A LOCALIZED SPACETIME PENROSE INEQUALITY AND HORIZON DETECTION WITH QUASI-LOCAL MASS

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Abstract. Our setting is a simply connected bounded domain with a smooth connected boundary, which arises as an initial data set for the general relativistic constraint equations satisfying the dominant energy condition. Assuming the domain to be admissible in a certain precise sense, we prove a localized spacetime Penrose inequality for the Liu-Yau and Wang-Yau quasi-local masses and the area of an outermost marginally outer trapped surface (MOTS). On the basis of this inequality, we obtain sufficient conditions for the existence and non-existence of a MOTS (along with outer trapped surfaces) in the domain, and for the existence of a minimal surface in its Jang graph, expressed in terms of various quasi-local mass quantities and the boundary geometry of the domain.

1. Introduction

For asymptotically flat, smooth, complete Riemannian manifolds with non negative scalar curvature, Schoen and Yau proved the fundamental Positive Mass Theorem [20] stating that the ADM mass of the Riemannian manifold is non-negative and zero if and only if the Riemannian manifold is Euclidean space. By an intricate deformation argument involving the so called Jang graph $\bar{\Omega}$ of Ω, they generalized this result to initial data sets satisfying the dominant energy condition. An initial data set (Ω, g, k) for Einstein equation consists of a Riemannian manifold Ω, with metric g, and a symmetric 2-tensor k denoting the second fundamental form satisfying the general relativistic constraint equations:

$$16\pi\mu = R_g + (\mathrm{Tr}_g k)^2 - |k|_g^2, \qquad 8\pi J = \mathrm{div}_g\left(k - (\mathrm{Tr}_g k)g\right), \tag{1.1}$$

where μ (scalar) and J (one-form) represent, respectively, the local energy and momentum density in Ω, and in this setting the dominant energy condition is the requirement that $\mu \geqslant |J|_g$.

In the Riemannian setting when $k = 0$, Bray [5] and Huisken-Ilmanen [8] indepedently proved a far reaching refinement of the original Positive Mass Theorem known as the Riemannian Penrose inequality. The result states that the ADM mass of any asymptotically flat manifold (Ω, g) with boundary $\partial\Omega$ composed of an outward minimizing minimal surface with area A should be at least $\sqrt{\frac{A}{16\pi}}$ with equality if and only if (Ω, g) is the Schwarzschild manifold. Huisken and Ilmanen prove this when A is the area of any connected component of $\partial\Omega$ using weak inverse mean curvature flow, whereas Bray's proof, which is based on a specific conformal flow of metrics, works when A is taken to be the sum of the areas of the connected components of $\partial\Omega$.

Combining the weak inverse mean curvature flow of Huisken and Ilmanen [8] and Miao's smoothening [16], Shi and Tam [18] were able to prove a quasi-local mass comparison theorem for a smooth simply connected domain Ω with non-negative scalar curvature and smooth connected mean convex boundary with positive Gauss curvature. In that setting they proved that the Brown York mass of $\partial\Omega$ is bounded below by the Hawking mass of the boundary of any connected minimizing hull E compactly contained in Ω. By taking ∂E to be an outward minimizing minimal surface, this gives a localized Riemannian Penrose inequality for the Brown-York mass. In their case, equality implies that Ω is a flat domain in $\mathbb{R}^3$. More recently, Shi, Wang, and Yu [19] proved a restricted version of the rigidity case for the Schwarzschild manifold, and analogous to Bray's ability to deal with multiple components, Miao and McCormick [15] proved a localized Riemannian Penrose inequality involving multiple components, which was later extended to manifolds with asymptotically cylindrical ends, along with charge and angular momentum, by Alaee, Khuri, and Yau [2]. The same authors [2] also prove a localized spacetime Penrose inequality involving the Liu-Yau $m_{LY}(\partial\Omega)$ and Wang-Yau $m_{WY}(\partial\Omega)$ mass of Ω where (Ω, g, k) is an initial data set satisfying dominant energy condition. The inner boundary of their domain is a MOTS (with possibly multiple components), and their inequalities include charge and angular momentum. The proof involves a new gluing procedure between the Jang graph of Ω and its Bartnik-Shi-Tam extension [17], along with a deformation of the Jang graph by Khuri [10], which itself is inspired by a Riemannian argument of Herzlich [7]. In virtue of being based on Herzlich's argument, their spacetime Penrose inequalities include an unwanted coefficient $\frac{\gamma}{1+\gamma} < 1$ in front of $\sqrt{\frac{A}{16\pi}}$, where the constant γ, said to be of Herzlich-type, is independent of the area of the MOTS.

The first goal of the present article is to prove a comparison theorem for the Wang-Yau, Liu-Yau, and Hawking quasi-local masses of an *admissible* domain, which we define as follows.

Definition 1.1. *A domain Ω is admissible if:*

(i) Ω is bounded, simply connected and has smooth connected, boundary $\partial\Omega$,
(ii) $\partial\Omega$ is untrapped $H_{\partial\Omega} > |Tr_{\partial\Omega} k|$,
(iii) (Ω, g, k) is an initial data set satisfying $\mu \geqslant |J|$,
(iv) the 1-form X related to the Jang graph $\bar{\Omega}$ of Ω is admissible as in Definition 2.3.

In this context, for any minimizing hull in a Jang graph $\bar{\Omega}$ of Ω, we show the following.

Theorem 1.2. *Let Ω be admissible and $\bar{E}$ be any connected minimizing hull contained in a Jang graph $\bar{\Omega}$ of Ω with $C^{1,1}$ boundary $\partial\bar{E}$. If the Gauss curvature of $\partial\Omega$ is positive, then*

$$m_{LY}(\partial\Omega) \geqslant m_H(\partial\bar{E}).$$

Moreover, if there is an admissible time function τ, then

$$m_{WY}(\partial\Omega) \geqslant m_H(\partial\bar{E}).$$

This inequality is very fruitful in that it immediately implies a localized spacetime Penrose inequality *without* a Herzlich-type constant. For although an outward minimizing minimal surface in the Jang graph (which is clearly a minimizing hull therein) does not in general project to a minimal surface in Ω, it is still the case that the area of these surfaces are larger than their projection in Ω, and so as a consequence we obtain a Penrose inequality for their projection in Ω, and, perhaps most remarkably, for any MOTS that is outward minimizing in the blow up Jang graph $\bar{\Omega}$, a blow up is an asymptotically cylindrical end in $\bar{\Omega}$ corresponds to MOTS in Ω.

Proposition 1.3. *Let Ω be admissible and S be a closed surface in Ω. Suppose that either*

(i) S is a MOTS and its projection in blow up Jang graph $\bar{\Omega}$ of Ω is outward minimizing, or
(ii) S has an outward minimizing minimal surface projection in a Jang graph $\bar{\Omega}$ of Ω.

If the Gauss curvature of $\partial\Omega$ is positive, then

$$m_{LY}(\partial\Omega) \geqslant \sqrt{\frac{|S|}{16\pi}}.$$

Moreover, if there is an admissible time function τ, then

$$m_{WY}(\partial\Omega) \geqslant \sqrt{\frac{|S|}{16\pi}}.$$

Over and above their intrinsic interest, Theorem 1.2 and Proposition 1.3 lead to black hole non-existence and existence results using quasi-local masses, where we note that the former is listed in Yau's open problem list, pg. 371-372 of [23] and the latter is listed in Bartnik's open problems in mathematical relativity, pg. 260 of [4] and it is related to the Thorne's well known hoop conjecture, which hypothesises that black holes form in Ω if the enclosed mass $M(\Omega)$ satisfies $M(\Omega) \geqslant aC(\partial\Omega)$ where a is a universal constant and $C(\partial\Omega)$ is some notion of the circumference of $\partial\Omega$. The precise definition of $M(\Omega)$ and $C(\partial\Omega)$ is left open and indeed a significant part of the problem lies in identifying these. Before stating these, let us recall some of the fundamental predecessors.

Based on their study of Jang's equation [21], Schoen and Yau [22] formulated conditions on a domain Ω that guarantee the existence of a MOTS in Ω. This was the first rigorous and general result in the direction of localized black hole existence. Yau [25] later refined the argument by weakening the condition imposed on the matter density in favour of a slightly stronger lower bound on the mean curvature of the geometry $\partial\Omega$, an improvement which was influential in proposing the Liu-Yau [12] and Wang-Yau definitions of quasi-local mass [24].

