

Non-Abelian Gauged Fracton Matter Field Theory: New Sigma Models, Superfluids and Vortices

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Abstract

By gauging a higher-moment polynomial degree global symmetry and a discrete charge conjugation (i.e., particle-hole) symmetry coupled to matter fields (two symmetries mutually non-commutative), we derive a new class of higher-rank tensor non-abelian gauge field theory with dynamically gauged fractonic matter fields: Non-abelian gauged fractons interact with a hybrid class of higher-rank (symmetric or generic non-symmetric) tensor gauge theory and anti-symmetric tensor topological field theory, generalizing [arXiv:1909.13879, 1911.01804]. We also apply a quantum phase transition similar to that between insulator v.s. superfluid/superconductivity (U(1) symmetry disordered phase described by a topological gauge theory or a disordered Sigma model v.s. U(1) global/gauge symmetry-breaking ordered phase described by a Sigma model with a U(1) target space underlying Goldstone modes): We can regard our tensor gauge theories as disordered phases, and we transition to their new ordered phases by deriving new Sigma models in continuum field theories. While one low energy theory is captured by degrees of freedom of rotor or scalar modes, another side of low energy theory has vortices and superfluids — we explore non-abelian vortices (two types of vortices mutually interacting non-commutatively beyond an ordinary group structure) and their Cauchy-Riemann relation.

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1 Introduction and Overview of Previous Works

Fracton order (see a recent review [1] in condensed matter) concerns new conservation laws imposed on the energetic excitations (such that the particle excitations are known as fractons) of quantum systems which have significant restrictions on their mobility:

1. Excitations cannot move without creating additional excitations (commonly known as fractons),
2. Excitations can only move in certain subdimensional or subsystem directions (for 0-dimensional excitations known as subdimensional particles).

The origins of such constraints are new conservation laws from conserved quantities of higher-moments, including dipole moments [2] (relevant for a vector global symmetry in field theory [3, 4]), quadrupole moments, or generalized multipole moment (relevant for the so-called the polynomial global symmetry [4–6] or the polynomial shift symmetries [7] in field theory), etc. The composite excitation of each mobility-restricted excitations are however mobile. The mobility constraint of fracton phases is also related to quantum glassy dynamics [8, 9]. Follow the previous work of Ref. [4, 6], motivated by the fracton order in condensed matter [1], we continue extending and developing this framework by including the dynamically gauged matter fields in the higher-rank tensor gauge theory in a $d + 1$ dimensional spacetime (e.g. $d + 1$ d, over a flat spacetime manifold M^{d+1} , and we should focus on Cartesian coordinates \mathbb{R}^{d+1}).¹ The important new ingredient in our present work is that the gauge structure can be non-commutative (i.e., the so-called non-abelian), while still *coupling to the matter fields* — thus a partial goal of our present work is to derive a new non-abelian tensor gauged fracton field theory (that has gauge interactions also *coupling with gauged matter fields*).

To recall, the field-theoretic models given in Ref. [4, 6] offer some unified features that we may summarize via examples that also connect to the literature:

1. *An ungauged matter field theory of higher-moment global symmetry without gauge fields* (e.g., an ungauged abelian theory with a degree-0 ordinary symmetry that encodes Schrödinger [10] or Klein-Gordon type field theory [11, 12] (see Sec. 2.1), with a degree-1 polynomial symmetry pioneered in Pretko’s work [2] (see Sec. 2.2) and its general higher-moment degree- $(m - 1)$ polynomial generalization [5, 6] (see Sec. 2.3)): For example, there is a field theory captured by the Lagrangian term with a covariant derivate term $(P_{i_1, \dots, i_m}(\Phi, \dots, \partial^m \Phi))$ or P_J given in Sec. 2.1 and Sec. 3.1 of Ref. [6]. The schematic path integrals \mathbf{Z} are:

$$\mathbf{Z} = \int [\mathcal{D}\Phi][\mathcal{D}\Phi^\dagger] \exp(i \int_{M^{d+1}} d^{d+1}x (P_{i_1, \dots, i_m}(\Phi, \dots, \partial^m \Phi)) (P^{i_1, \dots, i_m}(\Phi^\dagger, \dots, \partial^m \Phi^\dagger))). \quad (1.1)$$

The dynamical complex scalar fields

$$\Phi := \Phi(x) = \Phi(\vec{x}, t) \text{ and } \Phi^\dagger = \Phi^\dagger(x) = \Phi^\dagger(\vec{x}, t) \in \mathbb{C} \quad (1.2)$$

are summed over in a schematic path integral. The Lagrangian term and \mathbf{Z} are invariant under the global symmetry transformation.

2. *A pure abelian or non-abelian higher-rank tensor gauge theory (without coupling to gauged matter field):*

The abelian case is widely studied in various works in condensed matter literature: Ref. [13–17].

¹In this article, we follow the notations and definitions given in Ref. [4, 6]. We attempt to be succinct in this article, thus we will directly guide the readers to refer the sections/materials derived in Ref. [4] and Ref. [6].

But a non-abelian version of fractonic theory is not much explored. Recent progress on non-abelian fracton orders from Ref. [18–20] are mostly built from lattice models with a discrete gauge group (or a discrete gauge structure in general).

We will take an alternative route to non-abelian fracton via the field theory. A rank-2 non-abelian higher-rank tensor gauge theory with a continuous gauge structure is proposed firstly in Ref. [4]. The most general form of rank- m non-abelian higher-rank tensor gauge theory is given by a schematic path integral in Ref. [6]’s Sec. 2.1

$$\mathbf{Z}_{\text{rk-}m\text{-sym-}A}^{\text{asym-BF}} := \int \left(\prod_{I=1}^N [\mathcal{D}A_{I,i_1,\dots,i_m}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp(i \int_{M^{d+1}} d^{d+1}x \left(\sum_{I=1}^N |\hat{F}_{I,\mu,\nu,i_2,\dots,i_m}^c|^2 + \frac{2}{2\pi} \sum_{I=1}^N B_I dC_I \right)) \cdot \omega_{d+1}(\{C_I\}). \quad (1.3)$$

The cocycle $\omega_{d+1} \in \mathbb{H}^{d+1}((\mathbb{Z}_2^C)^N, \mathbb{R}/\mathbb{Z})$ is a group cohomology data [21] that we can take the continuum topological quantum field theory (TQFT) formulation of discrete gauge theory (see References therein [22–25] and the overview [4]). The I is an index for specifying the different copies/layers of tensor gauge theories, the cocycle ω_{d+1} couples different copies/layers of tensor gauge theories together. Thus the cocycle ω_{d+1} gives rise to the interlayer interaction effects. The index I may be neglected for simplicity below. The real-valued abelian gauge field strength $F_{\mu,\nu,i_2,\dots,i_m} \in \mathbb{R}$ is promoted into a new complex-valued non-abelian gauge field strength $\hat{F}_{\mu,\nu,i_2,\dots,i_m}^c \in \mathbb{C}$ after gauging a discrete charge conjugation \mathbb{Z}_2^C (i.e., particle-hole) symmetry [4, 6]:

$$\hat{F}_{\mu,\nu,i_2,\dots,i_m}^c := D_\mu^c A_{\nu,i_2,\dots,i_m} - D_\nu^c A_{\mu,i_2,\dots,i_m} := (\partial_\mu - ig_c C_\mu) A_{\nu,i_2,\dots,i_m} - (\partial_\nu - ig_c C_\nu) A_{\mu,i_2,\dots,i_m}, \quad (1.4)$$

while $|\hat{F}_{\mu,\nu,i_2,\dots,i_m}^c|^2 := \hat{F}_{\mu,\nu,i_2,\dots,i_m}^c \hat{F}^{\dagger c \mu,\nu,i_2,\dots,i_m}$. Here are the field contents:

- The $A \in \mathbb{R}$ can be chosen to be a fully-symmetric rank- m real-valued tensor gauge field.
 - The $B \in \mathbb{R}$ is a $(d-1)$ -th \mathbb{Z}_2 -cohomology class in terms of \mathbb{Z}_2 -discrete gauge theory, or in the continuum formulated as a $(d-1)$ -form (an anti-symmetric rank- $(d-1)$ tensor) real-valued gauge field. The B plays the role of a Lagrangian multiplier to set C to be flat.
 - The $C \in \mathbb{R}$ is a \mathbb{Z}_2 -cohomology class in terms of \mathbb{Z}_2 -discrete gauge theory, or in the continuum formulated as a 1-form (a rank-1 tensor) real-valued gauge field.
3. *An abelian gauge theory coupling to gauged matter field:* This is pioneered in Pretko’s [2] for the rank-2 tensor fields, while we can use the most general form for the rank- m tensor gauge field $A = A_{i_1,i_2,\dots,i_m}$ given in Ref. [6] Sec. 2.1’s and Sec. 3.1’s schematic path integral:

$$\mathbf{Z}_{\text{rk-}2\text{-sym-}\Phi} = \int [\mathcal{D}A] [\mathcal{D}\Phi] [\mathcal{D}\Phi^\dagger] \exp(i \int_{M^{d+1}} d^{d+1}x \left(|F_{\mu,\nu,i_2,\dots,i_m}|^2 + |D^A[\{\Phi\}]|^2 + V(|\Phi|^2) \right)). \quad (1.5)$$

$$|D^A[\{\Phi\}]|^2 := |R|^2 := (R)(R^\dagger) = (P - igA\mathcal{Q})(P^\dagger + igA\mathcal{Q}^\dagger).$$

The $D^A[\{\Phi\}] := R \equiv P - igA\mathcal{Q}$ is defined in Sec. 3.1 of Ref. [6]. Here P and \mathcal{Q} are polynomials of Φ and its differential of $\partial^\ell \Phi$ for some power of ℓ . Here P and \mathcal{Q} are uniquely determined by the polynomial $Q(x)$ in the higher-moment global symmetry

$$\Phi_I \rightarrow e^{iQ_I(x)} \Phi_I := e^{i(\Lambda_{I,i_1,\dots,i_{m-1}} x_{i_1} \dots x_{i_{m-1}} + \dots + \Lambda_{I,i,j} x_i x_j + \Lambda_{I,i} x_i + \Lambda_{I,0})} \Phi_I, \quad (1.6)$$

shown in Ref. [6]. We denote such a polynomial symmetry as $U(1)_{\text{poly}}$ following [6], see a review in Sec. 2.3.1

What else topics have not yet been done in the literature but should be formulated? We will focus on these two open issues:

1. *A non-abelian gauge theory coupling to gauged matter field:*

Previous works only did the abelian gauged matter theory, or the non-abelian gauge theory without coupling to (fractonic) matter fields [4, 6]. In Sec 2, we provide a systematic framework for non-abelian gauged fractonic matter field theories.

In order to facilitate such a non-abelian gauged matter formulation, we sometimes trade a single complex component $\Phi = \Phi_{\text{Re}} + i\Phi_{\text{Im}} \in \mathbb{C}$ into two real components $\begin{pmatrix} \Phi_{\text{Re}} \in \mathbb{R} \\ \Phi_{\text{Im}} \in \mathbb{R} \end{pmatrix}$. For the U(1) polynomial symmetry viewpoint, the $\Phi \in \mathbb{C}$ is more natural. The \mathbb{Z}_2^C (particle-hole or particle-anti-particle symmetry) transformation acts on $\Phi \rightarrow \Phi^\dagger$ in the complex U(1) basis, but the \mathbb{Z}_2^C acts on the 2-component field naturally as:

$$\begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \rightarrow \begin{pmatrix} \Phi_{\text{Re}} \\ -\Phi_{\text{Im}} \end{pmatrix}.$$

To introduce the non-abelian gauge coupling to the matter fields, we will need to introduce several new types of gauge derivatives.² For example, even for the simplest degree-0 polynomial symmetry with a rank-1 tensor gauge field (1-form gauge field A_μ), we require:

$$D_\mu^{c,\text{Im}}\Phi := \partial_\mu\Phi - ig_c C_\mu \Phi_{\text{Im}}, \quad (1.7)$$

$$D_\mu^A\Phi := (\partial_\mu - igA_\mu)\Phi = (\partial_\mu\Phi_{\text{Re}} + gA_\mu\Phi_{\text{Im}}) + i(\partial_\mu\Phi_{\text{Im}} - gA_\mu\Phi_{\text{Re}}), \quad (1.8)$$

$$D_\mu^{A,c,\text{Im}}\Phi := D_\mu^A\Phi - ig_c C_\mu \Phi_{\text{Im}} = (\partial_\mu - igA_\mu)\Phi - ig_c C_\mu \Phi_{\text{Im}}. \quad (1.9)$$

$$D_\mu^c := (\partial_\mu - ig_c C_{\mu,I}). \quad (1.10)$$

$$D_\mu^c A_{\nu,i_2,\dots,i_m} := (\partial_\mu - ig_c C_\mu) A_{\nu,i_2,\dots,i_m}. \quad (1.11)$$

The first line $D_\mu^{c,\text{Im}}$ has the C gauge field that only couples to the charged matter Φ_{Im} (the imaginary component) under \mathbb{Z}_2^C . The second line D_μ^A is the gauge covariant derivative of 1-form gauge field A_μ after gauging the ordinary 0-form U(1) symmetry (the degree-0 polynomial symmetry). The third line $D_\mu^{A,c,\text{Im}}\Phi$ shows the gauge derivative on Φ involving both the $U(1) \times \mathbb{Z}_2^C = O(2)$ gauge fields. The fourth and the fifth line shows that for the fields charged under \mathbb{Z}_2^C (e.g. the rank- m symmetric tensor $A_{\nu,i_2,\dots,i_m} \rightarrow -A_{\nu,i_2,\dots,i_m}$ is charged under \mathbb{Z}_2^C), then the gauge derivative is D_μ^c . The g_c is a \mathbb{Z}_2^C gauge coupling denoted explicitly for the convenience.

We will present explicit examples to gauge both higher-moment and charge conjugation global symmetries including the matter in Sec. 2.

2. *A new type of Sigma model:* We formulate a new type of Sigma model that can move between the ordered and disordered phases of these higher-rank non-abelian tensor field theories with fully gauged fractonic matter.³ Similar to the familiar quantum phase transition between insulator v.s. superfluid/superconductivity [27–29] (U(1) symmetry disorder described by a topological gauge theory or a disordered Sigma model v.s. U(1) global/gauge symmetry-breaking order described by a Sigma model with a U(1) target space with Goldstone modes), we can regard our tensor gauge theory as a disordered phase, and we drive to its new ordered phase by deriving a new Sigma model in terms of continuum field theory.

Very recently, the superfluid and vortices of an abelian version of pure fractonic theories (without gauge fields) are studied in [30]. Two new ingredients in our work, which are not present in Ref. [30], are the facts that we include the gauge field interactions (thus we include the additional long-range entanglements) and we also include the non-abelian gauge-matter interactions.

²Note that some of such gauge derivatives are *not gauge covariant* under the gauge transformations, due to the non-Gaussian nature and higher-moment terms appear already in the fractonic matter field theories. But we will be able to construct the new types of gauge covariant terms and gauge-invariant Lagrangian in Sec. 2.

