \mathbb{Z}_n$ from $\Omega_d^{\text{Spin}}(\text{B}\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma_n})$ with $n = 1, 2, 3, 6$. The Lie algebra of SM is known to be $su(3)\times su(2)\times u(1)$, but it is known that the global structure of gauge group allows a quotient group of

     $$\text{SU}(3)\times\text{SU}(2)\times\text{U}(1) \tag{1.6}$$

     as

     $$\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma_n}, \tag{1.7}$$

     which is well-explained, for example, in [53].[12]

[11] Before dynamically gauging Spin(10), SO(10)-GUT is one kind of such theory: an Spin(10)-chiral gauge theory with fermions in the half-integer (iso)spin-representation. So we may call this ungauged theory as SO(10)-GUT chiral fermion theory.

[12] More generally, the global structure of gauge group as G or G/Γ makes real physical differences in observables [54].

- Sec. 3: $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma_n}$ model from $\Omega_d^{\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4}(\text{B}\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma_n})$ with $n = 1,2,3,6$. This is an interesting group suggested by [44, 55].

2. Grand Unified Theories (GUT) in Sec. 4, Sec. 5 and Sec. 6:

- Sec. 4: Pati-Salam model $\frac{\text{Spin}\times\frac{\text{SU}(4)\times(\text{SU}(2)\times\text{SU}(2))}{\Gamma_n}}{\mathbb{Z}_2^F}$ with $n = 1, 2$.
- Sec. 5: SO(10) and SO(18) GUT from either $\Omega_d^{\frac{\text{Spin}(d)\times\text{Spin}(n)}{\mathbb{Z}_2^F}}$ and $\Omega_d^{\text{Spin}}(\text{BSO}(n))$.
- Sec. 6: SU(5) GUT from either $\Omega_d^{\text{Spin}}(\text{BSU}(5))$ or more generally $\Omega_d^{\text{Spin}}(\text{BSU}(n))$.

We provide an overview how this data can be used to constrain SMs, GUTs and BSMs in Conclusions in Sec. 7. After all these physics stories and inputs, and especially thanks to the readers are still staying in tune with us, now let us introduce some mathematics preliminary in Sec. 1.2.

1.2 Mathematics Preliminary

Adams spectral sequence is a mathematical tool to compute the homotopy groups of spectra. In particular, the homotopy group of the Madsen-Tillmann spectrum MTH is the bordism group Ω_d^H. We use Adams spectral sequence to compute several bordism groups related to Standard Models (SM), Grand Unified Theories (GUT) and beyond. We also compute the group $\text{TP}_d(H)$ classifying the topological phases (i.e., the topological terms in QFT or the topological phases of quantum matter) based on the computation of bordism groups and a short exact sequence.

We aim to compute the bordism group Ω_d^H and the group $\text{TP}_d(H)$ for several groups H, our computation is based on Adams spectral sequence, see [25, 26, 33] for primers.

The group $\text{TP}_d(H)$ is defined in [25] and classifies deformation classes of reflection positive invertible d-dimensional extended topological field theories with symmetry group H_d. The relation between $\text{TP}_d(H)$ and the bordism group Ω_d^H is the following short exact sequence

$$0 \to \text{Ext}^1(\Omega_d^H, \mathbb{Z}) \to \text{TP}_d(H) \to \text{Hom}(\Omega_{d+1}^H, \mathbb{Z}) \to 0. \tag{1.8}$$

We have the following Adams spectral sequence:

$$\text{Ext}_{\mathcal{A}_p}^{s,t}(\text{H}^*(Y, \mathbb{Z}_p), \mathbb{Z}_p) \Rightarrow \pi_{t-s}(Y)_p^\wedge \tag{1.9}$$

where $\mathcal{A}_p$ is the mod p Steenrod algebra, Y is any spectrum. For any finitely generated abelian group G, $G_p^\wedge = \lim_{n\to\infty} G/p^nG$ is the p-completion of G. In particular, the mod 2 Steenrod algebra $\mathcal{A}_2$ is generated by Steenrod squares Sq^i.

We will consider the case $Y = MTH$, the Madsen-Tillmann spectrum of some group H, since by the generalized Pontryagin-Thom isomorphism, we have

$$\Omega_d^H = \pi_d(MTH). \tag{1.10}$$

In particular, $MT\text{Spin} = M\text{Spin}$.

The mod 2 cohomology of the Thom spectrum $M\mathrm{Spin}$ is

$$\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2)=\mathcal{A}_2\otimes_{\mathcal{A}_2(1)}\{\mathbb{Z}_2\oplus M\} \tag{1.11}$$

where $\mathcal{A}_2(1)$ is the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2, M is a graded $\mathcal{A}_2(1)$ module with the degree i homogeneous part $M_i=0$ for $i<8$.

By the Frobenius reciprocity, we have

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathcal{A}_2\otimes_{\mathcal{A}_2(1)}L,\mathbb{Z}_2)=\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L,\mathbb{Z}_2). \tag{1.12}$$

If $MTH=M\mathrm{Spin}\wedge X$, then $\mathrm{H}^*(MTH,\mathbb{Z}_2)=\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2)\otimes\mathrm{H}^*(X,\mathbb{Z}_2)$ where all the cohomology are the reduced cohomology. So for $t-s<8$, we have

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathrm{H}^*(X,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow(\Omega^H_{t-s})^\wedge_2. \tag{1.13}$$

In the special case $H=\mathrm{Spin}\times G$ where G is a group, then $MTH=M\mathrm{Spin}\wedge(\mathrm{B}G)_+$, $\Omega^H_d=\Omega^{\mathrm{Spin}}_d(\mathrm{B}G)$, and $\mathrm{H}^*(MTH,\mathbb{Z}_2)=\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}G,\mathbb{Z}_2)$ where the cohomology $\mathrm{H}^*(\mathrm{B}G,\mathbb{Z}_2)$ is the ordinary cohomology. So for $t-s<8$, we have

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathrm{H}^*(\mathrm{B}G,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega^{\mathrm{Spin}}_{t-s}(\mathrm{B}G)^\wedge_2. \tag{1.14}$$

Throughout the article, we will use CS^V_{2n-1} to denote the Chern-Simons $2n-1$-form for the Chern class (if V is a complex vector bundle) or the Pontryagin class (if V is a real vector bundle). Note that $p_i(V)=(-1)^ic_{2i}(V\otimes\mathbb{C})$. The relation between the Chern-Simons form and the Chern class is

$$c_n(V)=\mathrm{d}\mathrm{CS}^V_{2n-1} \tag{1.15}$$

where d is the exterior differential and $c_n(V)$ is regarded as a closed differential form in de Rham cohomology.

There is also another kind of Chern-Simons form for Euler class [56], we denote it by $\mathrm{CS}^V_{2n-1,e}$, it satisfies

$$e_{2n}(V)=\mathrm{d}\mathrm{CS}^V_{2n-1,e}. \tag{1.16}$$

The relations between Pontryagin class, Euler class and Stiefel-Whitney class are

$$p_i(V)=w_{2i}(V)^2 \mod 2 \tag{1.17}$$

and

$$e_{2n}(V)=w_{2n}(V) \mod 2 \tag{1.18}$$

where $2n=\dim V$.

The relation between the signature and the first Pontryagin class of a 4-manifold M is

$$\sigma=\frac{1}{3}\int_M p_1(TM). \tag{1.19}$$

2 Standard Models

Now we consider the co/bordism classes relevant for Standard Model (SM) physics [3–5].

2.1 $\text{Spin} \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ model

We consider $H = \text{Spin} \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$.

We have

$$\text{H}^*(\text{BSU}(n), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, \ldots, c_n] \tag{2.1}$$

and

$$\text{H}^*(\text{BU}(n), \mathbb{Z}_2) = \mathbb{Z}_2[c_1, \ldots, c_n]. \tag{2.2}$$

We also have the Wu formula[13]

$$\text{Sq}^{2j}(c_i) = \sum_{k=0}^{j} \binom{i-k-1}{j-k} c_{i+j-k} c_k \text{ for } 0 \le j \le i. \tag{2.4}$$

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^*(\text{B}(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Spin}\times\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}. \tag{2.5}$$

By Künneth formula, we have

$$\text{H}^*(\text{B}(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3] \otimes \mathbb{Z}_2[c_2'] \otimes \mathbb{Z}_2[c_1'']. \tag{2.6}$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^*(\text{B}(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)), \mathbb{Z}_2)$ below degree 6 and the E_2 page are shown in Figure 1, 2. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 31 and 33.

[13] There is another Wu formula which will be used later:

$$\text{Sq}^i(x_{d-i}) = u_i x_{d-i} \tag{2.3}$$

on d-manifold M where $x_{d-i} \in \text{H}^{d-i}(M, \mathbb{Z}_2)$ and u_i is the Wu class.

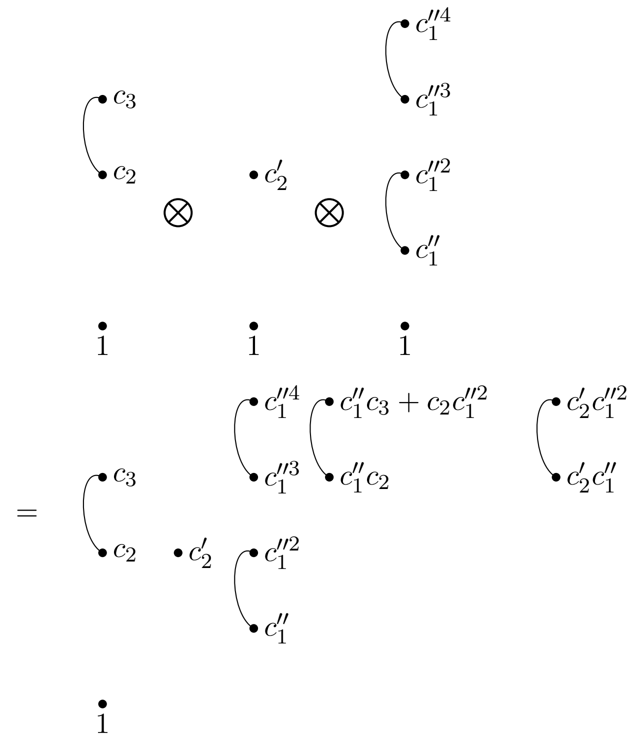


Figure 1: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)), \mathbb{Z}_2)$ below degree 6.

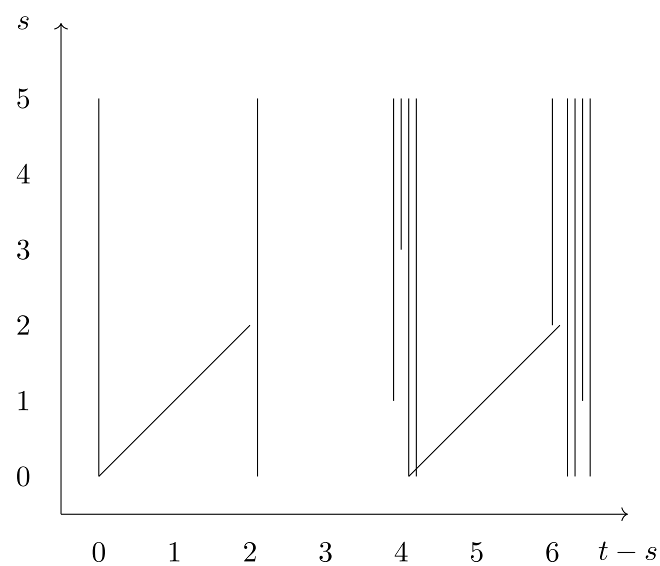


Figure 2: $\Omega_*^{\mathrm{Spin} \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)}$.

Bordism group		
d	$\Omega_d^{\mathrm{Spin}\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2$	$\tilde{\eta}$
2	$\mathbb{Z}\times\mathbb{Z}_2$	$c_1(\mathrm{U}(1))$, Arf
3	0	
4	$\mathbb{Z}^4$	$\frac{\sigma}{16}, \frac{c_1(\mathrm{U}(1))^2}{2}, c_2(\mathrm{SU}(2)), c_2(\mathrm{SU}(3))$
5	$\mathbb{Z}_2$	$c_2(\mathrm{SU}(2))\tilde{\eta}$
6	$\mathbb{Z}^5\times\mathbb{Z}_2$	$\frac{c_1(\mathrm{U}(1))(\sigma-F\cdot F)}{8}, c_1(\mathrm{U}(1))^3, c_1(\mathrm{U}(1))c_2(\mathrm{SU}(2)),$ $c_1(\mathrm{U}(1))c_2(\mathrm{SU}(3)), \frac{c_3(\mathrm{SU}(3))}{2}, c_2(\mathrm{SU}(2))\mathrm{Arf}$

Table 1: Bordism group. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. σ is the signature of manifold. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Note that $c_1(\mathrm{U}(1))^2 = \mathrm{Sq}^2 c_1(\mathrm{U}(1)) = (w_2+w_1^2)c_1(\mathrm{U}(1)) = 0 \mod 2$ on Spin 4-manifolds, $c_3(\mathrm{SU}(3)) = \mathrm{Sq}^2 c_2(\mathrm{SU}(3)) = (w_2+w_1^2)c_2(\mathrm{SU}(3)) = 0 \mod 2$ on Spin 6-manifolds.

Cobordism group		
d	TP_d $(\mathrm{Spin}\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1))$	generators
0	0	
1	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathrm{CS}_1^{\mathrm{U}(1)}, \tilde{\eta}$
2	$\mathbb{Z}_2$	Arf
3	$\mathbb{Z}^4$	$\frac{\mathrm{CS}_3^{TM}}{48}, \frac{1}{2}\mathrm{CS}_1^{\mathrm{U}(1)}c_1(\mathrm{U}(1)), \mathrm{CS}_3^{\mathrm{SU}(2)}, \mathrm{CS}_3^{\mathrm{SU}(3)}$
4	0	
5	$\mathbb{Z}^5\times\mathbb{Z}_2$	$\mu(\mathrm{PD}(c_1(\mathrm{U}(1)))), \mathrm{CS}_1^{\mathrm{U}(1)}c_1(\mathrm{U}(1))^2, \mathrm{CS}_1^{\mathrm{U}(1)}c_2(\mathrm{SU}(2)),$ $\mathrm{CS}_1^{\mathrm{U}(1)}c_2(\mathrm{SU}(3)), \frac{1}{2}\mathrm{CS}_5^{\mathrm{SU}(3)}, c_2(\mathrm{SU}(2))\tilde{\eta}$

Table 2: Topological phases ($\equiv$ TP) as a cobordism group, following Table 1. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\mathrm{PD}(c_1(\mathrm{U}(1))))$ is related to $\frac{c_1(\mathrm{U}(1))(\sigma-F\cdot F)}{8}$ in Table 1.

2.2 $\mathrm{Spin}\times\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_2}$ model

We consider $H = \mathrm{Spin}\times\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_2} = \mathrm{Spin}\times\mathrm{SU}(3)\times\mathrm{U}(2)$.

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3)\times\mathrm{U}(2)),\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}\times\mathrm{SU}(3)\times\mathrm{U}(2)}. \tag{2.7}$$

By Künneth formula, we have

$$\mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3)\times\mathrm{U}(2)),\mathbb{Z}_2) = \mathbb{Z}_2[c_2,c_3]\otimes\mathbb{Z}_2[c_1',c_2']. \tag{2.8}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{U}(2)), \mathbb{Z}_2)$ below degree 6 and the E_2 page are shown in Figure 3, 4. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 31 and 33.

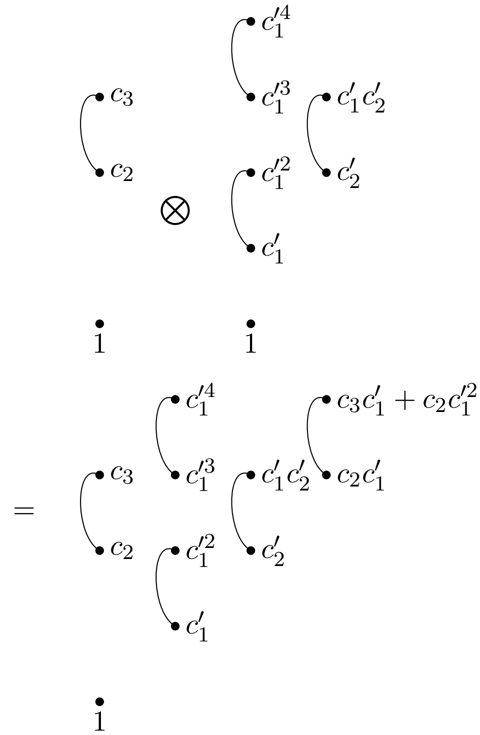


Figure 3: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{U}(2)), \mathbb{Z}_2)$ below degree 6.

