On The Physical Mathematics Dialogue

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October 27, 2023

Phys-i-cal Math-e-ma-tics, n.

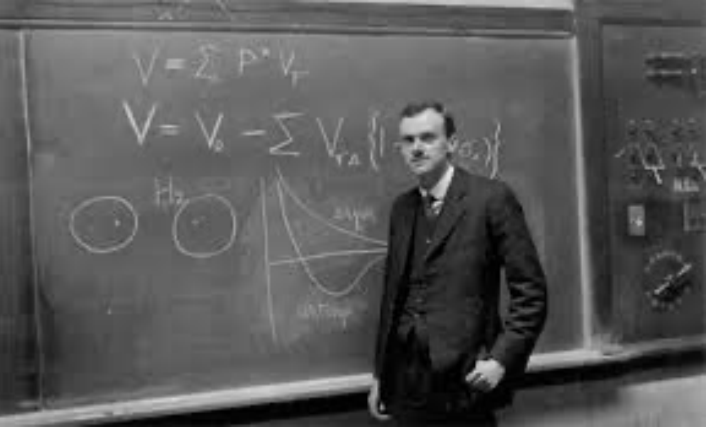
Pronunciation: Brit. /ˈfɪzɨkl ˌmaθ(ə)ˈmatɪks / , U.S. /ˈfɪzək(ə)l ˌmæθ(ə)ˈmædɪks/

Physical mathematics is a fusion of mathematical and physical ideas, motivated by the dual, *but equally central*, goals of

1. Elucidating the laws of nature at their most fundamental level,

together with

2. Discovering deep mathematical truths.



1931: Dirac's Paper on Monopoles

Quantised Singularities in the Electromagnetic Field

P.A.M. Dirac

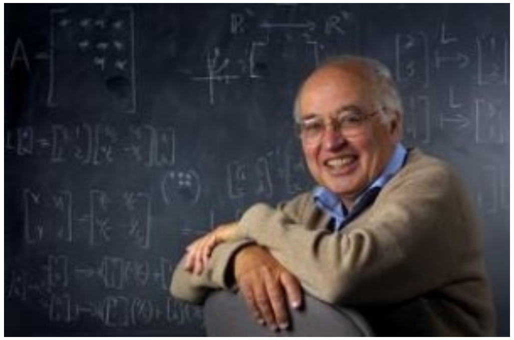
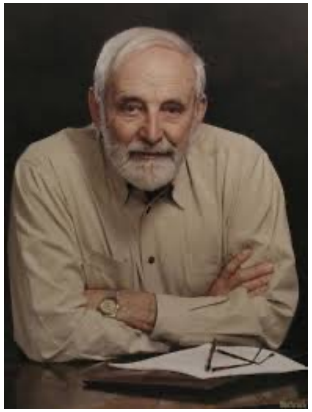
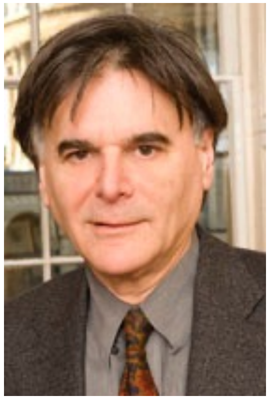
Received May 29, 1931

§ *1. Introduction*

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers

for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a

c. 1970: Some great mathematicians got interested in aspects of fundamental physics …..


While some great physicists started producing results requiring ever increasing mathematical sophistication, …..

Physical Mathematics Dialogue

In the past few decades a new field has emerged with its own distinctive character, its own aims and values, its own standards of proof.

One of the guiding principles is certainly the discovery of the ultimate foundations of physics.

This quest has led to ever more sophisticated mathematics…

A second guiding principle is that physical insights can lead to surprising and new results in mathematics

Such insights are a great success - just as profound and notable as an experimental confirmation of a theoretical prediction.

Today:

It’s best to discuss this by giving a paradigmatic example of the dialogue between physicists and mathematicians

Supersymmetric QFT & Differential Topology of Four-Manifolds

Beautiful story.

I’m still working on it.

But I’ve given that talk quite a lot.

Dan is getting bored with it.

A Panorama Of Physical Mathematics c. 2022

Ibrahima Bah[1], Daniel S. Freed [2], Gregory W. Moore[3], Nikita Nekrasov[4], Shlomo S. Razamat[5], Sakura Schäfer-Nameki[6]

November 10, 2022

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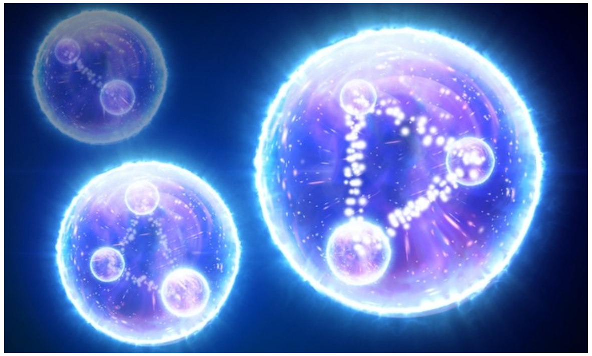
Abstract

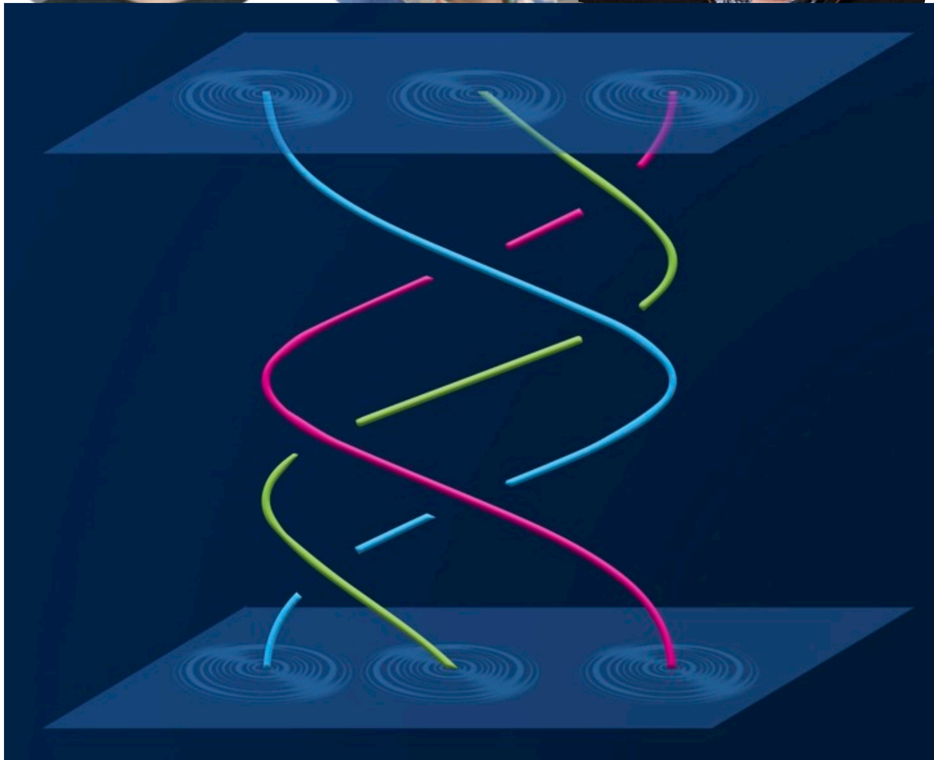
What follows is a broad-brush overview of the recent synergistic interactions between mathematics and theoretical physics of quantum field theory and string theory. The discussion is forward-looking, suggesting potentially useful and fruitful directions and problems, some old, some new, for further development of the subject. This paper is a much extended version of the Snowmass whitepaper on physical mathematics [1]. Version: November 10, 2022.