³In terms of the old Landau-Ginzburg paradigm, this is related to the Sigma model formulation of Landau-Ginzburg theory [26].

2 From Abelian Gauge Fractonic Theories to Non-Abelian Gauged Fractonic Matter Theories

2.1 Degree-0 polynomial symmetry to Schrödinger or Klein-Gordon type field theory

2.1.1 Global-covariant 1-derivative

Suppose we want to construct a field theory that preserves a degree-0 polynomial symmetry with a polynomial $Q(x) = \Lambda_0$. Then a complex scalar field transforms as $\Phi := \Phi(x) = \Phi(\vec{x}, t) \in \mathbb{C}$

$$\Phi \rightarrow e^{iQ(x)}\Phi = e^{i\Lambda_0}\Phi, \quad (2.1)$$

while its log transforms as

$$\log \Phi \rightarrow \log \Phi + i\Lambda_0. \quad (2.2)$$

Take the derivative $\partial_{x_i} := \partial_i$ respect to coordinates on both sides, we can eliminate $\partial_i\Lambda_0$ thus we get an invariant term:

$$\partial_i \log \Phi \rightarrow \partial_i \log \Phi, \quad (2.3)$$

This means $\partial_i \log \Phi$ is invariant under the global symmetry transformation. We can also define

$$\partial_i \log \Phi := \frac{P_i(\Phi, \partial\Phi)}{\Phi} = \frac{\partial_i\Phi}{\Phi} \quad (2.4)$$

Under $\Phi \rightarrow e^{iQ(x)}\Phi$, since the $\partial_i \log \Phi$ is invariant and the denominator $\Phi \rightarrow e^{iQ(x)}\Phi$ is covariant, so does the numerator $P_i(\Phi, \partial\Phi) = \partial_i\Phi \rightarrow e^{iQ(x)}\partial_i\Phi$ is covariant. Namely, both the numerator $P_i(\Phi, \partial\Phi)$ and Φ are global-covariant under $\Phi \rightarrow e^{iQ(x)}\Phi$, in order to maintain the $\partial_i \log \Phi$ to be invariant. Here $P_i(\Phi, \partial\Phi)$ denotes some functional P_i that depends on fields Φ or its derivative $\partial\Phi$.

For convenience, we will call such construction a *global-covariant 1-derivative*

$$P_i(\Phi, \partial\Phi) := \partial_i\Phi \quad (2.5)$$

to facilitate its further generalization later. We also construct a *globally invariant* Lagrangian

$$|P_i|^2 + V(|\Phi|^2) \quad (2.6)$$

that contains a potential term $V(|\Phi|^2)$ and a kinetic term⁴

$$|P_i|^2 := P_i(\Phi)P^i(\Phi^\dagger) = \partial_i\Phi\partial^i\Phi^\dagger. \quad (2.7)$$

In this way, based on the systematic method of Ref. [6], we can re-derive a Lagrangian formulation of Schrödinger equation in 1925 [10] and Klein-Gordon theory in 1926 [11, 12] for complex scalar fields.

⁴The raising and the lowering indices are merely used for contractions and summation, e.g. we sum over i indices in $\partial_i\Phi\partial^i\Phi^\dagger$. The conversion between the raising and lowering indices do not involve spacetime metrics, we only consider the flat spacetime.

2.1.2 Gauge-covariant 1-derivative

To gauge a degree-0 polynomial symmetry, we rewrite $Q(x)$ as a local gauge parameter $\eta(x)$,

$$\Phi \rightarrow e^{i\eta(x)}\Phi, \quad (2.8)$$

$$\partial_i \log \Phi \rightarrow \partial_i \log \Phi + i\partial_i \eta(x). \quad (2.9)$$

Then $\partial_i \log \Phi$ is no longer an invariant term. This implies that we can write a new gauge-covariant operator $D_i^A[\{\Phi\}]$ via combining P_i and A_i :

$$P_i(\Phi, \partial\Phi) := \partial_i \Phi \rightarrow e^{i\eta(x)}(P_i(\Phi, \partial\Phi) + i\partial_i \eta(x)). \quad (2.10)$$

$$A_i \rightarrow A_i + \frac{1}{g}\partial_i \eta. \quad (2.11)$$

$$D_i^A[\{\Phi\}] := P_i(\Phi, \partial\Phi) - igA_i\Phi = \partial_i \Phi - igA_i\Phi. \quad (2.12)$$

$$D_i^A[\{\Phi\}] \rightarrow e^{i\eta(x)}D_i^A[\{\Phi\}]. \quad (2.13)$$

To obtain a gauge invariant term, we can pair the gauge-covariant operator with its complex conjugation so to obtain a gauge invariant Lagrangian

$$|D_i^A[\{\Phi\}]|^2 + V(|\Phi|^2) = ((\partial_i - igA_i)\Phi)((\partial^i + igA^i)\Phi^\dagger) + V(|\Phi|^2). \quad (2.14)$$

2.1.3 Gauge-covariant non-abelian rank-2 field strength

Notice that in the pure matter theory Eq. (2.6) without gauge fields, we already have a degree-0 U(1) global symmetry and a \mathbb{Z}_2^C discrete charge conjugation (i.e., particle-hole) symmetry:

$$\Phi \mapsto \Phi^\dagger, \quad (2.15)$$

which makes Eq. (2.6) invariant. It is easy to see that the symmetry group structure is a non-abelian group

$$U(1) \times \mathbb{Z}_2^C = SO(2) \times \mathbb{Z}_2^C = O(2), \quad (2.16)$$

which acts on the Φ non-commutatively:

$$\begin{aligned} U_{\mathbb{Z}_2^C} U_{U(1)} \Phi &= U_{\mathbb{Z}_2^C}(e^{i\eta}\Phi) = e^{i\eta}\Phi^\dagger. \\ U_{U(1)} U_{\mathbb{Z}_2^C} \Phi &= U_{U(1)}(\Phi^\dagger) = e^{-i\eta}\Phi^\dagger. \end{aligned} \quad (2.17)$$

After we dynamically gauge the U(1) symmetry to obtain Eq. (2.14), we can still keep \mathbb{Z}_2^C discrete charge conjugation (i.e., particle-hole) symmetry intact which acts on gauge fields as

$$A_i \mapsto -A_i, \quad \eta(x) \mapsto -\eta(x). \quad (2.18)$$

If we fully gauge $U(1) \times \mathbb{Z}_2^C = O(2)$, we get a non-abelian O(2) gauge transformations which acts also on gauge fields non-commutatively:

$$\begin{aligned} U_{\mathbb{Z}_2^C} U_{U(1)} A_j &= U_{\mathbb{Z}_2^C}(A_j + \frac{1}{g}\partial_j \eta) = -A_j + \frac{1}{g}\partial_j \eta. \\ U_{U(1)} U_{\mathbb{Z}_2^C} A_j &= U_{U(1)}(-A_j) = -A_j - \frac{1}{g}\partial_j \eta. \end{aligned}$$

By promoting the global \mathbb{Z}_2^C to a local symmetry, we introduce a new 1-form \mathbb{Z}_2^C -gauge field C coupling to the 0-form symmetry \mathbb{Z}_2^C -charged object A_i with a new g_c coupling. The \mathbb{Z}_2^C local gauge transformation is:

$$A_i \rightarrow e^{i\gamma_c(x)}A_i, \quad C_i \rightarrow C_i + \frac{1}{g_c}\partial_i \gamma_c(x). \quad (2.19)$$

Note $A_{i_1} \in \mathbb{R}$ is in real-valued, so a generic $e^{i\gamma_c(x)}$ complexifies the A_i . Thus we restrict gauge transformation to be only \mathbb{Z}_2^C -gauged (not $U(1)^C$ -gauged)

$$e^{i\gamma_c(x)} := (-1)^{\gamma'_c(x)} \in \{\pm 1\}, \quad \gamma_c \in \pi\mathbb{Z}, \quad \gamma'_c \in \mathbb{Z}, \quad (2.20)$$

so $\gamma'_c(x)$ is an integer and $A_i \rightarrow \pm A_i$ stays in real. In the continuum field theory, the restrict gauge transformation is done by coupling to a level-2 BF theory (the \mathbb{Z}_2^C -gauge theory) [4, 6]. Thus $\gamma'_c(x)$ jumps between even or odd integers in \mathbb{Z} , while the \mathbb{Z}_2^C -gauge transformation can be suitably formulated on a lattice. We can also directly express the above on a triangulable spacetime manifold or a simplicial complex.

Approach 1: Gauge 0-form \mathbb{Z}_2^C -symmetry of 1-form gauge field A

Follow Ref. [4], we also define a new covariant derivative with respect to \mathbb{Z}_2^C :

$$D_i^c := (\partial_i - ig_c C_i). \quad (2.21)$$

We obtain a combined $O(2)$ gauge transformation of A_j ,

$$A_j \rightarrow e^{i\gamma_c(x)} A_j + (\pm)^\times \frac{1}{g} (D_j^c)(\eta(x)) := \begin{cases} V_{\mathbb{Z}_2} V_{U(1)} A_j = e^{i\gamma_c(x)} A_j + \frac{1}{g} D_j^c \eta, \\ V_{U(1)} V_{\mathbb{Z}_2} A_j = e^{i\gamma_c(x)} (A_j + \frac{1}{g} D_j^c \eta) \end{cases}. \quad (2.22)$$

We define that a new operation

$$(\pm)^\times \in \{+1, -1\}$$

via the above Eq. (2.22). Only when we perform a \mathbb{Z}_2^C first and then $U(1)$ gauge transformation second, and when $e^{i\gamma_c(x)} = -1$ we have $(\pm)^\times = -1$, otherwise all the other cases $(\pm)^\times = +1$. This factor $(\pm)^\times$ also captures the *non-abelian*-ness of the gauge structure.

Ref. [4, 6] defines a rank-2 *non-abelian* field strength as

$$\hat{F}_{i_1, i_2}^c := D_{i_1}^c A_{i_2} - D_{i_2}^c A_{i_1} := (\partial_{i_1} - ig_c C_{i_1}) A_{i_2} - (\partial_{i_2} - ig_c C_{i_2}) A_{i_1}, \quad (2.23)$$

with the locally flat \mathbb{Z}_2^C -gauge field C imposed by the level-2 BF theory $\frac{2}{2\pi} \int B dC$. It can be shown that \hat{F}_{i_1, i_2}^c is gauge-covariant under the $O(2)$ gauge transformation Eq. (2.22):

$$\hat{F}_{i_1, i_2}^c \rightarrow e^{i\gamma_c(x)} \hat{F}_{i_1, i_2}^c. \quad (2.24)$$

The above rank-2 field strength utilizes the viewpoint of gauging 0-form \mathbb{Z}_2^C -symmetry of 1-form gauge field A . However, because \hat{F}_{i_1, i_2}^c is a $O(2)$ field strength, we can write \hat{F}_{i_1, i_2}^c in a conventional way as a 2×2 matrix like Yang-Mills [31] did. We will pursue this alternative way in the next paragraph.

Approach 2: Non-abelian $O(2)$ field strength

Earlier from the $U(1)$ symmetry transformation, a single complex component

$$\Phi = \Phi_{\text{Re}} + i\Phi_{\text{Im}} \in \mathbb{C} \quad (2.25)$$

is a more natural view. From the $O(2) = \text{SO}(2) \times \mathbb{Z}_2^C$ view, the 2-component real scalar field ($\Phi_{\text{Re}} \in \mathbb{R}$, $\Phi_{\text{Im}} \in \mathbb{R}$) is natural such that the $U_{U(1)}$ and $U_{\mathbb{Z}_2^C}$ symmetry transforms the 2-component field as:

$$U_{U(1)} : \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix}. \quad (2.26)$$

$$U_{\mathbb{Z}_2^C} : \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \rightarrow \begin{pmatrix} \Phi_{\text{Re}} \\ -\Phi_{\text{Im}} \end{pmatrix}. \quad (2.27)$$

Let A gauge field be the generator of $U_{U(1)}$, and C gauge field be the generator of $U_{\mathbb{Z}_2^C}$. We can write down the non-abelian $O(2)$ gauge field X and field strength \hat{F}_X with a Lie algebra generator as:⁵

$$\begin{aligned} X &= \begin{pmatrix} 0 & A \\ -A & g_c C \end{pmatrix}, \\ \hat{F}_X &= dX - iXX = d \begin{pmatrix} 0 & A \\ -A & g_c C \end{pmatrix} - i \begin{pmatrix} 0 & A \\ -A & g_c C \end{pmatrix} \begin{pmatrix} 0 & A \\ -A & g_c C \end{pmatrix} = \begin{pmatrix} 0 & dA \\ -dA & g_c dC \end{pmatrix} - ig_c \begin{pmatrix} 0 & AC \\ AC & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & dA - ig_c AC \\ -dA - ig_c AC & 0 \end{pmatrix}. \end{aligned} \quad (2.28)$$

Here we use the fact that dC is locally flat.

The Yang-Mills $O(2)$ field strength kinetic term from \hat{F}_X is proportional to:

$$\begin{aligned} \text{Tr}[\hat{F}_X \wedge \star \hat{F}_X^\dagger] &= \text{Tr} \left[\begin{pmatrix} 0 & \partial_\mu A_\nu - ig_c A_\mu C_\nu \\ -\partial_\mu A_\nu - ig_c A_\mu C_\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & -\partial^\mu A^\nu + ig_c A^\mu C^\nu \\ \partial^\mu A^\nu + ig_c A^\mu C^\nu & 0 \end{pmatrix} \right] d^4x \\ &= 2 \left((\partial_\mu A_\nu)^2 + g_c (A_\mu C_\nu)^2 \right) d^4x. \end{aligned} \quad (2.29)$$

In comparison, the rank-2 non-Abelian field strength $\hat{F}_{\mu\nu}^c$ defined in Ref. [6] and Eq. (2.23) outputs

$$\begin{aligned} \hat{F}_{\mu\nu}^c \hat{F}^{c\mu\nu\dagger} &= ((\partial_\mu - ig_c C_\mu)A_\nu - (\partial_\nu - ig_c C_\nu)A_\mu) ((\partial^\mu + ig_c C^\mu)A^\nu - (\partial^\nu + ig_c C^\nu)A^\mu) \\ &\propto \partial_\mu A_\nu \partial^\mu A^\nu - (g_c)^2 (A_\mu C_\nu)^2. \end{aligned} \quad (2.30)$$

Thus two approaches agree on the Yang-Mills Lagrangian $\text{Tr}[\hat{F}_X \wedge \star \hat{F}_X^\dagger] \sim \hat{F}_{\mu\nu}^c \hat{F}^{c\mu\nu\dagger}$, up to a normalization constant.

