From string theory and Moonshine to vertex algebras

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Dedicated to the memory of John Horton Conway
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Disclaimers:

- This lecture includes a brief survey of the period prior to and soon after the creation of the theory of vertex algebras, and makes no claim of completeness – the survey is intended to highlight developments that reflect the speaker's own views (and biases) about the subject.
- As a short survey of early history, it will inevitably miss many of the more recent important or even towering results. Egs. geometric Langlands, braided tensor categories, conformal nets, applications to mirror symmetry, deformations of VAs, ...
- Emphases are placed on the mutually beneficial cross-influences between physics and vertex algebras in their concurrent early developments, and the lecture is aimed for a general audience.

Outline

The String Theory Universe

- 1968: Veneziano proposed a model (using the Euler beta function) to explain the '*st*-channel crossing' symmetry in 4-meson scattering, and the Regge trajectory (an angular momentum vs binding energy plot for the Coulumb potential).
- ∼1970: Nambu, Nielsen, and Susskind provided the first interpretation of the Veneziano amplitude in terms of a Fock space representation of infinitely many harmonic oscillators. The *n*-particle amplitude then became an *n*-point correlation function of certain 'vertex operators'

$$\boxed{:e^{ik\cdot\phi(z)}:}$$

on a Fock space representation of free bosonic fields $\phi_i(z)$ of a 'string' [Goddard et al 1972].

The String Theory Universe (cont.)

- This can be viewed as part of a **fundamental theory of strings**, producing fundamental particles of arbitrarily high spins as string resonances, including gravity. [Scherk et al, 1974].
- 1981: Polyakov's path integral formulation of the bosonic string theory.
- 1984: Belavin-Polyakov-Zamolodchikov's conformal bootstrap program – to systematically study 2d CFTs as **classical string vacua**, and models for **universal critical phenomena.**
  They introduced a powerful mathematical formalism – OPEs for 2d CFT observables.

The String Theory Universe (cont.)

Eg. The Virasoro OPE of the left moving 2d stress energy tensor of central charge c (conformal anomaly):

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots$$

- 1985 –: New generation of physicists to organize 2d CFTs using representations of **rational chiral algebras** (consisting of the left moving operators), and their fusion rules. A rational chiral algebra has only finite number of irreducible representations.

  Egs. The BPZ classified the Virasoro central charges and highest weights of the **rational minimal model CFTs** using Kac's determinant formula. (Cf. Y. Zhu's thesis and W. Wang's thesis).

The String Theory Universe (cont.)

Egs. The **Wess-Zumino-Witten theory** for loop groups of compact groups gave (unitary) rational CFTs.

Egs. Goddard-Kent-Olive's **coset constructions** from compact Lie groups gave (unitary) rational CFTs.

- Moore-Seiberg introduced the fundamental 'duality axioms' of CFTs. For the chiral algebra, they imply that matrix coefficients (or correlation functions) of left moving operators admit meromorphic continuations on $\mathbb{P}^1$. Most importantly, the operators are **formally** commutative after analytic continuations.

Eg. $\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4}$.

- 1978: McKay found evidence of the existence of an infinite dimensional $\mathbb{Z}$-graded representation $V^\natural$ of the hypothetical **Monster group** $\mathbb{M}$ (independently predicted by Fisher and Griess in 1973):– the coefficients of the q-series of j-function can be partitioned by dimensions of irreducible $\mathbb{M}$-modules:

$$j(q) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

$$\begin{aligned} 1 &= r_1 \\ 196884 &= r_1 + r_2 \\ 21493760 &= r_1 + r_2 + r_3 \\ 864299970 &= 2r_1 + 2r_2 + r_3 + r_4 \\ 20245856256 &= 3r_1 + 3r_2 + r_3 + 2r_4 + r_5 \\ &= 2r_1 + 3r_2 + 2r_3 + r_4 + r_6 \end{aligned}$$

The Moonshine Universe (cont.)

- Thompson: interpreted $j(q)$ as the graded- or q-trace

$$j(q) = qtr_{V^\natural} 1 := \sum_n tr_{V^\natural[n]} 1 q^n.$$

  This would later become the genus 1 partition function $tr\ q^{L_0 - \frac{c}{24}}$ of the **holomorphic vertex algebra** underlying $V^\natural$!
  Expectation: the q-trace of a general element $g \in \mathbb{M}$ should be interesting as well.

The Moonshine Universe (cont.)

- Conway-Norton computed leading terms of the hypothetical q-traces, and saw that they agree with q-series of certain special genus 0 modular functions. They gave a list of these functions: *the McKay-Thompson series* T_g.

