

The Local Langlands correspondence: from extended quotients to affine Hecke algebras

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The main actors

- G group of F -rational points of a connected reductive algebraic F -group, with F a non-archimedean local field (finite extension of \mathbb{Q}_p or of $\mathbb{F}_p((t))$). We will refer to G as a p -adic group.
- G^\vee complex reductive group with root datum dual to that of G

Examples:

G	Dynkin diagram	G^\vee
$GL_n(\mathbb{Q}_p)$		$GL_n(\mathbb{C})$
$SL_n(\mathbb{Q}_p)$		$PSL_n(\mathbb{C})$
$PGL_n(\mathbb{Q}_p)$		$SL_n(\mathbb{C})$
$Sp_{2n}(\mathbb{Q}_p)$		$SO_{2n+1}(\mathbb{C})$
$SO_{2n+1}(\mathbb{Q}_p)$		$Sp_{2n}(\mathbb{C})$
$SO_{2n}(\mathbb{Q}_p)$		$SO_{2n}(\mathbb{C})$
$G_2(\mathbb{Q}_p)$		$G_2(\mathbb{C})$

The Local Langlands Correspondence (LLC)

predicts a surjective map, satisfying several properties,

$$\left\{ \begin{array}{l} \text{irred. smooth} \\ \text{repres. } \pi \text{ of } G \end{array} \right\} / \text{iso.} \xrightarrow{\mathcal{L}} \left\{ \begin{array}{l} \text{L-parameters} \\ \text{i.e. cont. homomorphisms} \\ \varphi_\pi: W_F \times \text{SL}_2(\mathbb{C}) \rightarrow {}^L G \end{array} \right\} / G^\vee\text{-conj.},$$

where $W_F :=$ absolute Weil group of F and ${}^L G := G^\vee \rtimes W_F$.

The fibers of \mathcal{L} , called **L-packets**, are expected to be finite.

Remark

In order to obtain a bijection LLC between the group side and the Galois side, the map \mathcal{L} was later enhanced: on the Galois side, one considers **enhanced L-parameters**: (φ_π, ρ_π) , where the enhancement ρ_π is a representation of a certain component group attached to φ_π .

$\Phi_e(G) :=$ set of G^\vee -conjugacy classes of enhanced L-paramters.

A bijective LLC has been constructed in particular in the following cases:

- $G = F^\times$ Class field theory (first half of the 20th century);
- $G = \mathrm{GL}_n(F)$ Harris-Taylor (1998), Henniart (2000), Scholze (2010);
- $G = \mathrm{SL}_n(F)$ (and its inner twists) Hiraga-Saito $\mathrm{char}(F) = 0$ (2012); A.-Baum-Plymen-Solleveld $\mathrm{char}(F) > 0$ (2016).
- $G = \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n+1}(F)$ ($\mathrm{char}(F) = 0$) Arthur (2013);
- $G = \mathrm{G}_2(F)$ A.-Xu (2022), Gan-Savin (2022).

Spectral extended quotient

Let X be a space and Γ a finite group acting on X . For $x \in X$, let

$\Gamma_x \subset \Gamma$ be the fixator of x : $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$.

The (spectral) **extended quotient** of X by Γ is the quotient

$$X//\Gamma := \{(x, \tau) : x \in X, \tau \in \mathrm{Irr}(\Gamma_x)\} / \Gamma.$$

Example 1

Let T be a torus in a reductive group G and $W := N_G(T)/T$ the corresponding Weyl group, acting on T by conjugation, then we can consider the extended quotient $T//W$.

Example 2

G a p -adic group, L Levi subgroup of G and $\text{Irr}_{\text{cusp}}(L)$ set of isomorphism classes of **supercuspidal** irrep. of L .
The group $W(L) := N_G(L)/L$ acts on $\text{Irr}_{\text{cusp}}(L)$ and we can form the **panoramic (spectral) p -adic extended quotient**:

$$\text{Irr}_{\text{cusp}}(L)//W(L).$$

Example 3

G^\vee and L^\vee complex dual groups of G , L and $\Phi_{e,\text{cusp}}(L)$ the set of L^\vee -conjugacy classes of **cuspidal enhanced Langlands parameters** for L . The group $W(L^\vee) := N_{G^\vee}(L^\vee)/L^\vee$ acts on $\Phi_{e,\text{cusp}}(L)$ and we can form the **panoramic (spectral) Galois extended quotient**:

$$\Phi_{e,\text{cusp}}(L)//W(L^\vee).$$

Remark

The groups $W(L)$ and $W(L^\vee)$ are canonically isomorphic.