Gauge-covariant non-abelian rank-2 field strength Under the $O(2)$ gauge transformation,

$$A_j \rightarrow A'_j = e^{i\gamma_c(x)} A_j + (\pm)^{\times} \frac{1}{g} (D_j^c)(\eta(x))\eta, \quad C_j \rightarrow C'_j = C_j + \frac{1}{g_c} \partial_j \gamma_c(x),$$

we can explicitly check that the non-abelian rank-2 field strength $\hat{F}_{\mu\nu}^c$ is gauge covariant:

$$\begin{aligned} \hat{F}_{\mu\nu}^c &:= D_\mu^c A_\nu - D_\nu^c A_\mu := (\partial_\mu - ig_c C_\mu)A_\nu - (\partial_\nu - ig_c C_\nu)A_\mu \\ &\rightarrow e^{i\gamma_c} \hat{F}_{\mu\nu}^c + \frac{1}{g} (D_\mu^c D_\nu^c - D_\nu^c D_\mu^c) \eta = e^{i\gamma_c} \hat{F}_{\mu\nu}^c, \end{aligned} \quad (2.31)$$

where we list down the leading order omitting the potentially higher power of η and γ_c terms. Note that:

$$\begin{aligned} (D_\mu^c D_\nu^c - D_\nu^c D_\mu^c) &= (\partial_\mu - ig_c C_\mu)(\partial_\nu - ig_c C_\nu) - (\partial_\nu - ig_c C_\nu)(\partial_\mu - ig_c C_\mu) \\ &= (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) - ig_c (\partial_\mu C_\nu - \partial_\nu C_\mu) - ig_c (C_\nu \partial_\mu - C_\mu \partial_\nu) - ig_c (C_\mu \partial_\nu - C_\nu \partial_\mu) - g_c^2 (C_\mu C_\nu - C_\nu C_\mu) \\ &= -ig_c (\partial_\mu C_\nu - \partial_\nu C_\mu) = -ig_c (dC)_{\mu\nu} = 0, \end{aligned} \quad (2.32)$$

where we need to impose the locally flat condition for the \mathbb{Z}_2^C gauge field in the last equality.⁶ Thus the gauge covariance is true since we show $(D_\mu^c D_\nu^c - D_\nu^c D_\mu^c) = 0$.

⁵Readers may wonder whether the $O(2)$ Lie algebra generator needs to be traceless. There are however two facts:

- (1). It is known that Lie algebra generators of a semi simple Lie algebra must be traceless. A Lie algebra is semisimple if it is a direct sum of simple Lie algebras, i.e., non-abelian Lie algebras \mathfrak{g} whose only ideals are 0 and \mathfrak{g} itself. However, a one-dimensional Lie algebra (which is necessarily abelian) is by definition not considered a simple Lie algebra, although such an algebra has no nontrivial ideals. Thus, one-dimensional algebras are not allowed as summands in a semisimple Lie algebra.
- (2). The C is a discrete \mathbb{Z}_2^C 1-form gauge field so dC is locally flat. Later on we need to impose the condition to show gauge covariance of field strength. (In general we do not have to impose equations of motion to show gauge invariance, although in the case with the \mathbb{Z}_2^C -gauge field C , we do require its locally flatness for gauge covariance of \hat{F}_{i_1, i_2}^c or \hat{F}_X .)

⁶See also a related discussion but with a more explicit calculation on the gauge covariance of field strength in Ref. [4]

2.1.4 Non-abelian O(2) gauged matter: Polynomial invariant v.s. Yang-Mills method

Previous work [4, 6] does not couple to non-abelian gauge fields to matter field. In this work, we propose a systematic method to generate non-abelian gauged matter theories.

Approach 1: Polynomial invariant method — Covariant derivative on the log as an invariant

For any give complex field $\mathfrak{N} \in \mathbb{C}$, such that $\mathfrak{N} = \mathfrak{N}_{\text{Re}} + i\mathfrak{N}_{\text{Im}}$, where the imaginary $\mathfrak{N}_{\text{Im}} \rightarrow -\mathfrak{N}_{\text{Im}}$ is charged under \mathbb{Z}_2^C -symmetry, thus we define a new derivative

$$D_\mu^{c,\text{Im}}\mathfrak{N} := \partial_\mu\mathfrak{N}_{\text{Re}} + iD_\mu^c\mathfrak{N}_{\text{Im}} \equiv \partial_\mu\mathfrak{N}_{\text{Re}} + i(\partial_\mu - ig_c C_\mu)\mathfrak{N}_{\text{Im}}. \quad (2.33)$$

For example, for the complex scalar field $\Phi = \Phi_{\text{Re}} + i\Phi_{\text{Im}} \in \mathbb{C}$, with the real component $\Phi_{\text{Re}} \in \mathbb{R}$ and imaginary component $\Phi_{\text{Im}} \in \mathbb{R}$,

$$D_\mu^{c,\text{Im}}\Phi := \partial_\mu\Phi_{\text{Re}} + iD_\mu^c\Phi_{\text{Im}} \equiv \partial_\mu\Phi_{\text{Re}} + i(\partial_\mu - ig_c C_\mu)\Phi_{\text{Im}}. \quad (2.34)$$

Follow Eq. (2.8) and Eq. (2.9), in order to find an O(2) gauge covariant derivative, we aim to firstly design an invariant term under gauge transformations. First, we see that $D_\mu^{c,\text{Im}} \log \Phi$ is not invariant under a U(1) part of O(2) gauge transformation $\Phi \rightarrow e^{i\eta(x)}\Phi$,

$$D_\mu^{c,\text{Im}} \log \Phi \rightarrow D_\mu^{c,\text{Im}} \log \Phi + D_\mu^{c,\text{Im}}(i\eta(x)) = \frac{D_\mu^{c,\text{Im}}\Phi + D_\mu^{c,\text{Im}}(i\eta(x))\Phi}{\Phi} = \frac{D_\mu^{c,\text{Im}}\Phi + i(D_\mu^c\eta)\Phi}{\Phi}. \quad (2.35)$$

Here $\eta(x) \in \mathbb{R}$ is part of the complex phase of $e^{i\eta(x)}\Phi$, thus $D_\mu^{c,\text{Im}}(i\eta(x)) = i(D_\mu^c\eta)$. Now based on the same trick in Eq. (2.12), we can absorb the U(1) gauge transformation by introducing the 1-form A gauge field. The A transforms to A' under the U(1) part of Eq. (2.22):

$$A_j \rightarrow A'_j = A_j + \frac{1}{g}D_j^c\eta, \quad (2.36)$$

$$\begin{aligned} D_\mu^{A,c,\text{Im}} \log \Phi = \frac{D_\mu^{A,c,\text{Im}}\Phi}{\Phi} &\rightarrow D_\mu^{A',c,\text{Im}} \log \Phi + D_\mu^{A',c,\text{Im}}(i\eta(x)) = \frac{D_\mu^{A',c,\text{Im}}\Phi + i(D_\mu^c\eta)\Phi}{\Phi} \\ &= \frac{(\partial_\mu - ig(A_j + \frac{1}{g}D_j^c\eta))\Phi - ig_c C_\mu \Phi_{\text{Im}} + i(D_\mu^c\eta)\Phi}{\Phi} = \frac{D_\mu^{A,c,\text{Im}}\Phi}{\Phi}. \end{aligned} \quad (2.37)$$

Here $D_\mu^{A',c,\text{Im}}(i\eta(x)) = i(D_\mu^c\eta)$ because η is not charged under U(1) symmetry but only the U(1) gauge parameter of A itself. The above shows that $D_\mu^{A,c,\text{Im}} \log \Phi = \frac{D_\mu^{A,c,\text{Im}}\Phi}{\Phi}$ is a gauge invariant quantity under U(1) gauge transformation. We can show that it is also gauge invariant quantity under the full O(2) gauge transformation (including Eq. (2.22) and the definition of $(\pm)^\times \in \{+1, -1\}$ around Eq. (2.22)):

$$A_j \rightarrow A'_j = e^{i\gamma_c(x)}A_j + (\pm)^\times \frac{1}{g}(D_j^c)(\eta(x))\eta, \quad (2.38)$$

$$C_j \rightarrow C'_j = C_j + \frac{1}{g_c}\partial_j\gamma_c(x), \quad (2.39)$$

$$\Phi \rightarrow e^{(\pm)^\times i\eta(x)}\Phi_c = e^{(\pm)^\times i\eta(x)}(\Phi_{\text{Re}} + ie^{i\gamma_c(x)}\Phi_{\text{Im}}) = \begin{cases} e^{i\eta(x)}\Phi, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ e^{(\pm)^\times i\eta(x)}\Phi^\dagger, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \quad (2.40)$$

Let us focus on the case $\Phi \rightarrow e^{i\eta(x)}\Phi_c$ first, we have:

$$\begin{aligned}
D_\mu^{A,c,\text{Im}} \log \Phi &= \frac{D_\mu^{A,c,\text{Im}} \Phi}{\Phi} \rightarrow D_\mu^{A',c',\text{Im}} \log \Phi_c + D_\mu^{A',c',\text{Im}} (i\eta(x)) = \frac{D_\mu^{A',c',\text{Im}} \Phi_c + i(D_\mu^{c'} \eta) \Phi_c}{\Phi_c} \\
&= \frac{(\partial_\mu - ig(A_j + \frac{1}{g} D_j^{c'} \eta)) \Phi_c - ig_c C'_\mu e^{i\gamma_c} \Phi_{\text{Im}} + i(D_\mu^{c'} \eta) \Phi_c}{\Phi_c} \\
&= \frac{(\partial_\mu - ig(A_j + \frac{1}{g} D_j^{c'} \eta)) \Phi_c - ig_c (C_j + \frac{1}{g_c} \partial_j \gamma_c) e^{i\gamma_c} \Phi_{\text{Im}} + i(D_\mu^{c'} \eta) \Phi_c}{\Phi_c} \\
&= \begin{cases} \frac{D_\mu^{A,c,\text{Im}} \Phi}{\Phi}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ \frac{D_\mu^{A,c,\text{Im}}(\Phi^\dagger)}{\Phi^\dagger}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \tag{2.41}
\end{aligned}$$

Here some of the equalities hold when we focus on the leading order contribution for the gauge transformations. Note that $\partial_\mu \Phi_c$ can contribute a $\partial_\mu (e^{i\gamma_c} \Phi_{\text{Im}}) = i(\partial_\mu \gamma_c) e^{i\gamma_c} \Phi_{\text{Im}} + \dots$ that cancels with $-ig_c (\frac{1}{g_c} \partial_j \gamma_c) e^{i\gamma_c} \Phi_{\text{Im}}$. We can define a complex conjugation operator of $D_\mu^{A,c,\text{Im}}$ as :

$$D_\mu^{\dagger A,c,\text{Im}} := (D_\mu^{A,c,\text{Im}})^\dagger. \tag{2.42}$$

Similarly, we find under the gauge transformation $\Phi \rightarrow e^{i\eta(x)}\Phi_c$:

$$D_\mu^{\dagger A,c,\text{Im}} \log \Phi^\dagger \rightarrow \begin{cases} \frac{D_\mu^{\dagger A,c,\text{Im}} \Phi^\dagger}{\Phi^\dagger}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ \frac{D_\mu^{\dagger A,c,\text{Im}}(\Phi)}{\Phi}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \tag{2.43}$$

$$D_\mu^{A,c,\text{Im}} \log \Phi^\dagger \rightarrow \begin{cases} \frac{D_\mu^{A,c,\text{Im}} \Phi^\dagger}{\Phi^\dagger}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ \frac{D_\mu^{A,c,\text{Im}}(\Phi)}{\Phi}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \tag{2.44}$$

$$D_\mu^{\dagger A,c,\text{Im}} \log \Phi \rightarrow \begin{cases} \frac{D_\mu^{\dagger A,c,\text{Im}} \Phi}{\Phi}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ \frac{D_\mu^{\dagger A,c,\text{Im}}(\Phi^\dagger)}{\Phi^\dagger}, & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \tag{2.45}$$

Similar discussions follow after taking the factor $(\pm)^\times$ into the account. Since the denominators in Eq. (2.41), Eq. (2.43), Eq. (2.44), and Eq. (2.45) are all *gauge-covariant* or *complex conjugation gauge-covariant*:

$$\Phi \rightarrow e^{(\pm)^\times i\eta(x)} \Phi_c, \tag{2.46}$$

this means that the numerators are also *gauge-covariant* or *complex conjugation gauge-covariant*. We can construct the gauge invariant quantity via pairing the *gauge-covariant* term with its complex conjugation, and pairing the *complex conjugation gauge-covariant* term also with its complex conjugation. So we obtain:⁷

$$\begin{aligned}
&(D_\mu^{A,c,\text{Im}} \Phi)(D_{A,c,\text{Im}}^{\dagger \mu} \Phi^\dagger) + (D_\mu^{\dagger A,c,\text{Im}} \Phi)(D_{A,c,\text{Im}}^\mu \Phi^\dagger) \\
&= ((\partial_\mu - igA_\mu)\Phi - ig_c C_\mu \Phi_{\text{Im}})((\partial^\mu + igA^\mu)\Phi^\dagger + ig_c C^\mu \Phi_{\text{Im}}) \\
&+ ((\partial_\mu + igA_\mu)\Phi + ig_c C_\mu \Phi_{\text{Im}})((\partial^\mu - igA^\mu)\Phi^\dagger - ig_c C^\mu \Phi_{\text{Im}}).
\end{aligned} \tag{2.47}$$

⁷Here we are allowed to flip the sign $C \rightarrow -C$ since C is only a \mathbb{Z}_2 gauge field. More precisely, $\oint C = \frac{2\pi}{2}\mathbb{Z} = \pi\mathbb{Z} \pmod{2\pi}$, while $\oint C = -\oint C \pmod{2\pi}$. It may also look peculiar that the particle Φ and anti-particle Φ^\dagger both couples to the gauge field A with both ± 1 couplings. However, we may comfort the readers by reminding the fact that the particle-hole conjugation symmetry \mathbb{Z}_2^C is already gauged, thus \mathbb{Z}_2^C gauge field couples to both the particle Φ and anti-particle Φ^\dagger , via the Φ_{Im} part. Furthermore, the \mathbb{Z}_2^C gauge field can flip the sign of their U(1) gauge charge $+1 \leftrightarrow -1$. This seems to suggest that particle and anti-particle may share part of the degree of freedom. This reminds us the famous fact that Majorana fermion has the particle and anti-particle identified as the same, although we should beware that our particle Φ is bosonic instead.