GENERALIZED SYMMETRY

Simons Collaboration on Global Categorical Symmetry

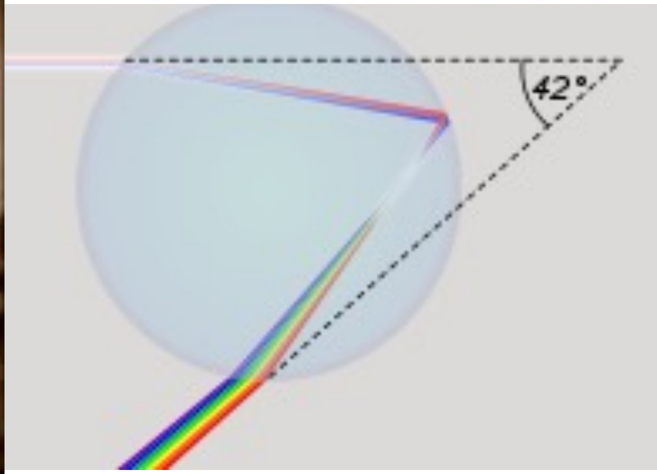


Coals to Newcastle



Somewhere Over The Rainbow

Descartes,
Newton,Huygens,
Young, Fresnel,



Supernumerary Arcs

1 Physical Mathematics

2 George Airy & George Stokes

3 Perturbative vs. Nonperturbative QFT

4 More About Airy & Stokes

5 Chern-Simons-Witten Gauge Theory

6 Stokes & Differential Equations

7 Summary & Future Directions

Airy's Integral [1838]

$$Ai(z) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(f(x;z))\ dx$$

$$f(x;z) = i\left(\frac{x^3}{3} + z\,x\right)$$

z ~ The distance from the main arc/λ.

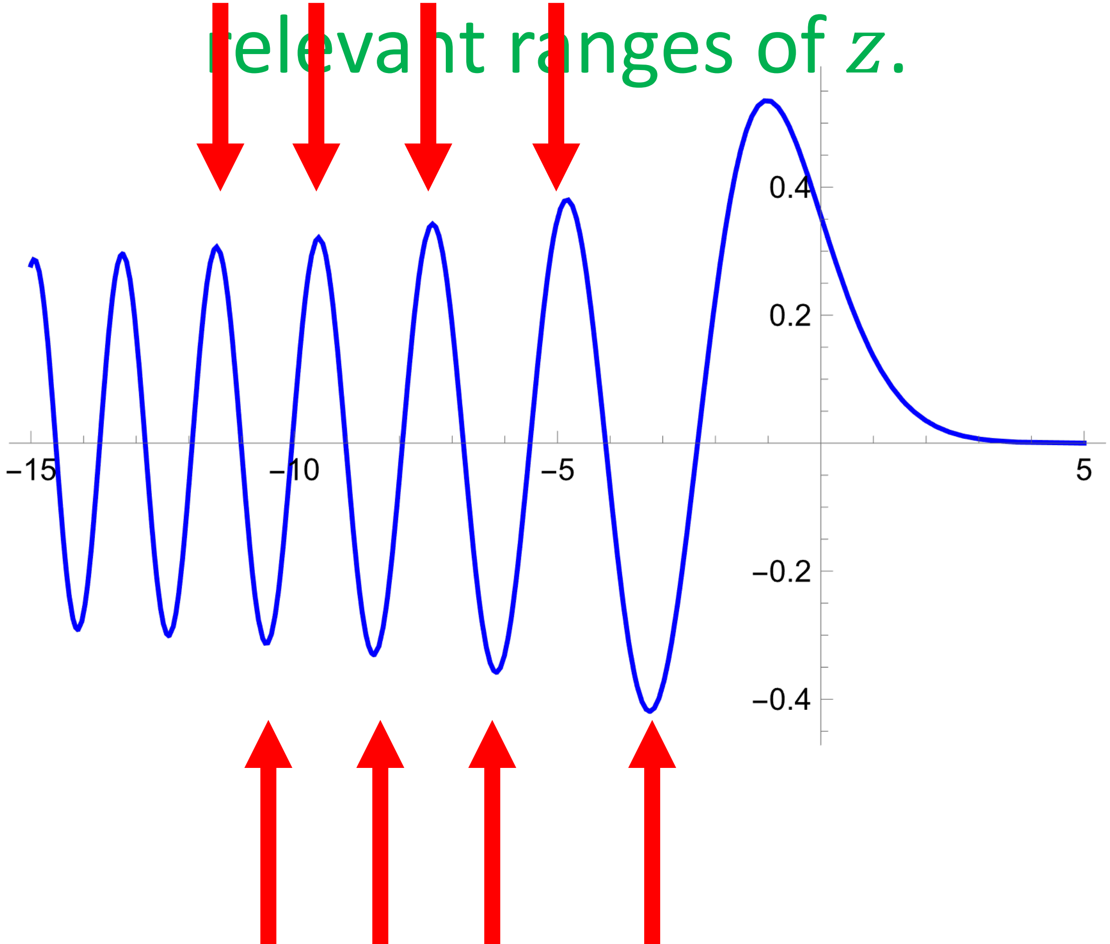
$\left(Ai(z)\right)^2 \propto$ Intensity of (monochromatic) light.

Wanted: Behavior for large $|z|$

But Airy couldn't do the integral (very well) ….

Stokes To The Rescue

Stokes [1850,1857,1889,1902] Invented the saddle point/steepest descent/stationary phase approximation to get a good approximation for the relevant ranges of z.



$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\big(f(x;z)\big)\ dx$$

$$f(x;z) = i\left(\frac{x^3}{3} + z\,x\right)$$

Saddle points = critical points: $d_x f = 0$

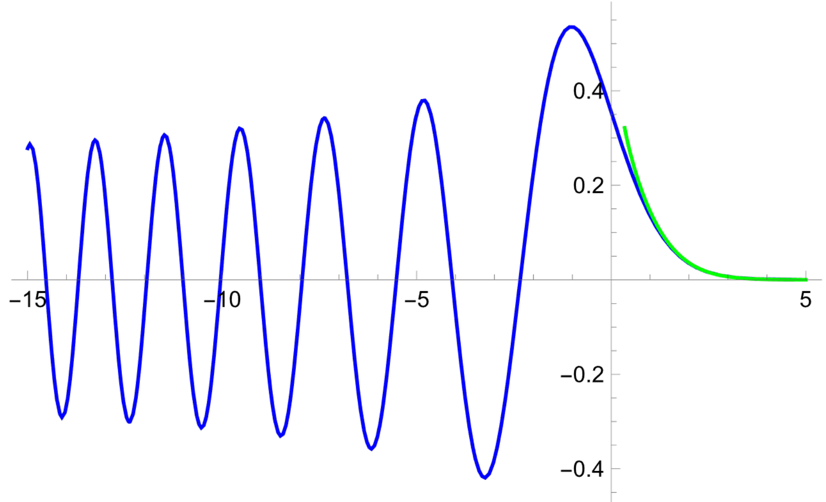
$$\Rightarrow \quad x^2 + z = 0 \qquad \Rightarrow \quad x_\pm = \pm\sqrt{-z}$$