The setting of their original theorem is an initial data set (Ω, g, k) with $\partial\Omega$ smooth, connected, outer untrapped, and they obtain a MOTS in Ω under the assumption that are $\mu - |J|_g \geqslant \Lambda$ in Ω, and that $\mathrm{Rad}(\Omega) \geqslant \sqrt{\frac{3}{2}}\frac{\pi}{\sqrt{\Lambda}}$, where $\mathrm{Rad}(\Omega)$ is a geometric quantity measuring the size of Ω. It has been an open question since their results to find analogous sufficient conditions involving a quasi-local mass of Ω.

Using quasi-local mass makes the problem significantly harder. The chief reason being that current definitions of quasi-local mass solely involve the boundary data of Ω (i.e., the metric induced on $\partial\Omega$ and the mean curvature vector of $\partial\Omega$) and as a result they tend to be highly non-coercive with respect to the interior geometry. It is straightforward to construct domains with the same boundary data (and thus quasi-local mass) but where one contains a minimal surface but the other one does not; see for instance the basic and non-exotic examples in Section 2 of [18], whereby domains are constructed so as to have the same boundary data but totally different interiors, some with minimal surfaces, and some without. The conclusion one draws is that versions of Schoen-Yau's result [22] involving quasi-local mass are difficult to obtain and will actually be impossible in certain cases.

This non-coercivity of boundary geometry goes some way in explaining why Schoen and Yau [22] resorted to a lower bound on $\mathrm{Rad}(\Omega)$, which measures the internal geometry of Ω, with a pointwise lower bound on $\mu - |J|_g$ inside Ω. In any case, perhaps the most striking aspect of their result is that their rather straightforward inequalities alone are sufficient to guarantee the existence of a MOTS, which moreover seems to indicate that $\mathrm{Rad}(\Omega)$ is particularly well-suited to the problem.

In spite of these non-coercivity issues, Shi and Tam [18] were able to give sufficient conditions for the existence of minimal surfaces in the Riemannian setting. Their deepest result in this direction involves an ingenious quantity $m_{ST}(\Omega)$, see Definition 2.2, recalled below and christened here as the Shi-Tam mass, which they show obeys $m_{BY}(\partial\Omega) \geqslant m_{ST}(\Omega)$ when Ω is absent of minimal surfaces. A minimal surface must exist upon reversing this inequality, and to our knowledge this is the most profound result in the direction of the hoop conjecture, despite being entirely Riemannian.

In trying to generalize their ideas to the spacetime case, one faces various additional obstacles (over and above the non-coercivity aforementioned) which are genuinely Lorentzian in nature, and seems to have been first observed by Iyer and Wald [9]. They constructed a foliation of a globally hyperbolic subset of the maximally extended Schwarzschild spacetime which gets arbitrarily close to the $r = 0$ black hole singularity but which has the property that none of its leaves contain either an outer trapped surface or a MOTS. So in spite of the region being filled with outer trapped surfaces, none of these register on the leaves of the foliation. This additional obstacle would seem to make the overall prospect of hoop conjecture type statements outside of symmetry all the more unlikely.

Here, using Theorem 1.2 and defining a new quasi-local mass $m^*(\Omega;\bar{\Omega})$, see Definition 2.4, which depends on geometry of Ω and $\bar{\Omega}$, we provide sufficient conditions for the existence of a MOTS along with outer trapped surfaces in Ω

Theorem 1.4. *Let Ω be admissible. If the Gauss curvature of $\partial\Omega$ is positive and for a Jang graph $\bar{\Omega}$ of Ω, we have either $m^*(\Omega;\bar{\Omega}) > m_{LY}(\Omega)$ or $m^*(\Omega;\bar{\Omega}) \geqslant \frac{1}{4}diam(\partial\Omega)$, then there is a MOTS and outer trapped surfaces in Ω. The same statement holds for $m_{WY}(\partial\Omega)$ provided there is an admissible τ on $\partial\Omega$.*

Note here that using quasi-local masses we are forcing the Jang solutions blow up. Using $m_{ST}(\Omega)$, we can also guarantee the existence of minimal seperating spheres in $\bar{\Omega}$.

Proposition 1.5. *Let Ω be admissible. If the Gauss curvature of $\partial\Omega$ is positive and there is an isoperimetric surface $\bar{V} \subset \bar{\Omega}$ with $m_H(\bar{V}) \geqslant m_{LY}(\partial\Omega)$ for a Jang graph $\bar{\Omega}$ of Ω, then there is a seperating outward minimizing minimal sphere in $\bar{\Omega}$. The same statement holds if there is an admissible time function τ and $m_H(\bar{V}) \geqslant m_{WY}(\partial\Omega)$.*

An *isoperimetric* surface $V \subset \Omega$ is a C^2 surface in (Ω, g) whose area is no more than any other C^2 surface enclosing the same volume. A wider class is that of *locally isoperimetric* surfaces. These minimize area given enclosed volume amongst local competitors, and are also sometimes referred to as *stable volume preserving CMC*.

With a different argument, one also obtains the following.

Proposition 1.6. *Let Ω be admissible. If the Gauss curvature of $\partial\Omega$ is positive and for a Jang graph $\bar{\Omega}$ of Ω we have either $m_{ST}(\bar{\Omega}) > m_{LY}(\partial\Omega)$ or $m_{ST}(\bar{\Omega}) \geqslant \frac{1}{4}diam(\partial\Omega)$, then there is a seperating outward minimizing minimal sphere in $\bar{\Omega}$. The analogous statement holds for $m_{WY}(\partial\Omega)$ if there is an admissible function τ on $\partial\Omega$.*

Lastly, we also note that our results along with that of [18] will lead to comparison theorems involving the spacetime Bartnik mass and the Liu-Yau or Wang-Yau mass, though we have not pursued this further. Finally, in the non-existence direction, our result is as follows.

Proposition 1.7. *Let Ω be admissible and $\bar{\Omega}$. Suppose that the sectional curvatures of all Jang graphs $\bar{\Omega}$ of Ω are bounded above by some constant C^2, $C > 0$. If the boundary has positive Gauss curvature and $m_{LY}(\partial\Omega) < \frac{1}{2C}$ then there is no MOTS in Ω and $\bar{\Omega}$ is diffeomorphic to a ball in $\mathbb{R}^3$. If $m_{WY}(\partial\Omega) < \frac{1}{2C}$, the same statement holds for $m_{WY}(\partial\Omega)$ provided there is an admissible τ on $\partial\Omega$.*

A Riemannian version of this statement for asymptotically flat Riemannian manifold with non-negative scalar curvature is due to Corvino [6] and one involving the Brown-York mass is by Alaee, Cabrera Pacheco, and McCormick [1].

Among the additional difficulties that we face in the spacetime context (over and above those mentioned) is the lack of scalar curvature non-negativity. This is crucial for the monotonicity of the Hawking mass under the weak inverse mean curvature flow, which indeed underpins the entire argument in [18]. Another issue that arises in the spacetime but not in the Riemannian context is that MOTS are not known to minimize some functional. Thus it is not *a priori* clear that notions like *minimizing hulls*, defined in [8] and crucial for the minimal surface detection argument in [18], will be of any use in the spacetime context.

The main ingredients of the proofs of our results combine ideas and results from Schoen and Yau [20, 21, 25], Huisken and Ilmanen's weak inverse mean curvature flow [8], the Bartnik-Shi-Tam extension of [17], Shi and Tam's new quasi-local mass [18], and the recent smoothening procedure of Alaee, Khuri and Yau [2], which itself is based on that of Miao [16].

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2. Definitions and Preliminaries

Let (Ω, g) be a compact Riemannian manifold. The Hawking mass $m_H(\Sigma)$ of a surface $\Sigma \subset \Omega$ is defined as follows.

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 dA_\sigma\right), \tag{2.1}$$

where σ is the induced metric on Σ and H is the mean curvature of Σ with respect to outward normal. Let (Ω, g) be a compact Riemannian manifold with boundary $\partial\Omega$ that has positive Gauss curvature and spacelike mean curvature vector $\vec{H} = H\nu - (\mathrm{Tr}_\Sigma k)n$, ν and n are unit spacelike and future-directed timelike normal to Σ, respectively. Then the Liu-Yau mass [12] is defined as

$$m_{LY}(\partial\Omega) = \frac{1}{8\pi}\int_{\partial\Omega}\left(H_0 - |\vec{H}|\right) dA_\sigma, \tag{2.2}$$

where H_0 is the mean curvature of isometric embedding of $\partial\Omega$ in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ and σ is the induced metric on $\partial\Omega$.