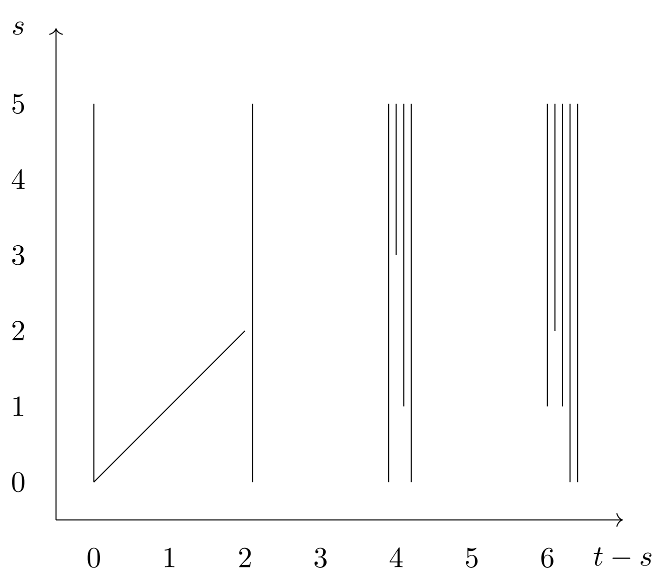


Figure 4: $\Omega_*^{\mathrm{Spin} \times \mathrm{SU}(3) \times \mathrm{U}(2)}$.

Bordism group		
d	$\Omega_d^{\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2}}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2$	$\tilde{\eta}$
2	$\mathbb{Z}\times\mathbb{Z}_2$	$c_1(\text{U}(2))$, Arf
3	0	
4	$\mathbb{Z}^4$	$\frac{\sigma}{16}, \frac{c_1(\text{U}(2))^2}{2}, c_2(\text{SU}(3)), c_2(\text{U}(2))$
5	0	
6	$\mathbb{Z}^5$	$\frac{c_1(\text{U}(2))(\sigma-F\cdot F)}{8}, \frac{c_3(\text{SU}(3))}{2}, \frac{c_1(\text{U}(2))c_2(\text{U}(2))}{2}, c_1(\text{U}(2))^3, c_1(\text{U}(2))c_2(\text{SU}(3))$

Table 3: Bordism group. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. σ is the signature of manifold. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Note that $c_1(\text{U}(2))^2 = \text{Sq}^2 c_1(\text{U}(2)) = (w_2+w_1^2)c_1(\text{U}(2)) = 0 \mod 2$ on Spin 4-manifolds, $c_3(\text{SU}(3)) = \text{Sq}^2 c_2(\text{SU}(3)) = (w_2+w_1^2)c_2(\text{SU}(3)) = 0 \mod 2$ on Spin 6-manifolds.

Cobordism group		
d	$\text{TP}_d(\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2})$	generators
0	0	
1	$\mathbb{Z}\times\mathbb{Z}_2$	$\text{CS}_1^{\text{U}(2)}, \tilde{\eta}$
2	$\mathbb{Z}_2$	Arf
3	$\mathbb{Z}^4$	$\frac{\text{CS}_3^{TM}}{48}, \frac{1}{2}\text{CS}_1^{\text{U}(2)}c_1(\text{U}(2)), \text{CS}_3^{\text{U}(2)}, \text{CS}_3^{\text{SU}(3)}$
4	0	
5	$\mathbb{Z}^5$	$\mu(\text{PD}(c_1(\text{U}(2)))), \frac{1}{2}\text{CS}_5^{\text{SU}(3)}, \frac{1}{2}\text{CS}_1^{\text{U}(2)}c_2(\text{U}(2)), \text{CS}_1^{\text{U}(2)}c_1(\text{U}(2))^2, \text{CS}_1^{\text{U}(2)}c_2(\text{SU}(3))$

Table 4: Topological phases ($\equiv$ TP) as a cobordism group, following Table 3. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\text{PD}(c_1(\text{U}(2))))$ is related to $\frac{c_1(\text{U}(2))(\sigma-F\cdot F)}{8}$ in Table 3.

2.3 $\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}$ model

We consider $H = \text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3} = \text{Spin}\times\text{U}(3)\times\text{SU}(2)$.

The localization of $M\text{Spin}$ at the prime 3 is the wedge sum of suspensions of the Brown-Peterson spectrum BP ($M\text{Spin}_{(3)} = BP\vee\Sigma^8 BP\vee\cdots$) and $\text{H}^*(BP,\mathbb{Z}_3) = \mathcal{A}_3/(\beta_{(3,3)})$ where $(\beta_{(3,3)})$ is the two-sided ideal generated by $\beta_{(3,3)}$, and $\beta_{(3,3)}$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_3\to\mathbb{Z}_9\to\mathbb{Z}_3$. Note that

$$\cdots\longrightarrow\Sigma^2\mathcal{A}_3\oplus\Sigma^6\mathcal{A}_3\oplus\cdots\longrightarrow\Sigma\mathcal{A}_3\oplus\Sigma^5\mathcal{A}_3\oplus\cdots\longrightarrow\mathcal{A}_3\longrightarrow\mathcal{A}_3/(\beta_{(3,3)}) \quad (2.9)$$

is an $\mathcal{A}_3$-resolution of $\mathcal{A}_3/(\beta_{(3,3)})$ (denoted by $P_\bullet$) where the differentials d_1 are induced by $\beta_{(3,3)}$.

The Adams chart of $\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_3), \mathbb{Z}_3)$ is shown in Figure 5. $P_\bullet \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{U}(3) \times \mathrm{SU}(2)), \mathbb{Z}_3)$ is a projective $\mathcal{A}_3$-resolution of $\mathrm{H}^*(BP, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{U}(3) \times \mathrm{SU}(2)), \mathbb{Z}_3)$ (since $P_\bullet$ is actually a free $\mathcal{A}_3$-resolution), the differentials d_1 are induced by $\beta_{(3,3)}$.

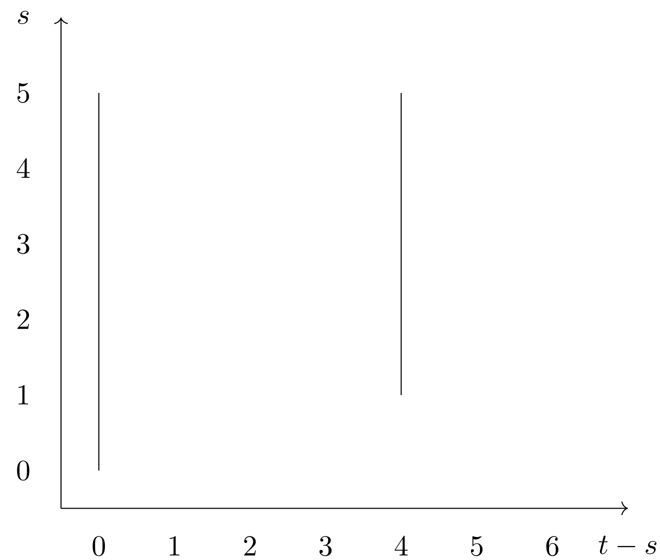


Figure 5: Adams chart of $\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_3), \mathbb{Z}_3)$.

We have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{U}(3) \times \mathrm{SU}(2)), \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow (\Omega_{t-s}^{\mathrm{Spin} \times \mathrm{U}(3) \times \mathrm{SU}(2)})_3^\wedge. \tag{2.10}$$

By Künneth formula, we have

$$\mathrm{H}^*(\mathrm{B}(\mathrm{U}(3) \times \mathrm{SU}(2)), \mathbb{Z}_3) = \mathbb{Z}_3[c_1, c_2, c_3] \otimes \mathbb{Z}_3[c_2']. \tag{2.11}$$

The Adams chart of $\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{U}(3) \times \mathrm{SU}(2)), \mathbb{Z}_3), \mathbb{Z}_3)$ is shown in Figure 6. There is no differential since the arrow of the differential d_r is of bidegree $(-1, r)$, while all lines are of interval 2 at degree $t - s$.

So there is actually no 3-torsion in $\Omega_d^{\mathrm{Spin} \times \mathrm{U}(3) \times \mathrm{SU}(2)}$.

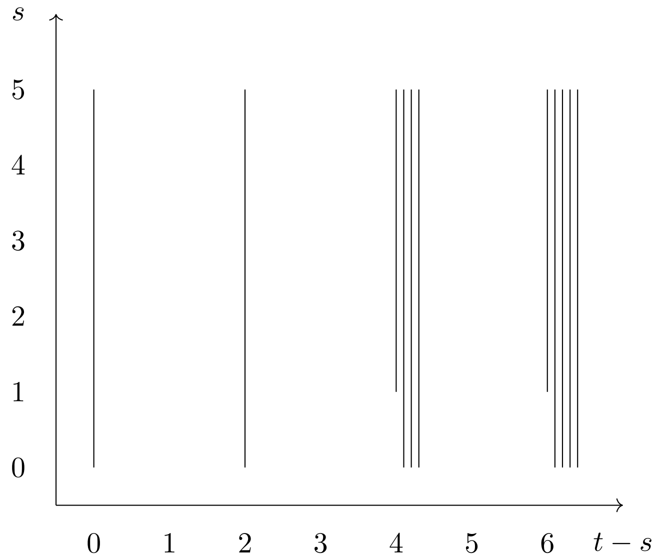


Figure 6: Adams chart of $\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{B}(\mathrm{U}(3)\times\mathrm{SU}(2)),\mathbb{Z}_3),\mathbb{Z}_3)$.

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{B}(\mathrm{U}(3)\times\mathrm{SU}(2)),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}\times\mathrm{U}(3)\times\mathrm{SU}(2)}. \tag{2.12}$$

By Künneth formula, we have

$$\mathrm{H}^*(\mathrm{B}(\mathrm{U}(3)\times\mathrm{SU}(2)),\mathbb{Z}_2)=\mathbb{Z}_2[c_1,c_2,c_3]\otimes\mathbb{Z}_2[c_2']. \tag{2.13}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}(\mathrm{U}(3)\times\mathrm{SU}(2)),\mathbb{Z}_2)$ below degree 6 and the E_2 page are shown in Figure 7, 8. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 31 and 33.

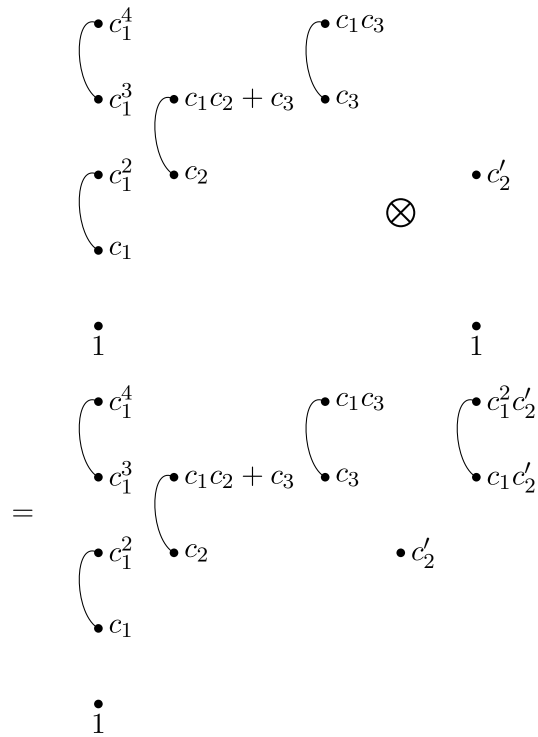


Figure 7: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}(\mathrm{U}(3) \times \mathrm{SU}(2)), \mathbb{Z}_2)$ below degree 6.

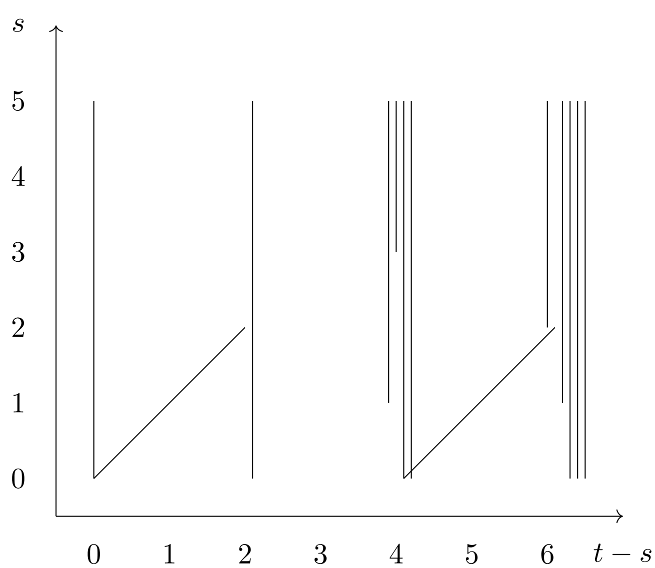


Figure 8: $\Omega_*^{\mathrm{Spin}\times\mathrm{U}(3)\times\mathrm{SU}(2)}$

Bordism group		
d	$\Omega_d^{\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2$	$\tilde\eta$
2	$\mathbb{Z}\times\mathbb{Z}_2$	$c_1(\text{U}(3))$, Arf
3	0	
4	$\mathbb{Z}^4$	$\frac{\sigma}{16}, \frac{1}{2}c_1(\text{U}(3))^2, c_2(\text{U}(3)), c_2(\text{SU}(2))$
5	$\mathbb{Z}_2$	$c_2(\text{SU}(2))\tilde\eta$
6	$\mathbb{Z}^5\times\mathbb{Z}_2$	$\frac{c_1(\text{U}(3))(\sigma-F\cdot F)}{8}, \frac{c_1(\text{U}(3))c_2(\text{U}(3))+c_3(\text{U}(3))}{2}, c_1(\text{U}(3))^3,$ $c_3(\text{U}(3)), c_1(\text{U}(3))c_2(\text{SU}(2)), c_2(\text{SU}(2))\text{Arf}$

Table 5: Bordism group. $\tilde\eta$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. σ is the signature of manifold. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Note that $c_1(\text{U}(3))^2 = \text{Sq}^2 c_1(\text{U}(3)) = (w_2+w_1^2)c_1(\text{U}(3)) = 0 \mod 2$ on Spin 4-manifolds, $c_1(\text{U}(3))c_2(\text{U}(3)) + c_3(\text{U}(3)) = \text{Sq}^2 c_2(\text{U}(3)) = (w_2+w_1^2)c_2(\text{U}(3)) = 0 \mod 2$ on Spin 6-manifolds.

Cobordism group		
d	$\text{TP}_d(\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3})$	generators
0	0	
1	$\mathbb{Z}\times\mathbb{Z}_2$	$\text{CS}_1^{\text{U}(3)}, \tilde\eta$
2	$\mathbb{Z}_2$	Arf
3	$\mathbb{Z}^4$	$\frac{\text{CS}_3^{TM}}{48}, \frac{1}{2}\text{CS}_1^{\text{U}(3)}c_1(\text{U}(3)), \text{CS}_3^{\text{U}(3)}, \text{CS}_3^{\text{SU}(2)}$
4	0	
5	$\mathbb{Z}^5\times\mathbb{Z}_2$	$\mu(\text{PD}(c_1(\text{U}(3)))), \frac{\text{CS}_1^{\text{U}(3)}c_2(\text{U}(3))+\text{CS}_5^{\text{U}(3)}}{2}, \text{CS}_1^{\text{U}(3)}c_1(\text{U}(3))^2,$ $\text{CS}_5^{\text{U}(3)}, \text{CS}_1^{\text{U}(3)}c_2(\text{SU}(2)), c_2(\text{SU}(2))\tilde\eta$

Table 6: Topological phases ($\equiv$ TP) as a cobordism group, following Table 5. $\tilde\eta$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\text{PD}(c_1(\text{U}(3))))$ is related to $\frac{c_1(\text{U}(3))(\sigma-F\cdot F)}{8}$ in Table 5.