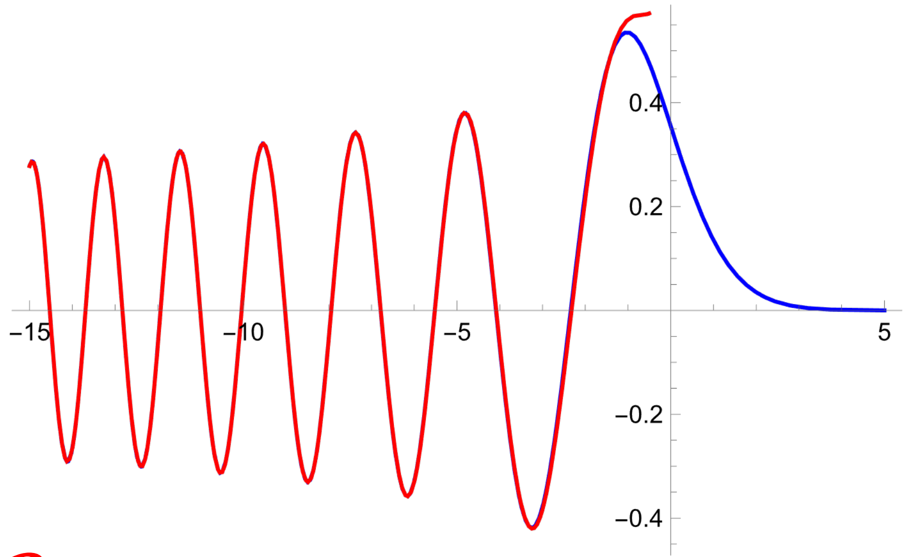
$$z > 0 \ \Rightarrow \quad x_\pm = \pm i\sqrt{|z|}$$

$$e^{f(x_\pm;z)} = e^{\mp\frac{2}{3} z^{\frac{3}{2}}}$$

$$z \to +\infty$$

$$\frac{1}{2\sqrt{\pi}}\, z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}}$$





$$z \to -\infty$$

$$\frac{1}{\sqrt{\pi}}\,(-z)^{-\frac{1}{4}} \sin\left(\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right)$$

Stokes wrote to Arthur Cayley, a pure mathematician, about his work 29th Oct. 1849

``Thomson [Lord Kelvin] and I are at present writing to each other about potentials. I think that potentials may throw light on the interpretation of $f\left(x+\sqrt{-1}\,y\right)$. *How horrible you would think it to prove, even in one's own mind, a proposition in pure mathematics by means of physics."*

(Quoted from A. O'Donnell, ``The work of G.G. Stokes in evaluating the Airy rainbow integral and its ramifications today,'')

`` *How horrible you would think it to prove, even in one's own mind, a proposition in pure mathematics by means of physics*.''

Today, that ``horrible idea''
is at the heart of physical mathematics.

In the context of (cohomological) TFT exact saddle point integrals have led to many important mathematical developments:
Donaldson/Seiberg-Witten, Gromov-Witten invariants, Floer Theory,…

Three Odd Things About Stokes' Computation

1. Why take x_+ with $e^{f(x_+;z)} = e^{-\frac{2}{3}z^{\frac{3}{2}}}$ and not the other saddle point with $e^{f(x_-;z)} = e^{\frac{2}{3}z^{\frac{3}{2}}}$?

Is there a principle, other than that we don't like the answer with x_-?

Three Odd Things About Stokes' Computation

2. $Ai(z)$ is a *single-valued* and *entire* function on the complex z-plane.
But the asymptotics look very different for the two ways of approaching ∞:

$z \to +\infty$ $\qquad$ $\dfrac{1}{2\sqrt{\pi}}\, z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}$

$z \to -\infty$ $\qquad$ Const. $(-z)^{-\frac{1}{4}}\left(e^{\frac{2}{3}z^{\frac{3}{2}}} - e^{-\frac{2}{3}z^{\frac{3}{2}}}\right)$

Three Odd Things About Stokes' Computation

3. Stokes got an even better approximation. For $z \to +\infty$

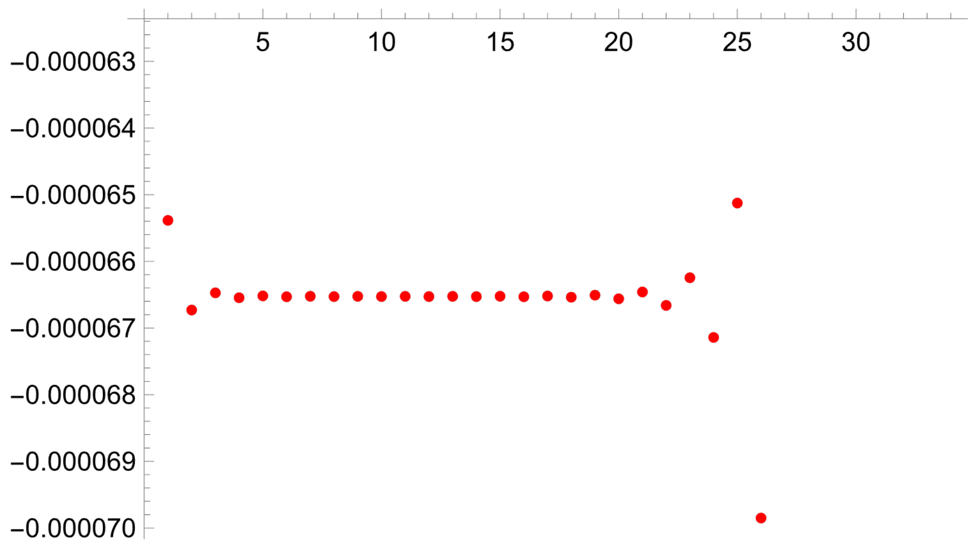
$$Ai(z) \sim \frac{1}{4\pi^{\frac{3}{2}}} \zeta^{-\frac{1}{6}} e^{-\zeta} \sum_{n=0}^{\infty} c_n \zeta^{-n}$$

$$\zeta = \frac{2}{3} z^{\frac{3}{2}} \qquad c_n = \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{(-2)^n n!} \sim n!$$

Series diverges: Has zero radius of convergence.

Stokes was confronting an example of an asymptotic series.

Fix z: $\mathrm{Err}(\mathrm{n}) := \quad Ai(z) - \frac{1}{4\pi^{\frac{3}{2}}} \zeta^{-\frac{1}{6}} e^{-\zeta} \sum_{j=0}^{n} c_j \zeta^{-j}$



$z = 4$

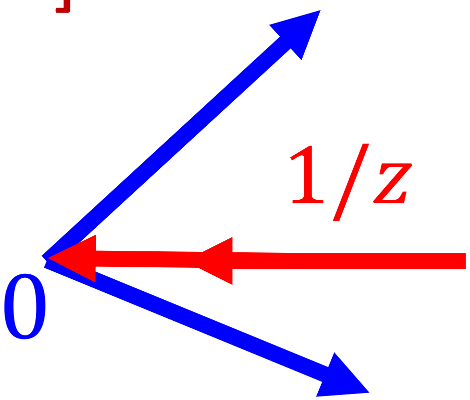
$n = 1, \ldots, 30$

For fixed z the best approximation has error $\sim \sqrt{|\zeta|} e^{-2\,|\zeta|}$

Modern definition of an asymptotic series [Poincare' [1886]…. See Dingle's book]

$1/z$

$f(z)$: an analytic function in some *wedge sector* around $z = \infty$



$$f(z) \sim \sum_{n=0}^{\infty} \frac{c_n}{z^n}$$

$$\lim_{|z|\to\infty} z^N \left(f(z) - \sum_{n=0}^{N-1} \frac{c_n}{z^n} \right) = 0$$

Why the wedge?

$e^{-z} \sim 0$ for $z \to \infty$ in RHP

$NOT\ for\ z \to \infty$ in LHP

Suppose $f(z)$ is entire.

$$\hat{f}(z) := f(z) + e^{-z} \quad \sim \quad f(z)$$

$z \to +\infty$ in the positive half-plane

$\hat{f}(z)$ and $f(z)$ are very different for $z \to \infty$ in the negative half-plane

Asymptotic Expansion and Analytic Continuation do not commute!