In [24], Wang and Yau defined a quasi-local energy for a spacelike 2-surface Σ embedded in a spacetime $N^{3,1}$. Let $\Sigma \hookrightarrow N^{3,1}$ be a spacelike 2-surface and suppose there is an isometric embedding of $\iota : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ with mean curvature vector $\vec{H}_0$ and a time function $\tau = -\left\langle \iota(\Sigma), \vec{T}_0 \right\rangle$, where $\vec{T}_0$ is the designated future timelike unit Killing field on $\mathbb{R}^{3,1}$. Then the 4-tuple $(\Sigma, \sigma, |\vec{H}|, \alpha_{\bar{e}_3})$ is called Wang-Yau data set, where σ is the induced metric on Σ, $\vec{H}$ is the mean curvature vector which is spacelike, and α is the connection one-form of the normal bundle of Σ and defined as

$$\alpha_{\bar{e}_3}(\cdot) = \langle {}^N\nabla_{(\cdot)}\bar{e}_3, \bar{e}_4\rangle. \tag{2.3}$$

Here $\{\bar{e}_3, \bar{e}_4\}$ is the unique orthonormal frame for the normal bundle of Σ in $N^{3,1}$ such that $\bar{e}_3$ is spacelike, $\bar{e}_4$ is future-directed timelike, and

$$\langle \vec{H}, \bar{e}_3\rangle > 0, \qquad \langle \vec{H}, \bar{e}_4\rangle = \frac{-\Delta\tau}{\sqrt{1+|\nabla\tau|^2}}. \tag{2.4}$$

Assuming $\vec{H}_0$ is spacelike, then the Wang-Yau quasi-local energy is defined to be

$$E_{WY}(\Sigma, \iota, \tau) = \frac{1}{8\pi}\int_\Sigma (\mathfrak{H}_0 - \mathfrak{H})\, dA_\sigma, \tag{2.5}$$

where the *generalized mean curvature* is

$$\mathfrak{H} = \sqrt{1+|\nabla\tau|^2}\langle \vec{H}, \bar{e}_3\rangle - \alpha_{\bar{e}_3}(\nabla\tau), \qquad \mathfrak{H}_0 = \sqrt{1+|\nabla\tau|^2}\hat{H}_0, \tag{2.6}$$

where $\hat{H}_0$ is the mean curvature of $\hat{\Sigma} \subset \mathbb{R}^3$ which is the orthogonal project of $\iota(\Sigma)$ with respect to $\vec{T}_0$. This definition depends on observer (ι, τ) and the Wang-Yau quasi-local energy is non-negative for *admissible* observers [**?**, 24]. An observer is *admissible* or the time function τ is *admissible* if the convexity condition

$$\left(1+|\nabla\tau|^2\right) K_{\hat{\sigma}} = K_\Sigma + \left(1+|\nabla\tau|^2\right)^{-1}\det(\nabla^2\tau) > 0, \tag{2.7}$$

where $K_{\hat{\sigma}}$ is the Gauss curvature of metric $\hat{\sigma} = \sigma + d\tau^2$ on $\hat{\Sigma}$, is satisfied, Σ arises as the untrapped boundary of a spacelike hypersurface $(\Omega, g, k) \hookrightarrow N^{3,1}$, and the generalized mean curvature is positive $\mathfrak{H}(e_3', e_4') > 0$ for the normal bundle frame $\{e_3', e_4'\}$ determined by the solution of Jang's equation, see Definition 5.1 of [24]. In analogy with special relativity, the Wang-Yau mass is defined as the infimum of energy over all admissible observers (ι, τ), that is

$$m_{WY}(\Sigma) = \inf_{(\iota,\tau)} E_{WY}(\Sigma, \iota, \tau) \tag{2.8}$$

We need following definition of minimizing hulls [8] in the next section.

Definition 2.1. *Let E be a set in Ω with locally finite perimeter. E is said to be a minimizing hull of Ω if $|\partial^* E \cap W| \leqslant |\partial^* F \cap W|$ for any set F with locally finite perimeter such that $F \supset E$ and $F \setminus E \subsetneq \Omega$ for any set $W \subset \Omega$ containing $F \setminus E$. Here $\partial^* E$ and $\partial^* F$ are the reduced boundaries of E and F respectively. E is said to be strictly minimizing hull if equality (for all W) implies $E \cap W = F \cap W$ a.e.*

Note that an *outward minimizing minimal surface* Σ is a boundary of minimizing hull with zero mean curvature $H_\Sigma = 0$. This surface represents black hole apparent horizon in Riemannian setting, which the second fundamental form of initial data set vanishes. However, for general initial data (Ω, g, k), a black hole apparent horizon is represented by a marginally outer trapped surface (MOTS) Σ in Ω and defined as a hypersurface embedded in Ω with $H_\Sigma + \mathrm{Tr}_\Sigma(k) = 0$.

We now come to the definition of the Shi-Tam mass $m_{ST}(\Omega)$, originally described in [18].

Definition 2.2. *Let $\Omega_1 \subsetneq \Omega_2 \subset \Omega$ such that Ω_1 and Ω_2 have smooth boundaries. Let $\mathcal{F}_{\Omega_2}$ be the family of connected minimizing hulls, with C^2 boundary, of Ω_2. Define*

$$m(\Omega_1; \Omega_2) = \sup_{E \in \mathcal{F}_{\Omega_2}, E \subset \Omega_1} m_H(E) \tag{2.9}$$

Then the Shi-Tam quasi-local mass is defined

$$m_{ST}(\Omega) \equiv \sup_{\Omega_1, \Omega_2} \alpha_{\Omega_1, \Omega_2} m(\Omega_1; \Omega_2), \tag{2.10}$$

where

$$\alpha^2_{\Omega_1;\Omega_2} = \min\left\{1, \frac{CK^{-2}\int_0^r \tau^{-1}\sin(K\tau)^2 d\tau}{|\partial\Omega_1|}\right\} \tag{2.11}$$

and $K > 0$ is an upper bound for sectional curvature of Ω_2 and C is an absolute positive constant.

Note that we remove the precompactness property of minimizing hull from above definition to include structure of blow up Jang graph. The constant $\alpha_{\Omega_1,\Omega_2}$ is related to Meeks and Yau [14] estimate for area of the minimal surface part of any strictly minimizing hull E' of E with respect to Ω in above definition. Shi and Tam [18, Theorem 2.4] show that $m_{ST}(\Omega)$ has various pleasant properties. If Ω has non-negative scalar curvature $R_g \geqslant 0$, and $\partial\Omega$ (which is smooth and connected) has positive mean curvature $H_{\partial\Omega} > 0$ and Gauss curvature $K_{\partial\Omega} > 0$, then $m_{ST}(\Omega) \geqslant 0$ and moreover equality is achieved if and only if Ω is a domain in flat $\mathbb{R}^3$.

We now consider definitions for initial data sets (Ω, g, k) following the classic arguments of Schoen and Yau [21]. Recall that in their proof of the spacetime positive mass theorem, they are faced with the issue that an initial data set satisfying the dominant energy condition $\mu \geqslant |J|_g$ that may not have a non-negative scalar curvature. They overcome this by considering deformations of the initial data $g \to \bar{g} = g + df^2$, where $f : \Omega \to \mathbb{R}$ is a solution of the Jang equation and $\bar{g}$ is the metric induced on the Jang graph of the initial data set. Upon proving existence of solution of the Jang's equation, they are able to conformally deform the Jang graph (without significantly affecting mass) to another initial data set with zero scalar curvature, which in turn eventually permits for an application of the Riemannian positive mass theorem [20]. In the current setting we consider the following Jang's equation with Dirichlet boundary condition:

$$\begin{cases} \left(g^{ij} - \frac{f^i f^j}{1+|\nabla f|_g^2}\right)\left(\frac{\nabla_{ij} f}{\sqrt{1+|\nabla f|_g^2}} - k_{ij}\right) = 0 & \text{in } \Omega \\ f = \tau & \text{on } \partial\Omega \end{cases}, \tag{2.12}$$

where $f^i = g^{ij} f_j$ and the covariant derivative ∇ is with respect to g. Assume the boundary is untrapped $H_{\partial\Omega} > |\mathrm{Tr}_{\partial\Omega} k|$. If there is no MOTS in Ω, the Dirichlet problem (2.12) has a unique smooth solution by Schoen and Yau [22], which we call it the Jang graph $\bar{\Omega}$ of Ω. Moreover, if there is a MOTS in Ω, there is no uniqueness for the Dirichlet problem (2.12) but there exist a smooth solution by Andersson and Metzger [3] which blow up in the form of a cylinder over MOTS, with $f \to \infty$ $(-\infty)$ at MOTS depending on whether it is a future (or past) MOTS.