Conjecture

The local Langlands correspondence induces a bijection

$$\text{Irr}_{\text{cusp}}(L)//W(L) \xleftrightarrow{1-1} \Phi_{e,\text{cusp}}(L)//W(L^\vee)$$

for any Levi subgroup L of G .

Notation

- \mathcal{G} complex (possibly disconnected) reductive group
- \mathcal{P} parabolic subgroup of \mathcal{G} (i.e., subgroup of G s.t. \mathcal{P}° is a parabolic subgroup of \mathcal{G}°)
- \mathcal{L} complement in \mathcal{P} of its unipotent radical \mathcal{U}
- $\text{Unip}_{\mathcal{G}}$ unipotent variety of \mathcal{G} , similarly, $\text{Unip}_{\mathcal{P}}$, $\text{Unip}_{\mathcal{L}}$
- $D_c^b(X)$ category of bounded constructible ℓ -adic sheaves on the algebraic stack X .

Geometric parabolic induction

We consider the correspondence of algebraic stacks

$$\text{Unip}_{\mathcal{L}}/\mathcal{L} \xleftarrow{\pi} \text{Unip}_{\mathcal{P}}/\mathcal{P} \xrightarrow{\iota} \text{Unip}_{\mathcal{G}}/\mathcal{G}$$

induced by the natural maps $\pi: \mathcal{P} \twoheadrightarrow \mathcal{L}$ and $\mathcal{P} \hookrightarrow \mathcal{G}$.

The functor $i_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}: D_c^b(\text{Unip}_{\mathcal{L}}/\mathcal{L}) \rightarrow D_c^b(\text{Unip}_{\mathcal{G}}/\mathcal{G})$ is defined by

$$i_{\mathcal{L},\mathcal{P}}^{\mathcal{G}} := \iota_! \circ \pi^*.$$

Definition of cuspidality

Let \mathcal{E} be an irreducible \mathcal{L} -equivariant local system on a unipotent class C in \mathcal{L} . We say that the pair (C, \mathcal{E}) is **cuspidal** if the intersection cohomology sheaf $\mathrm{IC}(C, \mathcal{E})$ does not occur in $i_{\mathcal{L}, \mathcal{P}}^{\mathcal{G}}(D_c^b(\mathrm{Unip}_{\mathcal{L}}/\mathcal{L}))$ for any proper Levi subgroup \mathcal{L} of \mathcal{G} .

Strategy

We will plug the above construction into the Galois side of the correspondence.

Notation

We set $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$, and define

$$S_{\varphi} := Z_{G^{\vee}}(\varphi(W'_F)) \quad (1)$$

if G is a pure inner twist of a quasi-split group. (There are variants for other cases.)

Definition

An **enhanced L -parameter** is a pair (φ, ρ) where φ is an L -parameter for G and $\rho \in \text{Irr}(\mathcal{S}_\varphi)$, with $\mathcal{S}_\varphi := \pi_0(\mathcal{S}_\varphi) = \mathcal{S}_\varphi / \mathcal{S}_\varphi^\circ$.

We set $\mathcal{G} = \mathcal{G}_\varphi := Z_{G^\vee}(\varphi(W_F))$. We have

$$\mathcal{S}_\varphi \simeq \pi_0(Z_{\mathcal{G}_\varphi}(u)), \quad \text{where } u = u_\varphi := \varphi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

Definition

An enhanced L -parameter $(\varphi, \rho) \in \Phi_e$ is called **cuspidal** if the following properties hold:

- φ is discrete (i.e., $\varphi(W_F \times \text{SL}_2(\mathbb{C}))$ is not contained in any proper Levi subgroup of G^\vee),
- (u_φ, ρ) is a *cuspidal pair* in \mathcal{G}_φ .