2.4 Spin $\times \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6}$ model

We consider $H = \text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6} = \text{Spin}\times\text{S}(\text{U}(3)\times\text{U}(2))$. This SM group is particularly interesting because:

$$\text{SO}(10)\supset\text{SU}(5)\supset\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6}. \tag{2.14}$$

Thus the SO(10) GUT and SO(5) GUT can be Higgs down to $\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6}$ SM. We have

$$(\text{SU}(3)\times\text{SU}(2)\times\text{U}(1))/\mathbb{Z}_6 = (\text{SU}(3)\times\text{U}(2))/\mathbb{Z}_3 = (\text{SU}(2)\times\text{U}(3))/\mathbb{Z}_2. \tag{2.15}$$

Let the hypercharge be:

$$\frac{Y}{2} = \mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}). \tag{2.16}$$

The group $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_6}$ is just the subgroup of SU(5) commuting with the group generated by Y.

The subgroup is $\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2)) = (A, B) \in \mathrm{U}(3) \times \mathrm{U}(2) | \det A \cdot \det B = 1$ and $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6 = \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2)) \subset \mathrm{SU}(5)$.

Since

$$\mathrm{H}^*(\mathrm{B}(\frac{\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)}{\mathbb{Z}_6}), \mathbb{Z}_3) = \mathrm{H}^*(\mathrm{B}(\frac{\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)}{\mathbb{Z}_3}), \mathbb{Z}_3), \tag{2.17}$$

similarly as the discussion in Sec. 2.3, there is no 3-torsion in $\Omega_d^{\mathrm{Spin}\times\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))}$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BS}(\mathrm{U}(3) \times \mathrm{U}(2)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}\times\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))}. \tag{2.18}$$

We have the following commutative diagram with exact columns

$$\begin{array}{ccc} \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2)) & \hookrightarrow & \mathrm{SU}(5) \\ \downarrow & & \downarrow \\ \mathrm{U}(3) \times \mathrm{U}(2) & \hookrightarrow & \mathrm{U}(5) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ \mathrm{U}(1) & & \mathrm{U}(1). \end{array} \tag{2.19}$$

So by Künneth formula, we have

$$\mathrm{H}^*(\mathrm{BS}(\mathrm{U}(3) \times \mathrm{U}(2)), \mathbb{Z}_2) = \mathbb{Z}_2[c_1, c_2, c_3] \otimes \mathbb{Z}_2[c_1', c_2']/(c_1 = c_1'). \tag{2.20}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BS}(\mathrm{U}(3) \times \mathrm{U}(2)), \mathbb{Z}_2)$ below degree 6 and the E_2 page are shown in Figure 9, 10. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 31 and 33.

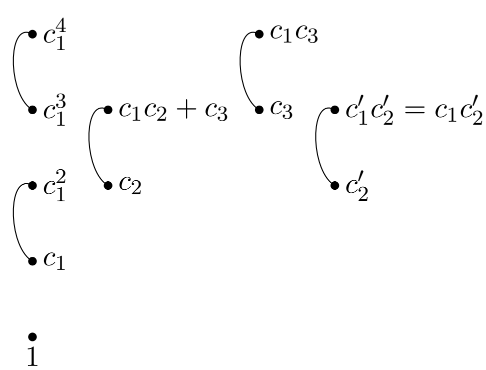


Figure 9: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BS}(\mathrm{U}(3) \times \mathrm{U}(2)), \mathbb{Z}_2)$ below degree 6.

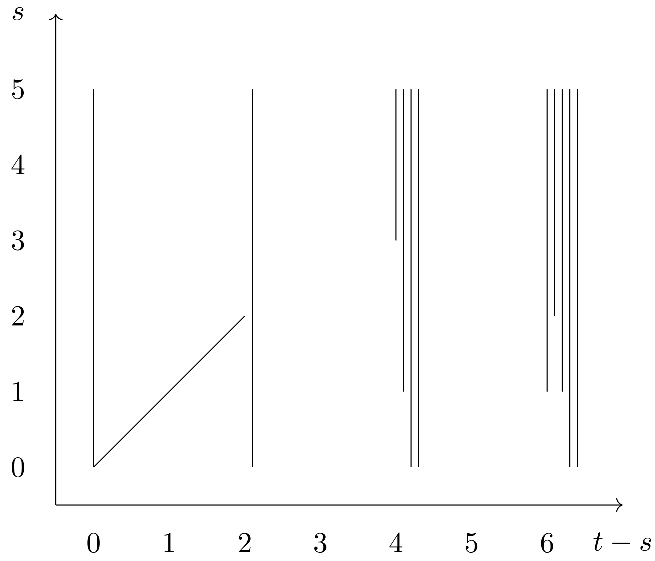


Figure 10: $\Omega_*^{\mathrm{Spin}\times \mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))}$.

Bordism group		
d	$\Omega_d^{\mathrm{Spin}\times\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_6}}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2$	$\tilde{\eta}$
2	$\mathbb{Z}\times\mathbb{Z}_2$	$c_1(\mathrm{U}(3))$, Arf
3	0	
4	$\mathbb{Z}^4$	$\frac{\sigma}{16}, \frac{1}{2}c_1(\mathrm{U}(3))^2, c_2(\mathrm{U}(3)), c_2(\mathrm{U}(2))$
5	0	
6	$\mathbb{Z}^5$	$\frac{c_1(\mathrm{U}(3))(\sigma-F\cdot F)}{8}, \frac{c_1(\mathrm{U}(3))c_2(\mathrm{U}(3))+c_3(\mathrm{U}(3))}{2}$, $c_1(\mathrm{U}(3))c_2(\mathrm{U}(2)), c_1(\mathrm{U}(3))^3, c_3(\mathrm{U}(3))$

Table 7: Bordism group. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. σ is the signature of manifold. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Here $c_1(\mathrm{U}(3))$ is identified with $c_1(\mathrm{U}(2))$. Note that $c_1(\mathrm{U}(3))^2 = \mathrm{Sq}^2 c_1(\mathrm{U}(3)) = (w_2+w_1^2)c_1(\mathrm{U}(3)) = 0 \mod 2$ on Spin 4-manifolds, $c_1(\mathrm{U}(3))c_2(\mathrm{U}(3)) + c_3(\mathrm{U}(3)) = \mathrm{Sq}^2 c_2(\mathrm{U}(3)) = (w_2+w_1^2)c_2(\mathrm{U}(3)) = 0 \mod 2$ on Spin 6-manifolds.

	Cobordism group	
d	$\mathrm{TP}_d(\mathrm{Spin} \times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_6})$	generators
0	0	
1	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathrm{CS}_1^{\mathrm{U}(3)}, \tilde{\eta}$
2	$\mathbb{Z}_2$	Arf
3	$\mathbb{Z}^4$	$\frac{\mathrm{CS}_3^{TM}}{48}, \frac{1}{2}\mathrm{CS}_1^{\mathrm{U}(3)} c_1(\mathrm{U}(3)), \mathrm{CS}_3^{\mathrm{U}(3)}, \mathrm{CS}_3^{\mathrm{U}(2)}$
4	0	
5	$\mathbb{Z}^5$	$\mu(\mathrm{PD}(c_1(\mathrm{U}(3)))), \frac{\mathrm{CS}_1^{\mathrm{U}(3)} c_2(\mathrm{U}(3)) + \mathrm{CS}_5^{\mathrm{U}(3)}}{2}$, $\mathrm{CS}_1^{\mathrm{U}(3)} c_2(\mathrm{U}(2)), \mathrm{CS}_1^{\mathrm{U}(3)} c_1(\mathrm{U}(3))^2, \mathrm{CS}_5^{\mathrm{U}(3)}$

Table 8: Topological phases ($\equiv$ TP) as a cobordism group, following Table 7. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. F is the characteristic 2-surface [57]. $F \cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\mathrm{PD}(c_1(\mathrm{U}(3))))$ is related to $\frac{c_1(\mathrm{U}(3))(\sigma - F\cdot F)}{8}$ in Table 7.

2.5 Comparison

In Ref. [51], using Atiyah-Hirzebruch spectral sequence, the authors compute $\Omega_d^{\mathrm{Spin}}(\mathrm{B}(\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\Gamma}))$ for $0 \leq d \leq 5$ and $\Gamma = 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$, but their results for the 3-torsion of $\Gamma = \mathbb{Z}_3$ and $\Gamma = \mathbb{Z}_6$ cases are different. Since $\mathrm{H}^*(\mathrm{B}(\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_3}), \mathbb{Z}_3) = \mathrm{H}^*(\mathrm{B}(\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_6}), \mathbb{Z}_3)$, using Adams spectral sequence, we find that the 3-torsion of $\Gamma = \mathbb{Z}_3$ and $\Gamma = \mathbb{Z}_6$ cases are the same. So our results are not in the full agreement with Ref. [51].

3 Standard Models with additional discrete symmetries

3.1 $\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ model

Inspired by the models of [44,55], below we consider the co/bordism classes relevant for Standard Models with additional discrete symmetries.[14]

We consider $H = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

We have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}(\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4) & \longrightarrow & \mathrm{B}\mathbb{Z}_2 \\ \downarrow & & \downarrow a^2 \\ \mathrm{BSO} & \xrightarrow{w_2} & \mathrm{B}^2\mathbb{Z}_2 \end{array} \tag{3.1}$$

[14] JW is grateful to Miguel Montero [44] for informing his unpublished note [55]. To make comparison, our approach is based on the Adams spectral sequence, while [44, 55] uses Atiyah-Hirzebruch spectral sequence (AHSS). Two approaches between ours [26,27] and Garcia-Etxebarria-Montero [44,55] are rather different. We will not make any attempts to interpret the data of these models and leave physics implications for future work, see [55, 58].

where a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$.

By [59], since there is a homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4) & \longrightarrow & *\times\mathrm{B}\mathbb{Z}_2 \\ \downarrow & & \downarrow \\ \mathrm{BO} & \xrightarrow{(w_1,w_2)} & K(\mathbb{Z}_2,1)\times K(\mathbb{Z}_2,2), \end{array} \tag{3.2}$$

which is equivalent to the homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4) & \longrightarrow & \mathrm{BSpin} \\ \downarrow & & \downarrow \\ \mathrm{BSO}\times\mathrm{B}\mathbb{Z}_2 & \xrightarrow{(\mathrm{Id},2\xi)} & \mathrm{BSO}, \end{array} \tag{3.3}$$

$MT(\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4) = M\mathrm{Spin}\wedge(\mathrm{B}\mathbb{Z}_2)^{2\xi} = M\mathrm{Spin}\wedge\Sigma^{-2}\mathbb{RP}_2^\infty$ where $2\xi:\mathrm{B}\mathbb{Z}_2\to\mathrm{BSO}$ is twice the sign representation, the final identification is by [60].

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}. \tag{3.4}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty,\mathbb{Z}_2)$ is shown in Figure 11.

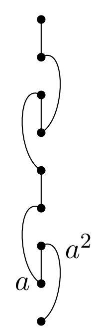


Figure 11: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty,\mathbb{Z}_2)$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty,\mathbb{Z}_2)\otimes\mathrm{H}^{*+2}(C\eta,\mathbb{Z}_2)$ is shown in Figure 12. Here $\eta:S^3\to S^2$ is the Hopf fibration, the mapping cone is $C\eta=S^2\cup_\eta e^4=\mathbb{CP}^2$. The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(C\eta,\mathbb{Z}_2)$ has two elements in degree 0 and 2 attached by a Sq^2.

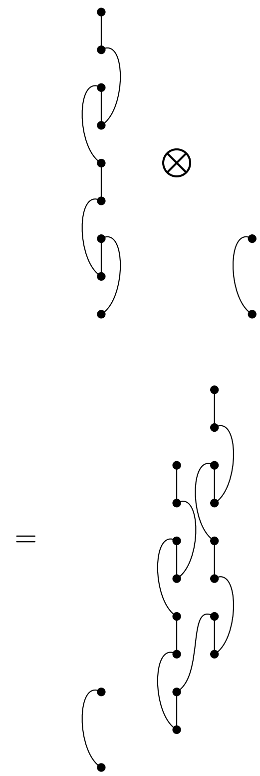


Figure 12: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes \mathrm{H}^{*+2}(C\eta, \mathbb{Z}_2)$.

Based on Figure 1 and 12, we obtain the $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)), \mathbb{Z}_2)$ below degree 6, as shown in Figure 13.

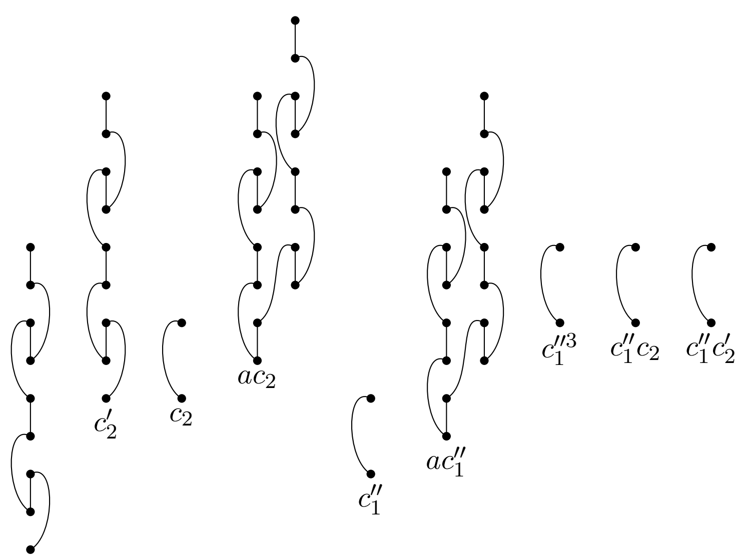


Figure 13: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)), \mathbb{Z}_2)$ below degree 6.

The E_2 page is shown in Figure 14. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 33, 34 and 35.

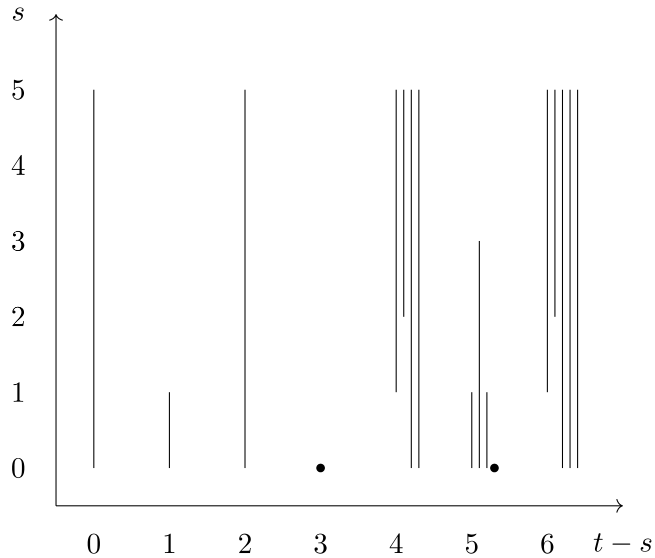


Figure 14: $\Omega_*^{\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}$.

Bordism group		
d	$\Omega_d^{\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_4$	η'
2	$\mathbb{Z}$	$c_1(\text{U}(1))$
3	$\mathbb{Z}_2$	$ac_1(\text{U}(1))$
4	$\mathbb{Z}^4$	$\frac{a^2c_1(\text{U}(1))+c_1(\text{U}(1))^2}{2}, c_2(\text{SU}(3)), c_2(\text{SU}(2)), \frac{\sigma-FF}{8}$
5	$\mathbb{Z}_2\times\mathbb{Z}_4^2\times\mathbb{Z}_{16}$	$ac_2(\text{SU}(3)), c_2(\text{SU}(2))\eta', c_1(\text{U}(1))^2\eta', \eta(\text{PD}(a))$
6	$\mathbb{Z}^5$	$c_1(\text{U}(1))\frac{\sigma-FF}{8}, \frac{a^2c_2(\text{SU}(3))+c_3(\text{SU}(3))}{2}, c_1(\text{U}(1))^3,$ $c_1(\text{U}(1))c_2(\text{SU}(3)), c_1(\text{U}(1))c_2(\text{SU}(2))$

Table 9: Bordism group. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. σ is the signature of manifold. η is the 4d eta invariant. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Note that $c_1(\text{U}(1))^2 = \text{Sq}^2c_1(\text{U}(1)) = (w_2+w_1^2)c_1(\text{U}(1)) = a^2c_1(\text{U}(1)) \mod 2$. $c_3(\text{SU}(3)) = \text{Sq}^2c_2(\text{SU}(3)) = (w_2+w_1^2)c_2(\text{SU}(3)) = a^2c_2(\text{SU}(3)) \mod 2$.