This is part of what's going on in Stokes' evaluation of Airy's integral – but there is so much more…

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3 Perturbative vs. Nonperturbative QFT

4 More About Airy & Stokes

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Stokes' asymptotic expansion has a very common analog in Quantum Field Theory.

What is QFT?

Quantum Field Theory:
The most successful framework for describing Nature in the history of Science.

OK. But What is QFT?

A table of correlation functions of local observables

Evaluation of Feynman path integrals

$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(\frac{x^3}{3} + z\,x\right)}\, dx$$

$$z = \hbar^{-\frac{2}{3}} \;\&\; x = \hbar^{-\frac{1}{3}} \phi$$

$$Z(\hbar) = \int_{-\infty}^{+\infty} d\phi\; e^{\frac{i}{\hbar} S(\phi)}$$

$$S(\phi) = \phi + \frac{\phi^3}{3}$$

Example of a 0-dimensional QFT

$$Z[\hbar; z] = \int d\phi_\alpha \; e^{\frac{i}{\hbar} S(\phi_\alpha; z)}$$

$$z \in \mathcal{M}$$

Many (possibly infinitely many) degrees of freedom ϕ_α

Semiclassical Analysis

Saddle point = solution ϕ_0

to equations of motion $\frac{\delta S}{\delta \phi}\big|_{\phi_0} = 0$

(e.g. saddle points $x_{\pm} = \pm\sqrt{-z}$ in the Airy integral)

Expand in quantum fluctuactions.

$$\phi = \phi_0 + \phi_q \qquad S[\phi; z] = S_{cl} + \phi_q \frac{\delta^2 S}{\delta^2 \phi}\Big|_{\phi_0} \phi_q + \Delta S$$

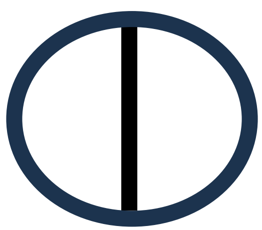
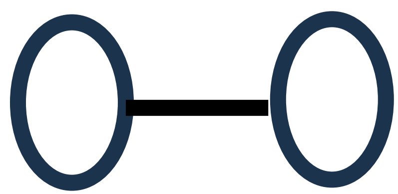
Expand $e^{-\mathrm{i}\,\Delta S/\hbar}$ as a power series in ϕ_q

and do the Gaussian integrals…

$$Z[\hbar; z\,] \sim \; e^{\frac{i}{\hbar}S_{cl}} \sqrt{2\pi i \frac{\hbar}{S''(\phi_0)}} \times \mathcal{S}$$

$$\mathcal{S} = 1 + \sum_{g \geq 2} \hbar^{g-1} \sum_{\Gamma \in \mathcal{G}_g} \frac{1}{|Sym\,\Gamma|} \prod_{v \in Vert(\Gamma)} c_v$$

$\mathcal{G}_g$: Set of (Feynman) graphs of Euler character $\chi = 1 - g$

$\mathcal{G}_2 = \{$  ,  ,  $\}$

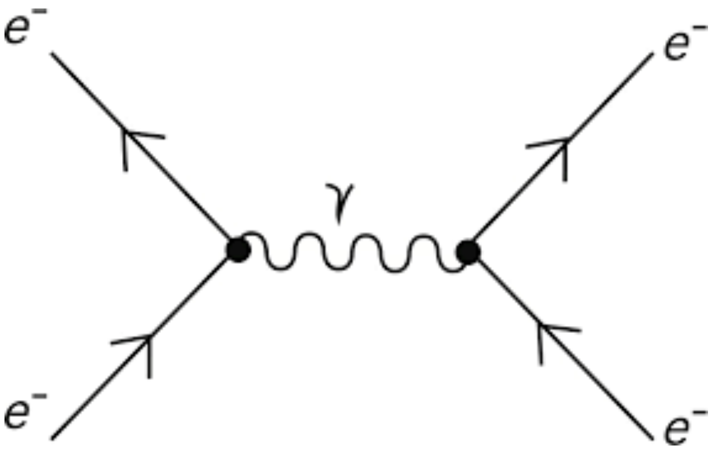
$$c_v = \frac{S^{(n(v))}(\phi_0)}{\left(\pm\sqrt{-S''(\phi_0)}\right)^{n(v)}}$$

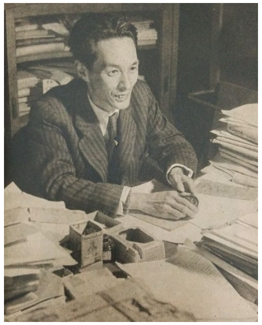
In a general QFT we face the same puzzles we encounter in Stokes' evaluation of the Airy integral

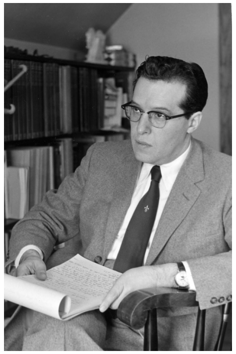
Which saddle points contribute?

The series in $\hbar$ is typically an asymptotic series.

Triumph of the perturbative approach: QED: Dyson, Feynman, Schwinger, Tomonaga (1946-1949)







$$\frac{g_e}{2} = 1 + \frac{\alpha}{2\pi} + \cdots$$

$$\left(\frac{g}{2}\right)_{expt} = 1.001\,159\,652\,180\,73\;(28\,ppt)$$

$$\left(\frac{g}{2}\right)_{thry} = 1.001\,159\,652\,177\,60\;(520\,ppt)$$

But doesn’t work so well for nonabelian gauge theories at large distance.

Perturbative analysis does not tell the whole story, in general.

Nonperturbative effects can be important

Central Issue In Much Of Modern Fundamental Physics

Physical mathematics dialogue of the past few decades has revealed special classes of theories or aspects of theories allowing nonperturbative analysis.

Typically involve ingenious and relentless pursuit of consequences of symmetry:

2d Rational Conformal Field Theory

Integrable models (Ising field theory, and relatives)