  The Conway-Norton Moonshine Conjecture:
  $\exists$ *a graded* $\mathbb{M}$*-module* $V^\natural$ *having the McKay-Thompson series* T_g *as its q-traces.*

- 1980: Griess announced his construction of $\mathbb{M}$ (the 'Friendly Giant'). It is the automorphism group of the Griess algebra, a commutative non-associative algebra of dimension 196,884. (Cf. Griess's Harvard CMSA May 6 lecture). This algebra would later become the weight 2 piece $V^\natural[2]$ of the more elaborate **Moonshine Vertex Operator Algebra** $V^\natural$**!**

The Moonshine Universe (cont.)

- 1980s: emergence of a representation theory for a class of interrelated infinite dimensional graded algebras, including the Virasoro, Kac-Moody algebras, W-algebras, their various 'coset' constructions,...
- 1980: The Frenkel-Kac construction of level 1 irreducible representations of affine Kac-Moody Lie algebras, using 'free bosonic vertex operators' $e_\alpha(z)$, $\alpha \in L$ (similar to physicists : $e^{ik\cdot\phi}(z)$:) where L=a weight lattice of ADE type.
- 1986: Borcherds axiomatized the notion of a **vertex algebra** (VA) by an infinite set of linear operator identities.

The Moonshine Universe (cont.)

- 1988: Frenkel-Lepowsky-Meurman gave a new definition (including the Virasoro) of what they called **vertex operator algebras** (VOA), based on a Jacobi identity of formal power series. FLM's and Borcherds's formulations are logically equivalent, but FLM's is technically and conceptually a bit easier to work with.
- FLM also gave a general Fock space construction of a lattice VOA V_L from any even lattice L. They also constructed the $\mathbb{Z}/2$ orbifold of V_L (a VOA counterpart of physicists' orbifold CFT), using the notion of twisted vertex operators. For the Leech lattice $L = \mathbb{L}$, its $\mathbb{Z}/2$ orbifold $V^\natural$ would yield the Moonshine VOA, with the correct genus 1 partition function $j(q)$.
  (The construction of $V^\natural$ will be sketched later using a slightly different formulation of vertex algebras.)

The Moonshine Universe (cont.)

- 1992: Borcherds announced his solution to the Conway-Norton Conjecture, the FLM construction of the Moonshine VOA $V^\natural$ playing a central role.

The Circle Algebra Formalism

[Lian-Zuckerman, H. Li, 1992-95]

- • Let $V = \oplus_{n\in\mathbb{Z}} V[n]$ be a $\mathbb{Z}$-graded vector space (always over $\mathbb{C}$), where $V[n]$ is the *weight n* subspace of V;
  • V^* is the graded dual of V.
  • $z, w, ...$ be formal variables (eventually taking values in $\mathbb{C}$), and are assigned weight -1;
  • k, m, n will typically mean arbitrary integers.
  • Let $A(z) := \sum_{n\in\mathbb{Z}} A(n)z^{-n-1}$ be a formal power series with coefficient in a whatever linear space. Its formal derivative is $\partial A(z) := \sum_n -(n+1)A(n)z^{-n-2}$. Its formal residue is the 0th coefficient $A(0) := \mathrm{Res}_z\, A(z)$.
  • The expression $(z-w)^k$ means the binomial expansion

$$(z-w)^k := \sum_{n\geq 0} \binom{k}{n} z^{k-n} w^n.$$

The Circle Algebra Formalism (cont.)

Thus $(w-z)^k$ means the same with z, w interchanged in the expansion. (Warning: $(z-w)^{-2} \neq (w-z)^{-2}$!)

- Let $\mathscr{CO} \equiv \mathscr{CO}(V)$ be the space of all weight homogeneous linear maps $a(z): V \to V((z))$, where $wt\ a(z) = k$ iff

$$a(n): V[m] \to V[m+k-n-1], \qquad \forall m, n \in \mathbb{Z}$$

.

- The nth circle product on $\mathscr{CO}$ is defined by the following non-associative operation of weight $-n-1$:

$$a(w) \circ_n b(w) = \mathrm{Res}_z a(z)b(w)(z-w)^n + \mathrm{Res}_z a(z)b(w)(-w+z)^n.$$

Note that $\mathscr{CO}$ is closed under the circle products. A subspace of $\mathscr{CO}$ containing 1 and is closed under the circle

The Circle Algebra Formalism (cont.)

products is called a **circle algebra.** The standard categorical notions of circle algebras and modules can be defined without any difficulty.

- We say that two operators $a, b \in \mathscr{C}\mathscr{O}$ **circle commute** if

$$\boxed{(z-w)^N[a(z), b(w)] = 0 \text{ for } N >> 0.}$$

- **Remark:** (1) In this case, for any $u^* \in V^*$, $v \in V$, both matrix coefficient series $\langle u^*, a(z)b(w)v\rangle$, $\langle u^*, b(w)a(z)v\rangle$ converge to the same rational function $f_{a,b}(z, w)$ with possible poles at $z = w$, $z = 0$ and $w = 0$, in the respective regions $|z| > |w|$, $|w| > |z|$. The converse is also true. (2) We often write an arbitrary matrix coefficient series simply as $\langle a(z)b(w)\rangle$ when the roles of the vectors u^*, v are not important.