	Cobordism group	
d	TP_d $(\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))$	generators
0	0	
1	$\mathbb{Z} \times \mathbb{Z}_4$	$\mathrm{CS}_1^{\mathrm{U}(1)}, \eta'$
2	0	
3	$\mathbb{Z}^4 \times \mathbb{Z}_2$	$\frac{a^2\mathrm{CS}_1^{\mathrm{U}(1)}+c_1(\mathrm{U}(1))\mathrm{CS}_1^{\mathrm{U}(1)}}{2}, \mathrm{CS}_3^{\mathrm{SU}(3)}, \mathrm{CS}_3^{\mathrm{SU}(2)}, \mu, ac_1(\mathrm{U}(1))$
4	0	
5	$\mathbb{Z}^5 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{16}$	$\mu(\mathrm{PD}(c_1(\mathrm{U}(1)))), \frac{a^2\mathrm{CS}_3^{\mathrm{SU}(3)}+\mathrm{CS}_5^{\mathrm{SU}(3)}}{2}$, $c_1(\mathrm{U}(1))^2\mathrm{CS}_1^{\mathrm{U}(1)}, c_1(\mathrm{U}(1))\mathrm{CS}_3^{\mathrm{SU}(3)}, c_1(\mathrm{U}(1))\mathrm{CS}_3^{\mathrm{SU}(2)}$, $ac_2(\mathrm{SU}(3)), c_2(\mathrm{SU}(2))\eta', c_1(\mathrm{U}(1))^2\eta', \eta(\mathrm{PD}(a))$

Table 10: Topological phases ($\equiv$ TP) as a cobordism group, following Table 9. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. η is the 4d eta invariant. F is the characteristic 2-surface [57]. $F \cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\mathrm{PD}(c_1(\mathrm{U}(1))))$ is related to $\frac{c_1(\mathrm{U}(1))(\sigma - F\cdot F)}{8}$ in Table 9.

3.2 $\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_2}$ model

We consider $H = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_2} = \mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(3) \times \mathrm{U}(2)$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{U}(2)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\mathrm{SU}(3)\times\mathrm{U}(2)}. \quad (3.5)$$

Based on Figure 3 and 12, we obtain the $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{U}(2)), \mathbb{Z}_2)$ below degree 6, as shown in Figure 15.

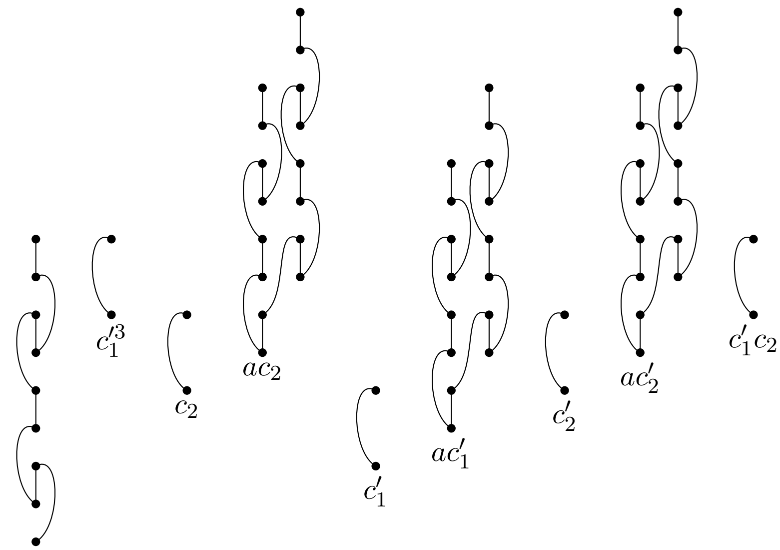


Figure 15: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathrm{SU}(3) \times \mathrm{U}(2)), \mathbb{Z}_2)$ below degree 6.

The E_2 page is shown in Figure 16. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 33, 34 and 35.

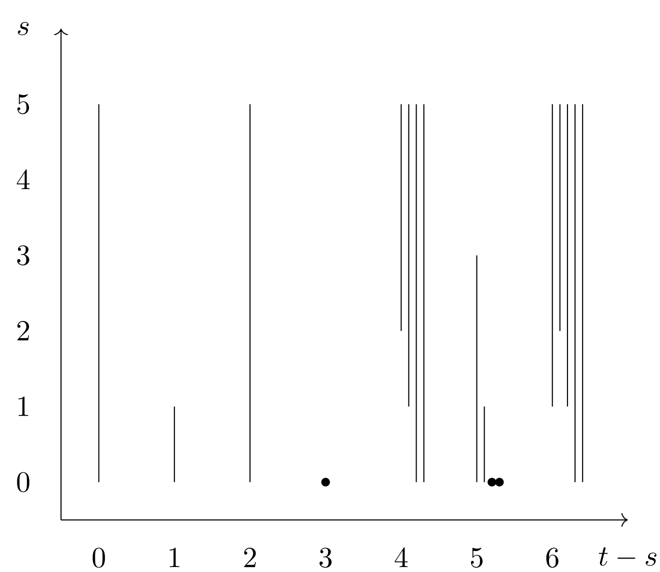


Figure 16: $\Omega_*^{\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \mathrm{SU}(3) \times \mathrm{U}(2)}$.

Bordism group		
d	$\Omega_d^{\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2}}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_4$	η'
2	$\mathbb{Z}$	$c_1(\text{U}(2))$
3	$\mathbb{Z}_2$	$ac_1(\text{U}(2))$
4	$\mathbb{Z}^4$	$\frac{a^2c_1(\text{U}(2))+c_1(\text{U}(2))^2}{2}, c_2(\text{SU}(3)), c_2(\text{U}(2)), \frac{\sigma-FF}{8}$
5	$\mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_{16}$	$ac_2(\text{SU}(3)), ac_2(\text{U}(2)), c_1(\text{U}(2))^2\eta', \eta(\text{PD}(a))$
6	$\mathbb{Z}^5$	$c_1(\text{U}(2))\frac{\sigma-FF}{8}, \frac{a^2c_2(\text{SU}(3))+c_3(\text{SU}(3))}{2}, \frac{a^2c_2(\text{U}(2))+c_1(\text{U}(2))c_2(\text{U}(2))}{2}$, $c_1(\text{U}(2))^3, c_1(\text{U}(2))c_2(\text{SU}(3))$

Table 11: Bordism group. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. η is the 4d eta invariant. σ is the signature of manifold. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Note that $c_1(\text{U}(2))^2=\text{Sq}^2c_1(\text{U}(2))=(w_2+w_1^2)c_1(\text{U}(2))=a^2c_1(\text{U}(2)) \mod 2$. $c_3(\text{SU}(3))=\text{Sq}^2c_2(\text{SU}(3))=(w_2+w_1^2)c_2(\text{SU}(3))=a^2c_2(\text{SU}(3)) \mod 2$. $c_1(\text{U}(2))c_2(\text{U}(2))=\text{Sq}^2c_2(\text{U}(2))=(w_2+w_1^2)c_2(\text{U}(2))=a^2c_2(\text{U}(2)) \mod 2$.

Cobordism group		
d	TP_d $(\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2})$	generators
0	0	
1	$\mathbb{Z}\times\mathbb{Z}_4$	$\text{CS}_1^{\text{U}(2)}, \eta'$
2	0	
3	$\mathbb{Z}^4\times\mathbb{Z}_2$	$\frac{a^2\text{CS}_1^{\text{U}(2)}+c_1(\text{U}(2))\text{CS}_1^{\text{U}(2)}}{2}, \text{CS}_3^{\text{SU}(3)}, \text{CS}_3^{\text{U}(2)}, \mu, ac_1(\text{U}(2))$
4	0	
5	$\mathbb{Z}^5\times\mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_{16}$	$\mu(\text{PD}(c_1(\text{U}(2)))), \frac{a^2\text{CS}_3^{\text{SU}(3)}+\text{CS}_5^{\text{SU}(3)}}{2}, \frac{a^2\text{CS}_3^{\text{U}(2)}+c_1(\text{U}(2))\text{CS}_3^{\text{U}(2)}}{2}$, $c_1(\text{U}(2))^2\text{CS}_1^{\text{U}(2)}, c_1(\text{U}(2))\text{CS}_3^{\text{SU}(3)}$, $ac_2(\text{SU}(3)), ac_2(\text{U}(2)), c_1(\text{U}(2))^2\eta', \eta(\text{PD}(a))$

Table 12: Topological phases ($\equiv$ TP) as a cobordism group, following Table 11. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. η is the 4d eta invariant. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4=M^3$, then $\mu(M^3)=((\sigma-FF)/8)(M^4)$, thus $\mu(\text{PD}(c_1(\text{U}(2))))$ is related to $\frac{c_1(\text{U}(2))(\sigma-F\cdot F)}{8}$ in Table 11.

3.3 Spin $\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}$ model

We consider $H=\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}=\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\text{U}(3)\times\text{SU}(2)$.

Since

$$\text{H}^*(M\text{Spin}\wedge(\text{B}\mathbb{Z}_2)^{2\xi},\mathbb{Z}_3)=\text{H}^*(M\text{Spin},\mathbb{Z}_3), \tag{3.6}$$

similarly as the discussion in Sec. 2.3, there is no 3-torsion in $\Omega_d^{\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\text{U}(3)\times\text{SU}(2)}$.

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathrm{H}^{*+2}(\mathbb{RP}^{\infty}_2,\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}(\mathrm{U}(3)\times \mathrm{SU}(2)),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \Omega_{t-s}^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times \mathrm{U}(3)\times\mathrm{SU}(2)}. \tag{3.7}$$

Based on Figure 7 and 12, we obtain the $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}^{\infty}_2,\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}(\mathrm{U}(3)\times \mathrm{SU}(2)),\mathbb{Z}_2)$ below degree 6, as shown in Figure 17.

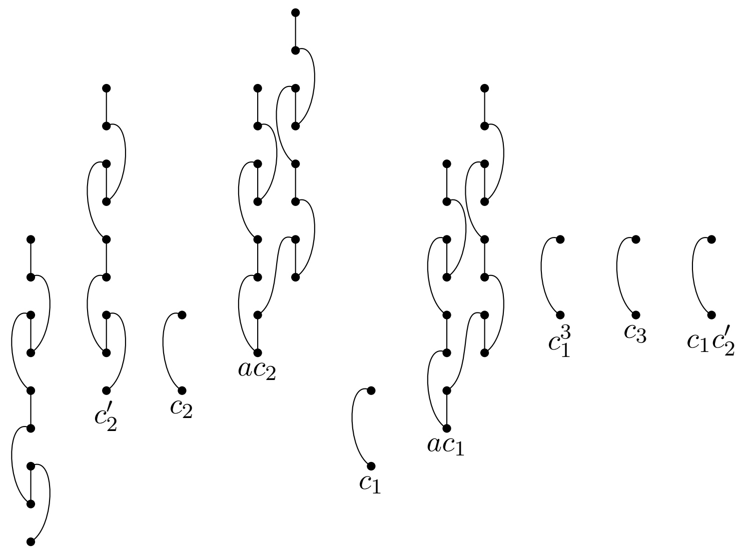


Figure 17: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}^{\infty}_2,\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}(\mathrm{U}(3)\times \mathrm{SU}(2)),\mathbb{Z}_2)$ below degree 6.

The E_2 page is shown in Figure 18. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 33, 34 and 35.

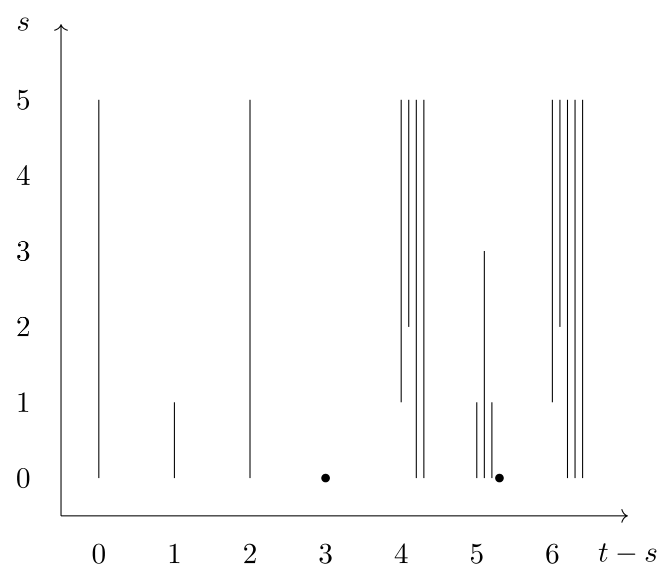


Figure 18: $\Omega_*^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times \mathrm{U}(3)\times\mathrm{SU}(2)}$.

Bordism group		
d	$\Omega_d^{\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_4$	η'
2	$\mathbb{Z}$	$c_1(\text{U}(3))$
3	$\mathbb{Z}_2$	$ac_1(\text{U}(3))$
4	$\mathbb{Z}^4$	$\frac{a^2c_1(\text{U}(3))+c_1(\text{U}(3))^2}{2}, c_2(\text{U}(3)), c_2(\text{SU}(2)), \frac{\sigma-FF}{8}$
5	$\mathbb{Z}_2\times\mathbb{Z}_4^2\times\mathbb{Z}_{16}$	$ac_2(\text{U}(3)), c_2(\text{SU}(2))\eta', c_1(\text{U}(3))^2\eta', \eta(\text{PD}(a))$
6	$\mathbb{Z}^5$	$c_1(\text{U}(3))\frac{\sigma-FF}{8}, \frac{a^2c_2(\text{U}(3))+c_1(\text{U}(3))c_2(\text{U}(3))+c_3(\text{U}(3))}{2},$ $c_1(\text{U}(3))^3, c_3(\text{U}(3)), c_1(\text{U}(3))c_2(\text{SU}(2))$

Table 13: Bordism group. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. η is the 4d eta invariant. σ is the signature of manifold. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Note that $c_1(\text{U}(3))^2 = \text{Sq}^2c_1(\text{U}(3)) = (w_2+w_1^2)c_1(\text{U}(3)) = a^2c_1(\text{U}(3)) \mod 2$. $c_1(\text{U}(3))c_2(\text{U}(3))+c_3(\text{U}(3)) = \text{Sq}^2c_2(\text{U}(3)) = (w_2+w_1^2)c_2(\text{U}(3)) = a^2c_2(\text{U}(3)) \mod 2$.

Cobordism group		
d	TP_d $(\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3})$	generators
0	0	
1	$\mathbb{Z}\times\mathbb{Z}_4$	$\text{CS}_1^{\text{U}(3)}, \eta'$
2	0	
3	$\mathbb{Z}^4\times\mathbb{Z}_2$	$\frac{a^2\text{CS}_1^{\text{U}(3)}+c_1(\text{U}(3))\text{CS}_1^{\text{U}(3)}}{2}, \text{CS}_3^{\text{U}(3)}, \text{CS}_3^{\text{SU}(2)}, \mu, ac_1(\text{U}(3))$
4	0	
5	$\mathbb{Z}^5\times\mathbb{Z}_2\times\mathbb{Z}_4^2\times\mathbb{Z}_{16}$	$\mu(\text{PD}(c_1(\text{U}(3)))), \frac{a^2\text{CS}_3^{\text{U}(3)}+c_1(\text{U}(3))\text{CS}_3^{\text{U}(3)}+\text{CS}_5^{\text{U}(3)}}{2},$ $c_1(\text{U}(3))^2\text{CS}_1^{\text{U}(3)}, \text{CS}_5^{\text{U}(3)}, c_1(\text{U}(3))\text{CS}_3^{\text{SU}(2)},$ $ac_2(\text{U}(3)), c_2(\text{SU}(2))\eta', c_1(\text{U}(3))^2\eta', \eta(\text{PD}(a))$

Table 14: Topological phases ($\equiv$ TP) as a cobordism group, following Table 13. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. η is the 4d eta invariant. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma-FF)/8)(M^4)$, thus $\mu(\text{PD}(c_1(\text{U}(3))))$ is related to $\frac{c_1(\text{U}(3))(\sigma-F\cdot F)}{8}$ in Table 13.