Approach 2: Yang-Mills method Let us cross-check the above result from a more conventional Yang-Mills method [31]. To start with, we observe that $(d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix}$ must be gauge covariant under the gauge transformation. Because under a generic gauge transformation $V_{\text{O}(2)}$, it demands $(d + X) \rightarrow (V_{\text{O}(2)}(d + X)V_{\text{O}(2)}^{-1})$ and $\begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \rightarrow (V_{\text{O}(2)} \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix})$, we show the gauge covariance of

$$(d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \rightarrow (V_{\text{O}(2)}(d + X)V_{\text{O}(2)}^{-1})(V_{\text{O}(2)} \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix}) = V_{\text{O}(2)}(d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix}. \quad (2.48)$$

Since all components of $A, C, \Phi_{\text{Re}}, \Phi_{\text{Im}} \in \mathbb{R}$ are reals, we can pair the gauge covariant term with its transpose (the Hodge dual \star) to obtain a gauge invariant Lagrangian term (again see footnote 7)

$$\begin{aligned} & (d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \wedge \star(d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \\ &= \left| \begin{pmatrix} \partial_\mu & gA_\mu \\ -gA_\mu & \partial_\mu - g_c C_\mu \end{pmatrix} \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \partial_\mu \Phi_{\text{Re}} + gA_\mu \Phi_{\text{Im}} \\ \partial_\mu \Phi_{\text{Im}} - gA_\mu \Phi_{\text{Re}} - g_c C_\mu \Phi_{\text{Im}} \end{pmatrix} \right|^2 \\ &= (\partial_\mu \Phi_{\text{Re}} + gA_\mu \Phi_{\text{Im}})^2 + (\partial_\mu \Phi_{\text{Im}} - gA_\mu \Phi_{\text{Re}} - g_c C_\mu \Phi_{\text{Im}})^2 \\ &= (\partial_\mu \Phi_{\text{Re}} + gA_\mu \Phi_{\text{Im}})^2 + (\partial_\mu \Phi_{\text{Im}} - gA_\mu \Phi_{\text{Re}})^2 + (g_c C_\mu \Phi_{\text{Im}})^2 - 2(\partial_\mu \Phi_{\text{Im}} - gA_\mu \Phi_{\text{Re}})(g_c C_\mu \Phi_{\text{Im}}) \\ &= D_\mu^A \Phi D_{A,c,\text{Im}}^\dagger \Phi^\dagger + (g_c C_\mu \Phi_{\text{Im}})^2 - 2(\partial_\mu \Phi_{\text{Im}} - gA_\mu \Phi_{\text{Re}})(g_c C_\mu \Phi_{\text{Im}}) = (D_\mu^{A,c,\text{Im}} \Phi)(D_{A,c,\text{Im}}^\dagger \Phi^\dagger). \end{aligned} \quad (2.50)$$

Thus we show (again see footnote 7)

$$\boxed{(D_\mu^{A,c,\text{Im}} \Phi)(D_{A,c,\text{Im}}^\dagger \Phi^\dagger) = (d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \wedge \star(d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix}}. \quad (2.51)$$

Similarly (again see footnote 7),

$$\boxed{(D_\mu^{A,c,\text{Im}} \Phi^\dagger)(D_{A,c,\text{Im}}^\dagger \Phi) = (d - X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} \wedge \star(d - X) \begin{pmatrix} \Phi_{\text{Re}} \\ \Phi_{\text{Im}} \end{pmatrix} = (d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ -\Phi_{\text{Im}} \end{pmatrix} \wedge \star(d + X) \begin{pmatrix} \Phi_{\text{Re}} \\ -\Phi_{\text{Im}} \end{pmatrix}}. \quad (2.52)$$

Thus we can construct an $\text{O}(2)$ gauged matter field theory contains a Lagrangian term Eq. (2.47) and a potential $V(|\Phi|^2)$ as:

$$\boxed{(D_\mu^{A,c,\text{Im}} \Phi)(D_{A,c,\text{Im}}^\dagger \Phi^\dagger) + (D_\mu^{\dagger A,c,\text{Im}} \Phi^\dagger)(D_{A,c,\text{Im}}^\mu \Phi) + V(|\Phi|^2)}. \quad (2.53)$$

The theory contains particle Φ and anti-particle Φ^\dagger pair together in an intricate way because the particle-hole \mathbb{Z}_2^C symmetry $\Phi \xleftrightarrow{\mathbb{Z}_2^C} \Phi^\dagger$ is also dynamically gauged. If the particle Φ has a gauge charge-1, then the anti-particle Φ^\dagger has a gauge charge-(-1) under the $\text{U}(1)$ gauge group. (However, see also footnote 7)

Follow [4, 6], we can consider the N -layers generalization of the theories with $(\mathbb{Z}_2^C)^N$ gauged, also by including the $\text{O}(2)$ -Yang Mills kinetic term Eq. (2.30) and the level-2 BF theory into the $\text{O}(2)$ gauge matter theory Eq. (2.53), we are allowed to introduce the twisted cocycle $\omega_{d+1} \in \mathbb{H}^{d+1}((\mathbb{Z}_2^C)^N, \mathbb{R}/\mathbb{Z})$ from a group cohomology data [21] to specify the interlayer interactions between N -layers. We can write down a schematic path integral:

$$\begin{aligned} \mathbf{Z}_{\text{rk-2-NAb-}\Phi} &= \int \left(\prod_{I=1}^N [\mathcal{D}A_I][\mathcal{D}B_I][\mathcal{D}C_I][\mathcal{D}\Phi_I][\mathcal{D}\Phi_I^\dagger] \right) \exp\left(i \int_{M^{d+1}} d^{d+1}x \left(\sum_{I=1}^N (|\hat{F}_{\mu\nu}^{c,I}|^2 \right. \right. \\ &\quad \left. \left. + (D_\mu^{A,c,\text{Im}} \Phi_I)(D_{A,c,\text{Im}}^\dagger \Phi_I^\dagger) + (D_\mu^{\dagger A,c,\text{Im}} \Phi_I^\dagger)(D_{A,c,\text{Im}}^\mu \Phi_I) + V(|\Phi_I|^2) \right) \right) \cdot \omega_{d+1}(\{C_I\}). \end{aligned} \quad (2.54)$$

2.2 Degree-1 polynomial symmetry to Pretko's field theory and non-abelian generalization

2.2.1 Global-covariant 2-derivative

Now we construct a field theory that preserves a degree-1 polynomial symmetry with a polynomial $Q(x) = (\Lambda_k x_k + \Lambda_0)$. A degree-1 polynomial symmetry transforms Φ and $\log \Phi$ as

$$\Phi \rightarrow e^{iQ(x)}\Phi = e^{i(\Lambda_k x_k + \Lambda_0)}\Phi, \quad (2.55)$$

$$\log \Phi \rightarrow \log \Phi + iQ(x) = \log \Phi + i(\Lambda_i x_i + \Lambda_0). \quad (2.56)$$

Take $\partial_{x_i}\partial_{x_j} := \partial_i\partial_j$ on both sides, we construct a globally invariant term,

$$\partial_i\partial_j \log \Phi \rightarrow \partial_i\partial_j \log \Phi. \quad (2.57)$$

We also define

$$\partial_i\partial_j \log \Phi := \frac{P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi)}{\Phi^2} = \frac{\Phi\partial_i\partial_j\Phi - (\partial_i\Phi)(\partial_j\Phi)}{\Phi^2}. \quad (2.58)$$

Under $\Phi \rightarrow e^{iQ(x)}\Phi$, since the denominator $\Phi^2 \rightarrow e^{i2Q(x)}\Phi^2$, so does the numerator $P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) \rightarrow e^{i2Q(x)}P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi)$ which we name

$$P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) := \Phi\partial_i\partial_j\Phi - (\partial_i\Phi)(\partial_j\Phi) \quad (2.59)$$

as a global-covariant 2-derivative term, in order to maintain the $\partial_i\partial_j \log \Phi$ to be invariant. The gauge-invariant Lagrangian contains

$$|P_{i,j}|^2 + V(|\Phi|^2) := P_{i,j}(\Phi)P^{i,j}(\Phi^\dagger) + V(|\Phi|^2) = (\Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi)(\Phi^\dagger\partial^i\partial^j\Phi^\dagger - \partial^i\Phi\partial^j\Phi^\dagger) + V(|\Phi|^2). \quad (2.60)$$

In this way, based on the systematic method of Ref. [6], we can re-derive a Lagrangian formulation of Pretko in 2018 [2], which are recently revisited in [3, 5] and [4] from other field theory perspectives.

2.2.2 Gauge-covariant 2-derivative

To gauge a degree-0 polynomial symmetry, we rewrite $Q(x)$ as a local gauge parameter $\eta(x)$,

$$\Phi \rightarrow e^{i\eta(x)}\Phi, \quad (2.61)$$

$$\partial_i\partial_j \log \Phi \rightarrow \partial_i\partial_j \log \Phi + i\partial_i\partial_j\eta(x). \quad (2.62)$$

Then $\partial_i\partial_j \log \Phi$ is no longer an invariant term. This implies that we can write a new gauge-covariant operator $D_{i,j}[\{\Phi\}]$ via combining $P_{i,j}$ and $A_{i,j}$:

$$P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) := (\Phi\partial_i\partial_j\Phi - (\partial_i\Phi)(\partial_j\Phi)) \rightarrow e^{i2\eta(x)}(P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) + i\partial_i\partial_j\eta(x)). \quad (2.63)$$

$$A_{i,j} \rightarrow A_{i,j} + \frac{1}{g}\partial_i\partial_j\eta. \quad (2.64)$$

$$D_{i,j}^A[\{\Phi\}] := P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) - igA_{i,j}\Phi^2 = (\Phi\partial_i\partial_j\Phi - (\partial_i\Phi)(\partial_j\Phi) - igA_{i,j}\Phi^2). \quad (2.65)$$

$$D_{i,j}^A[\{\Phi\}] \rightarrow e^{i2\eta(x)}D_{i,j}^A[\{\Phi\}]. \quad (2.66)$$

We shall call $D_{i,j}^A[\{\Phi\}]$ a gauge-covariant 2-derivative term.⁸ So a gauge invariant Lagrangian term can be obtained by complex-conjugate pairing the gauge-covariant operator as

$$|D_{i,j}^A[\{\Phi\}]|^2 + V(|\Phi|^2) = D_{i,j}^A[\{\Phi\}]D_A^{\dagger i,j}[\{\Phi^\dagger\}] + V(|\Phi|^2). \quad (2.67)$$

Thus we also reproduce Pretko's abelian gauge theory [2].

2.2.3 Gauge-covariant non-abelian $[U(1)_{x(d)} \times \mathbb{Z}_2^C]$ rank-3 field strength

Follow Ref. [4, 6], we define a non-abelian rank-3 field strength

$$\hat{F}_{\mu\nu\xi}^c := D_\mu^c A_{\nu\xi} - D_\nu^c A_{\mu\xi} := (\partial_\mu - ig_c C_\mu)A_{\nu\xi} - (\partial_\nu - ig_c C_\nu)A_{\mu\xi}. \quad (2.68)$$

Under the gauge transformation

$$\begin{aligned} A_{\mu\nu} &\rightarrow e^{i\gamma_c(x)} A_{\mu\nu} + (\pm)^\times \frac{1}{2g} (D_\mu^c D_\nu^c + D_\nu^c D_\mu^c)(\eta_\nu(x)) \\ &= \begin{cases} V_{\mathbb{Z}_2} V_{U(1)} A_{\mu\nu} = e^{i\gamma_c(x)} A_{\mu\nu} + \frac{1}{2g} (D_\mu^c D_\nu^c + D_\nu^c D_\mu^c) \eta_\nu, \\ V_{U(1)} V_{\mathbb{Z}_2} A_{\mu\nu} = e^{i\gamma_c(x)} (A_{\mu\nu} + \frac{1}{2g} (D_\mu^c D_\nu^c + D_\nu^c D_\mu^c) \eta_\nu) \end{cases} \cdot \\ C_\nu &\rightarrow C_\nu + \frac{1}{g_c} \partial_\nu \gamma_c(x). \end{aligned}$$

we can again show $\hat{F}_{\mu\nu\xi}^c$ is gauge-covariant:

$$\hat{F}_{\mu\nu\xi}^c \rightarrow e^{i\gamma_c(x)} \hat{F}_{\mu\nu\xi}^c + \frac{1}{2g} \left(D_\mu^c (D_\nu^c D_\xi^c + D_\xi^c D_\nu^c) - D_\nu^c (D_\mu^c D_\xi^c + D_\xi^c D_\mu^c) \right) \eta_\nu = e^{i\gamma_c(x)} \hat{F}_{\mu\nu\xi}^c. \quad (2.69)$$

This is true because under the locally flat $dC = 0$ condition, we had derived $(D_\mu^c D_\nu^c - D_\nu^c D_\mu^c) = 0$ in Eq. (2.32), furthermore

$$\begin{aligned} &\left(D_\mu^c (D_\nu^c D_\xi^c + D_\xi^c D_\nu^c) - D_\nu^c (D_\mu^c D_\xi^c + D_\xi^c D_\mu^c) \right) = (D_\mu^c D_\nu^c - D_\nu^c D_\mu^c) D_\xi^c + (D_\mu^c D_\xi^c D_\nu^c - D_\nu^c D_\xi^c D_\mu^c) \\ &= (D_\mu^c D_\xi^c D_\nu^c - D_\nu^c D_\xi^c D_\mu^c) = (\partial_\mu - ig_c C_\mu)(\partial_\xi - ig_c C_\xi)(\partial_\nu - ig_c C_\nu) - (\partial_\nu - ig_c C_\nu)(\partial_\xi - ig_c C_\xi)(\partial_\mu - ig_c C_\mu) \\ &= (\partial_\mu - ig_c C_\mu)(\partial_\xi \partial_\nu - ig_c C_\xi \partial_\nu - ig_c (\partial_\xi C_\nu) - ig_c C_\nu \partial_\xi - g_c^2 C_\xi C_\nu) - (\mu \leftrightarrow \nu) \\ &= -ig_c (\partial_\xi C_\nu)(\partial_\mu - ig_c C_\mu) - ig_c (\partial_\mu C_\xi)(\partial_\nu - ig_c C_\nu) - (\mu \leftrightarrow \nu) \\ &= ig_c ((dC)_{\nu\xi}(\partial_\mu - ig_c C_\mu) - (dC)_{\xi\mu}(\partial_\nu - ig_c C_\nu))|_{dC=0} = 0 \end{aligned} \quad (2.70)$$

The $(\mu \leftrightarrow \nu)$ are the term exchanging μ and ν respect to the previous term. The non-abelian field strength has firstly appeared in Ref. [4, 6]. The gauge-invariant non-abelian gauge field kinetic Lagrangian term corresponds to:

$$|\hat{F}_{\mu\nu\xi}^c|^2 := \hat{F}_{\mu\nu\xi}^c \hat{F}^{\dagger c\mu\nu\xi} \quad (2.71)$$

⁸Since $D_{i,j}^A[\{\Phi\}] \rightarrow e^{i2\eta(x)} D_{i,j}^A[\{\Phi\}]$ with a covariant factor of power 2 as $e^{i2\eta(x)}$, we may call this as ‘‘2-covariant’’ for the convenience.