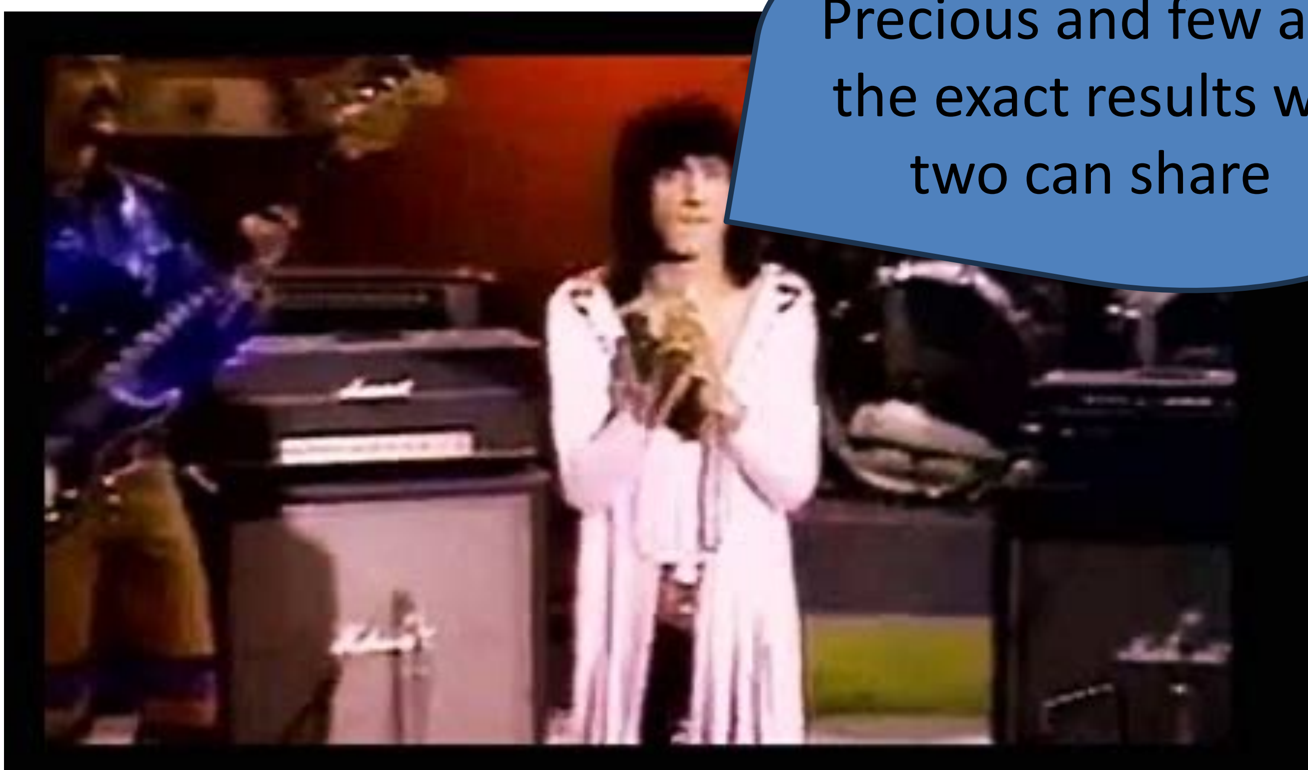
Solvable aspects of supersymmetric field theories

Anomalies

Precious & Few

Mathematically sound exact results in important cases like 4d Yang-Mills or QCD remains a major open problem

Value for humanity $\gg 10^6\ USD$



Precious and few are the exact results we two can share

Textbook Formulations Of QFT Are Inadequate

1. Putting theories with nontrivial RG flow on a firm mathematical basis has proven to be extremely difficult.

2. There is much more to a QFT than the correlation functions of the local operators.

A QFT is *NOT* equivalent to a table of correlation functions of local operators.

A full description involves nonlocal ``defects'' like Wilson-'t Hooft lines, and their generalizations.

3. Some QFT's are thought to have no description via an action principle involving fundamental field theoretic degrees of freedom.

4. Some QFT's have many different action principles involving totally different field theoretic degrees of freedom

5. Physical observables like S-matrix amplitudes encode causality and locality in highly nonobvious and nontrivial ways. (See Arkani-Hamed's talk.)

6. Many nontrivial field-theoretic phenomena have nontrivial geometrical reformulations.

So, what is QFT?

Functorial Approach

Atiyah, Segal [1988],…

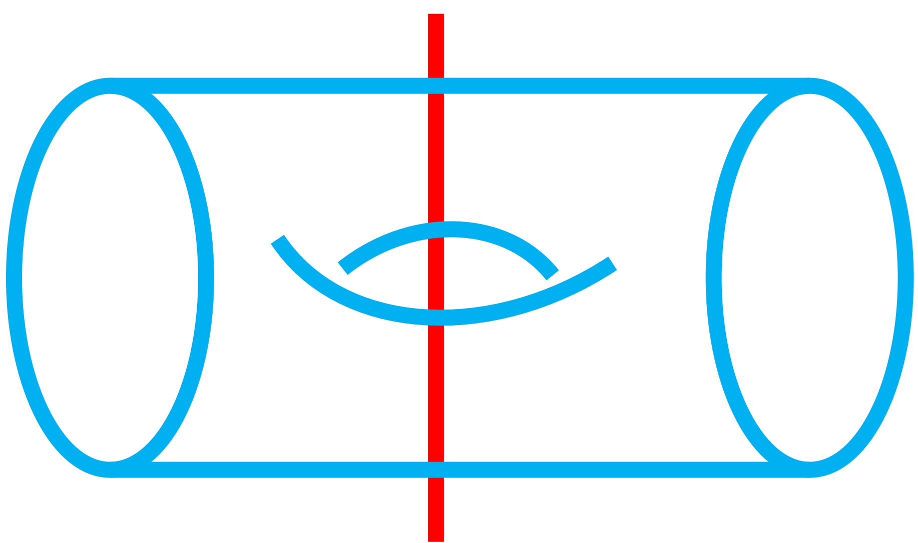
Physics tells us how states evolve from the past to the future.

A physical state of a system in n spacetime dimensions is:

A description of the way things are, at a fixed time, and is hence associated with a spatial slice, i.e. an $(n-1)$ −manifold: Space, at fixed time.

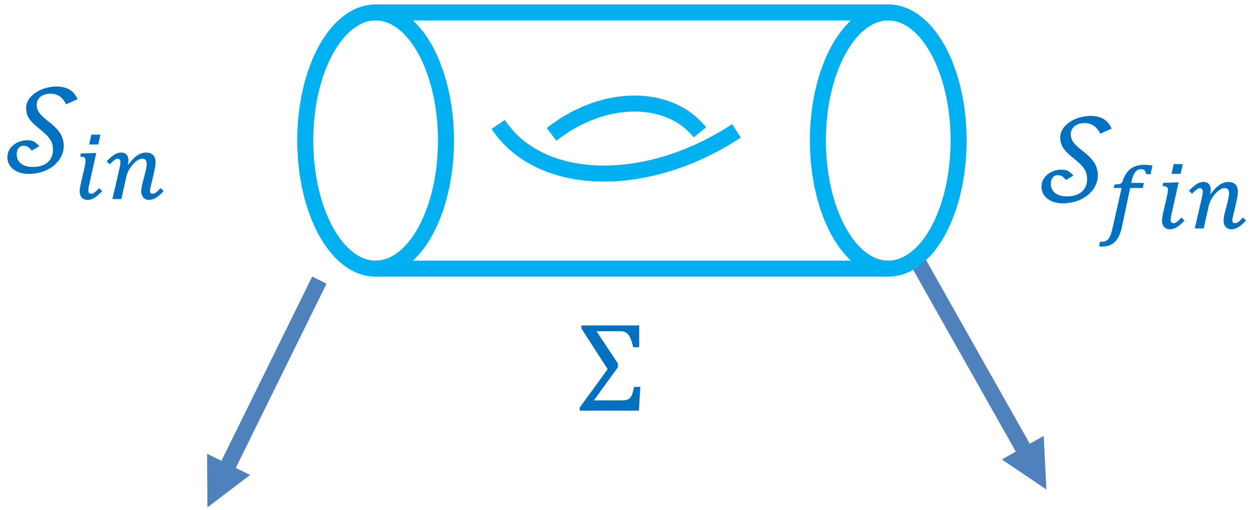
Functorial Approach

So spacetime evolution is described mathematically by a bordism from an initial spatial manifold to a final spatial manifold

$S_{initial}$  S_{final}

In QM the description of physical states involves Hilbert spaces.

So, to spatial manifolds we associate Hilbert spaces.



$$\mathcal{H}(\mathcal{S}_{in}) \Longrightarrow \mathcal{H}(\mathcal{S}_{fin})$$

Amplitude - a linear map

$$F(\Sigma): \mathcal{H}(\mathcal{S}_{in}) \rightarrow \mathcal{H}(\mathcal{S}_{fin})$$

DEFINITION:

An n −dimensional QFT is a monoidal functor from an n −dimensional bordism category to a monoidal n −category.

Many successes,

Leading experts Dan Freed, Mike Hopkins, …

Does QCD fit this framework?

Doesn’t address many of the points above.

So, what is QFT?

We know it when we see it.

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What about the choice of saddle point?