3.4 $\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6}$ model

We consider $H = \text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6} = \text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\text{S}(\text{U}(3)\times\text{U}(2))$.

Since

$$\text{H}^*(M\text{Spin}\wedge(\text{B}\mathbb{Z}_2)^{2\xi}, \mathbb{Z}_3) = \text{H}^*(M\text{Spin}, \mathbb{Z}_3), \tag{3.8}$$

and

$$\text{H}^*(\text{B}(\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6}), \mathbb{Z}_3) = \text{H}^*(\text{B}(\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}), \mathbb{Z}_3), \tag{3.9}$$

similarly as the discussion in Sec. 2.3, there is no 3-torsion in $\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))}$.

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BS}(\mathrm{U}(3)\times\mathrm{U}(2)),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))}. \tag{3.10}$$

Based on Figure 9 and 12, we obtain the $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BS}(\mathrm{U}(3)\times\mathrm{U}(2)),\mathbb{Z}_2)$ below degree 6, as shown in Figure 19.

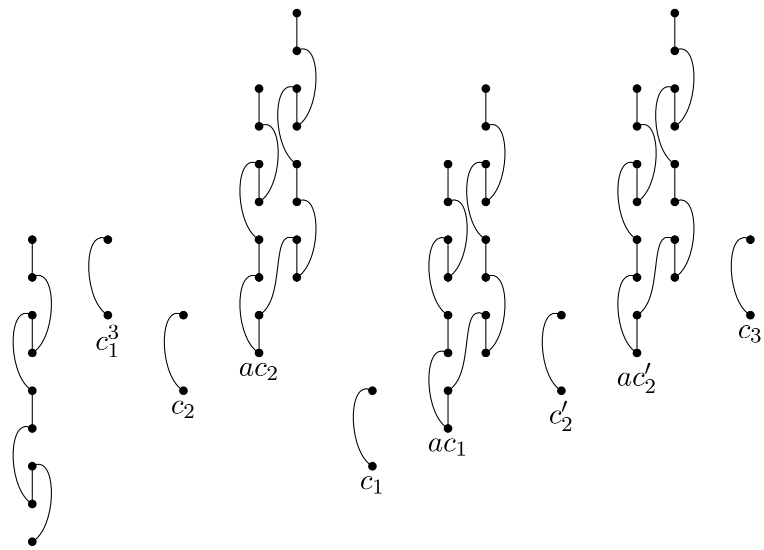


Figure 19: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(\mathbb{RP}_2^\infty,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BS}(\mathrm{U}(3)\times\mathrm{U}(2)),\mathbb{Z}_2)$ below degree 6.

The E_2 page is shown in Figure 20. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 33, 34 and 35.

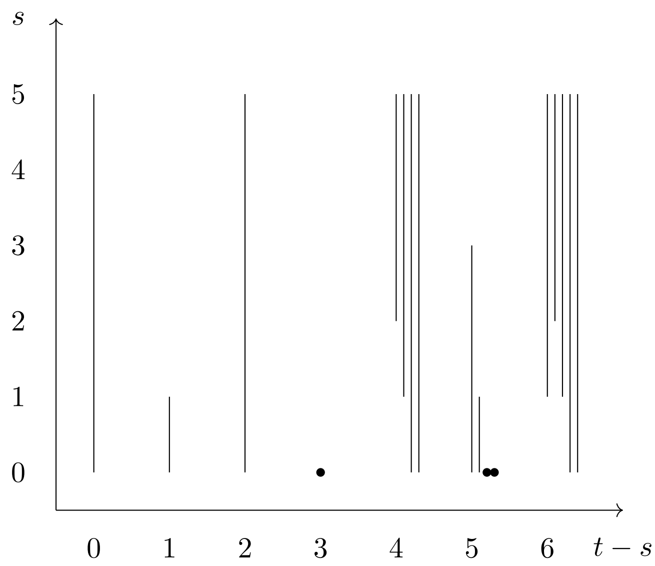


Figure 20: $\Omega_*^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times \mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))}$.

Bordism group		
d	$\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_6}}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_4$	η'
2	$\mathbb{Z}$	$c_1(\mathrm{U}(3))$
3	$\mathbb{Z}_2$	$ac_1(\mathrm{U}(3))$
4	$\mathbb{Z}^4$	$\frac{a^2c_1(\mathrm{U}(3))+c_1(\mathrm{U}(3))^2}{2}, c_2(\mathrm{U}(3)), c_2(\mathrm{U}(2)), \frac{\sigma-FF}{8}$
5	$\mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_{16}$	$ac_2(\mathrm{U}(3)), ac_2(\mathrm{U}(2)), c_1(\mathrm{U}(3))^2\eta', \eta(\mathrm{PD}(a))$
6	$\mathbb{Z}^5$	$c_1(\mathrm{U}(3))\frac{\sigma-FF}{8}, \frac{a^2c_2(\mathrm{U}(3))+c_1(\mathrm{U}(3))c_2(\mathrm{U}(3))+c_3(\mathrm{U}(3))}{2}, \frac{a^2c_2(\mathrm{U}(2))+c_1(\mathrm{U}(2))c_2(\mathrm{U}(2))}{2}, c_1(\mathrm{U}(3))^3, c_3(\mathrm{U}(3))$

Table 15: Bordism group. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. η is the 4d eta invariant. σ is the signature of manifold. F is the characteristic 2-surface [57]. $F\cdot F$ is the product of two F. Here $c_1(\mathrm{U}(3))$ is identified with $c_1(\mathrm{U}(2))$. Note that $c_1(\mathrm{U}(3))^2=\mathrm{Sq}^2c_1(\mathrm{U}(3))=(w_2+w_1^2)c_1(\mathrm{U}(3))=a^2c_1(\mathrm{U}(3)) \mod 2$. $c_1(\mathrm{U}(3))c_2(\mathrm{U}(3))+c_3(\mathrm{U}(3))=\mathrm{Sq}^2c_2(\mathrm{U}(3))=(w_2+w_1^2)c_2(\mathrm{U}(3))=a^2c_2(\mathrm{U}(3)) \mod 2$. $c_1(\mathrm{U}(2))c_2(\mathrm{U}(2))=\mathrm{Sq}^2c_2(\mathrm{U}(2))=(w_2+w_1^2)c_2(\mathrm{U}(2))=a^2c_2(\mathrm{U}(2)) \mod 2$.

Cobordism group		
d	TP_d $(\mathrm{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_6})$	generators
0	0	
1	$\mathbb{Z} \times \mathbb{Z}_4$	$\mathrm{CS}_1^{\mathrm{U}(3)}, \eta'$
2	0	
3	$\mathbb{Z}^4 \times \mathbb{Z}_2$	$\frac{a^2\mathrm{CS}_1^{\mathrm{U}(3)}+c_1(\mathrm{U}(3))\mathrm{CS}_1^{\mathrm{U}(3)}}{2}, \mathrm{CS}_3^{\mathrm{U}(3)}, \mathrm{CS}_3^{\mathrm{U}(2)}, \mu, ac_1(\mathrm{U}(3))$
4	0	
5	$\mathbb{Z}^5 \times \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}$	$\mu(\mathrm{PD}(c_1(\mathrm{U}(3)))), \frac{a^2\mathrm{CS}_3^{\mathrm{U}(3)}+c_1(\mathrm{U}(3))\mathrm{CS}_3^{\mathrm{U}(3)}+\mathrm{CS}_5^{\mathrm{U}(3)}}{2},$ $\frac{a^2\mathrm{CS}_3^{\mathrm{U}(2)}+c_1(\mathrm{U}(2))\mathrm{CS}_3^{\mathrm{U}(2)}}{2}, c_1(\mathrm{U}(3))^2\mathrm{CS}_1^{\mathrm{U}(3)}, \mathrm{CS}_5^{\mathrm{U}(3)},$ $ac_2(\mathrm{U}(3)), ac_2(\mathrm{U}(2)), c_1(\mathrm{U}(3))^2\eta', \eta(\mathrm{PD}(a))$

Table 16: Topological phases ($\equiv$ TP) as a cobordism group, following Table 15. η' is a $\mathbb{Z}_4$ valued 1d eta invariant. η is the 4d eta invariant. F is the characteristic 2-surface [57]. $F \cdot F$ is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\mathrm{PD}(c_1(\mathrm{U}(3))))$ is related to $\frac{c_1(\mathrm{U}(3))(\sigma - F\cdot F)}{8}$ in Table 15.

4 Pati-Salam models

Now we consider the co/bordism classes relevant for Pati-Salam GUT models [8] . There are actually two different cases for modding out different discrete normal subgroups.

4.1 $\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F}$ Pati-Salam model

We consider $H = \frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F}$.

Note that $\frac{\mathrm{SU}(4)}{\mathbb{Z}_2} = \mathrm{SO}(6)$, $\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbb{Z}_2} = \mathrm{SO}(4)$.

We have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}H & \longrightarrow & \mathrm{B}(\mathrm{SO}(6)\times\mathrm{SO}(4)) \\ \downarrow & & \downarrow{\scriptstyle w_2'+w_2''} \\ \mathrm{BSO} & \xrightarrow{w_2} & \mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{4.1}$$

Here $w_2' = w_2(\mathrm{SO}(6))$, $w_2'' = w_2(\mathrm{SO}(4))$.

Similarly as in [33], we can identify $\mathrm{B}H \to \mathrm{BO}$ with $\mathrm{BSpin}\times\mathrm{BSO}(6)\times\mathrm{BSO}(4) \to \mathrm{BO}$: $(W, V_1, V_2) \mapsto -W - (V_1 + V_2 - 10)$.

Hence $MTH = M\mathrm{Spin} \wedge \Sigma^{-6}M\mathrm{SO}(6) \wedge \Sigma^{-4}M\mathrm{SO}(4)$.

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathrm{H}^{*+6}(M\mathrm{SO}(6),\mathbb{Z}_2)\otimes\mathrm{H}^{*+4}(M\mathrm{SO}(4),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F}}. \tag{4.2}$$

$\mathrm{H}^{*+6}(M\mathrm{SO}(6),\mathbb{Z}_2)=\mathbb{Z}_2[w_2',w_3',w_4',w_5',w_6']U$, $\mathrm{H}^{*+4}(M\mathrm{SO}(4),\mathbb{Z}_2)=\mathbb{Z}_2[w_2'',w_3'',w_4'']V$. Here U and V are Thom classes with $\mathrm{Sq}^1U=0$, $\mathrm{Sq}^2U=w_2'U$, $\mathrm{Sq}^1V=0$ and $\mathrm{Sq}^2V=w_2''V$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+6}(M\mathrm{SO}(6),\mathbb{Z}_2)\otimes\mathrm{H}^{*+4}(M\mathrm{SO}(4),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 21, 22. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 32, 36, 37, 39 and 42.

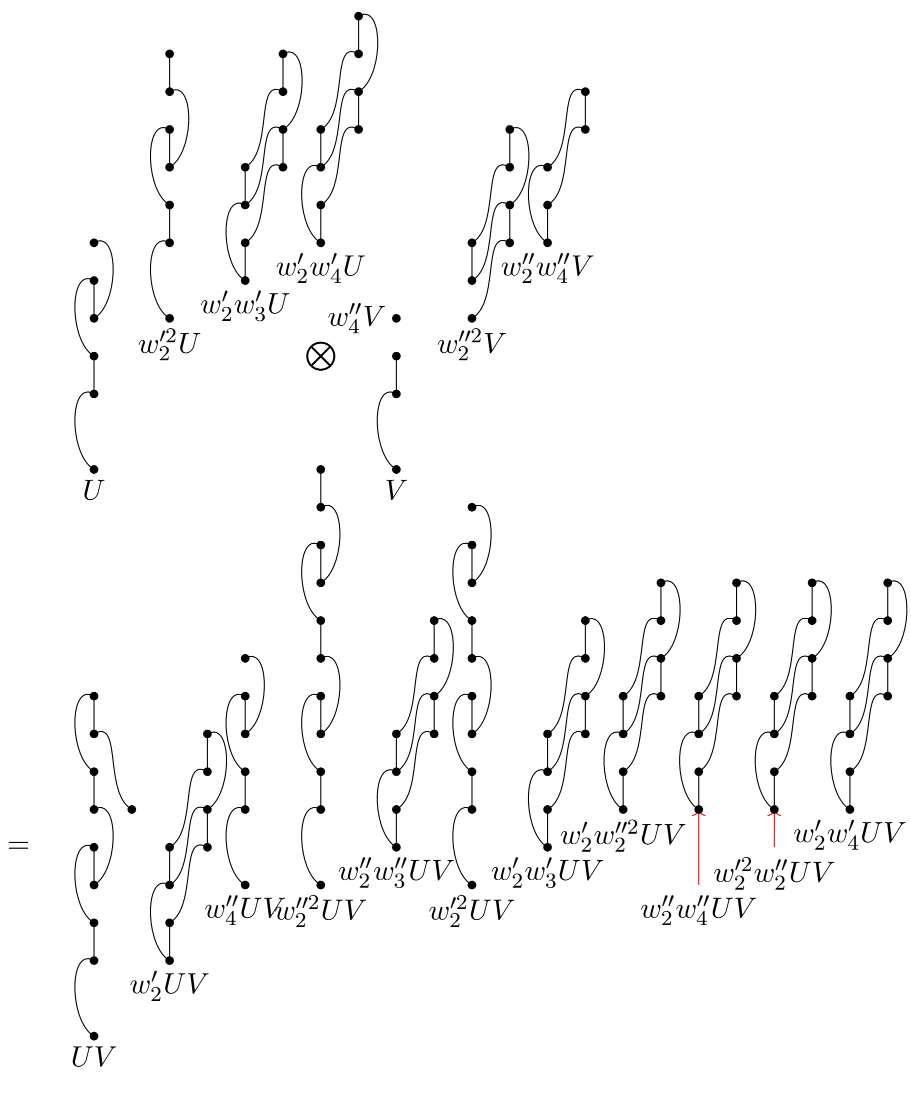


Figure 21: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+6}(M\mathrm{SO}(6),\mathbb{Z}_2)\otimes\mathrm{H}^{*+4}(M\mathrm{SO}(4),\mathbb{Z}_2)$ below degree 6.

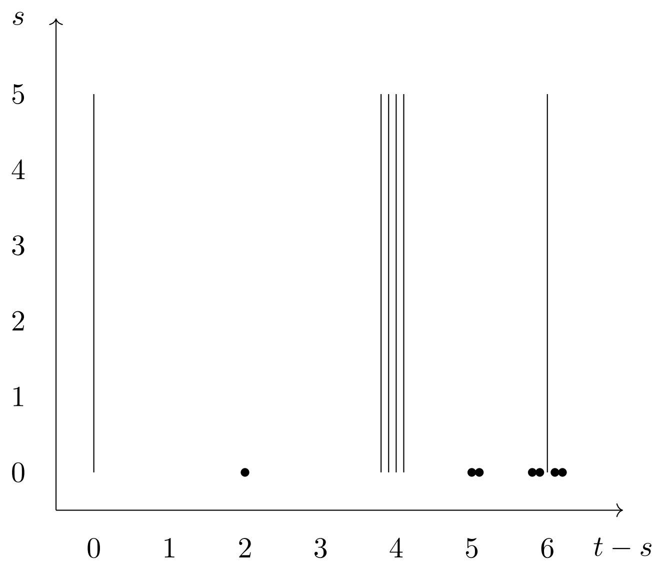


Figure 22: $\Omega_*^{\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F}}$.