Following [12, 24], in the case of the Wang-Yau mass, we set $f = \tau$ on $\partial\Omega$ which τ is admissible time function and for the Liu-Yau mass, we set $f = \tau = 0$ on $\partial\Omega$ which means the metrics $\bar{g}$ and g are the same on the boundary. Moreover the equation implies that the scalar curvature of the Jang metric is weakly nonnegative, that is

$$\bar{R} = 2\left(\mu - J(w)\right) + |h-k|_{\bar{g}}^2 + 2|X|_{\bar{g}}^2 - 2\mathrm{div}_{\bar{g}} X \geqslant 2|X|_{\bar{g}} - 2\mathrm{div}_{\bar{g}} X, \tag{2.13}$$

where h is second fundamental form of the graph $t = f(x)$ in the product manifold $(\Omega \times \mathbb{R}, g + dt^2)$, and w, X are 1-forms given by

$$h_{ij} = \frac{\nabla_{ij} f}{\sqrt{1+|\nabla f|_g^2}}, \qquad w_i = \frac{f_i}{\sqrt{1+|\nabla f|_g^2}}, \qquad X_i = \frac{f^j}{\sqrt{1+|\nabla f|_g^2}}\left(h_{ij} - k_{ij}\right). \tag{2.14}$$

We denote the Jang graph by $(\bar{\Omega}, \overline{g}, \bar{k} \equiv h - k, X)$. Let $(\mathrm{div}_{\bar{g}}\, X)_+$ be positive part of $\mathrm{div}_{\bar{g}}\, X$, then we have the following definition for admissible one-form X.

Definition 2.3. *Let $(\bar{\Omega}, \overline{g}, \bar{k}, X)$ be a Jang graph of an initial data set (Ω, g, k). The one-form X is admissible if there exist constants $d_1 > 0$, depending on the Sobolev constant of asymptotically flat extension manifold $(\bar{\Omega} \cup M_+, \overline{g} \cup g_+)$, and $d_2 > 0$, depending on $L^1(\partial\bar{\Omega})$-norm of $X(\bar{\nu})$, such that*

$$\left(\int_{\bar{\Omega}} (\mathrm{div}_{\bar{g}}\, X)_+^{3/2}\, dV_{\bar{g}}\right)^{2/3} < d_1, \qquad \left(\int_{\bar{\Omega}} (\mathrm{div}_{\bar{g}}\, X)_+^{6/5}\, dV_{\bar{g}}\right)^{5/6} < d_2, \tag{2.15}$$

and $X(\bar{\nu}) > 0$ on $\partial\bar{\Omega}$.

With these definitions, we can also define a different mass $m^*(\Omega, \bar{\Omega})$, where $\bar{\Omega}$ stands for a Jang graph over Ω.

Definition 2.4. *Let $\bar{\Omega}_1 \subsetneq \bar{\Omega}_2 \subset \bar{\Omega}$ with smooth boundaries, C be an absolute positive constant, and K be an upper bound of the sectional curvature of $\bar{\Omega}_2$. Define*

$$m^*(\Omega; \bar{\Omega}) \equiv \sup_{\bar{\Omega}_1, \bar{\Omega}_2} \alpha^*_{\bar{\Omega}_1, \bar{\Omega}_2} m^*(\bar{\Omega}_1; \bar{\Omega}_2), \tag{2.16}$$

where

$$\alpha^{*2}_{\bar{\Omega}_1, \bar{\Omega}_2} \equiv \min\left\{1, \frac{CK^{-2}\int_0^r \tau^{-1}\sin(K\tau)^2 d\tau}{|\partial\Omega_1|^{\beta_{\Omega,\bar{\Omega}}}}\right\}, \qquad \beta_{\Omega,\bar{\Omega}} \equiv \frac{\mathrm{Rad}(\bar{\Omega})}{\mathrm{Rad}(\Omega)}. \tag{2.17}$$

and

$$m^*(\bar{\Omega}_1;\bar{\Omega}_2) \equiv \sup_{\bar{E}\in F^*_{\bar{\Omega}_2}, \bar{E}\subset\bar{\Omega}_1} m_H(\partial\bar{E}), \tag{2.18}$$

where $F^*_{\bar{\Omega}_2}$ *is the family of connected minimizing hulls, with* C^2 *boundary, of* $\bar{\Omega}_2$ *such that for any* $\bar{E}\in F^*_{\bar{\Omega}_2}$ *we have* $\partial\bar{E}\cap\partial\bar{V}\neq\varnothing$ *for some connected strictly minimizing hull* $\bar{V}$ *of* $\bar{\Omega}$ *with connected boundary.*

By $\mathrm{Rad}(\Omega)$ we mean the definition of [22], which goes as follows. Let Ω be a domain with boundary $\partial\Omega$ and Γ be a simple closed curve in Ω that bounds a disc. Let $N_r(\Gamma)$ be set of all points within a r radius of Γ. Define $\mathrm{Rad}(\Omega,\Gamma)$ to be

$$\mathrm{Rad}(\Omega,\Gamma) \equiv \sup\{r : \mathrm{dist}(\partial\Omega,\Gamma) > r \text{ and } \Gamma \text{ does not bound a disc in } N_r(\Gamma)\} \tag{2.19}$$

Then $\mathrm{Rad}(\Omega) = \sup_\Gamma \mathrm{Rad}(\Omega,\Gamma)$. In particular, $\mathrm{Rad}(\Omega)$ may be described as the radius of the largest torus that can be embedded in Ω. For example for a ball with radius R in $\mathbb{R}^3$, $\mathrm{Rad}(\Omega) = R/2$ and for a torus $\mathbb{S}^2_R\times(-L,L)$, $\mathrm{Rad}(\Omega) = \min\{\pi R/2, L\}$. Note that this definition is constructed such that the radius of a blow up Jang graph $\bar{\Omega}$ for a bounded domain Ω stays finite. Moreover, this new quantity $m^*(\Omega;\bar{\Omega})$ is non-negative, and $m^*(\bar{\Omega}_1;\bar{\Omega}_2) \geqslant m(\bar{\Omega}_1;\bar{\Omega}_2)$ since the latter involves more competitors. If the Jang graph does not blow up, since $\mathrm{Rad}(\Omega)\leqslant\mathrm{Rad}(\bar{\Omega})$, the constant satisfies $\beta_{\Omega,\bar{\Omega}}\geqslant 1$. When blow up occurs, there is no known general relation between $\mathrm{Rad}(\Omega)$ and $\mathrm{Rad}(\bar{\Omega})$.

3. Proof of Main Results

We start with an admissible domain Ω and consider solutions to Jang's equation with prescribed boundary data $\bar{g} = \hat{\sigma}$ on $\partial\bar{\Omega}$. Solutions to this Dirichlet boundary problem exist by Schoen-Yau [21] and each defines a graph $\bar{\Omega}$ over Ω with the properties aforementioned. By admissible condition $X(\nu) > 0$ on $\partial\bar{\Omega}$, we construct a Bartnik-Shi-Tam extension [2, 17] to $\partial\bar{\Omega}$, denoted $(M_+, g_+, k_+ = 0, X_+ = 0)$, which an asymptotically flat Riemannian manifold with zero scalar curvature and boundary $\partial M_+ = \partial\bar{\Omega}$ has mean curvature $\bar{H} > 0$ and induced metric $\hat{\sigma} = g_+|_{\partial M_+}$. By Shi and Tam [17], we have the following inequality for the ADM mass of extension.

$$\frac{1}{8\pi}\int_{M_+}\left(\hat{H}_0 - \bar{H}\right) dA_{\hat{\sigma}} \geqslant m_{ADM}(g_+), \tag{3.1}$$

where $\hat{H}_0$ is the mean curvature of isometric embedding of $(\partial\bar{\Omega},\hat{\sigma})$ in Euclidean space $\mathbb{R}^3$. In contrast to our setting, the extension in [2, 12, **?**] has mean curvature $\bar{H} - X(\bar{\nu})$ for boundary $\partial\bar{\Omega}$. Next, we attach the Jang graph $(\bar{\Omega},\bar{g},\bar{k},X)$ to this Bartnik-Shi-Tam extension and denote it by

$$(\bar{\mathbf{M}},\bar{\mathbf{g}},\bar{\mathbf{k}},\mathbf{X}) = (\overline{\Omega}\cup M_+, \bar{g}\cup g_+, \bar{k}\cup k_+, X\cup X_+) \tag{3.2}$$

which will in general have corner along $\partial\bar{\Omega}$. Since the mean curvature is the same along boundary $\partial\bar{\Omega}$, we modify general gluing developed by Alaee, Khuri, and Yau [2] for the Jang graph and have the following result.