The Circle Algebra Formalism (cont.)

(3) Circle commutativity of a, b makes computing all their circle products very easy. For then $\langle a(z) \circ_n b(z) \rangle$ coincides with **analytic residue** at $z = w$ of the function $(z - w)^n f_{a,b}(z, w)$. So we can read off the circle products by expanding $f_{a,b}$ in powers of $(z - w)$:

$$f_{a,b}(z, w) = \sum_n \langle a(w) \circ_n b(w) \rangle (z - w)^{-n-1}$$

(4) This also makes precise physicists' notion of a 'short distance expansion' or OPE of the product $a(z)b(w)$ on the Riemann sphere, which they would formally write as

$$\begin{aligned} a(z)b(w) &= a(w) \circ_N b(w)(z - w)^{-N-1} + \cdots \\ &\quad + a(w) \circ_0 b(w)(z - w)^{-1} + :a(z)b(w): . \end{aligned}$$

The Circle Algebra Formalism (cont.)

Our *n*th circle product then corresponds to their operator valued contour integral over a circle C centered at $z = w$:

$$\oint_C a(z)b(w)(z-w)^n dz.$$

(5) Since the Wick product term $:a(z)b(w):$ is universal (i.e. it has the same form without pole at $z = w$ for any two operators; in fact $a \circ_{-n-1} b = \frac{1}{n!} : \partial^n a\, b :$), one often drops it from the expansion and write only the polar terms like

$$a(z)b(w) \sim a(w) \circ_N b(w)(z-w)^{-N-1} + \cdots + a(w) \circ_0 b(w)(z-w)^{-1}$$

We follow the same convention here, and refer to this (now) precise formula as **the OPE of a, b.**

The Circle Algebra Formalism (cont.)

- **Lemma**:(Dong) *If $a, b, c \in \mathscr{CO}$ pairwise circle commute, then each $a \circ_n b$ and c circle commute.*
- Therefore any set of operators that pairwise circle commute generates a CCA. This makes it quite easy to construct CCAs: one needs only to find operators that circle commute.
- Eg. Let V be any highest weight module of the Heisenberg algebra satisfying the commutator relations

$$[a(n), a(m)] = n\delta_{n+m,0}1.$$

Then an easy OPE calculation of $a(z) = \sum_n a(n)z^{-n-1} \in \mathscr{CO}(V)$ yields

$$[a(z), a(w)] = (z-w)^{-2} - (w-z)^{-2}$$

The Circle Algebra Formalism (cont.)

implying that $(z-w)^N[a(z), a(w)] = 0$ for $N \geq 2$. Hence the operator $a(z)$ generates a CCA $\mathscr{A}$ with the OPE

$$a(z)a(w) \sim (z-w)^{-2}.$$

- **The left regular module.** For a CCA $\mathscr{A}$, define $\rho_{\mathscr{A}} : \mathscr{A} \to \mathscr{C}\mathscr{O}(\mathscr{A})$, $a \mapsto \hat{a}(\zeta)$

$$\hat{a}(\zeta)b = \sum_n (a \circ_n b)\zeta^{-n-1}$$

If a is homogeneous, then $\hat{a}$ is also homogeneous of the same weight. Moreover $\rho_{\mathscr{A}}$ is a CA homomorphism, hence defines an $\mathscr{A}$-module structure on the space $\mathscr{A}$, which we call the **left regular module.**

The Circle Algebra Formalism (cont.)

- **Theorem:** (L-Zuckerman, H. Li) *If $\mathscr{A}$ is a CCA, then the structure* $(\mathscr{A}, 1, \partial, \rho_{\mathscr{A}})$ *is a vertex algebra in the sense of Borcherds and FLM. The converse is also true: a VA naturally gives a CCA.*

- This shows that CCA and VA are also logically equivalent notions. One difference however is that CCAs are technically much easier to work with (thanks to the idea of **OPE**'s). Another advantage of CCAs is that it makes it easier to view them as **higher analogues of commutative algebras** – one equipped with infinitely many non-associative products $\circ_n$, combining to form a commutative operator product in matrix coefficients. (In fact, commutative algebras form a full subcategory).