Bordism group		
d	$\Omega_d^{\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F}}$	generators
0	$\mathbb{Z}$	
1	0	
2	$\mathbb{Z}_2$	w_2' (or w_2'')
3	0	
4	$\mathbb{Z}^4$	(p_1' from $w_2'^2$, e_4' from $w_4'+w_2'w_2''$, p_1'' from $w_2''^2$, e_4'' from w_4'')
5	$\mathbb{Z}_2^2$	$(w_2'w_3', w_2''w_3'')$
6	$\mathbb{Z}\times\mathbb{Z}_2^4$	(e_6' from w_6', $w_2'w_2''^2, w_2''w_4'', w_2'^2w_2''$(or $w_2'^3+w_3'^2$)$, w_2'w_4'$)

Table 17: Bordism group. Here e_i is the Euler class, p_i is the Pontryagin class. Since $w_2+w_1^2=0$ on 2-manifold, we have $w_2=w_2'+w_2''=w_1^2=0$ on oriented 2-manifold. Since $\mathrm{Sq}^2(w_2'^2)=w_3'^2=(w_2+w_1^2)w_2'^2=(w_2'+w_2'')w_2'^2$ on oriented 6-manifold by Wu formula, we have $w_2''w_2'^2=w_2'^3+w_3'^2$. Since $w_2'^2=\mathrm{Sq}^2(w_2')=(w_2+w_1^2)w_2'=(w_2'+w_2'')w_2'$ on oriented 4-manifold by Wu formula, we have $w_2'w_2''=0$ on oriented 4-manifold. On a 4-manifold, the oriented bundle of rank 6 splits as the direct sum of an oriented bundle of rank 4 and a trivial plane bundle, the Euler class e_4' is the Euler class of the subbundle of rank 4.

Cobordism group		
d	$\mathrm{TP}_d(\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F})$	generators
0	0	
1	0	
2	$\mathbb{Z}_2$	w_2' (or w_2'')
3	$\mathbb{Z}^4$	$\mathrm{CS}_3^{\mathrm{SO}(6)}, \mathrm{CS}_3^{\mathrm{SO}(4)}, \mathrm{CS}_{3,e}^{\mathrm{SO}(6)}, \mathrm{CS}_{3,e}^{\mathrm{SO}(4)}$
4	0	
5	$\mathbb{Z}\times\mathbb{Z}_2^2$	$\mathrm{CS}_{5,e}^{\mathrm{SO}(6)}, w_2'w_3', w_2''w_3''$

Table 18: Topological phases ($\equiv$ TP) as a cobordism group, following Table 17.

4.2 $\frac{\mathrm{Spin}\times\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2^F}$ Pati-Salam model

We consider $H=\frac{\mathrm{Spin}\times\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2^F}$, let $G=\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}$, then we have a fibration

$$\begin{array}{ccc} \mathrm{B}G & & \\ \downarrow & & \\ \mathrm{BSO}(6)\times\mathrm{BSO}(4) & \xrightarrow{w_2'+w_2''} & \mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{4.3}$$

Here $w_2'=w_2(\mathrm{SO}(6))$, $w_2''=w_2(\mathrm{SO}(4))$.

We have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}H & \longrightarrow & \mathrm{B}G \\ \downarrow & & \downarrow{\scriptstyle w_2'=w_2''} \\ \mathrm{BSO} & \xrightarrow{w_2} & \mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{4.4}$$

So $MTH=M\mathrm{Spin}\wedge\Sigma^{-10}MG$.

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+10}(MG,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\frac{\mathrm{Spin}\times\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2^F}}. \tag{4.5}$$

There is a fibration

$$\begin{array}{ccc} \mathrm{B}\mathbb{Z}_2 & \longrightarrow & \mathrm{B}G \\ & & \downarrow \\ & & \mathrm{BSO}(6)\times\mathrm{BSO}(4). \end{array} \tag{4.6}$$

So we have the Serre spectral sequence

$$\mathrm{H}^p(\mathrm{BSO}(6)\times\mathrm{BSO}(4),\mathrm{H}^q(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2))\Rightarrow\mathrm{H}^{p+q}(\mathrm{B}G,\mathbb{Z}_2). \tag{4.7}$$

Let a be the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$, then we have the differentials $d_2(a) = w_2' + w_2''$, $d_3(a^2) = \mathrm{Sq}^1 d_2(a) = w_3' + w_3''$, $d_5(a^4) = \mathrm{Sq}^2 d_3(a^2) = w_2'w_3' + w_5' + w_2''w_3''$ and so on. So in $\mathrm{H}^*(\mathrm{B}G,\mathbb{Z}_2)$, we have $w_2' = w_2''$, $w_3' = w_3''$ and $w_5' = 0$.

So below degree 6, we have

$$\mathrm{H}^{*+10}(MG,\mathbb{Z}_2) = ((\mathbb{Z}_2[w_2', w_3', w_4', w_5', w_6'] \otimes \mathbb{Z}_2[w_2'', w_3'', w_4''])/(w_2' = w_2'', w_3' = w_3'', w_5' = 0))U \quad (4.8)$$

where U is the Thom class with $\mathrm{Sq}^1 U = 0$, $\mathrm{Sq}^2 U = w_2'U = w_2''U$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+10}(MG,\mathbb{Z}_2)$ below degree 6 and the E_2 page are shown in Figure 23, 24. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 31, 32, 33, 38, 40 and 41.

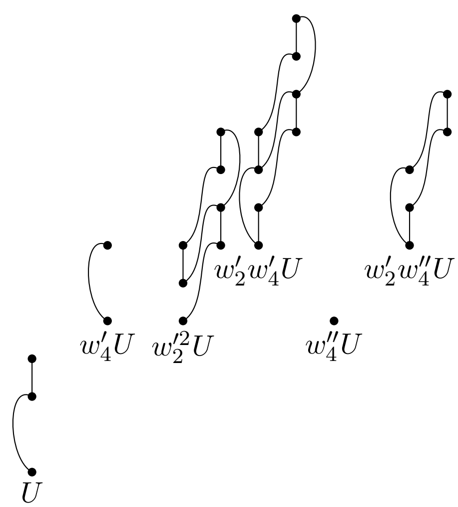


Figure 23: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+10}(MG,\mathbb{Z}_2)$ below degree 6.

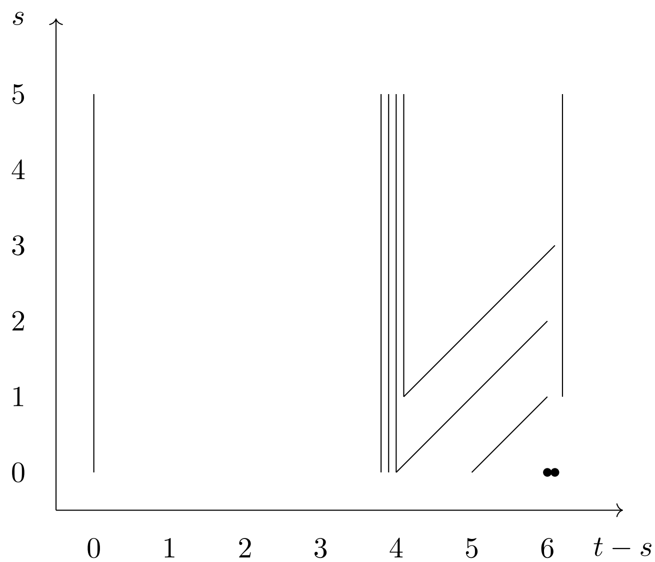


Figure 24: $\Omega_*^{\frac{\text{Spin}\times\text{SU}(4)\times(\text{SU}(2)\times\text{SU}(2))}{\mathbb{Z}_2^F}}$.

Bordism group		
d	$\Omega_d^{\frac{\text{Spin}\times\text{SU}(4)\times(\text{SU}(2)\times\text{SU}(2))}{\mathbb{Z}_2^F}}$	generators
0	$\mathbb{Z}$	
1	0	
2	0	
3	0	
4	$\mathbb{Z}^4$	(p'_1 from $w_2'^2$, e'_4 from w'_4, e''_4 from w''_4, $?_1$)
5	$\mathbb{Z}_2^3$	$(w'_2w'_3, w''_4\tilde{\eta}, ?_2)$
6	$\mathbb{Z}\times\mathbb{Z}_2^5$	($\frac{e'_6}{2}$ from w'_6, $w'_2w'_4, w'_2w''_4, w''_4\text{Arf}, w'_2w'_3\tilde{\eta}, ?_3$)

Table 19: Bordism group. Here e_i is the Euler class, p_i is the Pontryagin class. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. Since $\text{Sq}^2w'_4 = w'_2w'_4 + w'_6 = (w_2 + w_1^2)w'_4 = w'_2w'_4$ on oriented 6-manifold by Wu formula, we have $e'_6 = w'_6 = 0 \mod 2$. On a 4-manifold, the oriented bundle of rank 6 splits as the direct sum of an oriented bundle of rank 4 and a trivial plane bundle, the Euler class e'_4 is the Euler class of the subbundle of rank 4. There are 3 undetermined bordism invariants which we mark them as $?_1, ?_2, ?_3$. The $?_1$ in 4d is $\mathbb{Z}$ valued and we read from the Adams chart that there is an extension: $\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}_2$ where the middle $\mathbb{Z}$ is $?_1$, the first $\mathbb{Z}$ is $\frac{\sigma-FF}{8}$, and the last $\mathbb{Z}_2$ is $w_3\tilde{\eta}$. The $?_2$ in 5d is $\mathbb{Z}_2$ valued and we read from the Adams chart that it looks like $w_3\text{Arf}$. The $?_3$ in 6d is $\mathbb{Z}_2$ valued and we read from the Adams chart that it looks like $\frac{\sigma(\text{PD}(w_2))}{16} \mod 2$.

Cobordism group		
d	$\mathrm{TP}_d(\frac{\mathrm{Spin}\times \mathrm{SU}(4)\times(\mathrm{SU}(2)\times \mathrm{SU}(2))}{\mathbb{Z}_2^F})$	generators
0	0	
1	0	
2	0	
3	$\mathbb{Z}^4$	$\mathrm{CS}_3^{\mathrm{SO}(6)}, \mathrm{CS}_{3,e}^{\mathrm{SO}(6)}, \mathrm{CS}_{3,e}^{\mathrm{SO}(4)}, ?_4$
4	0	
5	$\mathbb{Z}\times\mathbb{Z}_2^3$	$\frac{1}{2}\mathrm{CS}_{5,e}^{\mathrm{SO}(6)}, w_2'w_3', w_4''\tilde{\eta}, ?_2$

Table 20: Topological phases ($\equiv$ TP) as a cobordism group, following Table 19. The $?_4$ in 3d is related to the $?_1$ in 4d in Table 19. The $?_2$ in 5d is explained in Table 19.

5 SO(10), SO(18) and SO(n) Grand Unifications

Now we consider the co/bordism classes relevant for Fritzsch-Minkowski SO(10) GUT [7]. There are actually two cases, depending on whether the gauged SO(10) GUT allows gauge-invariant fermions, or whether the gauged SO(10) GUT only allows gauge-invariant bosons.

5.1 $\Omega_d^{\mathrm{Spin}}(\mathrm{BSO}(n))$ for $n \geq 7$: SO(10) and SO(18)

For $t-s<8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BSO}(n),\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BSO}(n)). \tag{5.1}$$

We have

$$\mathrm{H}^*(\mathrm{BSO}(n),\mathbb{Z}_2)=\mathbb{Z}_2[w_2',w_3',\dots,w_n']. \tag{5.2}$$

We also have the Wu formula

$$\mathrm{Sq}^j w_i' = \sum_{k=0}^{j}\binom{i-k-1}{j-k} w_{i+j-k}' w_k' \text{ for } 0\leq j\leq i. \tag{5.3}$$

For $n\geq 7$, the $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BSO}(n),\mathbb{Z}_2)$ below degree 6 is shown in Figure 25.

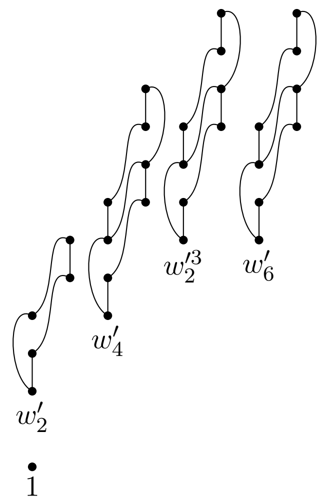


Figure 25: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BSO}(n), \mathbb{Z}_2)$ below degree 6 for $n \geq 7$.

The E_2 page is shown in Figure 26. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 31, 32 and 38.

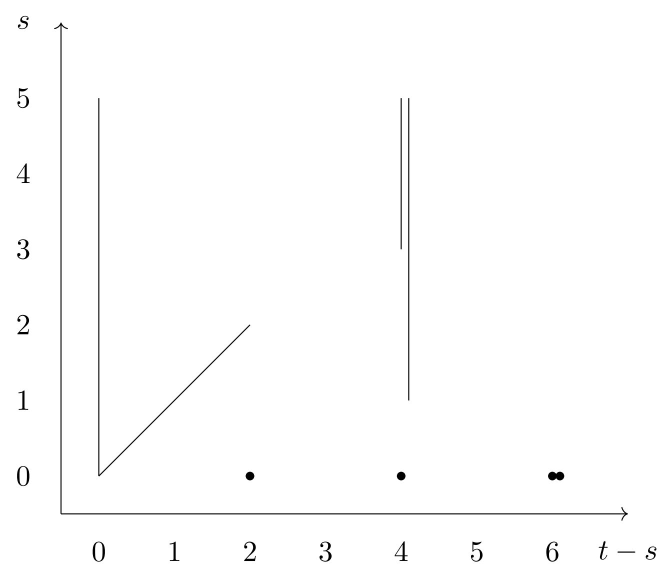


Figure 26: $\Omega_*^{\mathrm{Spin}}(\mathrm{BSO}(n))$ for $n \geq 7$.

Bordism group		
d	$\Omega_d^{\mathrm{Spin}}(\mathrm{BSO}(n))$ for $n \geq 7$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2$	$\tilde{\eta}$
2	$\mathbb{Z}_2^2$	Arf, w_2'
3	0	
4	$\mathbb{Z}^2 \times \mathbb{Z}_2$	$\frac{\sigma}{16}$, $\frac{p_1'}{2}$, w_4'
5	0	
6	$\mathbb{Z}_2^2$	$w_2'^3$, w_6'

Table 21: Bordism group. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. σ is the signature of manifold.

Cobordism group		
d	$\mathrm{TP}_d(\mathrm{Spin} \times \mathrm{SO}(n))$ for $n \geq 7$	generators
0	0	
1	$\mathbb{Z}_2$	$\tilde{\eta}$
2	$\mathbb{Z}_2^2$	Arf, w_2'
3	$\mathbb{Z}^2$	$\frac{1}{48}\mathrm{CS}_3^{TM}$, $\frac{1}{2}\mathrm{CS}_3^{\mathrm{SO}(n)}$
4	$\mathbb{Z}_2$	w_4'
5	0	

Table 22: Topological phases ($\equiv$ TP) as a cobordism group, following Table 21. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. The TM is the spacetime tangent bundle.

5.2 $\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}$: $\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F}$ and $\frac{\mathrm{Spin}\times\mathrm{Spin}(18)}{\mathbb{Z}_2^F}$

Let $H = \frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}$, we have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}H & \longrightarrow & \mathrm{BSO}(n) \\ \downarrow & & \downarrow{\scriptstyle w_2'} \\ \mathrm{BSO} & \xrightarrow{w_2} & \mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{5.4}$$

Here $w_2' = w_2(\mathrm{SO}(n))$.

There is a homotopy equivalence $f : \mathrm{BSO}\times\mathrm{BSO}(n) \xrightarrow{\sim} \mathrm{BSO}\times\mathrm{BSO}(n)$ by $(V, W) \mapsto (V - W + n, W)$. Note that $f^*(w_2) = w_2(V - W) = w_2(V) + w_1(V)w_1(W) + w_2(W) = w_2 + w_2'$. Then we have the following homotopy pullback

$$\begin{tikzcd} \mathrm{B}H \arrow[r, "\sim"] \arrow[d] & \mathrm{BSpin}\times\mathrm{BSO}(n) \arrow[d] & \\ \mathrm{BSO}\times\mathrm{BSO}(n) \arrow[r, "f"] \arrow[d, "(V,W)\mapsto V"'] \arrow[rr, bend right, "w_2+w_2'"'] & \mathrm{BSO}\times\mathrm{BSO}(n) \arrow[r, "w_2+0"] \arrow[dl, "(V,W)\mapsto V+W-n"] & \mathrm{B}^2\mathbb{Z}_2 \\ \mathrm{BSO} & & \end{tikzcd} . \tag{5.5}$$

This implies that $\mathrm{B}H \sim \mathrm{BSpin} \times \mathrm{BSO}(n)$.