The **cuspidality conjecture** (special case of the above for $L = G$)

The local Langlands correspondence restricts to a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(G) \xleftrightarrow{1-1} \Phi_{e,\mathrm{cusp}}(G).$$

State of art

The cuspidality conjecture is known to hold for all the Levi subgroups (including the groups themselves) of

- general linear groups and split classical p -adic groups [Moussaoui, 2017],
- inner forms of linear groups and of special linear groups, and quasi-split unitary p -adic groups [A-Moussaoui-Solleveld, 2018],
- the p -adic group G_2 [A-Xu, 2022],
- pure inner forms of quasi-split classical p -adic groups [A-Moussaoui-Solleveld, 2022].

Property C(L)

There is a bijection $\mathfrak{L}_L: \text{Irr}_{\text{cusp}}(L) \xrightarrow{1-1} \Phi_{e,\text{cusp}}(L)$ such that

$$\mathfrak{L}_L \circ \text{Ad}(w) = \text{Ad}(w^\vee) \circ \mathfrak{L}_L \quad \text{for any } w \in W_G(L)$$

where $w \mapsto w^\vee$ denotes the canonical bijection $W(L) \rightarrow W(L^\vee)$.

Remark

(Cuspidality conjecture) \Leftrightarrow (Property C(G) is satisfied with $\mathfrak{L}_G = \text{LLC}$).

A Langlands correspondence between panoramic extended quotients:

If Property C(L) is satisfied for L a Levi subgroup of G, then \mathfrak{L}_L induces a bijection

$$\text{Irr}_{\text{cusp}}(L) // W_G(L) \xleftrightarrow{1-1} \Phi_{e,\text{cusp}}(L) // W_{G^\vee}(L^\vee). \quad (2)$$

Theorem [A-Moussaoui-Solleveld, 2018]

For any G , there is a bijection:

$$\Phi_e(G) \xleftrightarrow{1-1} \bigsqcup_{L \in \mathcal{L}(G)} (\Phi_{e, \text{cusp}}(L) // W_{G^\vee}(L^\vee))_{\mathfrak{L}}.$$

Theorem [Solleveld, 2020]

For any G , there is a bijection:

$$\text{Irr}(G) \xleftrightarrow{1-1} \bigsqcup_{L \in \mathcal{L}(G)} (\text{Irr}_{\text{cusp}}(L) // W_G(L))_{\mathfrak{L}}.$$

Consequence

If Property $C(L)$ is satisfied for all the Levi subgroups L of G (including $L = G$), and if the bijection (2) is “compatible with the twists”, then we get a bijection ([hopefully the LLC](#))

$$\text{Irr}(G) \xleftrightarrow{1-1} \Phi_e(G).$$

Notation

- $\mathfrak{X}_{\text{nr}}(L)$ group of **unramified** characters of L (a character is unramified if it is trivial on every compact subgroup of L).
- $\mathfrak{s} = \mathfrak{s}_G := [L, \sigma]_G$ the G -conjugacy class of the pair $(L, \mathfrak{X}_{\text{nr}}(L) \cdot \sigma)$, where $\sigma \in \text{Irr}_{\text{cusp}}(L)$
- $\text{Irr}^{\mathfrak{s}}(G)$: set of (isomorphism classes of) irreducible representations of G whose supercuspidal support lies in \mathfrak{s}
- $\mathfrak{X}_{\text{nr}}(L, \sigma) := \{\chi \in \mathfrak{X}_{\text{nr}}(L) : \sigma \otimes \chi \cong \sigma\}$
- the bijection $\mathfrak{X}_{\text{nr}}(L)/\mathfrak{X}_{\text{nr}}(L, \sigma) \rightarrow \text{Irr}^{\mathfrak{s}_L}(L)$, $\chi \mapsto \sigma \otimes \chi$ endows $\text{Irr}^{\mathfrak{s}_L}(L)$ with the structure of a complex torus $T^{\mathfrak{s}}$
- $W^{\mathfrak{s}} := N_G(\mathfrak{s})/L$ acts on $T^{\mathfrak{s}}$ by automorphisms of algebraic varieties
- $\mathfrak{B}(G)$ set of such classes \mathfrak{s} .