The Moonshine VOA $\mathscr{A}^\natural$ (after FLM)

- Let L be any even lattice. Put

$$\boxed{V_L := F_L \otimes \mathbb{C}[L].}$$

$F_L := \mathbb{C}[\alpha(-1), \alpha(-2), ...|\alpha \in L]$ is the polynomial space in the variables $\alpha(-1), \alpha(-2), ...$, each linear in $\alpha \in L$. This is the vacuum Fock space representation of the Heisenberg algebra with commutator relations

$$[\alpha(n), \beta(m)] = n\langle\alpha, \beta\rangle\delta_{m+n,0}1$$

where $\alpha(n) \curvearrowright F_L$ by the usual annihilation/creation operators.
$\mathbb{C}[L] := \oplus_{\alpha\in L}\mathbb{C}e^\alpha$ is the group algebra of L.
(Note: L=the 'momentum lattice' parametrizing the winding modes of the bosonic string on the real torus $L^*_{\mathbb{R}}/L^*$.)

The Moonshine VOA $\mathscr{A}^\natural$ (after FLM) (cont.)

- View $\alpha(n) \in \mathrm{End}\, V_L$: for $n \neq 0$, they act on the F_L factor, and $\alpha(0)$ acts the $\mathbb{C}[L]$ factor by

$$\alpha(0)e^\beta = \langle \alpha, \beta \rangle e^\beta.$$

Define the operator on V_L:

$$e_\alpha(z) := \pm e^\alpha z^{\alpha(0)} \exp\left(-\sum_{n<0} \frac{\alpha(n)}{n} z^{-n}\right) \exp\left(-\sum_{n>0} \frac{\alpha(n)}{n} z^{-n}\right)$$

(the physics counterpart of the vertex operator $: e^{ik\cdot\phi} :$.) The sign $\pm$ is defined according to which subspace $F_L \otimes e^\beta \subset V_L$, $e_\alpha(z)$ acts on, determined by a 2-cocycle defining a group extension of L by $\mathbb{Z}/2$, a technicality we omit here.

The Moonshine VOA $\mathscr{A}^\natural$ (after FLM) (cont.)

- **Lemma:** *If* $\langle\alpha,\beta\rangle \geq 0$ *then* $e_\alpha(z), e_\beta(w)$ *strictly commute. Otherwise* $(z-w)^{-\langle\alpha,\beta\rangle}[e_\alpha(z), e_\beta(w)] = 0$. *Hence the operators* $e_\alpha(z) \in \mathscr{CO}(V_L)$ *generate a CCA* $\mathscr{A}_L$.
- To construct the Moonshine VOA $V^\natural$, let $L = \mathbb{L}$ **the Leech lattice**, and consider the involution $\sigma : L \to L$, $\alpha \mapsto -\alpha$. Let $\mathscr{A}_o := \mathscr{A}_L^\sigma$ be the induced fixed point subalgebra. Then $\mathscr{A}_L$ has a module consisting of 'twisted' vertex operators [see FLM 1988, Ginzparg 1990], and therefore its σ invariant subspace $\mathscr{A}_1$ is an $\mathscr{A}_o$-module. FLM defined

$$\boxed{V^\natural \equiv \mathscr{A}^\natural := \mathscr{A}_o \oplus \mathscr{A}_1.}$$

The Moonshine VOA $\mathscr{A}^\natural$ (after FLM) (cont.)

- The space of operators $\mathscr{A}^\natural$ is naturally a CCA. First, $\mathscr{A}_o$ acts on $\mathscr{A}_o$ and $\mathscr{A}_1$ through their natural $\mathscr{A}_o$-module structures. Second, as $\mathscr{A}_o$-modules $\mathscr{A}_o, \mathscr{A}_1$ are cyclically generated by $a_o := 1 \in \mathscr{A}_o$, and a twisted operator $a_1 \in \mathscr{A}_1$ of the lowest weight in $\mathscr{A}_1$. The latter is also an intertwining operator $a_1(\zeta) : \mathscr{A}_o \to \mathscr{A}_1((\zeta))$ satisfying the condition that

$$\langle b_o(z) a_1(w) \rangle = \langle a_1(w) b_o(z) \rangle$$

after analytic continuation, for all $b_o(z) \in \mathscr{A}_o$. This condition determines uniquely the action of $\mathscr{A}_1$ on $\mathscr{A}_o$. In fact, this condition determines all circle products $b_o \circ_n b_1$ for $b_o \in \mathscr{A}_o, b_1 \in \mathscr{A}_1$, and that they are elements of $\mathscr{A}_1$. Third, the OPE $a_1(z)a_1(w)$ [see Chapter 9 of FLM] shows that $a_1 \circ_n a_1 \in \mathscr{A}_o$. Therefore $\mathscr{A}^\natural$ is closed under the circle products, and contains $a_o = 1$, hence form a CCA.

The Moonshine VOA $\mathscr{A}^\natural$ (after FLM) (cont.)