Lemma 3.1. *There exists a smooth deformation* $(\bar{\mathbf{M}},\bar{\mathbf{g}}_\delta,\bar{\mathbf{k}}_\delta,\mathbf{X}_\delta)$ *which differs from the original data* $(\bar{\mathbf{M}},\bar{\mathbf{g}},\bar{\mathbf{k}},\mathbf{X})$ *only on a* δ*-tubular neighborhood of* $\partial\bar{\Omega}$*, that is* $\mathcal{O}_\delta = \left[-\frac{\delta}{2},\frac{\delta}{2}\right]\times\partial\bar{\Omega}$*, and satisfies*

$$\left(\bar{\mathbf{R}}_\delta - 2|\mathbf{X}_\delta|^2_{\bar{\mathbf{g}}_\delta} - |\bar{\mathbf{k}}_\delta|^2_{\bar{\mathbf{g}}_\delta}\right)(t,x) = O(1), \qquad (t,x)\in\mathcal{O}_\delta \tag{3.3}$$

$$\mathrm{div}_{\bar{\mathbf{g}}_\delta}\mathbf{X}_\delta(t,x) = O(\delta^{-1/3}), \qquad (t,x)\in\mathcal{O}_\delta \tag{3.4}$$

where $\bar{\mathbf{R}}_\delta$ *is the scalar curvature of* $\bar{\mathbf{g}}_\delta$*, and* $O(1)$ *is a constant depends on the initial data set and independent of* δ*.*

Proof. It is shown in [16, Section 3], if the mean curvature along the corner is the same, then

$$\bar{\mathbf{R}}_\delta(t,x) = O(1), \qquad \text{as } \delta \to 0, \tag{3.5}$$

where the deformation region for the metric is a δ-tubular neighborhood $(t,x) \in \mathcal{O}_\delta = \left[-\frac{\delta}{2}, \frac{\delta}{2}\right] \times \partial\bar{\Omega}$. Next, near the corner surface $\partial\bar{\Omega}$, the one form is

$$\mathbf{X} = \mathbf{X}_t dt + \mathbf{X}_i dx^i, \tag{3.6}$$

where t is the geodesic normal coordinate and x^i are coordinates on $\partial\bar{\Omega}$. We denote the deformation by

$$\mathbf{X}_\delta = \mathbf{X}_{\delta t} dt + \mathbf{X}_{\delta i} dx^i. \tag{3.7}$$

The deformation $\mathbf{X}_{\delta i}$ is similar to [2, Lemma 5.1] and its norm and tangential derivative are bounded. However, $\mathbf{X}_{\delta t}$ is different as following. By definition of $\mathbf{X}$, we have

$$\mathbf{X}_t = X(\bar{\nu}), \qquad t < 0, \qquad \text{and} \qquad \mathbf{X}_t = 0, \qquad t > 0, \tag{3.8}$$

Let $\varsigma(t) > 0$ be a smooth cut-off function defined as

$$\varsigma(t) = \begin{cases} 1 & t \leqslant -\frac{\delta}{2} \\ |\varsigma'(t)| \leqslant \delta^{-1/3} & -\frac{\delta}{2} < t < -\frac{\delta^2}{200} \\ 0 & t \geqslant -\frac{\delta^2}{200} \end{cases}. \tag{3.9}$$

Define the deformation $\mathbf{X}_{\delta t}(t,x) = \varsigma(t) X(\bar{\nu})(t,x)$ for all $t \in (-\delta, \delta)$. Then the divergence of the smoothed 1-form is

$$\begin{aligned} \operatorname{div}_{\bar{\mathbf{g}}_\delta} \mathbf{X}_\delta(t,x) &= \left(\partial_t \mathbf{X}_{\delta t} + \frac{1}{2}\mathbf{X}_{\delta t}\partial_t \log\det\gamma_\delta + \frac{1}{\sqrt{\det\gamma_\delta}}\partial_i\left(\sqrt{\det\gamma_\delta}\gamma_\delta^{ij}\mathbf{X}_{\delta j}\right)\right)(t,x) \\ &= \varsigma(t)' X(\bar{\nu})(t,x) + O(1) \\ &= O(\delta^{-1/3}), \qquad (t,x) \in \mathcal{O}_\delta. \end{aligned} \tag{3.10}$$

where γ_δ is deformed metric on $\partial\bar{\Omega}$ defined in equation (11) of [16]. The smoothing of $\bar{\mathbf{k}}_\delta$ is similar to [2, Lemma 5.1]. □

This deformation readies our initial data for a conformal deformation which will further improve the scalar curvature of the deformed Jang graph, and moreover which will do so without changing the ADM mass by much.

Proposition 3.2. *Given the deformation $(\bar{\mathbf{M}}, \bar{\mathbf{g}}_\delta, \bar{\mathbf{k}}_\delta, \mathbf{X}_\delta)$ of Lemma 3.1 and let $\bar{F} \subsetneq \bar{\Omega} \subset \bar{\mathbf{M}}$. For sufficiently small d_1 depending on the Sobolev constant of asymptotically flat extension manifold $(\bar{\Omega} \cup M_+, \overline{g} \cup g_+)$, there exist a C^2 positive function $u_\delta \geqslant 1$ such that the conformal metric $\hat{\mathbf{g}}_\delta = u_\delta^4 \bar{\mathbf{g}}_\delta$ has non-negative scalar curvature and satisfies $m_{ADM}(g_+) + \lim_{\delta\to 0}\epsilon_{\delta,d_2} \geqslant \lim_{\delta\to 0} m_{ADM}(\hat{\mathbf{g}}_\delta)$ for some small positive constant ϵ_{δ,d_2} which depends on d_2.*

Proof. Let $\kappa = K_{\delta-} + 2\left(\operatorname{div}_{\bar{\mathbf{g}}_\delta} \mathbf{X}_\delta\right)_+$ and $K_\delta = \bar{\mathbf{R}}_\delta - 2|\mathbf{X}_\delta|^2_{\bar{\mathbf{g}}_\delta} + 2\operatorname{div}_{\bar{\mathbf{g}}_\delta} \mathbf{X}_\delta - |\bar{\mathbf{k}}_\delta|^2_{\bar{\mathbf{g}}_\delta}$. Moreover, since $\varsigma(t)' \leqslant 0$ in Lemma 3.1 and $X(\bar{\nu}) > 0$ on $\mathcal{O}_\delta$, we have

$$\begin{cases} K_{\delta-} = 0 & \text{outside } \mathcal{O}_\delta \\ |K_{\delta-}| \leqslant C_1\delta^{-1/3} & \text{inside } \mathcal{O}_\delta. \end{cases}, \tag{3.11}$$

where C_1 depends on initial data sets and independent of δ. Combining Lemma 3.1 and Definition 2.3, we have

$$\left(\int_{\bar{\Omega}} (\operatorname{div}_{\bar{\mathbf{g}}_\delta} X_\delta)_+^{3/2}\, dV_{\bar{\mathbf{g}}_\delta}\right)^{2/3} < d_1, \qquad \left(\int_{\bar{\Omega}} (\operatorname{div}_{\bar{\mathbf{g}}_\delta} X_\delta)_+^{6/5}\, dV_{\bar{\mathbf{g}}_\delta}\right)^{5/6} < d_2, \tag{3.12}$$

for sufficiently small δ. Next we consider the following PDE

$$\begin{cases} \Delta_{\bar{\mathbf{g}}_\delta} w_\delta + \frac{1}{8}\kappa w_\delta = -\frac{1}{8}\kappa & \bar{\mathbf{M}}\setminus\bar{F} \\ w_\delta = 0 & \infty \\ \bar{\nu}(w_\delta) = 0 & \partial\bar{F} \end{cases}. \tag{3.13}$$