Recall Stokes' evaluation of $Ai(z)$:

One saddle contributes for $z \to +\infty$

Two saddles contribute for $z \to -\infty$

Generalize:

$$\mathcal{I}(k;\gamma;z) = \int_\gamma d\phi \ e^{k\, S(\phi;z)}$$

$S(\phi;z)$: Holomorphic function on a noncompact Kähler manifold X.

Example: $X = \mathbb{C}^\ell$ and $S(\phi;z)$ polynomial in ϕ.

$\gamma \subset X$: Noncompact real cycle:
Half the dimension of X

e.g. $X = \mathbb{C}$, γ comes in from ∞ and goes out to ∞

$k = \frac{1}{\hbar}$ in QM $k = \frac{1}{\lambda}$ in optics;

What cycles γ should we use so the integral is well defined?

What are the $k \to \infty$ asymptotics?

$$\mathcal{I}(k;\gamma;z) = \int_\gamma d\phi \; e^{\,k\,S(\phi;z)}$$

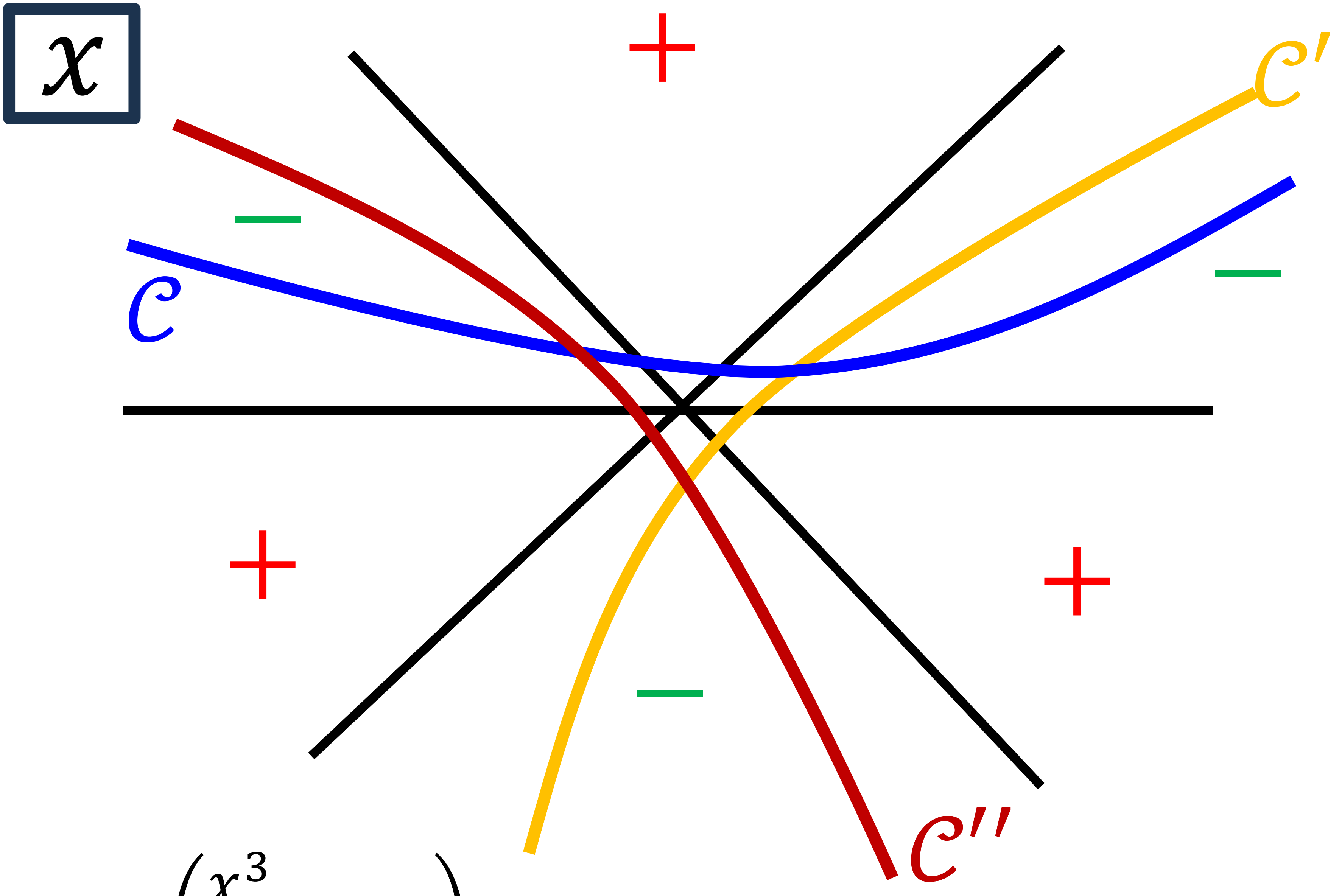
$$\left|e^{k\,S(\phi;z)}\right| = e^{h} \qquad\qquad h = Re\big(k\,S(\phi;z)\big)$$

When $\phi \to \infty$ along γ want $h \to -\infty$ and not $h \to +\infty$ for convergence.

Cauchy: Integral depends only on ***HOW*** γ behaves at $\phi \to \infty$

$$[\gamma] \in H_\ell(X, X_{-\infty}) \qquad\qquad \ell = \dim_{\mathbb{C}} X$$

$X_{-\infty} = \{\, \phi \in X \mid h(\phi) < -T \,\}$ sufficiently large T



$$S = i\left(\frac{x^3}{3} + z\,x\right)$$

$$\mathcal{C} + \mathcal{C}' + \mathcal{C}'' = 0$$

Morse Theory Indomitable!

Good basis of cycles from the downward Morse flows:

$$\frac{d\phi^i}{dt} = -g^{ij}\partial_j h \qquad h = Re\big(k\, S(\phi; z)\big)$$

$\mathcal{Cr} = \{\ p \mid dh(p) = 0\ \}$: Isolated Morse critical points

$\mathcal{D}_p(k,z) = \{\ \phi_0 \mid \phi_0$ flows to p for $t \to -\infty\ \}$

Steepest descent paths from $p \in \mathcal{Cr}$:
Submanifold of X of half-dimension

Lefshetz Thimbles

$$h = Re\big(k\,S(\phi;z)\big) \Rightarrow \text{Morse Flow}$$

= Hamiltonian Flow for $H = Im\big(k\,S(\phi;z)\big)$

$\mathcal{D}_p(k,z) =$ Steepest descent manifold

= Stationary phase manifold $\Rightarrow$

$k \rightarrow \infty$ asymptotics given by Feynman expansion:

$$\mathcal{J}\big(k;\mathcal{D}_p(k,z);z\big)$$

$$\sim e^{k\,S(p;z)} \left(\frac{2\pi}{-kS''}\right)^{\frac{1}{2}} \left[1+\mathcal{O}\left(\frac{1}{k}\right)\right]$$

Lefshetz thimbles form a basis of the relative homology

$$[\gamma] = \sum_{p \in \mathcal{C}r} n_p \, \left[\mathcal{D}_p(k,z)\right]$$

$$\mathcal{J}(k;\gamma;z) = \sum_{p \in \mathcal{C}r} n_p \, \mathcal{J}\big(k;\mathcal{D}_p(k,z);z\big)$$

Lefshetz Thimbles Jump Across Stokes Walls

Vary $(k, z) \in \mathbb{C} \times \mathcal{M}$

The thimbles $\mathcal{D}_p(k, z)$ vary smoothly (parallel transport by Gauss-Manin connection)

EXCEPT …. Across Stokes' walls!