2.2.4 Non-abelian $[U(1)_{x(d)} \times \mathbb{Z}_2^C]$ gauged matter: Polynomial invariant method

Follow the first approach in Sec. 2.1.4, we construct non-abelian gauged matter theory. By generalizing Eq. (2.58), we consider

$$\begin{aligned}
D_\mu^{c,\text{Im}} D_\nu^{c,\text{Im}} \log \Phi &= D_\mu^{c,\text{Im}} ((D_\nu^{c,\text{Im}} \Phi) \Phi^{-1}) = D_\mu^{c,\text{Im}} ((D_\nu^{c,\text{Im}} \Phi) \frac{\Phi^\dagger}{|\Phi|^2}) \\
&= \left(\frac{\Phi (D_\mu^{c,\text{Im}} D_\nu^{c,\text{Im}} \Phi)}{\Phi^2} \right) + (D_\nu^{c,\text{Im}} \Phi) (D_\mu^{c,\text{Im}} \Phi^{-1}) \\
&= \left(\frac{\Phi (D_\mu^{c,\text{Im}} D_\nu^{c,\text{Im}} \Phi)}{\Phi^2} \right) + (D_\nu^{c,\text{Im}} \Phi) (D_\mu^{c,\text{Im}} \frac{\Phi_{\text{Re}} - i \Phi_{\text{Im}}}{\Phi_{\text{Re}}^2 + \Phi_{\text{Im}}^2}) \\
&= \left(\frac{\Phi (D_\mu^{c,\text{Im}} D_\nu^{c,\text{Im}} \Phi)}{\Phi^2} \right) + \frac{-(D_\mu^{c,\text{Im}} \Phi) (D_\nu^{c,\text{Im}} \Phi) + g_c C_\mu \left(\frac{-2i \Phi_{\text{Im}}^2 \Phi}{|\Phi|^2} \right) (D_\nu^{c,\text{Im}} \Phi)}{\Phi^2}. \tag{2.72}
\end{aligned}$$

In Sec. 2.2.2, we had learned that for the abelian gauge sector, we require to introduce a symmetric tensor gauge field $A_{\mu,\nu}$ in order to cancel the gauge transformation $\partial_i \partial_j \eta$ between Eq. (2.63) and Eq. (2.64). Thus, we also symmetrize the above equation⁹ in order to naturally couple to a symmetric tensor gauge field later:

$$\frac{\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+}{2} \log \Phi = \frac{(\Phi \frac{\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+}{2} \Phi - D_\mu^{c,\text{Im}} \Phi D_\nu^{c,\text{Im}} \Phi + g_c \left(\frac{-i \Phi_{\text{Im}}^2}{\Phi^\dagger} \right) ((C_\mu D_\nu^{c,\text{Im}} + C_\nu D_\mu^{c,\text{Im}}) \Phi))}{\Phi^2}. \tag{2.74}$$

The $[U(1)_{x(d)} \times \mathbb{Z}_2^C]$ gauged transformations are:

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = e^{i\gamma_c(x)} A_{\mu\nu} + (\pm)^\times \frac{1}{2g} (D_\mu^c D_\nu^c + D_\nu^c D_\mu^c) (\eta_\nu(x)) = e^{i\gamma_c(x)} A_{\mu\nu} + (\pm)^\times \frac{1}{2g} (\{D_\mu^c, D_\nu^c\}_+ \eta_\nu(x)). \tag{2.75}$$

$$C_j \rightarrow C'_j = C_j + \frac{1}{g_c} \partial_j \gamma_c(x). \tag{2.76}$$

$$\Phi \rightarrow e^{(\pm)^\times i\eta(x)} \Phi_c = e^{(\pm)^\times i\eta(x)} (\Phi_{\text{Re}} + i e^{i\gamma_c(x)} \Phi_{\text{Im}}) = \begin{cases} e^{i\eta(x)} \Phi, & \text{if } \gamma_c \in \pi \mathbb{Z}_{\text{even}}, \\ e^{(\pm)^\times i\eta(x)} \Phi^\dagger, & \text{if } \gamma_c \in \pi \mathbb{Z}_{\text{odd}}. \end{cases} \tag{2.77}$$

Under the $[U(1)_{x(d)} \times \mathbb{Z}_2^C]$ gauged transformation, $\frac{\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+}{2} \log \Phi$ is not gauge invariant, thus the numerator in Eq. (2.74), $(\Phi \frac{\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+}{2} \Phi - D_\mu^{c,\text{Im}} \Phi D_\nu^{c,\text{Im}} \Phi + g_c \left(\frac{-i \Phi_{\text{Im}}^2}{\Phi^\dagger} \right) ((C_\mu D_\nu^{c,\text{Im}} + C_\nu D_\mu^{c,\text{Im}}) \Phi))$, is also not gauge-covariant. This result is what we should expect, because there is a variant term $i \frac{\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+}{2} \eta$ in this non-abelian theory, similar to the variant term $i \partial_i \partial_j \eta(x)$ appears in the abelian version of Eq. (2.63). However, the symmetric tensor gauge field can cancel such a gauge variant term exactly.

So we define a new non-abelian gauge-covariant 2-derivative for this non-abelian theory (generalizing the abelian case in Sec. 2.2.2):

$$D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi\}] := \left(\Phi \frac{\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+}{2} \Phi - D_\mu^{c,\text{Im}} \Phi D_\nu^{c,\text{Im}} \Phi + g_c \left(\frac{-i \Phi_{\text{Im}}^2}{\Phi^\dagger} \right) ((C_\mu D_\nu^{c,\text{Im}} + C_\nu D_\mu^{c,\text{Im}}) \Phi) - i g A_{\mu\nu} \Phi^2 \right). \tag{2.78}$$

⁹Below we use the notation $\{J_1, J_2\}_+ := J_1 J_2 + J_2 J_1$ to define the anti-commutator, e.g.

$$\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+ := D_\mu^{c,\text{Im}} D_\nu^{c,\text{Im}} + D_\nu^{c,\text{Im}} D_\mu^{c,\text{Im}}. \tag{2.73}$$

Similarly, we have:

$$\begin{aligned}
D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi^\dagger\}] &:= (\Phi^\dagger \frac{\{D_\mu^{c,\text{Im}\dagger}, D_\nu^{c,\text{Im}\dagger}\}_+}{2} \Phi^\dagger - D_\mu^{c,\text{Im}\dagger} \Phi^\dagger D_\nu^{c,\text{Im}\dagger} \Phi^\dagger + g_c (\frac{+i\Phi_{\text{Im}}^2}{\Phi}) ((C_\mu D_\nu^{c,\text{Im}\dagger} + C_\nu D_\mu^{c,\text{Im}\dagger}) \Phi^\dagger) + ig A_{\mu\nu} \Phi^{\dagger 2}), \\
D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi\}] &:= (\Phi \frac{\{D_\mu^{c,\text{Im}\dagger}, D_\nu^{c,\text{Im}\dagger}\}_+}{2} \Phi - D_\mu^{c,\text{Im}\dagger} \Phi D_\nu^{c,\text{Im}\dagger} \Phi + g_c (\frac{+i\Phi_{\text{Im}}^2}{\Phi^\dagger}) ((C_\mu D_\nu^{c,\text{Im}\dagger} + C_\nu D_\mu^{c,\text{Im}\dagger}) \Phi) - ig A_{\mu\nu} \Phi^2), \\
D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi^\dagger\}] &:= (\Phi^\dagger \frac{\{D_\mu^{c,\text{Im}}, D_\nu^{c,\text{Im}}\}_+}{2} \Phi^\dagger - D_\mu^{c,\text{Im}} \Phi^\dagger D_\nu^{c,\text{Im}} \Phi^\dagger + g_c (\frac{-i\Phi_{\text{Im}}^2}{\Phi}) ((C_\mu D_\nu^{c,\text{Im}} + C_\nu D_\mu^{c,\text{Im}}) \Phi^\dagger) + ig A_{\mu\nu} \Phi^{\dagger 2}). \quad (2.79)
\end{aligned}$$

Such that Eq. (2.78) to Eq. (2.79) are gauge 2-covariant (see footnote 8) or gauge 2-covariant to its complex conjugate field Φ^\dagger under the gauge transformations Eq. (2.75), Eq. (2.76) and Eq. (2.77):

$$D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi\}] \rightarrow \begin{cases} e^{i2\eta(x)} D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ e^{i(\pm)^\times 2\eta(x)} D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi^\dagger\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \quad (2.80)$$

$$D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi^\dagger\}] \rightarrow \begin{cases} e^{-i2\eta(x)} D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi^\dagger\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ e^{-i(\pm)^\times 2\eta(x)} D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \quad (2.81)$$

$$D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi\}] \rightarrow \begin{cases} e^{i2\eta(x)} D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ e^{i(\pm)^\times 2\eta(x)} D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi^\dagger\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \quad (2.82)$$

$$D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi^\dagger\}] \rightarrow \begin{cases} e^{-i2\eta(x)} D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi^\dagger\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{even}}, \\ e^{-i(\pm)^\times 2\eta(x)} D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi\}], & \text{if } \gamma_c \in \pi\mathbb{Z}_{\text{odd}}. \end{cases} \quad (2.83)$$

Follow [4, 6], we can consider the N -layers generalization of the theories with $(\mathbb{Z}_2^C)^N$ gauged, also by including the $[\text{U}(1)_{x(d)} \times \mathbb{Z}_2^C]$ -gauge kinetic term Eq. (2.71) and the level-2 BF theory into the gauge matter theory, again we are allowed to introduce the twisted cocycle $\omega_{d+1} \in \text{H}^{d+1}((\mathbb{Z}_2^C)^N, \mathbb{R}/\mathbb{Z})$ from a group cohomology data [21] to specify the interlayer interactions between N -layers. We can write down a schematic path integral (see also footnote 7):

$$\begin{aligned}
\mathbf{Z}_{\text{rk-3-NAb-}\Phi} &= \int \left(\prod_{I=1}^N [\mathcal{D}A_I] [\mathcal{D}B_I] [\mathcal{D}C_I] [\mathcal{D}\Phi_I] [\mathcal{D}\Phi_I^\dagger] \right) \exp(i \int_{M^{d+1}} \sum_{I=1}^N (d^{d+1}x (|\hat{F}_{\mu\nu\xi}^{c,I}|^2 \\
&+ D_{\mu,\nu}^{A,c,\text{Im}}[\{\Phi_I\}] D_{A,c,\text{Im}}^{\dagger\mu,\nu}[\{\Phi_I^\dagger\}] + D_{\mu,\nu}^{\dagger A,c,\text{Im}}[\{\Phi_I\}] D_{A,c,\text{Im}}^{\mu,\nu}[\{\Phi_I^\dagger\}] + V(\{|\Phi_I|^2\}) + \frac{2}{2\pi} B_I dC_I)) \cdot \omega_{d+1}(\{C_I\}). \quad (2.84)
\end{aligned}$$

The new ingredient in our present work beyond the previous Ref. [4, 6] is that now the matter fields directly interact with non-abelian gauge fields.

2.3 Degree-(m-1) polynomial symmetry to non-abelian higher-rank tensor gauged matter theory

In this subsection, we outline a generalization of previous Sec. 2.1 and Sec. 2.2 to a general degree-(m-1) polynomial symmetry and by gauging it and the particle-hole \mathbb{Z}_2^C symmetry to obtain a non-abelian higher-rank tensor gauged matter field theory.

2.3.1 Global symmetry: $\prod_{M=1}^{m-1} \text{U}(1)_{x \binom{M}{m}}$

Follow Ref. [6], a degree (m-1)-polynomial symmetry acts on the complex scalar $\Phi(x) \in C$:

$$\Phi \rightarrow e^{iQ(x)} \Phi = e^{i(\Lambda_{i_1, \dots, i_{m-1}} x_{i_1} \dots x_{i_{m-1}} + \dots + \Lambda_{i,j} x_i x_j + \Lambda_i x_i + \Lambda_0)} \Phi. \quad (2.85)$$

Different Λ_{\dots} introduce degree U(1) degrees of freedom. Different U(1) symmetries for different degrees and different Λ_{\dots} commute. We denote such a degree (m-1)-polynomial symmetry structure with several U(1) symmetry groups as¹⁰

$$\boxed{U(1)_{x_{\binom{n}{m-1}}} \times \cdots \times U(1)_{x_{(n)}} \times U(1) = \prod_{M=1}^{m-1} U(1)_{x_{\binom{n}{M}}}}. \quad (2.86)$$

The sub-indices of U(1) specifies which Λ_{\dots} degree of freedom contributes such a U(1). Note that $0 \leq M \leq m-1$. For example, for the degree-1 polynomial symmetry with different Λ_j , we denote

$$U(1)_{x_{(n)}} := \prod_{j=1}^n U(1)_{x_j},$$

each for different Λ_j . In general, we denote

$$U(1)_{x_{\binom{n}{M}}} := \prod_{\{j_1, \dots, j_M\}} U(1)_{x_{j_1, \dots, j_M}}, \quad (2.87)$$

each for different $\Lambda_{j_1, \dots, j_M}$.

2.3.2 \mathbb{Z}_2^C charge-conjugation (particle-hole) symmetry

In addition to the polynomial symmetry in Sec. 2.3.1, as noticed in [4, 6], we have a \mathbb{Z}_2^C charge-conjugation (particle-hole) symmetry. It acts on the complex scalar Φ switching from a particle to an anti-particle. The \mathbb{Z}_2^C symmetry persists even after we gauge the abelian polynomial-symmetry, which also acts on the rank-m abelian symmetric tensor A_{i_1, \dots, i_m} and the gauge parameter $\eta_v(x)$ for $\Phi \rightarrow e^{i\eta_v} \Phi$:

$$\begin{aligned} \Phi &\mapsto \Phi^\dagger, \\ A_{i_1, \dots, i_m} &\mapsto -A_{i_1, \dots, i_m}, \\ \eta_v(x) &\mapsto -\eta_v(x). \end{aligned}$$

The U(1) degree (m-1)-polynomial symmetry does not commute with \mathbb{Z}_2^C symmetry. Abbreviate Eq. (2.86)'s $\prod_{M=1}^{m-1} U(1)_{x_{\binom{n}{M}}}$ as $U(1)_{\text{poly}}$ symmetry

$$\begin{aligned} U_{\mathbb{Z}_2^C} U_{U(1)_{\text{poly}}} \Phi &= U_{\mathbb{Z}_2^C} (e^{iQ(x)} \Phi) = e^{iQ(x)} \Phi^\dagger, \\ U_{U(1)_{\text{poly}}} U_{\mathbb{Z}_2^C} \Phi &= U_{U(1)_{\text{poly}}} (\Phi^\dagger) = e^{-iQ(x)} \Phi^\dagger. \end{aligned} \quad (2.88)$$

So we have indeed a non-abelian/non-commutative global symmetry structure:

$$\boxed{\left(\prod_{M=1}^m U(1)_{x_{\binom{n}{M}}} \right) \rtimes \mathbb{Z}_2^C := U_{U(1)_{\text{poly}}} \rtimes \mathbb{Z}_2^C}. \quad (2.89)$$

¹⁰Each of U(1) factors represents a U(1) group. However the product of these U(1) act differently: the left most U(1) acts globally as 0-form symmetry, the left second most $U(1)_{x_{(n)}}$ acts a vector global symmetry, etc. Thus we call it a global symmetry structure (instead of a global symmetry group), because each U(1) has different physical meanings associated to space(/time) coordinates. Furthermore, we also refer to its gauging as a *gauged structure* (not necessarily the same as the conventional *gauge group*).