The partition of $\text{Irr}(G)$ into Bernstein series

We have [Bernstein, 1984]:

$$\text{Irr}(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \text{Irr}^{\mathfrak{s}}(G). \quad (3)$$

Moreover, for every $\mathfrak{s} \in \mathfrak{B}(G)$ [Solleveld, 2020]:

$$\text{Irr}^{\mathfrak{s}}(G) \xleftrightarrow{1-1} (T^{\mathfrak{s}}/W^{\mathfrak{s}})_{\mathfrak{h}}. \quad (4)$$

Example

Suppose G is F -split and let $\mathfrak{s}^1 := [T, \text{triv}]_G$, with T an F -split maximal torus. We have

$$\text{Irr}^{\mathfrak{s}^1}(G) = \{\text{Iwahori-spherical irreps. of } G\} / \sim.$$

Slogan: There is a similar decomposition on the Galois side.

Notation

- L^\vee Langlands dual group of L
- If G is F -split: $\mathfrak{X}_{\text{nr}}({}^L L) := \{\zeta: W_F/I_F \rightarrow Z_{L^\vee}^\circ\}$, where I_F is the inertia group of F . In general, $\mathfrak{X}_{\text{nr}}({}^L L) :=$
- $\mathfrak{s}^\vee = \mathfrak{s}_{G^\vee}^\vee := [L^\vee \rtimes W_F, (\varphi_0, \rho_0)]_{G^\vee}$ the G^\vee -conjugacy class of $(L^\vee \rtimes W_F, \mathfrak{X}_{\text{nr}}({}^L L) \cdot (\varphi_0, \rho_0))$, where $(\varphi_0, \rho_0) \in \Phi_{e, \text{cusp}}(L)$
- $\Phi_e^{\mathfrak{s}^\vee}(G)$ enhanced L -parameters whose cuspidal support lies in \mathfrak{s}^\vee
- $W^{\mathfrak{s}^\vee} := N_{G^\vee}(\mathfrak{s}^\vee)/L^\vee$ acts on $\Phi_e^{\mathfrak{s}^\vee}({}^L L)$
- $\mathfrak{B}^\vee(G)$ the set of such \mathfrak{s}^\vee .

Theorem [A-Moussaoui-Solleveld, 2018]

The set $\Phi_e(G)$ is partitioned into series à la Bernstein as

$$\Phi_e(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)} \Phi_e^{\mathfrak{s}^\vee}(G). \quad (5)$$

Moreover, for every $\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)$, we have

$$\Phi_e^{\mathfrak{s}^\vee}(G) \xleftrightarrow{1-1} (\Phi_e^{\mathfrak{s}^\vee}_{L^\vee}(L) // W^{\mathfrak{s}^\vee})_{L_{\mathfrak{h}}}. \quad (6)$$

Theorem

If G is

- an inner form of $GL_n(F)$ [A-Baum-Plymen-Solleveld, 2019],
- the exceptional group of type G_2 [A-Xu, 2022],
- a pure inner form of a quasi-split classical p -adic group [A-Moussaoui-Solleveld, 2022],

then, for every $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$ and for $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, \text{LLC}(\sigma)]_{G^\vee}$,

$$\text{Irr}^{\mathfrak{s}}(G) \xrightarrow{1-1} \text{Irr}^{\mathfrak{s}^L}(L) // W^{\mathfrak{s}} \xrightarrow{1-1} \Phi_e^{\mathfrak{s}^{\vee L^\vee}}(L) // W^{\mathfrak{s}^\vee} \xrightarrow{1-1} \Phi_e^{\mathfrak{s}^\vee}(G)$$

coincides with the LLC, and the following diagram is commutative

$$\begin{array}{ccc} \text{Irr}^{\mathfrak{s}}(G) & \xrightarrow[1-1]{\text{LLC}} & \Phi_e^{\mathfrak{s}^\vee}(G) \\ \text{Sc} \downarrow & & \downarrow \text{Sc} \\ \text{Irr}^{\mathfrak{s}^L}(L) & \xrightarrow[\text{LLC}]{1-1} & \Phi_e^{\mathfrak{s}^{\vee L^\vee}}(L) \end{array}$$

Theorem [A-Moussaoui-Solleveld, 2018]

There exist a (twisted) extended affine Hecke algebra $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ such that

$$\Phi_e^{\mathfrak{s}^\vee}(G) \xleftrightarrow{1-1} \text{Irr}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)).$$

The Bernstein decomposition [Bernstein, 1984]

The category $\mathfrak{R}(G)$ of smooth representations of a p -adic group G is a direct product

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G) \quad (7)$$

of the full subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, where $\mathfrak{R}^{\mathfrak{s}}(G)$ is the subcategory of $\mathfrak{R}(G)$ whose objects are the representations π such that every irreducible G -subquotient of π has its supercuspidal support in \mathfrak{s} .