Following Lemma 3.2 and Lemma 3.3 of [20], it is enough to prove an $L^6(\bar{\mathbf{M}}\setminus F)$-bound for w_δ to show existence of a positive $C^2(\bar{\mathbf{M}}\setminus F)$ solution w_δ such that $w_\delta = \frac{A_\delta}{|x|} + O(|x|^{-2})$ as $|x| \to \infty$. We multiply the PDE in (3.13) by w_δ and integrate by parts. Then using (3.11), (3.12), Hölder inequality, and Young inequality, we have

$$\begin{aligned} &\int_{\bar{\mathbf{M}}\setminus\bar{F}} |\nabla w_\delta|^2 dV_{\bar{\mathbf{g}}_\delta} = \frac{1}{8}\int_{\bar{\mathbf{M}}\setminus\bar{F}} \kappa w_\delta^2 dV_{\bar{\mathbf{g}}_\delta} - \frac{1}{8}\int_{\bar{\mathbf{M}}\setminus\bar{F}} \kappa w_\delta dV_{\bar{\mathbf{g}}_\delta} \\ &\leqslant \frac{1}{8}C_1\delta^{1/3}|\partial\bar{\Omega}|^{2/3}\left(\int_{\bar{\mathbf{M}}\setminus\bar{F}} w_\delta^6 dV_{\bar{\mathbf{g}}_\delta}\right)^{1/3} + \frac{1}{8}d_1\left(\int_{\bar{\mathbf{M}}\setminus\bar{F}} w_\delta^6 dV_{\bar{\mathbf{g}}_\delta}\right)^{1/3} \\ &+ \frac{1}{32 l_1}C_1^2\delta|\partial\bar{\Omega}|^{5/3} + \frac{l_1}{8}\left(\int_{\bar{\mathbf{M}}\setminus\bar{F}} w_\delta^6 dV_{\bar{\mathbf{g}}_\delta}\right)^{1/3} + \frac{1}{32 l_1}d_2^2 + \frac{l_1}{8}\left(\int_{\bar{\mathbf{M}}\setminus\bar{F}} w_\delta^6 dV_{\bar{\mathbf{g}}_\delta}\right)^{1/3} \\ &\leqslant \frac{1}{8}\left(C_1\delta^{1/3}|\partial\bar{\Omega}|^{2/3} + d_1 + 2l_1\right)\left(\int_{\bar{\mathbf{M}}\setminus\bar{F}} w_\delta^6 dV_{\bar{\mathbf{g}}_\delta}\right)^{1/3} + \frac{1}{32 l_1}C_1^2\delta|\partial\bar{\Omega}|^{5/3} + \frac{1}{32 l_1}d_2^2, \end{aligned} \tag{3.14}$$

for some arbitrary constant $l_1 > 0$. Recall the Sobolev inequality [20, Lemma 3.1]

$$\left(\int_{\bar{\mathbf{M}}\setminus\bar{F}} w_\delta^6 dV_{\bar{\mathbf{g}}_\delta}\right)^{1/3} \leqslant C_\delta \int_{\bar{\mathbf{M}}\setminus\bar{F}} |\nabla w_\delta|^2 dV_{\bar{\mathbf{g}}_\delta}, \tag{3.15}$$

where C_δ is the Sobolev constant and it is uniformly close to Sobolev constant of $\bar{\mathbf{g}}$. Since $\delta > 0$ and $l_1 > 0$ are sufficiently small, we choose d_1 such that

$$\frac{1}{8}\left(C_1\delta^{1/3}|\partial\bar{\Omega}|^{2/3} + d_1 + 2l_1\right) \leqslant \frac{1}{3C_\delta}. \tag{3.16}$$

Combining equations (3.14), (3.15), and (3.16), we get the L^6 bound for w_δ. Then, we define $u_\delta = w_\delta + 1 \geqslant 1$. Moreover, the scalar curvature of the conformal metric $\hat{\mathbf{g}}_\delta = u_\delta^4\bar{\mathbf{g}}_\delta$ is

$$\begin{aligned} R(\hat{\mathbf{g}}_\delta) &= u_\delta^{-5}\left(R(\bar{\mathbf{g}}_\delta)u_\delta - \frac{1}{8}\Delta_{\bar{\mathbf{g}}_\delta}u_\delta\right) \\ &= u_\delta^{-5}\left(K_\delta u_\delta + 2|\mathbf{X}_\delta|^2_{\bar{\mathbf{g}}_\delta}u_\delta - 2\operatorname{div}_{\bar{\mathbf{g}}_\delta}\mathbf{X}_\delta u_\delta + |\bar{\mathbf{k}}_\delta|^2_{\bar{\mathbf{g}}_\delta}u_\delta - \frac{1}{8}\Delta_{\bar{\mathbf{g}}_\delta}u_\delta\right) \\ &= u_\delta^{-5}\left(K_{\delta+}u_\delta + 2|\mathbf{X}_\delta|^2_{\bar{\mathbf{g}}_\delta}u_\delta + 2\left(\operatorname{div}_{\bar{\mathbf{g}}_\delta}\mathbf{X}_\delta\right)_- u_\delta + |\bar{\mathbf{k}}_\delta|^2_{\bar{\mathbf{g}}_\delta}u_\delta\right) \geqslant 0. \end{aligned} \tag{3.17}$$

Next we compute the ADM mass of the conformal metric

$$m_{ADM}(\hat{\mathbf{g}}_\delta) = m_{ADM}(\bar{\mathbf{g}}_\delta) + 2A_\delta = m_{ADM}(g_+) + 2A_\delta, \tag{3.18}$$

where $1 \leqslant u_\delta = 1 + \frac{A_\delta}{|x|} + O(|x|^{-2})$ and

$$\begin{aligned} 0 \leqslant A_\delta &= \frac{1}{32\pi}\int_{\bar{\mathbf{M}}\setminus\bar{F}} \kappa u_\delta dV_{\bar{\mathbf{g}}_\delta} \\ &\leqslant \frac{1}{32\pi}C_1\delta^{1/3}|\partial\bar{\Omega}|^{2/3} + \frac{1}{32\pi}d_2\left(\int_{\bar{\Omega}\setminus\bar{F}}|u_\delta|^6 dV_{\bar{\mathbf{g}}_\delta}\right)^{1/6}. \end{aligned} \tag{3.19}$$

For sufficiently small δ, we choose d_2 so that $A_\delta < \epsilon_{\delta,d_2}$ for some small constant ϵ_{δ,d_2}. Therefore, d_1 depends on Sobolev constant and d_2 is a measure for difference of ADM mass of the conformal metric and Bartnik-Shi-Tam extension. In particular, we have the following relation

$$\lim_{\delta\to 0} m_{ADM}(\hat{\mathbf{g}}_\delta) \leqslant m_{ADM}(g_+) + \lim_{\delta\to 0}\epsilon_{\delta,d_2}. \tag{3.20}$$

This complete the proof. □

We now have an asymptotically flat Riemannian manifold $(\bar{\mathbf{M}}\setminus\bar{F}, \hat{\mathbf{g}}_\delta)$ for $\bar{F}\subsetneq\bar{\Omega}$ and by running the inverse mean curvature flow in $\bar{\mathbf{M}}\setminus\bar{F}$, we can prove Theorem 1.2.

Proof of Theorem 1.2. Assume Ω is admissible. Let $\bar{E}\subsetneq\bar{\Omega}$ be a minimizing hull and without loss of generality we assume $m_H(\partial\bar{E}) > 0$. Let $\theta > 0$ be given. We can find a connected set $\bar{F}\supset\bar{E}$ with smooth boundary $\partial\bar{F}$ such that $\bar{F}\subsetneq\bar{\Omega}$ and

$$|\partial\bar{F}|_{\bar{g}} - \theta \leqslant |\partial\bar{E}|_{\bar{g}} \leqslant |\partial\bar{F}|_{\bar{g}} + \theta, \qquad m_H(\partial\bar{E}) \leqslant m_H(\partial\bar{F}) + \theta, \qquad m_H(\partial\bar{F}) > 0. \tag{3.21}$$

Then we attach the Jang graph and obtain a complete Riemannian manifold

$$(\bar{\mathbf{M}}, \bar{\mathbf{g}}, \bar{\mathbf{k}}, \mathbf{X}) = (\overline{M}\cup M_+, \bar{g}\cup g_+, \bar{k}\cup k_+, X\cup X_+),$$

with ADM mass $m_{ADM}(g_+)$. We follow Lemma 3.1 and Proposition 3.2, to obtain smooth Riemannian $(\bar{\mathbf{M}}\setminus\bar{F}, \hat{\mathbf{g}}_\delta)$ with non-negative scalar curvature. Let $\bar{F}'$ be strictly minimizing hull containing $\bar{F}$ in $(\bar{\mathbf{M}}, \hat{\mathbf{g}}_\delta\cup\bar{g}_+)$.