$MTH = \mathrm{Thom}(\mathrm{B}H; -V)$, where V is the induced virtual bundle (of dimension 0) by the map $\mathrm{B}H \to \mathrm{BO}$.

We can identify $\mathrm{B}H \to \mathrm{BO}$ with $\mathrm{BSpin} \times \mathrm{BSO}(n) \xrightarrow{V - V_n + n} \mathrm{BSO} \hookrightarrow \mathrm{BO}$.

The spectrum MTH is homotopy equivalent to $\mathrm{Thom}(\mathrm{BSpin} \times \mathrm{BSO}(n); -(V - V_n + n))$, which is $M\mathrm{Spin} \wedge \Sigma^{-n} M\mathrm{SO}(n)$.

We consider $H = \frac{\mathrm{Spin} \times \mathrm{Spin}(10)}{\mathbb{Z}_2^F}$, $MTH = M\mathrm{Spin} \wedge \Sigma^{-10} M\mathrm{SO}(10)$.

We have $w_2 = w_2'$, namely $w_2(TM) = w_2(\mathrm{SO}(10))$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+10}(M\mathrm{SO}(10), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\frac{\mathrm{Spin} \times \mathrm{Spin}(10)}{\mathbb{Z}_2^F}}.$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+10}(M\mathrm{SO}(10), \mathbb{Z}_2)$ below degree 6 and the E_2 page are shown in Figure 27, 28. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 32 and 39.

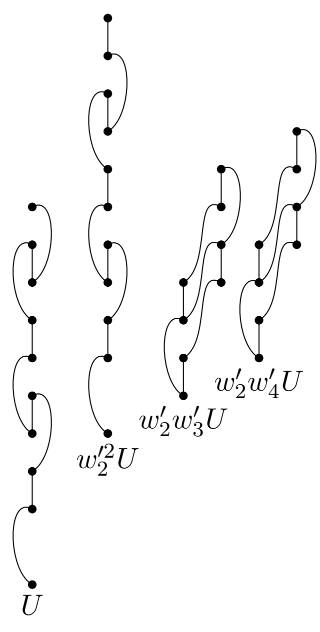


Figure 27: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+10}(M\mathrm{SO}(10), \mathbb{Z}_2)$ below degree 6.

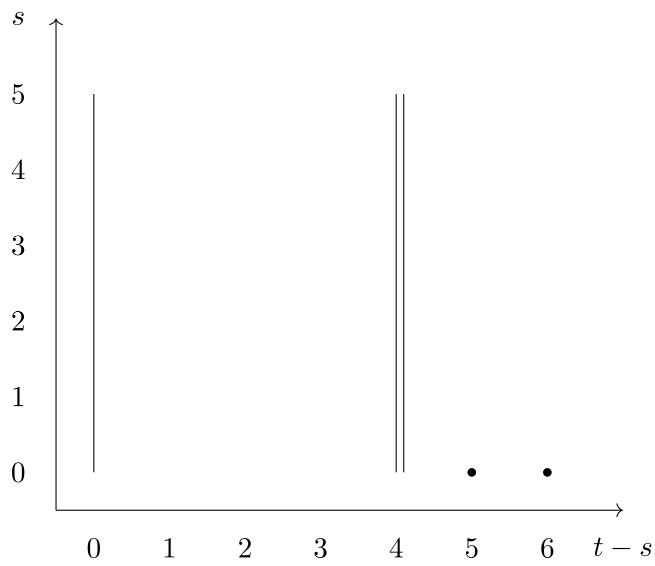


Figure 28: $\Omega_*^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F}}$.

Bordism group		
d	$\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F}}$	generators
0	$\mathbb{Z}$	
1	0	
2	0	
3	0	
4	$\mathbb{Z}^2$	p_1' (from $w_2'^2$), e_4' (from w_4')
5	$\mathbb{Z}_2$	$w_2'w_3'$
6	$\mathbb{Z}_2$	$w_2'w_4'$

Table 23: Bordism group. Here e_i is the Euler class, p_i is the Pontryagin class. On a 4-manifold, the oriented bundle of rank 10 splits as the direct sum of an oriented bundle of rank 4 and a trivial bundle of rank 6, the Euler class e_4' is the Euler class of the subbundle of rank 4.

Actually $\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}} = \Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n+1)}{\mathbb{Z}_2^F}}$ for $n \geq 7$ and $0 \leq d \leq 6$.

Cobordism group		
d	$\mathrm{TP}_d(\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F})$	generators
0	0	
1	0	
2	0	
3	$\mathbb{Z}^2$	$\mathrm{CS}_3^{\mathrm{SO}(10)}, \mathrm{CS}_{3,e}^{\mathrm{SO}(10)}$
4	0	
5	$\mathbb{Z}_2$	$w_2'w_3'$

Table 24: Topological phases ($\equiv$ TP) as a cobordism group, following Table 23.

6 SU(5) and SU(n) Grand Unifcations: Spin × SU(n): Spin × SU(5)

Now we consider the co/bordism classes relevant for Georgi-Glashow SU(5) GUT [6].

We consider $H = \text{Spin} \times \text{SU}(5)$, $MTH = M\text{Spin} \wedge (\text{BSU}(5))_+$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(\text{H}^*(\text{BSU}(5), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{\text{Spin}\times\text{SU}(5)}_{t-s}.$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^*(\text{BSU}(5), \mathbb{Z}_2)$ below degree 6 and the E_2 page are shown in Figure 29, 30. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Figure 31 and 33.

Figure 29: The $\mathcal{A}_2(1)$-module structure of $\text{H}^*(\text{BSU}(5), \mathbb{Z}_2)$ below degree 6.

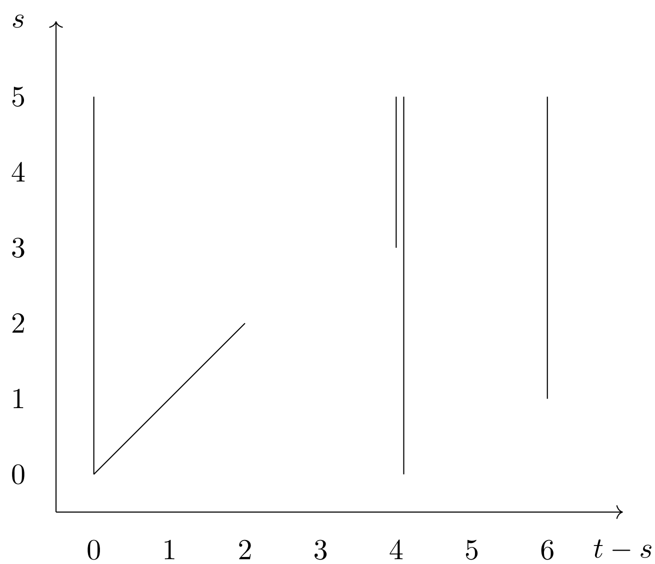


Figure 30: $\Omega^{\text{Spin}\times\text{SU}(5)}_*$.

Bordism group		
d	$\Omega_d^{\text{Spin}\times\text{SU}(5)}$	generators
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2$	$\tilde\eta$
2	$\mathbb{Z}_2$	Arf
3	0	
4	$\mathbb{Z}^2$	$\frac{\sigma}{16}, c_2$
5	0	
6	$\mathbb{Z}$	$\frac{c_3}{2}$

Table 25: Bordism group. $\tilde\eta$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. σ is the signature of manifold. Note that $c_3 = \text{Sq}^2 c_2 = (w_2 + w_1^2)c_2 = 0 \mod 2$ on Spin 6-manifolds.

Actually $\Omega_d^{\text{Spin}\times\text{SU}(n)} = \Omega_d^{\text{Spin}\times\text{SU}(n+1)}$ for $n \geq 3$ and $0 \leq d \leq 6$.

Cobordism group		
d	$\text{TP}_d(\text{Spin}\times\text{SU}(5))$	generators
0	0	
1	$\mathbb{Z}_2$	$\tilde\eta$
2	$\mathbb{Z}_2$	Arf
3	$\mathbb{Z}^2$	$\frac{1}{48}\text{CS}_3^{TM}$, $\text{CS}_3^{\text{SU}(5)}$
4	0	
5	$\mathbb{Z}$	$\frac{1}{2}\text{CS}_5^{\text{SU}(5)}$

Table 26: Topological phases ($\equiv$ TP) as a cobordism group, following Table 25. $\tilde\eta$ is a mod 2 index of 1d Dirac operator. Arf is 2d Arf invariant. The TM is the spacetime tangent bundle.

7 Conclusions, and Explorations on Non-Perturbative and Topological Sectors of BSM

7.1 Summary

- In Sec. 2, for $\Gamma = 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$, we compute the bordism groups $\Omega_d^{\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma}}$ and the bordism invariants for $0 \leq d \leq 6$. We also determine the group $\text{TP}_d(\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma})$ and the topological terms for $0 \leq d \leq 5$.

  We find that there is only 2-torsion in these bordism groups, and the bordism groups $\Omega_d^{\text{Spin}\times\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}$ and $\Omega_d^{\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}}$ are isomorphic, while $\Omega_d^{\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2}}$ and $\Omega_d^{\text{Spin}\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_6}}$ are isomorphic.

  We use the 3d Rokhlin invariant and Chern-Simons forms to express the topological terms.

- In Sec. 3, for $\Gamma = 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$, we compute the bordism groups $\Omega_d^{\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma}}$ and the bordism invariants for $0 \leq d \leq 6$. We also determine the group $\text{TP}_d(\text{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times\frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\Gamma})$ and the topological terms for $0 \leq d \leq 5$.

We find that there is only 2-torsion in these bordism groups, and the bordism groups $\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}$ and $\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_3}}$ are isomorphic, while $\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_2}}$ and $\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4\times \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathbb{Z}_6}}$ are isomorphic.

We also use the 3d Rokhlin invariant and Chern-Simons forms to express the topological terms. Compared to Sec. 2, there are new bordism invariants. For example, in 1d, $\tilde{\eta}$ becomes η' which is $\mathbb{Z}_4$ valued; in 4d, $\frac{\sigma}{16}$ becomes $\frac{\sigma - F\cdot F}{8}$ where F is a characteristic surface [57]; in 5d, there is a new $\eta(\mathrm{PD}(a))$ since there is an induced Pin$^+$ structure on PD(a) where η is the 4d eta invariant, PD is the Poincaré dual.

- In Sec. 4, we compute the bordism groups $\Omega_d^{\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F}}$ and $\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2^F}}$ for $0\leq d\leq 6$. We also determine the group $\mathrm{TP}_d(\frac{\mathrm{Spin}\times\frac{\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2}}{\mathbb{Z}_2^F})$ and $\mathrm{TP}_d(\frac{\mathrm{Spin}\times\mathrm{SU}(4)\times(\mathrm{SU}(2)\times\mathrm{SU}(2))}{\mathbb{Z}_2^F})$ for $0\leq d\leq 5$.

  We find that Euler class appears in the bordism invariants and the Chern-Simons form for Euler class appears in the topological terms.

- In Sec. 5, we compute the bordism groups $\Omega_d^{\mathrm{Spin}}(\mathrm{BSO}(n))$ and $\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}}$ for $n\geq 7$ and $0\leq d\leq 6$. We also determine the group $\mathrm{TP}_d(\mathrm{Spin}\times\mathrm{SO}(n))$ and $\mathrm{TP}_d(\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F})$ for $n\geq 7$ and $0\leq d\leq 5$.

  We find that for $n\geq 7$, $\Omega_d^{\mathrm{Spin}}(\mathrm{BSO}(n))=\Omega_d^{\mathrm{Spin}}(\mathrm{BSO}(n+1))$ and $\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}}=\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n+1)}{\mathbb{Z}_2^F}}$ for $0\leq d\leq 6$.

  We also find that Euler class appears in the bordism invariants of $\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}}$ and the Chern-Simons form for Euler class appears in the topological terms of $\mathrm{TP}_d(\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F})$.

- In Sec. 6, we compute the bordism groups $\Omega_d^{\mathrm{Spin}\times\mathrm{SU}(5)}$ for $0\leq d\leq 6$. We also determine the group $\mathrm{TP}_d(\mathrm{Spin}\times\mathrm{SU}(5))$ for $0\leq d\leq 5$.

  We find that for $n\geq 3$, $\Omega_d^{\mathrm{Spin}\times\mathrm{SU}(n)}=\Omega_d^{\mathrm{Spin}\times\mathrm{SU}(n+1)}$ for $0\leq d\leq 6$.

7.2 Constraints on Quantum Dynamics

We have discussed and summarized potential anomalies and topological terms in SM, GUT and BSM. There are actually two versions of anomalies we are speaking of: One is the Anomaly (2)for the ungauged SM, GUT and BSM with the G is simply a global symmetry. Another is the Anomaly (3)for the gauged SM, GUT and BSM with the $\mathbb{G}_{\text{internal}}$ dynamically gauged inside G. In more details,

I. For ungauged SM, GUT and BSM, we can use the 't Hooft anomaly (Anomaly (2)) to the gappability of these models' matter field sectors. For ungauged SM, GUT and BSM, we can simply have matter field contents (e.g. fermions: quarks and leptons) without dynamical gauge fields. In fact, Ref. [45–47] use the all anomaly free conditions to support that a non-perturbative definition (lattice regularization) of SO(10) GUT is doable, by checking. By anomaly free, there exists non-perturbative interactions for gapping the mirror world chiral fermions [49]. This mirror-fermion gapping can help to get rid of the mirror world chiral fermion doublers, surpassing the Nielsen-Ninomiya fermion doubling theorem (which is only true for the free non-interacting systems). The fact that all anomaly free gapless theories can be deformed to a fully gapped trivial vacuum is also consistent with the concept of Seiberg's deformation class [61]. More details can be found in [49].

II. For gauged SM, GUT and BSM, we can use the dynamical gauge anomaly matching conditions (Anomaly (3)) to rule out inconsistent theories. Importantly, depending on the matter contents and their representations in $\mathbb{G}_{\text{internal}}$, we may gain or loss some global symmetries. For example, for fermions in the adjoint representation, we can have a 1-form center symmetry for the gauge theory. Thus, we should beware potentially additional new higher 't Hooft anomalies (see [26,27]) for gauged SM, GUT and BSM can help us to constrain quantum dynamics (also more discussions below).

We use the path integral and the action to understand the basic kinematics and the global symmetry of the QFTs. We can apply the spacetime geometric topology properties to constrain QFTs, such as doing the spacetime surgery for QFTs [62–64]. We can also determine the anomalies of QFTs at UV. However, given the potentially complete anomalies, we can constrain the IR dynamics by UV-IR anomaly matching. The consequence of anomaly matching implies that the IR theories with 't Hooft anomalies in G-symmetry must be matched by at least one of the following scenarios:

1. Symmetry-breaking:
   - (say discrete or continuous G-symmetry breaking).

2. Symmetry-preserving:
   - Degenerate ground states (like the "Lieb-Schultz-Mattis theorem").
   - Gapless, e.g., conformal field theory (CFT),
   - Intrinsic topological orders (Symmetry-preserving TQFT),

3. Symmetry-extension [65]: Symmetry-extension is a rather exotic possibility, which does not occur naturally without fine-tuning or artificial designed, explained in [65]. However, the symmetry-extension approach is a useful intermediate stepstone, to construct another earlier scenario: *symmetry-preserving TQFT*, via gauging the extended-symmetry [65].

In more details, suppose there are mixed anomalies between the ordinary 0-form global symmetries and higher-form symmetries (say 1-form symmetries) in the gauged SM, GUT, and BSM as in Model II, we have further refinement of possibilities:

(i) Ordinary 0-form symmetry broken (spontaneously or explicitly).

(ii) 1-form center discrete $\mathbb{Z}^e_{N,[1]}$ symmetry broken (spontaneously or explicitly) as *deconfinement*:

  (2-i) 1-form $\mathbb{Z}^e_{N,[1]}$-symmetry breaking and deconfined TQFTs, i.e., *topological order* in condensed matter terminology.

  (2-ii) 1-form $\mathbb{Z}^e_{N,[1]}$-symmetry breaking and deconfined gapless theories (e.g. CFTs).