2.3.3 Non-Abelian/non-commutative gauge structure: $U(1)_{x \binom{m-1}{n}} \times \mathbb{Z}_2^C$

Even after we gauge the $U(1)$ polynomial symmetry, we can still observe the gauge transformation of $[U_{U(1)_{\text{poly}}}]$ does not commute with the \mathbb{Z}_2^C global symmetry transformation, which both can act on the Φ and the rank- m symmetric tensor A respectively:

$$\begin{aligned} U_{\mathbb{Z}_2^C}[U_{U(1)_{\text{poly}}}] \Phi &= U_{\mathbb{Z}_2^C}(e^{i\eta v} \Phi) = e^{i\eta v} \Phi^\dagger. \\ [U_{U(1)_{\text{poly}}}] U_{\mathbb{Z}_2^C} \Phi &= U_{\mathbb{Z}_2^C}(\Phi^\dagger) = e^{-i\eta v} \Phi^\dagger. \\ U_{\mathbb{Z}_2^C}[U_{U(1)_{\text{poly}}}] A_{i_1, \dots, i_m} &= U_{\mathbb{Z}_2^C}(A_{i_1, \dots, i_m} + \frac{1}{g} \partial_\mu \partial_\nu \eta v) = -A_{i_1, \dots, i_m} + \frac{1}{g} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-1}} \partial_{i_m} \eta v. \\ [U_{U(1)_{\text{poly}}}] U_{\mathbb{Z}_2^C} A_{i_1, \dots, i_m} &= U_{\mathbb{Z}_2^C}(-A_{i_1, \dots, i_m}) = -A_{i_1, \dots, i_m} - \frac{1}{g} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-1}} \partial_{i_m} \eta v. \end{aligned}$$

By gauging the degree- $(m-1)$ polynomial symmetry and keep *only* the rank- m symmetric tensor A_{i_1, \dots, i_m} , we are left with a non-abelian/non-commutative gauge structure

$$\boxed{U(1)_{x \binom{m-1}{n}} \times \mathbb{Z}_2^C}. \quad (2.90)$$

2.3.4 Non-abelian $[U(1)_{x \binom{m-1}{n}} \times \mathbb{Z}_2^C]$ gauged matter: Polynomial invariant method

We propose a polynomial invariant method to generalize the procedure of Sec. 2.2.4 from a degree-1 polynomial to a generic degree (here a degree- $(m-1)$ polynomial). First, we determine,

$$D_{i_1}^{c, \text{Im}} D_{i_2}^{c, \text{Im}} \cdots D_{i_m}^{c, \text{Im}} \log \Phi := \frac{P_{i_1, \dots, i_m}^c(\Phi, \dots, (D^{c, \text{Im}})^m \Phi)}{\Phi^m}. \quad (2.91)$$

For example, the degree-1 case is obtained in Eq. (2.72). The degree-2 case is obtained in Ref. [6].¹¹ The functional P^c should be a generalization of the result obtained in Ref. [6]. The derivative $(D^{c, \text{Im}})$ involves the coupling to a 1-form C gauge field. Moreover, when we turn off the C gauge field, we reduce $D_{i_1}^{c, \text{Im}} D_{i_2}^{c, \text{Im}} \cdots D_{i_m}^{c, \text{Im}} \log \Phi$ to a previous formula obtained in Ref. [6]:

$$\partial_{i_1} \cdots \partial_{i_m} \log \Phi := \frac{P_{i_1, \dots, i_m}(\Phi, \dots, \partial^m \Phi)}{\Phi^m}.$$

¹¹For a degree-2 polynomial symmetry: $\Phi \rightarrow e^{iQ(x)} \Phi = e^{i(\Lambda_{i,j} x_i x_j + \Lambda_i x_i + \Lambda_0)} \Phi$, we construct a covariant 3-derivative (triple-derivative) below. First, $\log \Phi \rightarrow \log \Phi + iQ(x) = \log \Phi + i(\Lambda_{i,j} x_i x_j + \Lambda_i x_i + \Lambda_0)$. we take $\partial_{x_i} \partial_{x_j} \partial_{x_k} := \partial_i \partial_j \partial_k$ on both sides $\partial_i \partial_j \partial_k \log \Phi = \frac{P_{i,j,k}(\Phi, \dots, \partial^3 \Phi)}{\Phi^3} \rightarrow \partial_i \partial_j \partial_k \log \Phi$, which is globally invariant under the degree-2 polynomial symmetry:

$$\frac{P_{i,j,k}(\Phi, \dots, \partial^3 \Phi)}{\Phi^3} = \frac{\Phi^2 (\partial_i \partial_j \partial_k \Phi) - 3\Phi (\partial_k \Phi \partial_i \partial_j \Phi) + 2(\partial_i \Phi)(\partial_j \Phi)(\partial_k \Phi)}{\Phi^3}.$$

We use the symmetrized tensor notation: $T_{(i_1 i_2 \dots i_k)} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{i_{\sigma_1} i_{\sigma_2} \dots i_{\sigma_k}}$, with parentheses (ijk) around the indices being symmetrized. The S_k is the symmetric group of k elements. Since the denominator $\Phi^3 \rightarrow e^{i3Q(x)} \Phi^3$, so does the numerator $P_{i,j,k}(\Phi, \dots, \partial^3 \Phi) \rightarrow e^{i3Q(x)} P_{i,j,k}(\Phi, \dots, \partial^3 \Phi)$, which we call the denominator and numerator are 3-covariant. Lagrangian thus contains $|P_{i,j,k}|^2 := P_{i,j,k}(\Phi) P^{i,j,k}(\Phi^\dagger)$ [6].

A generic $U(1)_{x^{\binom{m-1}{n}} \times \mathbb{Z}_2^C}$ gauge transformation contains,

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = e^{i\gamma_c(x)} A_{\mu\nu} + (\pm)^\times \frac{1}{(m!)g} (D_{i_1}^c D_{i_2}^c \dots D_{i_m}^c)(\eta_\nu(x)). \quad (2.92)$$

$$C_j \rightarrow C'_j = C_j + \frac{1}{g_c} \partial_j \gamma_c(x). \quad (2.93)$$

$$\Phi \rightarrow e^{(\pm)^\times i\eta(x)} \Phi_c = e^{(\pm)^\times i\eta(x)} (\Phi_{\text{Re}} + i e^{i\gamma_c(x)} \Phi_{\text{Im}}) = \begin{cases} e^{i\eta(x)} \Phi, & \text{if } \gamma_c \in \pi \mathbb{Z}_{\text{even}}, \\ e^{(\pm)^\times i\eta(x)} \Phi^\dagger, & \text{if } \gamma_c \in \pi \mathbb{Z}_{\text{odd}}. \end{cases} \quad (2.94)$$

Here $(D_{i_1}^c D_{i_2}^c \dots D_{i_m}^c) := (D_{i_1}^c D_{i_2}^c \dots D_{i_m}^c + D_{i_2}^c D_{i_1}^c \dots D_{i_m}^c + \dots)$ yields a symmetrization over the subindices under the lower bracket (i_1, \dots, i_m) , the permutation $(m!)$ -terms. The $P_{i_1, \dots, i_m}^c(\Phi, \dots, (D^{c, \text{Im}})^m \Phi)$ is not gauge covariant under the generic gauge transformation. But we can append the A gauge field to make it gauge covariant:

$$D_{i_1, \dots, i_m}^{A, c, \text{Im}}[\{\Phi\}] := P_{i_1, \dots, i_m}^c(\Phi, \dots, (D^{c, \text{Im}})^m \Phi) - i g A_{i_1, \dots, i_m} \Phi^m, \quad (2.95)$$

where we implicitly sum over all possible indices as $\sum_{\{i_1, \dots, i_m\}}$ over both the left and right hand sides. The special case when $m = 1$ is given in Eq. (3.14) and $m = 2$ is given in Eq. (2.78). The non-abelian gauge covariant rank- $(m+1)$ field strength is already obtained and defined in Ref. [6]:

$$\hat{F}_{\mu, \nu, i_2, \dots, i_m}^c := D_\mu^c A_{\nu, i_2, \dots, i_m} - D_\nu^c A_{\mu, i_2, \dots, i_m} := (\partial_\mu - i g_c C_\mu) A_{\nu, i_2, \dots, i_m} - (\partial_\nu - i g_c C_\nu) A_{\mu, i_2, \dots, i_m}$$

We can include the ingredients of non-abelian gauge theory coupling to the newly obtained gauged matter sectors (see footnote 7)

$$\begin{aligned} \mathbf{Z}_{\text{rk}-(m+1)\text{-NAb-}\Phi} &= \int \left(\prod_{I=1}^N [\mathcal{D}A_I] [\mathcal{D}B_I] [\mathcal{D}C_I] [\mathcal{D}\Phi_I] [\mathcal{D}\Phi_I^\dagger] \right) \exp\left(i \int_{M^{d+1}} \sum_{I=1}^N \left(d^{d+1}x (|\hat{F}_{\mu, \nu, i_2, \dots, i_m}^{c, I}|^2 \right. \right. \\ &+ D_{i_1, \dots, i_m}^{A, c, \text{Im}}[\{\Phi_I\}] D_{A, c, \text{Im}}^{\dagger i_1, \dots, i_m}[\{\Phi_I^\dagger\}] + D_{i_1, \dots, i_m}^{\dagger A, c, \text{Im}}[\{\Phi_I\}] D_{A, c, \text{Im}}^{i_1, \dots, i_m}[\{\Phi_I^\dagger\}] + V(|\Phi_I|^2) \left. \left. \right) + \frac{2}{2\pi} B_I dC_I \right) \cdot \omega_{d+1}(\{C_I\}), \end{aligned} \quad (2.96)$$

so we derive a new non-abelian gauged matter field theory. The setup and notations are directly generalized from Sec. 2.2.4.

3 New Sigma Models in a Family and Two Types of Vortices

Section 2 proposes a family of non-abelian gauged matter field theories. In this section, we study the ‘‘dualized’’ theory – instead of using the matter field degrees of freedom, we try to incorporate the vortex degrees of freedom into the field theory.

To start with, there are at least two types of vortex degrees of freedom that we can identify.

1. The complex scalar matter field can be written as:

$$\Phi(x) = \sqrt{\rho(x)} \exp(i\phi(x)) \in \mathbb{C}, \quad (3.1)$$

$$\rho(x) \in \mathbb{R}_{\geq 0}, \quad (3.2)$$

$$\phi(x) \in [0, 2\pi) + 2\pi\mathbb{Z}. \quad (3.3)$$

So we can Eq. (2.96) replace the path integral measure $\int [\mathcal{D}\Phi_I] [\mathcal{D}\Phi_I^\dagger]$ to $\int [\mathcal{D}\rho] [\mathcal{D}\phi]$ (up to some phase space volume factor), also substitute $\Phi_I = \sqrt{\rho_I} \exp(i\phi_I)$ and $\Phi_I^\dagger = \sqrt{\rho_I} \exp(-i\phi_I)$.¹² The $\mathbb{R}_{\geq 0}$ takes the non-negative real values. When there are N layers, $I = 1, \dots, N$, there are N flavors of vortex fields ϕ_I .

¹²We will capture the integer non-smooth singular part of $\frac{1}{2\pi} d\phi = n \in \mathbb{Z}$ via Cauchy-Riemann relation, the winding number and topological degree theory in Sec. 4.

2. The anti-symmetric tensor TQFT sector (the level-2 BF theory as a \mathbb{Z}_2 gauge theory twisted by Dijkgraaf-Witten group cohomology topological terms)

$$\int \prod_{I=1}^N [\mathcal{D}B_I][\mathcal{D}C_I] \exp(i \int_{M^{d+1}} (\sum_{I=1}^N \frac{2}{2\pi} B_I dC_I)) \cdot \omega_{d+1}(\{C_I\})$$

can also be regarded as the *disordered phase* of a sigma model given by another scalar field θ_I . The derivations of sigma models governing the ordered-disorder phases relevant for twisted Dijkgraaf-Witten type TQFTs ¹³ had been studied in [32, 33, 35], here we will implement the procedure done in [32]. We write

$$\theta_I = \theta_{s,I} + \theta_{v,I}, \quad (3.10)$$

¹³ To recall the approach of Ref. [32, 33] and their generalization:

- The ordered phase of sigma models describes the weak fluctuations around the symmetry-breaking phases.
- The disorder phases of sigma models describes the strong fluctuations around the symmetry-restored phases as continuum formulations of TQFTs, SETs or SPTs of Dijkgraaf-Witten type.

The U(1) spontaneously symmetry breaking phase has a superfluid ground state, which is an ordered phase respect to ϕ with an order parameter

$$\langle \exp(i\theta) \rangle \neq 0.$$

It is well-known that if we disorder the U(1) spontaneously symmetry breaking (superfluid) state, we can obtain an disordered phase known as a gapped insulator [27–29]. Our approach is basically along this logical thinking, except that we generalize the approach by:

- Disorder the ordered phase (U(1) symmetry breaking superfluid) to a disordered phase of gapped topological order (e.g. the \mathbb{Z}_N -gauge theory, where the \mathbb{Z}_1 -gauge theory means a trivial gapped insulator, and the \mathbb{Z}_2 -gauge theory means the low energy theory of deconfined \mathbb{Z}_2 -toric code, \mathbb{Z}_2 -spin liquid or \mathbb{Z}_2 -superconductor). (Beware that Ref. [32, 33] only consider the case of a superfluid-insulator transition for $N = 1$, here we consider a superfluid-topological-order transition for a generic N .)
- Follow [32, 33], introducing additional topological multi-kink Berry phase specified by the cocycle of cohomology group

$$\omega_{d+1}(\{C_I\}) \simeq \omega_{d+1}(\{d\theta_I\})$$

to the superfluid.