As in Theorem 3.1 of [18], $\bar{F}$ and $\bar{F}'$ are connected, and by simple connectedness and the Seifert van Kampen theorem, $\partial\bar{\Omega}$ is homeomorphic to S^2. Now we show that $\bar{F}'$ is connected. By the definition of $\bar{F}'$, we know that $\partial\bar{F}'$ seperates $\bar{\Omega}\cup\bar{M}$. Now suppose that $\partial F'$ has at least two components A and B, and consider in this case a closed curve γ intersecting A and B. By simple connectedness, there exists a map $f: D^2\to\bar{\Omega}\cup\bar{M}$ with $f(\partial D^2)$ a homeomorphism onto γ. Generically, $f^{-1}(A)$ is a compact properly embedded 1-manifold in D^2, which it clearly is not since γ only intersects A once.

Returning to our extension and the flow, since the scalar curvature of $\hat{\mathbf{g}}_\delta$ has been made non-negative, the weak inverse mean curvature flow emanating from $\partial\bar{F}'$ produces

$$\begin{aligned}
m_{ADM}(\hat{\mathbf{g}}_\delta) &\geqslant \sqrt{\frac{|\partial\bar{F}'|_{\hat{\mathbf{g}}_\delta}}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\partial\bar{F}'}\hat{H}_\delta dA_{\hat{\mathbf{g}}_\delta}\right)\\
&\geqslant \sqrt{\frac{|\partial\bar{F}'|_{\hat{\mathbf{g}}_\delta}}{|\partial\bar{F}|_{\bar{g}}}}\sqrt{\frac{|\partial\bar{F}|_{g}}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\partial\bar{F}}\bar{H}dA_{\bar{\mathbf{g}}}\right)\\
&= \sqrt{\frac{|\partial\bar{F}'|_{\hat{\mathbf{g}}_\delta}}{|\partial\bar{F}|_{\bar{g}}}}m_H(\partial F)\\
&\geqslant \sqrt{\frac{|\partial\bar{F}'|_{\hat{\mathbf{g}}_\delta}}{|\partial\bar{F}|_{\bar{g}}}}\left(m_H(\partial\bar{E})-\theta\right)\\
&\geqslant \sqrt{\frac{|\partial\bar{F}'|_{\bar{\mathbf{g}}}}{|\partial\bar{F}|_{\bar{g}}}}m_H(\partial\bar{E})-\theta\sqrt{\frac{|\partial\bar{F}'|_{\hat{\mathbf{g}}_\delta}}{|\partial\bar{F}|_{\bar{g}}}}\\
&\geqslant \sqrt{\frac{|\partial\bar{E}|_{\bar{g}}}{|\partial\bar{F}|_{\bar{g}}}}m_H(\partial\bar{E})-\theta C\\
&\geqslant \sqrt{\frac{|\partial\bar{E}|_{\bar{g}}}{|\partial E|_{\bar{g}}+\theta}}m_H(\partial\bar{E})-\theta C.
\end{aligned}\tag{3.22}$$

The first inequality follows from the Geroch monotonicity of inverse mean curvature flow. The second inequality used the fact that mean curvature of $\partial\bar{F}'$ in $\hat{\mathbf{g}}_\delta$ is zero on $\partial\bar{F}'\setminus\partial F$ and the mean curvature of $\partial\bar{F}'$ and $\partial\bar{F}$ are the same $\hat{H}_\delta=\bar{H}u_\delta^{-2}$ a.e. on $\partial\bar{F}'\cap\partial\bar{F}$. The third inequality follows from equation (3.21) and the fourth inequality follows from $u_\delta\geqslant 1$. To get the fifth inequality we combine the following two facts. First, following the proof of Theorem 3.1 of [18], we obtain $|\partial\bar{F}'|_{\bar{\mathbf{g}}}\geqslant|\partial\bar{E}|_{\bar{g}}$. Second, if there is no blow-up in the Jang graph, u_δ is bounded on compact manifold $\bar{\Omega}$ and we get $\sqrt{\frac{|\partial\bar{F}'|_{\hat{\mathbf{g}}_\delta}}{|\partial\bar{F}|_{\bar{g}}}}\leqslant C$ for some constant $C<\infty$. On the other hand, if there is a blow-up in the Jang graph and we have asymptotically cylindrical end [20, 3], by analyzing eigenvalues of the operator $\Delta_{\bar{\mathbf{g}}_\delta}+\frac{1}{8}\kappa$, we observe that $u_\delta(t,x)\sim a_0+\sum_{i=1}^\infty a_i e^{-\sqrt{\lambda_i}t}\xi_i(x)$, where $a_0>1$ and a_i are constants and λ_i and $\xi_i(x)$ are eigenvalue and eigenfunction of the Laplacian operator at the cylindrical end, respectively. Therefore, u_δ is bounded on $\bar{\Omega}$ and we have $\sqrt{\frac{|\partial\bar{F}'|_{\hat{\mathbf{g}}_\delta}}{|\partial\bar{F}|_{\bar{g}}}}\leqslant C$. Clearly, the last inequality follows from equation (3.21). Hence, as $\theta\to 0$ and $\delta\to 0$ we obtain

$$\lim_{\delta\to 0} m_{ADM}(\hat{\mathbf{g}}_\delta)\geqslant m_H(\partial\bar{E}).\tag{3.23}$$

Combining this with Proposition 3.2, we have

$$m_{ADM}(g_+)+\lim_{\delta\to 0}\epsilon_{\delta,d_2}\geqslant m_H(\partial\bar{E}).\tag{3.24}$$

By Wang and Yau [24] and admissible condition $X(\bar\nu) > 0$, we have

$$\begin{aligned} m_{WY}(\partial\Omega) &\geqslant \int_{\partial\bar\Omega} \left(\hat H_0 - \left(\bar H - X(\nu)\right)\right) dA_{\hat\sigma} \\ &\geqslant \int_{\partial\bar\Omega} \left(\hat H_0 - \bar H\right) dA_{\hat\sigma} + \int_{\partial\bar\Omega} X(\bar\nu) dA_{\hat\sigma} \\ &\geqslant m_{ADM}(g_+) + \int_{\partial\bar\Omega} X(\bar\nu) dA_{\hat\sigma} \end{aligned} \tag{3.25}$$

where the last inequality follows from (3.1). We choose d_2 such that

$$\int_{\partial\bar\Omega} X(\bar\nu) dA_{\hat\sigma} \geqslant \lim_{\delta\to 0} \epsilon_{\delta,d_2}. \tag{3.26}$$

Together with (3.24) and (3.25), the desired inequality for the Wang-Yau quasi-local mass in Theorem 1.2 is achieved. A similar argument leads to the inequality for the Liu-Yau quasi-local mass. □

Proof of Proposition 1.3. If the projection of MOTS S is outward minimizing surface $\bar S$ in the blow up Jang graph, then applying Theorem 1.2, we have

$$m_{WY}(\partial\Omega) \geqslant m_H(\bar S) = \sqrt{\frac{|\bar S|}{16\pi}} \tag{3.27}$$

But since $|\bar S| \geqslant |S|$, we get the result. Similarly, the inequality follows for outward minimizing minimal surfaces in a Jang graph.

□

Proof of Theorem 1.4. The argument is by contradiction and roughly follows from combing the radius definition of Schoen and Yau [22] and Theorem 3.2 of [18], with the main difference being that $m^*(\Omega;\bar\Omega)$ does not detect minimal surfaces, unlike $m_{ST}(\Omega)$ in Riemannian setting.

Suppose that there is no MOTS in Ω. Since $\partial\Omega$ is untrapped, then by Proposition 2 of [22] there exist a unique smooth bounded Jang graph $\bar\Omega$ solution to the Dirichlet problem (2.12). Now consider some $\bar\Omega_1 \subsetneq \bar\Omega_2 \subset \bar\Omega$ with smooth boundaries, and let $\bar E$ be a connected minimizing hull of $\bar\Omega_2$ with C^2 boundary such that $\bar E \subset \bar\Omega_1$. Wanting to prove that $m_{LY}(\partial\Omega) \geqslant m^*(\Omega;\bar\Omega)$, it suffices to show that $m_{LY}(\partial\Omega) \geqslant \alpha^*_{\bar\Omega_1,\bar\Omega_2} m_H(\partial\bar E)$.