- Finally by a tour de force calculation, FLM proved that the vertex algebra $V^\natural \equiv \mathscr{A}^\natural$ has the correct genus 1 partition function or q-trace of 1 with respect to the Virasoro structure $T(z)$ above: $tr\ q^{T_0 - \frac{c}{24}} = j(q)$ where T is the standard Viasoro structure $T(z) = \frac{1}{2}\sum : \alpha_i(z)\alpha_i^*(z) :$ of central charge $24 = \mathrm{rk}\, L$, where the α_i, α_i^* are dual bases of the lattice $L \equiv L^*$.
- The weight 2 subspace $\mathscr{A}^\natural[2]$ equipped with the circle product $\circ_1$ recovers the 196,884 dimensional Griess algebra. It is also known that $\mathscr{A}^\natural$ is a 'holomorphic' VOA: its left regular module is its unique simple module [Dong 1993].

Holomorphic VOAs

- $V^\natural$ is an example of a **holomorphic** VOA.
- Holomorphic means it is rational and its module category has only one simple object, namely $V^\natural$.
- If $\mathscr{V}$ is holomorphic, its central charge c is a positive integer divisible by 8 [Schellekens 1988, Zhu's thesis 1990].
- For $c = 8$, only example is lattice VOA for the E_8 root lattice. For $c = 16$, only examples are lattice VOAs for $E_8 \oplus E_8$, and D_{16}^+ [Dong-Mason 2004].

Holomorphic VOAs (cont.)

- For $c = 24$, things are a bit more interesting. Assuming further that $\mathscr{V}$ is CFT type (i.e. $\mathscr{V}[0] = \mathbb{C}$), then the q-trace must be

  $$\chi(\mathscr{V}, q) = q^{-1} + \dim \mathscr{V}[1] + 196884q + 21493760q^2 + \cdots ,$$

  i.e., the *j*-function up to additive constant $\dim \mathscr{V}[1]$, a consequence of genus 1 modular invariance [Schellekens 1988].

Holomorphic VOAs (cont.)

- **Note**: If weight 1 subspace $\mathscr{V}[1] \neq 0$, it is either abelian of rank 24 (in which case $\mathscr{V} = V_{\mathbb{L}}$, $\mathbb{L}$=Leech lattice), or $\mathscr{V}[1]$ is a semisimple Lie algebra $\mathfrak{g}$ [Lian 1988, Schellekens 1988].
- If $\text{rank}(\mathfrak{g}) = 24$ and $\mathfrak{g}$ is semisimple, then $\mathscr{V} = V_L$ where L is a Niemeier lattice characterized by the root system of $\mathfrak{g}$. There are 23 such possibilities.
- If $\text{rank}(\mathfrak{g}) < 24$, Schellekens further considered possible structures of $\mathfrak{g}$ such that a simple affine vertex algebra of $\hat{\mathfrak{g}}$ admits an extension having the character $j(q) + \dim \mathscr{V}[1]$. (Cf. Frenkel-Kac, Kac-Peterson on structures of unitary $\hat{\mathfrak{g}}$-modules). There are exactly 46 such possibilities.

Holomorphic VOAs (cont.)

- There are thus $1 + 23 + 46 = 70$ possibilities in total, together with $V^\natural$ which has $V^\natural[1] = 0$. Schellekens [1988] then conjectured that there are exactly 71 holomorphic VOAs with $c = 24$.
- Due to the efforts of many people over the last 26 years (van Ekeren, Lam, Möller, Scheithauer, Shimakura, and others), this is now a theorem, with just one exception: uniqueness of the Moonshine!
- For $\mathscr{V}[1] \neq 0$, $\mathscr{V}$ exists and is uniquely determined by the Lie algebra $\mathscr{V}[1]$.
- For $\mathscr{V}[1] = 0$, *is $V^\natural$ the only one*, as conjectured by Igor Frenkel ~1987?

VOAs and commutative algebras

- VOAs are a natural generalization of commutative algebras.
- There are also several functors from the category of VOAs to the category of commutative algebras.
- Given a VOA $\mathscr{V}$, consider vector space $C(\mathscr{V}) \subseteq \mathscr{V}$ spanned by all elements of the form $a \circ_{-2} b = :(\partial a)b: \mid a, b \in \mathscr{V}\}$.
- Set $R_{\mathscr{V}} = \mathscr{V}/C(\mathscr{V})$. Map $\mathscr{V} \to R_{\mathscr{V}}$ sends $a \mapsto \bar{a}$.
- $R_{\mathscr{V}}$ has commutative, associative product $\bar{a} \cdot \bar{b} = \overline{:ab:}$. It is called **Zhu's commutative algebra**.
- **Eg**: For the Heisenberg VOA $\mathscr{H}$ with generating field α, $R_{\mathscr{H}} = \mathbb{C}[\bar{\alpha}]$.
- More generally, if $\mathscr{V}$ is *strongly generated* by a set $\{\alpha_i\}$, then $R_{\mathscr{V}}$ is generated by the corresponding elements $\{\bar{\alpha}_i\}$.