To comprehend our formalism, here we overview this approach using field theory [32]. We start from the superfluid state in a d -spacetime dimension described by a bosonic U(1) quantum phase θ kinetic term and a superfluid compressibility coefficient χ , the partition function \mathbf{Z} is:

$$\mathbf{Z} = \int [\mathcal{D}\theta] \exp(- \int d^d x \frac{\chi}{2} (\partial_\mu \theta_s + \partial_\mu \theta_v)^2). \quad (3.4)$$

The $\theta = \theta_s + \theta_v$ with a smooth piece θ_s and a singular vortex piece θ_v for the bosonic phase θ . We emphasize that the θ_v is essential to capture the vortex core, see Sec. 4. We introduce an auxiliary field j^μ and apply the Hubbard-Stratonovich technique [34],

$$\mathbf{Z} = \int [\mathcal{D}\theta][\mathcal{D}j^\mu] \exp(- \int d^d x \frac{1}{2\chi} (j_t^\mu)^2 - i j^\mu (\partial_\mu \theta_s + \partial_\mu \theta_v)). \quad (3.5)$$

By integrating out the smooth part $\int [D\theta_s]$, we obtain a constraint $\delta(\partial_\mu j^\mu)$ into the path integral measure. Naively, in the anti-symmetric tensor differential form notation, the constraint seems in disguise

$$d(\star j) = 0, \quad \text{or,} \quad 2\pi \oint d(\star j) = \mathbb{Z},$$

and the solution in disguise is $j = \frac{1}{2\pi}(\star dB)$. (Here we choose a normalization convention.) However, we imagine the procedure is the *N-fold vortex of superfluid becomes a trivial object (instead of a 1-fold vortex of superfluid)* that can be created or annihilated for free from the \mathbb{Z}_N -gauge theory vacuum. Instead we may impose a revised constraint

$$2\pi \oint d(\star j) = N\mathbb{Z}. \quad (3.6)$$

Note that the $2\pi N$ on the right hand side means that N -fold of 2π vortices become to be identified as a trivial zero vortex (none vortex). The solution is, with \star the Hodge star,

$$j = \frac{N}{2\pi}(\star dB). \quad (3.7)$$

where $\theta_{s,I}$ describes the smooth (s) part while the $\theta_{v,I}$ describes the singular vortex (v) part. See the footnote 13 and [32], the exterior derivative of the vortex field should be identified as the 1-form C gauge field as:

$$d\theta_v = C. \quad (3.11)$$

In this way, the TQFT sector of the theory (as a disordered phase of some sigma model) can be re-written as a sigma model with the vortex field θ_v degree of freedom:

$$\int \left(\prod_{I=1}^N [\mathcal{D}B_I][\mathcal{D}\theta_{s,I}][\mathcal{D}\theta_{v,I}] \exp(i \int_{M^{d+1}} \left(i \left(\sum_{I=1}^N \frac{\chi_I}{2} (d\theta_{s,I} + d\theta_{v,I}) \wedge \star(d\theta_{s,I} + d\theta_{v,I}) \right. \right. \right. \\ \left. \left. \left. + \frac{2}{2\pi} \sum_{I=1}^N B_I d(d\theta_{v,I}) \right) \right) \cdot \omega_{d+1}(\{d\theta_{v,I}\}). \quad (3.12)$$

where $\omega_{d+1}(\{d\theta_{v,I}\})$ is mapped to a multi-kink Berry phase topological term $\exp(i \int_{M^{d+1}} \#(d\theta_{v,1}) \wedge \cdots \wedge (d\theta_{v,N}))$ [32, 33].

We replace and redefine a new derivative on the right hand side by substituting $C = d\theta_v$ (or $C_\mu = \partial_\mu \theta_v$).¹⁴

$$D_\mu^{c, \text{Im}} \Phi \rightarrow D_\mu^{d\theta_v, \text{Im}} \Phi := \partial_\mu \Phi - i g_c (\partial_\mu \theta_v) \Phi_{\text{Im}}, \quad (3.13)$$

We can define a generic form $j^\mu = \frac{N}{2\pi(d-2)!} \epsilon^{\mu\mu_2 \cdots \mu_d} \partial_{\mu_2} B_{\mu_3 \cdots \mu_d}$, with an anti-symmetric tensor B with a total spacetime dimension d (most conveniently, we may consider 2d space or 2+1d spacetime in order to implement a winding number in Sec. 4), to satisfy this constraint. To disorder the superfluid, we have to make the θ_v -angle strongly fluctuates — namely we should take the $\chi \rightarrow \infty$ limit to achieve large $|\delta\theta_v|^2 \gg 1$, the disordered limit of superfluid. Plug in Eq. (3.7), the partition function becomes:

$$\mathbf{Z} = \int [\mathcal{D}\theta_v][\mathcal{D}B] \exp(+ \int i \frac{N}{2\pi} B \wedge (d^2\theta_v)).$$

Hereafter we may compensate the dropped \pm -sign by a field-redefinition. Although naively $d^2 = 0$, due to the singularity core of θ_v , the $(d^2\theta_v)$ can be nonzero, see Sec. 4, which implies that (at least for the 2-dimensional space mapping to a deformed S^1 -circle as a target space):

$$\frac{1}{2\pi} d^2\theta_v = n \pmod{N}, \text{ thus } n \in \mathbb{Z}_N. \quad (3.8)$$

Thus, $(d^2\theta_v)$ describes the vortex core density and the vortex current, which we denote

$$\frac{1}{2\pi} d^2\theta_v = \star j_{\text{vortex}}.$$

In addition, Noether theorem leads to the conservation of the vortex current: the continuity equation

$$d \star j_{\text{vortex}} = 0,$$

this implies that

$$\star j_{\text{vortex}} = dC/(2\pi)$$

for some 1-form gauge field C . We can thus define the singular part of bosonic phase

$$d\theta_v = C$$

as a 1-form gauge field, to describe the vortex core. The partition function in the disordered state away from the superfluid, now becomes that of an gapped insulator (for $N = 1$) or topologically ordered state with a topological level- N BF action as a \mathbb{Z}_N -gauge theory:

$$\mathbf{Z} = \int [\mathcal{D}b][\mathcal{D}a] \exp\left(\frac{i}{2\pi} \int B \wedge dC\right) = \int [\mathcal{D}b][\mathcal{D}a] \exp\left(i \int \frac{d^d x}{2\pi(d-2)!} \epsilon^{\mu\mu_2 \cdots \mu_d} B_{\mu\mu_2\mu_3 \cdots} \partial_{\mu_{d-1}} C_{\mu_d}\right). \quad (3.9)$$

¹⁴We also have the gauge transformation descended from 1-form C gauge field,

$$\partial_\nu \theta_{v,I} \rightarrow \partial_\nu \theta_{v,I} + \frac{1}{g_c} \partial_\nu \gamma_{c,I}(x),$$

$$D_\mu^{A,c,\text{Im}}\Phi \rightarrow D_\mu^{A,d\theta_v,\text{Im}}\Phi := D_\mu^A\Phi - ig_c(\partial_\mu\theta_v)\Phi_{\text{Im}} = (\partial_\mu - igA_\mu)\Phi - ig_c(\partial_\mu\theta_v)\Phi_{\text{Im}}. \quad (3.14)$$

$$D_\mu^c A_{\nu,i_2,\dots,i_m} \rightarrow D_\mu^{d\theta_v} A_{\nu,i_2,\dots,i_m} := (\partial_\mu - ig_c(\partial_\mu\theta_v))A_{\nu,i_2,\dots,i_m}. \quad (3.15)$$

$$\hat{F}_{\mu,\nu,i_2,\dots,i_m}^c \rightarrow \hat{F}_{\mu,\nu,i_2,\dots,i_m}^{d\theta_v} := D_\mu^{d\theta_v} A_{\nu,i_2,\dots,i_m} - D_\nu^{d\theta_v} A_{\mu,i_2,\dots,i_m}. \quad (3.16)$$

We can either include or omit the I index for these operators.

By combining two kinds of vortex degrees of freedom from the vortex 1 of ϕ and the vortex 2 of θ , thus we can rewrite Eq. (2.96) into a sigma model-like expression for a non-abelian gauged fractonic matter theory:

$$\begin{aligned} \mathbf{Z}_{\text{rk}-(m+1)\text{-NAb-vortex}}^{\text{Sigma model}} &= \int \left(\prod_{I=1}^N [\mathcal{D}A_I][\mathcal{D}B_I][\mathcal{D}\theta_{s,I}][\mathcal{D}\theta_{v,I}][\mathcal{D}\rho_I][\mathcal{D}\phi_I] \right) \\ \exp(i \int_{M^{d+1}} \sum_{I=1}^N &\left(d^{d+1}x (|\hat{F}_{\mu,\nu,i_2,\dots,i_m}^{d\theta_v,I}|^2 + \sum_{I=1}^N \frac{\chi_I}{2} |d\theta_{s,I} + d\theta_{v,I}|^2 \right. \\ &+ D_{i_1,\dots,i_m}^{A,d\theta_v,\text{Im}}[\{\sqrt{\rho_I} \exp(i\phi_I)\}] D_{A,d\theta_v,\text{Im}}^{\dagger i_1,\dots,i_m}[\{\sqrt{\rho_I} \exp(-i\phi_I)\}] + D_{i_1,\dots,i_m}^{\dagger A,d\theta_v,\text{Im}}[\{\sqrt{\rho_I} \exp(i\phi_I)\}] D_{A,d\theta_v,\text{Im}}^{i_1,\dots,i_m}[\{\sqrt{\rho_I} \exp(-i\phi_I)\}] \\ &\left. + V(\{|\rho_I|\}) + \frac{2}{2\pi} B_I d(d\theta_{v,I}) \right) \cdot \omega_{d+1}(\{d\theta_{v,I}\}). \end{aligned} \quad (3.17)$$

Here we have substituted Eq. (3.1), Eq. (3.11) and $|\Phi_I|^2 = |\rho_I|$. This Eq. (3.17) is the most generic form of sigma model – a part of its phase diagram gives rise to the TQFT (when the $\theta_{v,I}$ vortices disordered), while the other part of its phase diagram gives rise to the spontaneously symmetry breaking superfluid like phases. We emphasize that $d(d\theta_{v,I}) = d^2\theta_{v,I}$ is nonzero and can be related to a quantized number such as a winding number at the core of the vortex field, see Sec. 4.

Here B is only a Lagrange multiplier. Here we also have not yet replaced the symmetric tensor gauge field A to any kinds of vortex degrees of freedom in Eq. (3.17). As some of the readers may wonder, and it is tempting to ask this question: whether the gauge field A can be “dualized” into some new vortex degrees of freedom. However, we will not attempt to attack this issue and leave this as an open question for future work.

4 Cauchy-Riemann Relation, Winding Number, and Topological Degree Theory

Here we derive a relation used in the previous section, relating the vortex degrees of freedom to a winding number, via the Cauchy-Riemann relation and topological degree theory, at least for the 2-dimensional space mapping to a deformed S^1 -circle as a target space. On the complex plane

$$z := x + iy = r \exp(i\varphi), \quad (4.1)$$

we define the Hodge star operator \star (for the differential form, this is the Hodge dual) on the 1-form as

$$\star(fdx + gdy) = (-gdx + fdy), \quad (4.2)$$

for some generic functions f and g .

$$\theta_{v,I} \rightarrow \theta_{v,I} + \frac{1}{g_c} \gamma_{c,I}(x).$$

Then we have $\star dz = -i dz$ with $dz = dx + i dy$. We can compute that

$$d \star df = (\Delta f)(dx \wedge dy), \quad (4.3)$$

with the Laplacian or Laplace operator $\Delta \equiv \nabla^2$.

If $f = u + iv$ is holomorphic dependent on z independent of \bar{z} (namely the Cauchy-Riemann equation), $df = f' dz$ and also $i \star df = f' dz$ with $f' = \frac{df}{dz}$, so we see that $dv = \star du$.

Take $f = \log z = \log(r \exp(i\varphi)) = \log r + i\varphi$, then $u = \log r$ and $v = \varphi$, where $z = r e^{i\varphi}$ is the polar coordinate, then we have

$$dd\varphi = ddv = d \star du = (\Delta \log r)(dx \wedge dy) = 2\pi\delta_0 \quad (4.4)$$

where

$$\delta_0 \equiv \delta(\mathbf{r})(dx \wedge dy)$$

is the delta function at the origin 0 of the polar coordinate.¹⁵ Hence we derive that

$$\frac{1}{2\pi} dd\varphi = \delta_0. \quad (4.6)$$

For a general S^1 valued function ϕ defined outside a singular point p , we may assume $p = 0$ at the origin (without loss of generality).

Applications to two types of vortices:

1. For the fractonic phase field ϕ : We can always write the phase of the fractonic matter field in Eq. (3.1) as

$$\phi := \phi_s + \phi_v := \phi_s + n\varphi \quad (4.7)$$

where ϕ_s captures the smooth (s) part and ϕ_v captures the singular vortex (v) part, while n is the winding number. In terms of the degree theory, we only focus on $\Sigma^2 \rightarrow S^1$, specifically here we consider $\Sigma^2 = \mathbb{R}^2 - \{0\}$, as a punctured 2-plane mapping to a circle S^1 (or the U(1) target space). We can possibly generalize this result to other target spaces. The first term ϕ_s extends smoothly over 0 is known as the smooth fluctuation. Then using the previous result we see that

$$\frac{1}{2\pi} dd\phi = \frac{1}{2\pi} dd(\phi_s + \phi_v) = \frac{1}{2\pi} dd(\phi_v) = n \frac{1}{2\pi} dd(\varphi) = n \delta_0, \text{ with } n \in \mathbb{Z}. \quad (4.8)$$

Thus importantly, we can identify the solution of vortex core equation $\frac{1}{2\pi} d^2\phi_v = n \delta_0$ with $n \in \mathbb{Z}$ as the winding number.

¹⁵Here we use the facts about the fundamental solutions of Laplace's or Poisson's equations by solving the source of Dirac delta function $\delta(r)$, see for example [36]. The volume of D -dimensional ball B^D is $V_D \equiv \frac{\pi^{D/2}}{\Gamma(D/2+1)}$, where the gamma function obeys $\Gamma(n+1) = n\Gamma(n)$ for general $n \in \mathbb{C}$, $\Gamma(1/2) = \sqrt{\pi}$, while $\Gamma(n) = (n-1)!$ for any positive integer n . The area of the boundary of the D -dimensional ball B^D is the hypervolume of the $(D-1)$ -dimensional sphere, denoted as $A_{B^D} \equiv A_{S^{D-1}} = DV_D \equiv \frac{D\pi^{D/2}}{\Gamma(D/2+1)} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$. Given the vector $\mathbf{r} \in \mathbb{R}^D$ and the radial $r = |\mathbf{r}|$, then we have the following properties for Laplacian $\Delta = \nabla^2$:

$$\begin{cases} D = 2, & \nabla^2 \left(\ln(r) \right) = 2\pi\delta^{(2)}(\mathbf{r}). \\ D = 3, & \nabla^2 \left(r^{-1} \right) = -4\pi\delta^{(3)}(\mathbf{r}). \\ \dots \\ D \geq 3, & \nabla^2 \left(r^{-(D-2)} \right) = -(D-2)A_{S^{D-1}}\delta^{(D)}(\mathbf{r}). \end{cases} \quad (4.5)$$

In general $\delta_0 \equiv \delta(\mathbf{r})(\text{vol}_D) = \delta(\mathbf{r})(dx_1 \wedge \dots \wedge dx_D)$.