Theorem ([A.-Moussaoui-Solleveld], [A.-Xu])

If G an inner form of $GL_n(F)$, the group G_2 , or a pure inner form of a quasi-split classical p -adic group, then

$$\mathfrak{R}^{\mathfrak{s}}(G) \stackrel{\text{Morita}}{\sim} \text{Mod}(\mathcal{H}(G, \mathfrak{s})),$$

where $\mathcal{H}(G, \mathfrak{s})$ is an affine Hecke algebra, which is isomorphic to $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$.

The Bernstein decomposition of the Hecke algebra of G

Let $\mathcal{H}(G)$ be the convolution algebra of locally constant, compactly supported functions $f: G \rightarrow \mathbb{C}$. By letting G act on $\mathcal{H}(G)$ by left translation, we obtain a decomposition

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}, \quad (8)$$

with $\mathcal{H}(G)^{\mathfrak{s}} \in \mathfrak{R}^{\mathfrak{s}}(G)$. The spaces $\mathcal{H}(G)^{\mathfrak{s}}$ are two-sided ideals of $\mathcal{H}(G)$.

A version with C^* -algebras

- From the point of view of noncommutative geometry, for the study of tempered representations of G , it is interesting to use the **reduced C^* -algebra** $C_r^*(G)$ of G (i.e. the completion of $\mathcal{H}(G)$ in the algebra of bounded linear operators on the Hilbert space $L^2(G)$).
- The spectrum of $C_r^*(G)$ coincides with the tempered dual $\text{Irr}^{\text{temp}}(G)$ of G .
- We have the following decomposition of $C_r^*(G)$:

$$C_r^*(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} C_r^*(G)^{\mathfrak{s}}.$$

Conjecture (topological K -theory version of the ABPS Conjecture)

Let $\mathfrak{s} \in \mathfrak{B}(G)$ and let $T_{\text{un}}^{\mathfrak{s}}$ be the set of unitary representations in $T^{\mathfrak{s}}$, a $W^{\mathfrak{s}}$ -stable compact real subtorus. There exists a canonical isomorphism

$$K_{W^{\mathfrak{s}}}^*(T_{\text{un}}^{\mathfrak{s}}) \rightarrow K(C_r^*(G)^{\mathfrak{s}})$$

where $K_{W^{\mathfrak{s}}}^j(T_{\text{un}}^{\mathfrak{s}})$ is the classical topological equivariant K -theory for the group $W^{\mathfrak{s}}$ acting on $T_{\text{un}}^{\mathfrak{s}}$.

Remark

If G is a classical group, then $C_r^*(G)^{\mathfrak{s}}$ is Morita equivalent to the reduced C^* -completion of $\mathcal{H}(G, \mathfrak{s})$ (i.e. the closure of $\mathcal{H}(G, \mathfrak{s})$ in the algebra of bounded linear operators on the Hilbert space completion of $\mathcal{H}(G, \mathfrak{s})$).

Theorem [A., 2023]

Let $(G, G') = (\mathrm{Sp}_{2n}(F), \mathrm{O}_{2n'}(F))$, viewed as a dual pair in $\mathrm{Sp}_{2nn'}(F)$, and fix $\mathfrak{s} \in \mathfrak{B}(G)$.

- 1 The images by the Howe correspondence of all the $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G)$ belong to a unique Bernstein series $\mathrm{Irr}^{\theta(\mathfrak{s})}(G')$ of G' .
- 2 The Howe correspondence for (G, G') induces a correspondence between simple modules of the extended affine Hecke algebras $\mathcal{H}(G_n, \mathfrak{s})$ and $\mathcal{H}(G'_m, \theta(\mathfrak{s}))$, and hence a **correspondence $\theta^{\mathfrak{s}}$ between $\Phi_e^{\mathfrak{s}}(G)$ and $\Phi_e^{\theta(\mathfrak{s})}(G')$** .
- 3 When $n' = n$ or $n' = n + 1$, the Howe correspondence induces a correspondence between simple modules of $C_{\mathbb{R}}^*(G)^{\mathfrak{s}}$ and $C_{\mathbb{R}}^*(G)^{\theta(\mathfrak{s})}$.

Thank you very much for your attention!