VOAs and commutative algebras (cont.)

- **Zhu's finiteness condition**: $\mathscr{V}$ is called C_2*-cofinite* or *lisse* if $R_{\mathscr{V}}$ is finite-dimensional as a vectors space.
- If $\mathscr{V}$ is lisse, it has finitely many simple modules.
- This plays a key role in Zhu's modularity theorem: he proves that if $\mathscr{V}$ is C_2-cofinite and rational, the span of the characters of its modules forms a representation of $SL(2, \mathbb{Z})$.
- For many years, it was conjectured that rationality and C_2-cofiniteness are equivalent.
- This turns out to be false. First examples of C_2-cofinite, non-rational VOAs are the **triplet algebras** (Kausch, Adamovic, Milas).
- It is expected that any rational VOA is C_2-cofinite, and this is an important open problem.

VOAs and commutative algebras (cont.)

- **Def**: (Arakawa) $\tilde{X}_{\mathscr{V}} = \text{Spec}(R_{\mathscr{V}})$ is called the **associated scheme** of $\mathscr{V}$, and $X_{\mathscr{V}} = \text{Specm}(R_{\mathscr{V}}) = (\tilde{X}_{\mathscr{V}})_{\text{red}}$ is called the **associated variety.**
- Geometric properties of $X_{\mathscr{V}}$ have important implications for the representation theory of $\mathscr{V}$.
- **Def**: (Arakawa, Kawasetsu) $\mathscr{V}$ is called **quasi-lisse** if $X_{\mathscr{V}}$ has finitely many symplectic leaves as a Poisson variety w.r.t. the Poisson bracket $\{a, b\} = a \circ_0 b$.
- **Note:** Here, the notion of symplectic leaves for Poisson variety was developed earlier [Brown-Gordon 2003] for a large class of algebras, including symplectic reflection algebras of [Etingof-Ginzburg 2002]. The stratification by symplectic leaves can give deep insights into the representation theory of the algebra in question.

VOAs and commutative algebras (cont.)

- **Eg**: (Arakawa) If $\mathfrak{g}$ is a simple Lie algebra and k is an admissible level for $\mathfrak{g}$, the simple affine VOA $L_k(\mathfrak{g})$ is quasi-lisse (one shows that the associated variety lies in the nilpotent cone of $\mathfrak{g}$.)
- The quasi-lisse condition is a natural generalization of the lisse condition.
- **Thm**: (Arakawa, Kawasetsu) If $\mathscr{V}$ is quasi-lisse, it has finitely many simple ordinary modules.

Vertex Algebra Hilbert Problems

- **Thm**: (Hilbert) If G is a reductive group and A is a finitely generated commutative $\mathbb{C}$-algebra, then the invariant ring A^G is finitely generated.
- Many foundational results in commutative algebra were introduced by Hilbert in connection with this problem (basis theorem, Nullstellensatz, syzygy theorem, etc.)
- **Idea of proof**: Since G is reductive, A^G is a *summand* of A, i.e., a subalgebra which is a direct summand.
- Any summand of a finitely generated, graded commutative ring is finitely generated.
- **Vertex algebra Hilbert problem** asks the analogous question: given a strongly finitely generated (SFG) vertex algebra $\mathscr{V}$ and a reductive group G of automorphisms of $\mathscr{V}$, is $\mathscr{V}^G$ also SFG?

Vertex Algebra Hilbert Problems (cont.)

- In general, this is **false**. For example, if $\mathscr{V}$ is the abelian vertex algebra $\mathbb{C}[\alpha, \partial\alpha, \partial\alpha^2, \dots]$ with $\mathbb{Z}_2$-action $\partial^i\alpha \mapsto -\partial^i\alpha$, then $\mathscr{V}^{\mathbb{Z}_2}$ is not SFG.
- It is expected to hold if $\mathscr{V}$ is **simple**, for any reductive G.
- Recently, Linshaw and Creutzig have shown that this is true for a large class of simple VOAs. This includes
  1. Free field algebras. These are VOAs like the Heisenberg algebra where the only nontrivial fields in the OPEs are the constant terms.
  2. Affine vertex algebra $V^k(\mathfrak{g})$ for any simple Lie superalgebra $\mathfrak{g}$, at generic level k.
  3. $\mathscr{W}$-algebra $\mathscr{W}^k(\mathfrak{g}, f)$ at generic level k, associated to any simple Lie superalgebra $\mathfrak{g}$, and any nilpotent element $f \in \mathfrak{g}$ [Cf. Drinfeld-Sokolov reduction of BRST complex].
  4. Any tensor product of finitely many VOAs of the above types.