2. For the \mathbb{Z}_2^C -gauge phase field θ : Similarly, for the vortex associated to the C gauge field,

$$\frac{1}{2\pi} dd\theta = \frac{1}{2\pi} dd(\theta_s + \theta_v) = \frac{1}{2\pi} dd(\theta_v) = n_{\theta_v} \delta_0, \text{ with } n_{\theta_v} \in \mathbb{Z}. \quad (4.9)$$

So if the Cauchy-Riemann relation can be applied, we can rewrite:

$$\frac{2}{2\pi} B_I d(d\theta_{v,I}) = B_I n_{\theta_v} \delta_0$$

to a term associated to the winding number n_{θ_v} . We can adjust δ_0 from the location of the origin to other points p as δ_p for all the above discussions to indicate the locations of vortices at points p .

5 Conclusions

We have proposed a systematic framework to obtain a family of non-abelian gauged fractonic matter field theories in Sec. 2. We have derived a new family of Sigma models with two types of vortices in Sec. 3 that can interplay and transient between the disordered phase (with higher-rank tensor non-abelian gauge theories coupled to fractonic matter) and ordered phase (with superfluids and vortex excitations) of Sigma models. We formulate two types of vortices, one is associated to the fractonic matter fields ($d(d\phi)$), the other is associated to the 1-form C gauge field ($d(d\theta_v)$). These two types of vortices mutually interact non-commutatively when they communicate via the higher-rank tensor gauge field A as a propagator. We apply the Cauchy-Riemann relation and topological degree theory to capture the winding number for the two types of vortices in Sec. 4.

Here we make some extended comments and also list down some open questions.

1. *Reduction to the abelian case*: When the 1-form gauge field $C = 0$, we reduce to the abelian tensor gauged matter theory. If we further turn off the tensor gauge field A , then our sigma model should reduce to a simplified special case of the abelian fractonic superfluid models in Ref. [30]. When we turn on the gauge field C , there are nontrivial couplings between matter sectors and anti-matter sectors, because the particle-hole symmetries are dynamically gauged.
2. Our Sigma model Eq. (3.17) contains the target space with radius size $\sqrt{\rho}$.
If the $\sqrt{\rho}$ is fixed, then we have a fixed radius S^1 target space, we only have the fluctuation around the ϕ fields (e.g. Goldstone modes, superfluid, or vortices).
If the $\sqrt{\rho}$ also fluctuates, the volume and radius size of Sigma model target space change, so we have more interesting dynamics for the Sigma model. Its dynamics and low energy fates are interesting, but perplexing and challenging, which are important questions for the future.
3. We had mentioned *two types* of vortices mutually interact via tensor gauge field A , causing some kind of non-abelian vortex behavior altogether. Moreover, it is tempting to know whether we can also dualize tensor gauge field A to represent the third type of vortex from A .
4. Our formulation of Sigma models may have applications to superfluid, supersolid, quantum melting transition, and elasticity studied in the recent fracton literature (for selected references, quantum crystal disclinations and dislocations, see [37–42] and citations therein).
5. In the present literature, there are *three different routes* to obtain non-abelian fracton orders:
 - (1) Gauge the charge conjugation (i.e., particle-hole) \mathbb{Z}_2^C -symmetry and $U(1)_{\text{poly}}$ polynomial symmetry [4,6],
 - (2) Gauge the permutation symmetry of N -layer systems [19,20],
 - (3) Couple to non-abelian TQFT/topological order [18,43,44] and [4,6].

Given a N -layer systems, there is a larger non-abelian group structure that we can explore. In the previous work [4, 6] and our present work, we focus on the finite abelian group $(\mathbb{Z}_2^C)^N$ by gauging N -layer of particle-hole symmetries, and consider a non-abelian gauge structure: $U(1)_{\text{poly}} \times (\mathbb{Z}_2^C)^N$. In fact, a natural larger group is including also the S_N permutation symmetry group of N -layers [19, 20]. The $(\mathbb{Z}_2^C)^N$ and S_N form a short exact sequence via a group extension:

$$1 \rightarrow (\mathbb{Z}_2^C)^N \rightarrow G_{\text{nAb}} \rightarrow S_N \rightarrow 1. \quad (5.1)$$

The G_{nAb} is related to the *hyperoctahedral group* in mathematics. An overall larger non-abelian group structure including both G_{nAb} and $U(1)_{\text{poly}}$, that mutually are non-commutative, possibly can be studied also via field theories or lattice models in the future.

6. Ref. [6] points out possible proper tools for studying these field theories include algebraic variety and affine-geometry/manifold. One motivation is to explore these theories and more general models on general affine manifolds beyond the Euclidean spacetime. This is left for future work.
7. Quantization and quantum path integral: Although we write down schematic path integral forms, our analysis is mostly based on equations of motion (EOM) and semi-classical or mean-field analyses. The most challenging question may be to explore the full quantum nature of the path integral we proposed, or study the quantization of these field theories. However, since we do have a *quantum mechanical definition* of the theory on a regularized lattice with an energy cutoff:
 - the $U(1)$ tensor field theory on the lattice [13] [1, 45], and
 - the topological gauge theory from group cohomology (of Dijkgraaf-Witten type) on the lattice [46–48],

so we believe that our field theories (as the interplay between the two models) should be a promising quantum theory, and *quantum field theories* can be made to be mathematically rigorously well-defined.

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References

- [1] R. M. Nandkishore and M. Hermele, *Fractons*, *Ann. Rev. Condensed Matter Phys.* **10** 295–313 (2019), [[arXiv:1803.11196](#)].
- [2] M. Pretko, *The Fracton Gauge Principle*, *Phys. Rev.* **B98** 115134 (2018), [[arXiv:1807.11479](#)].
- [3] N. Seiberg, *Field Theories With a Vector Global Symmetry*, [arXiv:1909.10544](#).
- [4] J. Wang and K. Xu, *Higher-Rank Tensor Field Theory of Non-Abelian Fracton and Embeddon*, [arXiv:1909.13879](#).

- [5] A. Gromov, *Towards classification of Fracton phases: the multipole algebra*, *Phys. Rev.* **X9** 031035 (2019), [[arXiv:1812.05104](#)].
- [6] J. Wang, K. Xu and S.-T. Yau, *Higher-Rank Tensor Non-Abelian Field Theory: Higher-Moment or Subdimensional Polynomial Global Symmetry, Algebraic Variety, Noether's Theorem, and Gauge*, [arXiv:1911.01804](#).
- [7] T. Griffin, K. T. Grosvenor, P. Horava and Z. Yan, *Scalar Field Theories with Polynomial Shift Symmetries*, *Commun. Math. Phys.* **340** 985–1048 (2015), [[arXiv:1412.1046](#)].
- [8] C. Chamon, *Quantum Glassiness in Strongly Correlated Clean Systems: An Example of Topological Overprotection*, "*Phys. Rev. Lett.*" **94** 040402 (2005 Jan), [[arXiv:cond-mat/0404182](#)].
- [9] C. Castelnovo and C. Chamon, *Topological quantum glassiness*, *Philosophical Magazine* **92** 304–323 (2012 Jan), [[arXiv:1108.2051](#)].
- [10] E. Schrödinger, *An Undulatory Theory of the Mechanics of Atoms and Molecules*, *Physical Review* **28** 1049–1070 (1926 Dec).
- [11] O. Klein, *Quantentheorie und fünfdimensionale Relativitätstheorie*, *Zeitschrift für Physik* **37** 895–906 (1926 Dec).
- [12] W. Gordon, *Der Comptoneffekt nach der Schrödingerschen Theorie*, *Zeitschrift für Physik* **40** 117–133 (1926 Jan).
- [13] A. Rasmussen, Y.-Z. You and C. Xu, *Stable Gapless Bose Liquid Phases without any Symmetry*, *arXiv e-prints* arXiv:1601.08235 (2016 Jan), [[arXiv:1601.08235](#)].
- [14] M. Pretko, *Subdimensional Particle Structure of Higher Rank $U(1)$ Spin Liquids*, *Phys. Rev.* **B95** 115139 (2017), [[arXiv:1604.05329](#)].
- [15] M. Pretko, *Generalized Electromagnetism of Subdimensional Particles: A Spin Liquid Story*, *Phys. Rev.* **B96** 035119 (2017), [[arXiv:1606.08857](#)].
- [16] M. Pretko, *Higher-Spin Witten Effect and Two-Dimensional Fracton Phases*, *Phys. Rev.* **B96** 125151 (2017), [[arXiv:1707.03838](#)].
- [17] K. Slagle, A. Prem and M. Pretko, *Symmetric Tensor Gauge Theories on Curved Spaces*, *Annals Phys.* **410** 167910 (2019), [[arXiv:1807.00827](#)].
- [18] A. Prem, S.-J. Huang, H. Song and M. Hermele, *Cage-Net Fracton Models*, *Phys. Rev.* **X9** 021010 (2019), [[arXiv:1806.04687](#)].
- [19] D. Bulmash and M. Barkeshli, *Gauging fractons: immobile non-Abelian quasiparticles, fractals, and position-dependent degeneracies*, [arXiv:1905.05771](#).
- [20] A. Prem and D. J. Williamson, *Gauging permutation symmetries as a route to non-Abelian fractons*, [arXiv:1905.06309](#).
- [21] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, *Commun. Math. Phys.* **129** 393 (1990).
- [22] J. Wang, X.-G. Wen and S.-T. Yau, *Quantum Statistics and Spacetime Surgery*, [arXiv:1602.05951](#).
- [23] P. Putrov, J. Wang and S.-T. Yau, *Braiding Statistics and Link Invariants of Bosonic/Fermionic Topological Quantum Matter in 2+1 and 3+1 dimensions*, *Annals Phys.* **384** 254–287 (2017), [[arXiv:1612.09298](#)].

- [24] J. Wang, K. Ohmori, P. Putrov, Y. Zheng, Z. Wan, M. Guo et al., *Tunneling Topological Vacua via Extended Operators: (Spin-)TQFT Spectra and Boundary Deconfinement in Various Dimensions*, *PTEP* **2018** 053A01 (2018), [[arXiv:1801.05416](#)].
- [25] J. Wang, X.-G. Wen and S.-T. Yau, *Quantum Statistics and Spacetime Topology: Quantum Surgery Formulas*, *Annals Phys.* **409** 167904 (2019), [[arXiv:1901.11537](#)].
- [26] V. L. Ginzburg and L. D. Landau, *On the Theory of superconductivity*, *Zh. Eksp. Teor. Fiz.* **20** 1064–1082 (1950).
- [27] M. P. A. Fisher and D. H. Lee, *Correspondence between two-dimensional bosons and a bulk superconductor in a magnetic field*, *Phys. Rev. B* **39** 2756–2759 (1989 Feb).
- [28] C. Dasgupta and B. I. Halperin, *Phase transition in a lattice model of superconductivity*, *Phys. Rev. Lett.* **47** 1556–1560 (1981 Nov).
- [29] D. R. Nelson, *Vortex entanglement in high- T_c superconductors*, *Phys. Rev. Lett.* **60** 1973–1976 (1988 May).
- [30] J.-K. Yuan, S. Chen and P. Ye, *Fractonic superfluids, topological vortices, and quantum fluctuations*, [arXiv:1911.02876](#).
- [31] C. N. Yang and R. L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, *Phys. Rev.* **96** 191–195 (1954 Oct).
- [32] Z.-C. Gu, J. C. Wang and X.-G. Wen, *Multi-kink topological terms and charge-binding domain-wall condensation induced symmetry-protected topological states: Beyond Chern-Simons/BF theory*, *Phys. Rev.* **B93** 115136 (2016), [[arXiv:1503.01768](#)].
- [33] P. Ye and Z.-C. Gu, *Topological quantum field theory of three-dimensional bosonic Abelian-symmetry-protected topological phases*, *Phys. Rev.* **B93** 205157 (2016), [[arXiv:1508.05689](#)].
- [34] J. Hubbard, *Calculation of Partition Functions*, "*Phys. Rev. Lett.*" **3** 77–78 (1959 Jul).
- [35] P. Ye and Z.-C. Gu, *Vortex-Line Condensation in Three Dimensions: A Physical Mechanism for Bosonic Topological Insulators*, *Phys. Rev.* **X5** 021029 (2015), [[arXiv:1410.2594](#)].
- [36] L. C. Evans, *Partial differential equations*. American Mathematical Society, Providence, R.I., 2010.
- [37] M. Pretko and L. Radzihovsky, *Fracton-Elasticity Duality*, *Phys. Rev. Lett.* **120** 195301 (2018 May), [[arXiv:1711.11044](#)].
- [38] M. Pretko and L. Radzihovsky, *Symmetry-Enriched Fracton Phases from Supersolid Duality*, *Phys. Rev. Lett.* **121** 235301 (2018 Dec), [[arXiv:1808.05616](#)].
- [39] L. Radzihovsky and M. Hermele, *Fractons from vector gauge theory*, *arXiv e-prints* arXiv:1905.06951 (2019 May), [[arXiv:1905.06951](#)].
- [40] M. Pretko, Z. Zhai and L. Radzihovsky, *Crystal-to-Fracton Tensor Gauge Theory Dualities*, *arXiv e-prints* arXiv:1907.12577 (2019 Jul), [[arXiv:1907.12577](#)].
- [41] A. Gromov and P. Surowka, *On duality between Cosserat elasticity and fractons*, [arXiv:1908.06984](#).
- [42] Y. You, Z. Bi and M. Pretko, *Emergent fractons and algebraic quantum liquid from plaquette melting transitions*, [arXiv:1908.08540](#).
- [43] S. Vijay and L. Fu, *A Generalization of Non-Abelian Anyons in Three Dimensions*, [arXiv:1706.07070](#).

- [44] H. Song, A. Prem, S.-J. Huang and M. A. Martin-Delgado, *Twisted Fracton Models in Three Dimensions*, *Phys. Rev.* **B99** 155118 (2019), [[arXiv:1805.06899](https://arxiv.org/abs/1805.06899)].
- [45] M. Pretko, X. Chen and Y. You, *Fracton Phases of Matter*, *Int. J. Mod. Phys. A* **35** 2030003 (2020), [[arXiv:2001.01722](https://arxiv.org/abs/2001.01722)].
- [46] Y. Hu, Y. Wan and Y.-S. Wu, *Twisted quantum double model of topological phases in two dimensions*, *Phys. Rev. B* **87** 125114 (2013 Mar.), [[arXiv:1211.3695](https://arxiv.org/abs/1211.3695)].
- [47] Y. Wan, J. C. Wang and H. He, *Twisted Gauge Theory Model of Topological Phases in Three Dimensions*, *Phys. Rev.* **B92** 045101 (2015), [[arXiv:1409.3216](https://arxiv.org/abs/1409.3216)].
- [48] H. Song, A. Prem, S.-J. Huang and M. A. Martin-Delgado, *Twisted Fracton Models in Three Dimensions*, *Phys. Rev.* **B99** 155118 (2019), [[arXiv:1805.06899](https://arxiv.org/abs/1805.06899)].
- [49] J. C. Wang, *Higher-Rank Tensor Non-Abelian Gauge Field Theory of Fracton and Embeddon*, *Quantum Matter workshop, Quantum Matter in Mathematics and Physics at Harvard CMSA, December 3rd, 2019* <https://www.youtube.com/watch?v=77vkcOrvW8k> (2019).