(iii) 1-form center discrete $\mathbb{Z}^e_{N,[1]}$ symmetry unbroken as *confinement*:

  (3-i) 1-form symmetry-extended invertible TQFT: The exotic scenario of symmetry-extended invertible TQFT is systematically studied in Ref. [65]. This idea is generalized to higher symmetries for higher-symmetry-extended invertible TQFT in Ref. [66].

(3-ii) 1-form symmetry-preserving TQFT: In fact, under certain criteria, there is a no go theorem to match some specific anomalies by this scenario. Hinted by the obstruction of constructing the symmetry-extended TQFTs [18, 66], Cordova-Ohmori [67] proved a no go theorem for TQFTs preserving both the 1-form symmetry and time-reversal symmetry while saturating the 4d SU(N) YM anomaly with $\theta = \pi$.

(iv) Full symmetry-preserving gapless theory (e.g. CFTs).

In summary, based on the kinematics, global symmetries (ordinary or higher symmetries), anomalies and spacetime topology constraints, we may suggest new quantum dynamics constraints.

7.3 Anomaly Cancellation or Anomaly-Matched Hidden Sectors for BSM

The previous section also has phenomenology implications of the gauged SM, GUT, and BSM as in Model II.

- Suppose we prove that the theory is fully 't Hooft anomaly free (for global symmetries, not the dynamical gauge anomalies), then there is an application for gapping the mirror world chiral fermions (see Ref. [45–47, 49] and References therein).
- Suppose we prove that the theory has a certain 't Hooft anomaly (for global symmetries, not the dynamical gauge anomalies), then we discover either of the following (see more details in):

1. Anomaly-Matched Hidden Topological Sectors and Topological Field Theories:
2. Anomaly-Matched Hidden Gapless Sectors: This may be more surprising for phenomenology grounds.
3. Anomaly-Matched Symmetry-Breaking Sectors: Global symmetries can be spontaneously broken.
4. All Anomaly-Free conditions are matched: This suggests the relation between the gapping criteria and defining fermion/gauge theories in particular the challenge of defining chiral fermion/gauge theories [45–47, 49].
5. Swapland Implications and Defects: In the recent work of McNamara-Vafa [50], by combing the ideas of Quantum Gravity/String Landscape and Swampland (e.g.Ref. [68, 69], see a review [70]) together with no internal global symmetry for quantum gravity [71–74], Ref. [50] argues that all the cobordism classes in the full Quantum Gravity (QG) must be vanished as in the trivial class. Effectively, the effective cobordism classes of QG $\Omega_k^{\tilde{\text{QG}}}$ must be vanished. This gives several powerful constraints for the cobordism class data in any dimension. If we find any anomalies or topological terms in lower dimensions, say $k = 1, 2, 3, 4, 5$ in our case, this suggests that in the full QG, their cobordism classes must be cancelled by some other new objects/excitations/extended operators. Two particular interesting possibilities are [50]:

(1). Symmetry is broken: Mathematically, this suggests a map

$$\Omega_k^{\tilde{\text{QG}}} \to \Omega_k^{\tilde{\text{QG}}+\text{defects}}, \tag{7.1}$$

which means the cancellation of cobordism classes via new additional symmetry defects.

(2). Symmetry is gauged: Mathematically, this suggests a map

$$\Omega_k^{\tilde{\text{QG}}+\text{gauge sectors}} \to \Omega_k^{\tilde{\text{QG}}}, \tag{7.2}$$

which means the cancellation of cobordism classes via additional new gauge sectors.

We will leave the further detail explorations of the above predictions in upcoming works [24, 27, 58].

8 Acknowledgements

The authors are listed in the alphabetical order by the standard convention. During the completion of this manuscript, we become aware a recent work has obtained related results [51] on global anomalies of SMs and GUTs. JW is grateful to Miguel Montero [44] for informing his unpublished note [55], and thanks colleagues for warmly encouragements [58]. Major tools of this work were built in Fall 2018 in a prior work Ref. [26], and the major parts of calculations of this work were done in Fall 2018 - Spring 2019, when ZW was at USTC and when JW was at Institute for Advanced Study. ZW acknowledges previous supports from NSFC grants 11431010 and 11571329. ZW is supported by the Shuimu Tsinghua Scholar Program. JW was supported by NSF Grant PHY-1606531. This work is also supported by NSF Grant DMS-1607871 "Analysis, Geometry and Mathematical Physics" and Center for Mathematical Sciences and Applications at Harvard University.

A The correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page

In this Appendix, we list the correspondence between $\mathcal{A}_2(1)$-module L and its E_2 page used in our computation before.

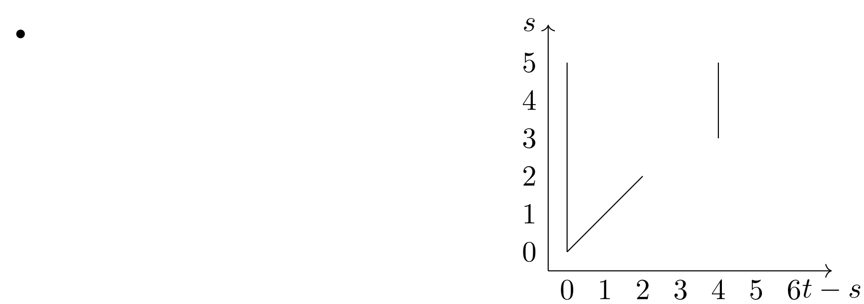


Figure 31: The $\mathcal{A}_2(1)$-module $L = \mathbb{Z}_2$ and its E_2 page.

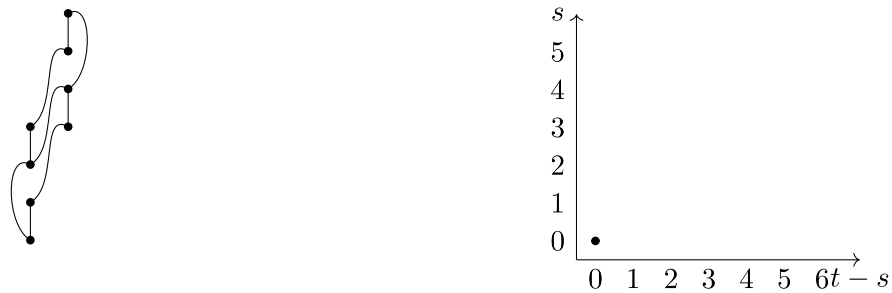


Figure 32: The $\mathcal{A}_2(1)$-module $L = \mathcal{A}_2(1)$ and its E_2 page.

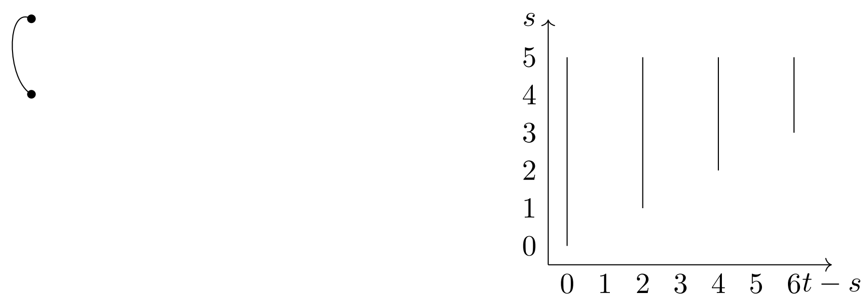


Figure 33: The $\mathcal{A}_2(1)$-module $L = \mathrm{H}^{*+2}(C\eta, \mathbb{Z}_2)$ and its E_2 page.

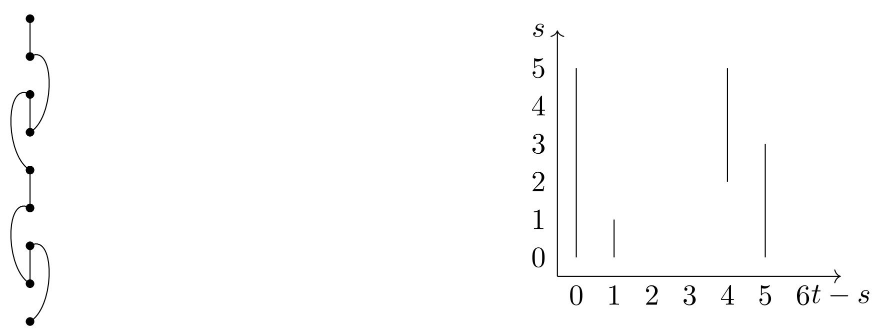


Figure 34: The $\mathcal{A}_2(1)$-module $L = \mathrm{H}^{*+2}(\mathbb{RP}^\infty_2, \mathbb{Z}_2)$ and its E_2 page.

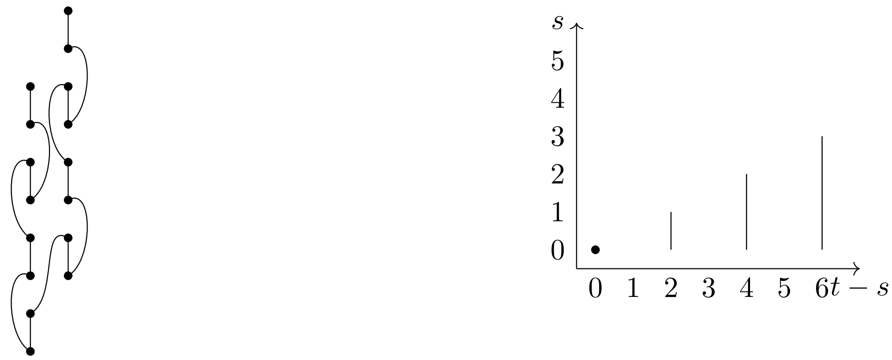


Figure 35: The $\mathcal{A}_2(1)$-module L_1 which appears in Figure 12 and its E_2 page.

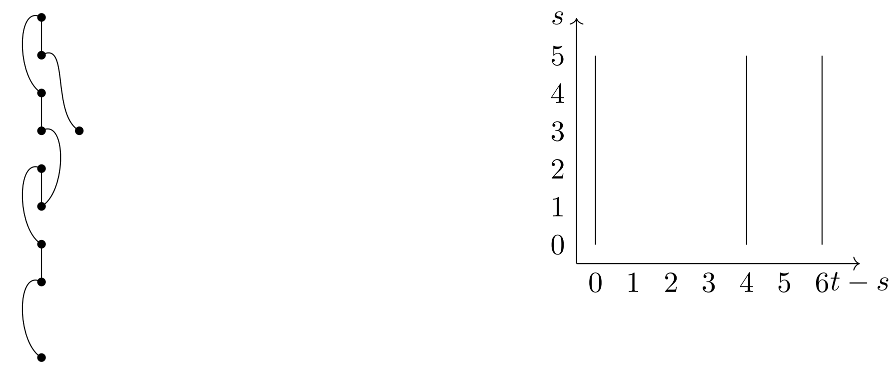


Figure 36: The $\mathcal{A}_2(1)$-module L_2 which appears in Figure 21 and its E_2 page.

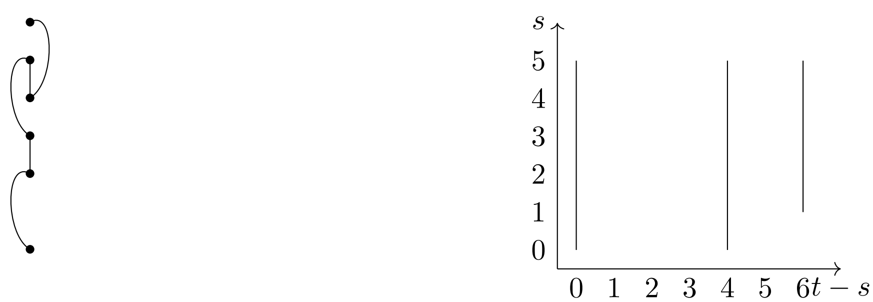


Figure 37: The $\mathcal{A}_2(1)$-module L_3 which appears in Figure 21 and its E_2 page.

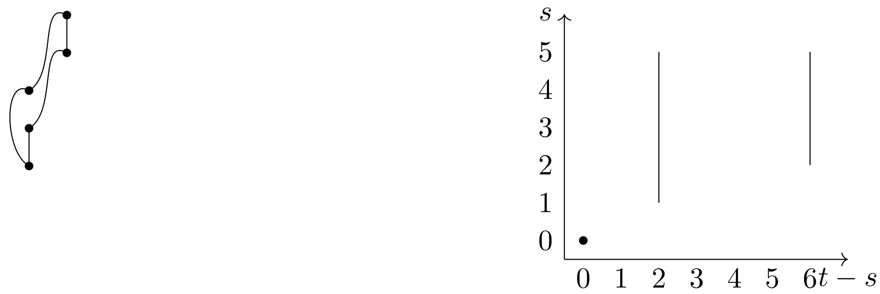


Figure 38: The $\mathcal{A}_2(1)$-module L_4 which appears in Figure 23 and 25 and its E_2 page.

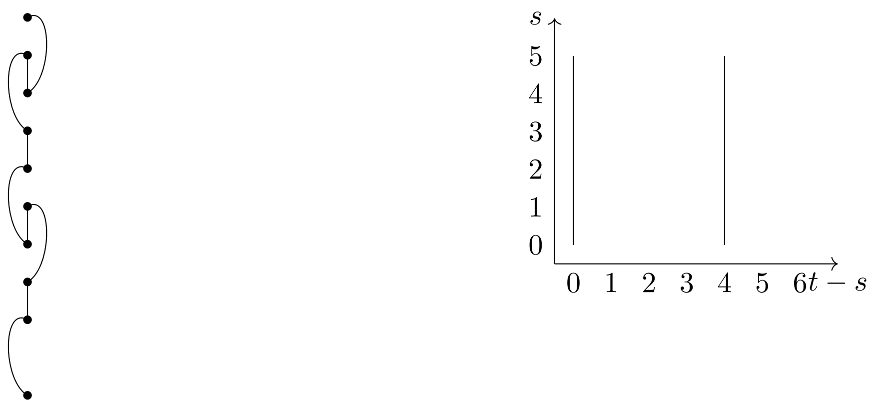


Figure 39: The $\mathcal{A}_2(1)$-module L_5 which appears in Figure 21 and 27 and its E_2 page.

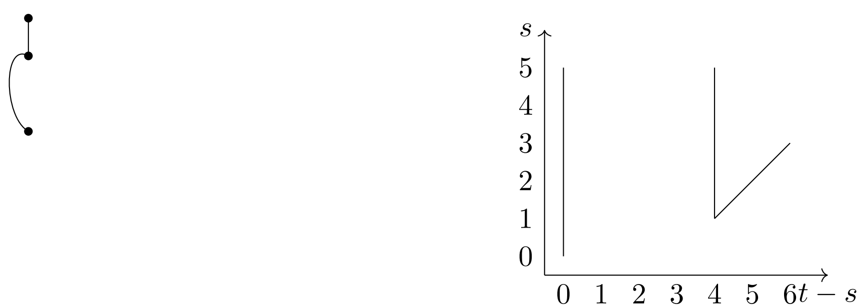
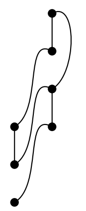


Figure 40: The $\mathcal{A}_2(1)$-module L_6 which appears in Figure 23 and its E_2 page.



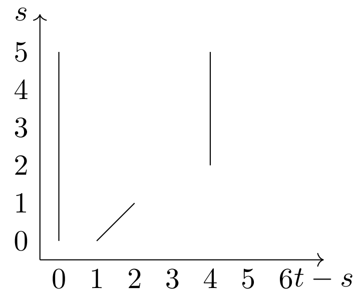
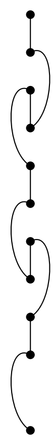


Figure 41: The $\mathcal{A}_2(1)$-module L_7 which appears in Figure 23 and its E_2 page.



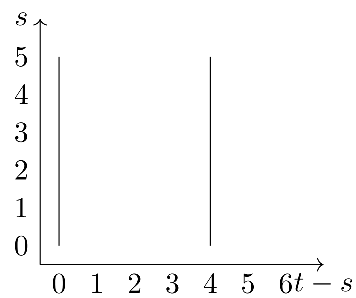


Figure 42: The $\mathcal{A}_2(1)$-module L_8 which appears in Figure 21 and 27 and its E_2 page.

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