Vertex Algebra Hilbert Problems (cont.)

- All the above VOAs admit a certain limit in which the problem can be handled using **classical invariant theory.**
- Proof is constructive starting from the **first and second fundamental theorems of invariant theory** (FFT and SFT) for G and some finite-dimensional G-module V.
- **Recall**: FFT for G and V is the explicit generating set for the invariant ring $R = \mathbb{C}[\bigoplus_{i\geq 0} V_i]^G$, where each $V_i \cong V$. SFT is the explicit generating set for the ideal of relations among these generators.
- FFT and SFT are known only in a few example, including
  1. Standard representations of classical groups (Weyl, 1939),
  2. Adjoint representation of classical groups (Procesi, 1976),
  3. 7-dimensional respresentation of G_2 (Schwarz, 1988).
- In such cases, explicit minimal strong generating sets for invariant VOAs can be given.

Vertex Algebra Hilbert Problems (cont.)

- This has applications to the **coset construction** of GKO and Frenkel-Zhu.
- Given a VOA $\mathscr{V}$ and a subVOA $\mathscr{A}$, the coset $\mathscr{C} = \text{Com}(\mathscr{A}, \mathscr{V})$, which is the set of elements of $\mathscr{V}$ which strictly commute with $\mathscr{A}$, is another subalgebra of $\mathscr{V}$.
- Homomorphism $\mathscr{A} \otimes \mathscr{C} \hookrightarrow \mathscr{V}$ is a **conformal embedding**, which implies that there are strong connections between the representation theory of the three VOAs $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{V}$.
- If $\mathscr{A}$ is an affine VOA $V^k(\mathfrak{g})$, then $\text{Com}(V^k(\mathfrak{g}), \mathscr{V}) = \mathscr{V}^{\mathfrak{g}[t]}$, which is also an invariant theory problem.
- Often, strong generating type of $\mathscr{V}^{\mathfrak{g}[t]}$ is the same as that of $(\mathscr{V}')^G$ for some VOA $\mathscr{V}'$, where G is a group with Lie algebra $\mathfrak{g}$.

Vertex Algebra Hilbert Problems (cont.)

- Consider the $\mathscr{W}$-algebra $\mathscr{W}^k(\mathfrak{g}, f)$ for some simple Lie superalgebra $\mathfrak{g}$ and nilpotent $f \in \mathfrak{g}$.
- **Thm**: (Creutzig, Linshaw, 2020) If $\mathscr{W}^k(\mathfrak{g}, f)$ has affine subVOA $V^\ell(\mathfrak{g}')$ for a reductive Lie algebra $\mathfrak{g}'$, then

$$\mathscr{C}^k = \mathrm{Com}(V^\ell(\mathfrak{g}'), \mathscr{W}^k(\mathfrak{g}, f)) = \mathscr{W}^k(\mathfrak{g}, f)^{\mathfrak{g}'[t]},$$

  is SFG for generic levels k.
- **Application:** This allows an explicit descriptions of many new and different coset constructions of W-algebras that are SFG in VA invariant theory.

Vertex Algebra Hilbert Problems (cont.)

- Egs. There is an important class of VOAs $Y_{L,M,N}[\psi]$ recently defined by Gaiotto and Rapcak as cosets of certain $\mathscr{W}$-algebras and $\mathscr{W}$-superalgebras by affine subVOAs. Based on considerations from gauge theory, they are expected to satisfy a symmetry known as **triality**: there are nontrivial isomorphisms between three different Y-algebras after suitable level shifts.
- This was proven by Creutzig and Linshaw in the case when one of the labels L, M, N is zero using above methods.

The Moonshine Cohomology & the Monster

- In 1995, L-Zuckerman proposed to study a cohomological construction based on two key ingredients: the so-called Lian-Zuckerman algebra and the Moonshine VOA $\mathscr{A}^\natural$. For any VOA $\mathscr{V}$, put

$$M^*(\mathscr{V}) := H^{\frac{\infty}{2}+*}(Vir, \mathbb{C}c, \mathscr{A}^\natural \otimes \mathscr{V})$$

where the right side is the Feigin semi-infinite cohomology of the Virasoso algebra relative to its center, with coefficient in the module $\mathscr{A}^\natural \otimes \mathscr{V}$. We call this graded cohomology group the **Moonshine Cohomology Functor**. Note that this group is trivial unless $\mathscr{V}$ has central charge 2.

The Moonshine Cohomology & the Monster (cont.)