Let $\bar E'$ be the strictly minimizing hull of $\bar E$ in $\bar\Omega$, which we know exists since $\partial\bar\Omega$ is mean convex. By definition we have that $\partial\bar E \cap \partial\bar V \neq \varnothing$ for some connected strictly minimizing hull $\bar V$ of $\bar\Omega$ and $\partial\bar E'$ is connected. By the definition of strictly minimizing hulls and the Regularity Theorem 1.3 of [8], we know that $\partial\bar E'$ is $C^{1,1}$, and that $H_{\partial\bar E'\setminus\partial\bar E} = 0$ and the mean curvature H of $\partial\bar E$ and $\partial\bar E'$ are equal when $\partial\bar E$ and $\partial\bar E'$ intersect. Using Theorem 1.2, it follows that

$$m_{LY}(\partial\Omega) \geqslant m_H(\partial\bar E') = \sqrt{\frac{|\partial\bar E'|}{16\pi}}\left(1 - \int_{\partial\bar E'} H^2 dA\right) = \sqrt{\frac{|\partial\bar E'|}{|\partial\bar E|}} m_H(\partial\bar E) \tag{3.28}$$

In general, it could be that $\partial\bar E$ and $\partial\bar E'$ do not intersect. This only occurs when there is an outward minimizing minimal surface outside of $\bar E$. When such a jump occurs, then is no general relation between the mean curvature or indeed area of $\partial\bar E$ and $\partial\bar E'$ (apart from the fact that $H = 0$ on $\partial\bar E'$). This scenario is forbidden by the fact that $\partial\bar E \cap \partial\bar V \neq \varnothing$ for some connected strictly minimizing hull $\bar V$ of $\bar\Omega$.

Therefore there are only two cases, either $\partial\bar E' \subset \bar\Omega_2$ or $\bar\Omega\setminus\bar\Omega_2 \cap \partial\bar E' \neq \varnothing$. The former case occurs when $\bar E'$ is actually also a strictly minimizing hull with respect to $\bar\Omega_2$, and in that case since $\bar E$ is also minimizing with respect to $\bar\Omega_2$, it follows by definition that $|\partial\bar E'| \geqslant |\partial\bar E|$, which in turn yields

$m_{LY}(\partial\Omega) \geqslant m_H(\partial\bar{E}) \geqslant m^*(\Omega;\bar{\Omega})$, where we have used that $\alpha^*{}_{\bar{\Omega}_1,\bar{\Omega}_2} \leqslant 1$.

If $\partial E' \nsubseteq \bar{\Omega}_2$, then we are in the setting of the Theorem 3.2 of [18], which due to Meeks and Yau [14] we can estimate minimal surface area part of $\partial\bar{E}'$. In this case, we have

$$\partial\bar{E}' \cap \bar{\Omega}_1 \supset \partial\bar{E}' \cap \partial\bar{E} \neq \varnothing. \tag{3.29}$$

Therefore, the distance d of $\bar{\Omega}_1$ and $\partial\bar{\Omega}_2$ is large enough such that there exist a x in minimal part of $\partial\bar{E}'$ with $d(x,\partial\bar{\Omega}_2) = \frac{d}{2}$. By Lemma 2.4 of [14], for any ball $B_x(r)$ centred at x with radius $r = \min\{\iota, \frac{d}{2}\}$, where ι is the infinmum of the injectivity of points in $\{x;\ d(x,\partial\bar{\Omega}_2) > \frac{d}{4}\}$, we have

$$\begin{aligned} |\partial\bar{E}'| &\geqslant |\partial\bar{E}' \cap B_x(r)| \geqslant CK^{-2}\int_0^r \tau^{-1}\sin(K\tau)^2 d\tau \\ &= \alpha^*{}^2_{\bar{\Omega}_1,\bar{\Omega}_2}|\partial\bar{\Omega}_1|^{\beta_{\Omega,\bar{\Omega}}} \\ &\geqslant \alpha^*{}^2_{\bar{\Omega}_1,\bar{\Omega}_2}|\partial\bar{E}|. \end{aligned} \tag{3.30}$$

The second inequality is by Meeks and Yau [14] and the last inequality follows from $\beta_{\Omega,\bar{\Omega}} \geqslant 1$ and the fact that $\bar{E}$ is a minimizing hull with respect to $\bar{\Omega}_2$. Hence combining this with inequality (3.28), we obtain $m_{LY}(\partial\Omega) \geqslant m^*(\Omega;\bar{\Omega})$ which is a contradiction. The inequality involving $m_{WY}(\partial\Omega)$ follows similarly.

Now to obtain the inequality involving $\mathrm{diam}(\partial\Omega)$ and $m_{LY}(\partial\Omega)$, we use a standard Minkowski formula in $\mathbb{R}^3$. Let S be the smallest sphere that circumscribes and envelops Ω in $\mathbb{R}^3$. Let $\mathbf{x_0}$ be its centre and R its radius. Using the formula for $m_{LY}(\partial\Omega)$ and equation (ii), Lemma 6.2.9, pg. 136 of Klingenberg [11], we have by Gauss-Bonnet theorem

$$m^*(\Omega;\bar{\Omega}) \leqslant m_{LY}(\partial\Omega) < \frac{1}{8\pi}\int_{\partial\Omega} H_0 \leqslant \frac{R}{8\pi}\int_{\partial\Omega} K_{\partial\Omega} = \frac{1}{2}R \leqslant \frac{1}{4}\mathrm{diam}(\partial\Omega),$$

where we use the positivity of Gauss curvature $K_{\partial\Omega}$ in the third inequality. This is a contradiction and complete the proof. □

Before proving Proposition 1.5, we recall the following Lemma from [18, Lemma 3.6].

Lemma 3.3. *Let (Ω, g) be a compact Riemannian 3-manifold with smooth mean convex boundary $\partial\Omega$ such that Gauss curvature of $\partial\Omega$ is positive. Suppose $E \subset\subset \Omega$ such that ∂E is a isoperimetric surface. Then either Ω contains a outward minimizing minimal surface or E is a minimizing hull.*

The proof of Propositions 1.5 and 1.6 are as follows.

Proof of Proposition 1.5. By Lemma 3.3 and Theorem 1.2, a isoperimetric surface $\bar{V}$ in $\bar{\Omega}$ satisfying $m_H(\bar{V}) \geqslant m_{LY}(\partial\Omega)$ is not a minimizing hull in $\bar{\Omega}$, therefore, there is a outward minimizing minimal surface $\bar{S}$ in $\bar{\Omega}$. □

Proof of Proposition 1.6. Combining Theorem 1.2 and the argument of Theorem 3.2 of [18], if there are no minimal surfaces in $\bar{\Omega}$, then $m_{LY}(\partial\Omega) \geqslant m_{ST}(\bar{\Omega})$. Flipping the inequality yields the desired result and the diameter bound is given below. The argument for $m_{WY}(\partial\Omega)$ is identical if there exists an admissible function τ. □

Finally, the proof of the Proposition 1.7 is similar to [6, Theorem 2.3].

Proof of Proposition 1.7. Assume $\bar{\Omega}$ is not diffeomorphic to a ball in $\mathbb{R}^3$. By Meeks-Simon-Yau [13, Theorem 1], there exist an outward minimizing minimal surface $\bar{S} = \mathbb{S}^2$ in $\bar{\Omega}$. For a given point

$p \in \bar{S}$ let $\{e_1, e_2\}$ be a basis of $T_p\bar{S}$ in which the second fundamental form II is diagonal and denote the principal curvatures at p by κ_i. The Gauss equation implies

$$K_{\bar{S}} = Rm_{\bar{S}}(e_1, e_2, e_1, e_2) = Rm_{\bar{\Omega}}(e_1, e_2, e_1, e_2) + \kappa_1\kappa_2. \tag{3.31}$$

where $K_{\bar{S}}$ is the Guass curvature of $\bar{S}$. Since $\bar{S}$ is minimal, $\kappa_1\kappa_2 \leqslant 0$ at p and therefore we have

$$K_{\bar{S}} \leqslant C^2. \tag{3.32}$$

By the Gauss–Bonnet Theorem and assumption $m_{LY}(\partial\Omega) < \frac{1}{2C}$, we have

$$4\pi = \int_{\bar{S}} K_{\bar{S}} \leqslant |\bar{S}|C^2 < \frac{|\bar{S}|}{4m_{LY}(\partial\Omega)^2}. \tag{3.33}$$

Combining this and Penrose inequality in Proposition 1.3 for outward minimizing minimal surface $\bar{S}$ in $\bar{\Omega}$, we have the following contradiction

$$m_{LY}(\partial\Omega) < \left(\frac{|\bar{S}|}{16\pi}\right)^{1/2} \leqslant m_{LY}(\partial\Omega). \tag{3.34}$$

Therefore, $\bar{\Omega}$ has no minimal surfaces. Furthermore, by Meeks-Simon-Yau [13], $\bar{\Omega}$ is diffeomorphic to a ball in $\mathbb{R}^3$. This means there is no blow-up in any Jang graphs of Ω and therefore, there is no MOTS in Ω. This also shows that the solution of Jang's equation must be unique. □

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