- **Theorem:**(L-Zuckerman 1991-1994, 2003) *For any VOA $\mathscr{V}$ of central charge 2, $M^*(\mathscr{V})$ is a Batalin-Vilkovinsky algebra. In particular $M^0(\mathscr{V})$ is naturally a commutative algebra with respect to Wick product, and $M^1(\mathscr{V})$ a Lie algebra with respect to the BV bracket.*
- **Eg.** If $\mathscr{V} = \mathscr{V}_{II_{1,1}}$, the lattice VOA for the unimodular rank 2 hyperbolic lattice, then $M^0(\mathscr{V}) = \mathbb{C}$ and $M^1(\mathscr{V})$ turns out to be Borcherds's Monster Lie algebra [See L-Zuckerman 2003 details].

The Moonshine Cohomology & the Monster (cont.)

- **Note:** (1) The BV algebra structure of the Feigin cohomology was a result of the theory of *topological vertex algebras* introduced and developed in the early 90s [L-Zuckerman 1991-1994], together with Witten's construction of the 'ground ring' of the $c = 1$ 2d gravity theory.
  (2) The Fischer-Griess Monster group F_1 acts as a group of natural transformations of M^*.
  (3) Frenkel-Kostrikin 2010 use the Moonshine functor to give the first cohomological realization of certain quantum groups.
- **Conjecture/Speculation:** (L-Zuckerman 1995) *F_1 is the full automorphism group of M^*.*

A Theory of Chiral Schemes?

- **Question:** *Does the geometric notion of schemes and variaties carry over to CCAs (or VOAs), as higher commutative algebras?* If so, this amounts to enlarging the category of classical schemes, since ordinary CAs form a full subcategory of CCAs.
- **Notations:** $\mathscr{A}, \mathscr{B}, \ldots$ will denote a CCA. If J, K are subspaces of $\mathscr{A}$, then JK is the span of all circle products $a \circ_n b$, with $a \in J, b \in K, n \in \mathbb{Z}$.
- **Definition/Observation:** Let $\mathscr{A}$ be a CCA. A subspace $I \subset \mathscr{A}$ is left ideal if $\mathscr{A} I \subset I$, and likewise for right ideal. If $\mathscr{A}$ is conformal, then any left ideal is a right ideal, and vice versa. The hypothesis can be weakened, but we assume this here for simplicity, and any ideal is then 2-sided. An ideal $I \subsetneq \mathscr{A}$ is **prime** if $JK \subset I$ implies that $J \subset I$ or $K \subset I$, for any ideals J, K.

A Theory of Chiral Schemes? (cont.)

- As in the classical case, the set of ideals is closed under addition, multiplication, and these operations on ideals are commutative and distributive (i.e. $I(J+K) = IJ + IK$). Also maximal ideals are prime.
- Given an ideal I, let $V(I) = \{P \subset I | P \text{ prime}\}$. We call $V(I)$ the closed set associated with I. Again as in the classical case, we have

$$V(IJ) = V(I) \cup V(J), \qquad V(\sum_\alpha I_\alpha) = \cap_\alpha V(I_\alpha).$$

  Thus we can form an affine scheme $\text{Spec}\,\mathscr{A}$, using prime ideals of $\mathscr{A}$. Moreover, a morphism of CCAs induces a morphism of affine schemes. One can also define the notion of general schemes. We can also use the weight

A Theory of Chiral Schemes? (cont.)

structure of CCA to define a notion of graded ideals and hence projective schemes. I call this such a scheme a 'chiral scheme'.

- **Eg.** If $\mathscr{A}$ is a simple CCA, i.e. it has no proper ideals, then $\mathrm{Spec}\,\mathscr{A}$ consists of just one point (0), which is prime.
- **Eg.** If $\mathscr{H}$ is the Heisenberg CCA generated by one bosonic field $\alpha(z)$, then $\mathscr{H}$ is linearly isomorphic to the polynomial space $\mathbb{C}[\alpha, \partial\alpha, ...]$. But since $\mathscr{H}$ is simple, $\mathrm{Spec}\,\mathscr{H} = (0)$.
- **Eg.** Let γ be a weight zero commutative field, i.e. $\gamma(z)\gamma(w) \sim 0$. Then the CCA $\mathscr{A} = \mathbb{C}[\gamma, \partial\gamma, ...]$ is a differential commutative algebra. So we expect $\mathrm{Spec}\,\mathscr{A}$ to be a jet scheme over $\mathbb{C} = \mathrm{Spec}\,\mathbb{C}[\gamma]$.

A Theory of Chiral Schemes? (cont.)

- Question: What is the precise relationship between the 'chiral scheme' of $\mathscr{A}$ and the associated scheme of Zhu's commutative algebra? Note that if $\mathscr{A}$ is weight bounded below by 0, then both the associated scheme and the chiral scheme project onto the classical scheme $\mathsf{Spec}\,\mathscr{A}[0]$. So, both are some kinds of bundles of the same classical scheme.