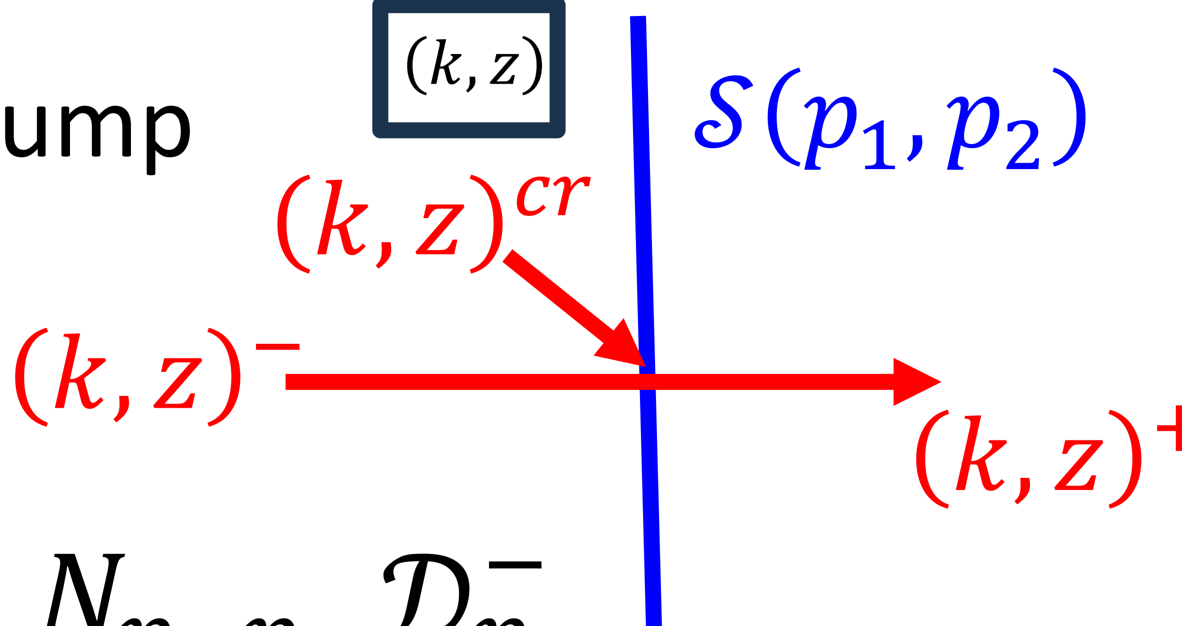
$\mathcal{S}(p_1, p_2) =$ Set of (k, z) such that:

1. $Im(k\, S(p_1; z)) = Im\left(k\, S(p_2; z)\right)$

2. $Re(k\, S(p_1; z)) > Re\left(k\, S(p_2; z)\right)$

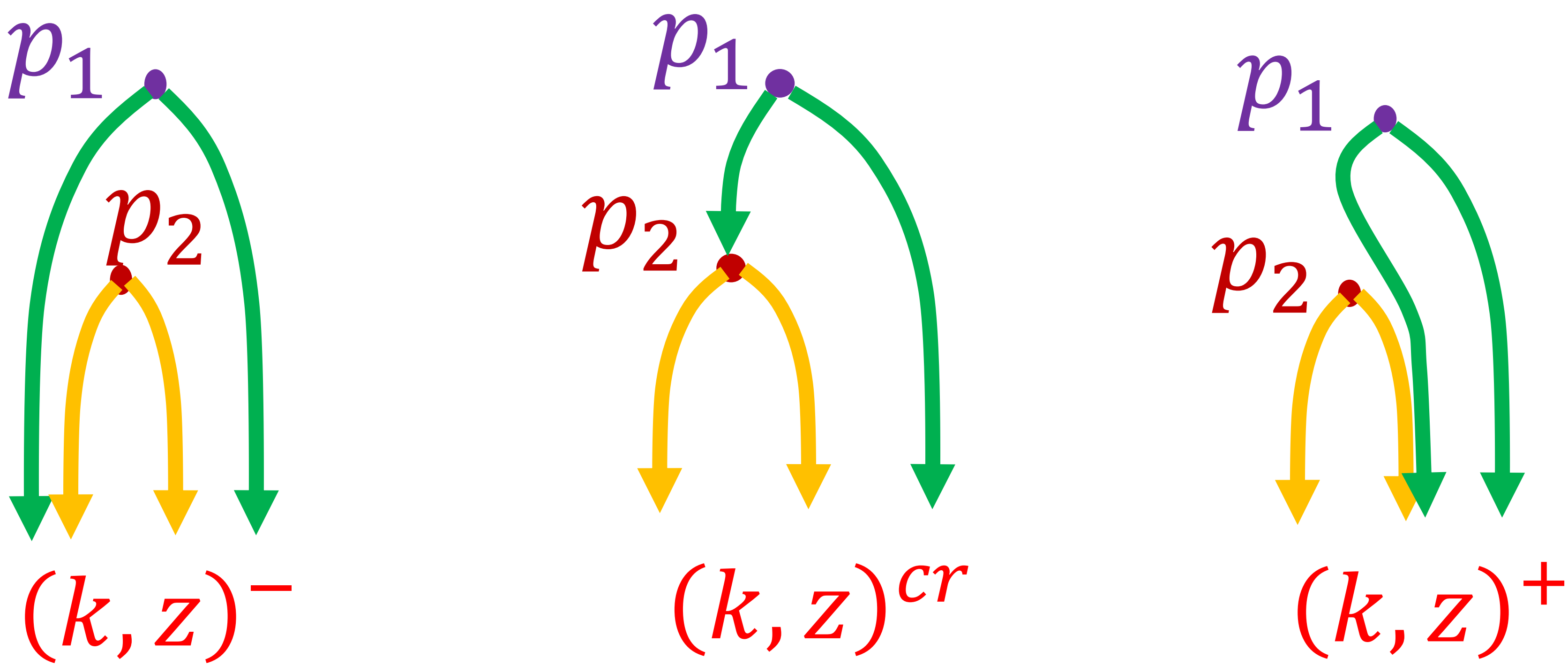
3. There is a Morse flow $p_1 \to p_2$

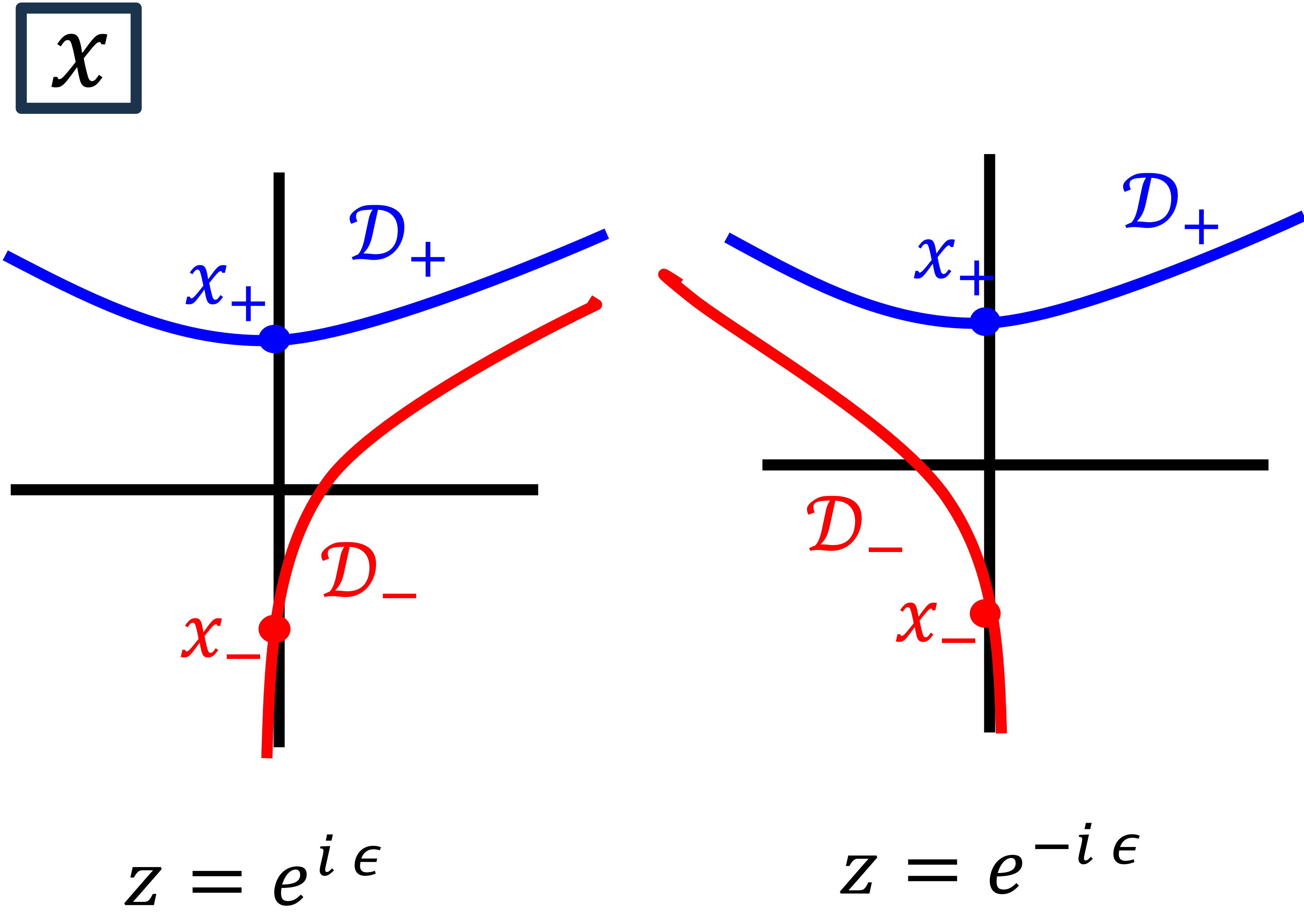
See How They Jump



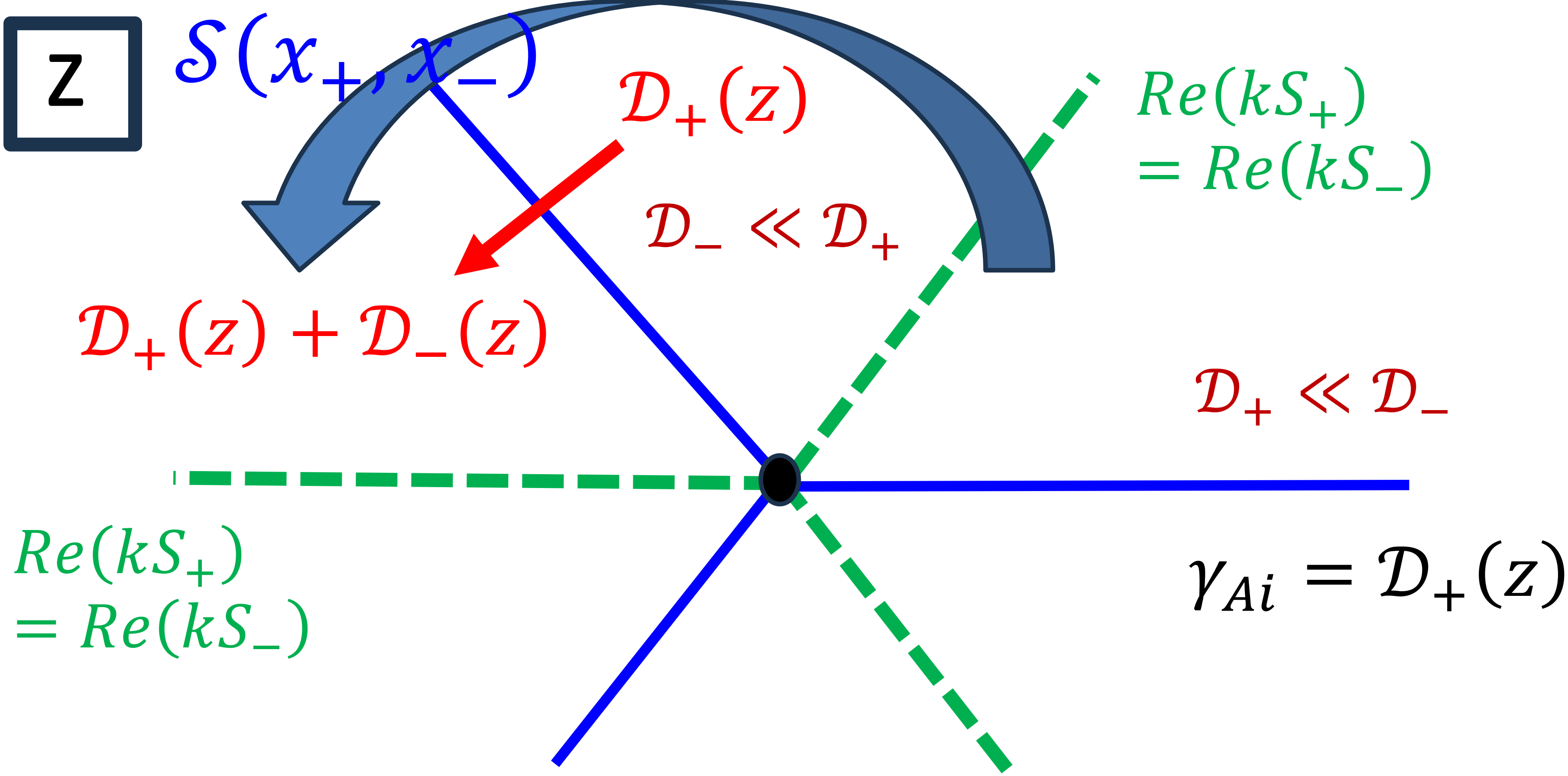
$$\mathcal{D}^+_{p_2} \cong \mathcal{D}^-_{p_2}$$

$$\mathcal{D}^+_{p_1} \cong \mathcal{D}^-_{p_1} + N_{p_1,p_2}\,\mathcal{D}^-_{p_2}$$





x
$\mathcal{D}_+$
x_+
$\mathcal{D}_-$
x_-
$z = e^{i\epsilon}$
$\mathcal{D}_+$
x_+
$\mathcal{D}_-$
x_-
$z = e^{-i\epsilon}$



Anti-Stokes walls:

Two thimbles exchange dominance.

March 19, 1857 letter to his fiancée

``When the cat’s away the mice may play. You are the cat and I am the mouse. I have been doing what I guess you won’t let me do when we are married, sitting up till 3 o’clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy’s, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill …. ’’

Quoted from M.V. Berry, ``Smoothing A Victorian Singularity’’

$$\gamma_{Ai} = [\mathbb{R}] = n_+(z)[\mathcal{D}_+(z)] + n_-(z)[\mathcal{D}_-(z)]$$

$n_{\pm}(z)$: Discontinuous in z

That is Stokes' great discovery:
Stokes' phenomenon

One model for ``BPS state wall-crossing''
Continues to be a hot topic of current research
in supersymmetric QFT

1. Physical Mathematics
2. George Airy & George Stokes
3. Perturbative vs. Nonperturbative QFT
4. More About Airy & Stokes
5. Chern-Simons-Witten Gauge Theory
6. Stokes & Differential Equations
7. Summary & Future Directions

Airy Integral of the 21st Century: Chern-Simons-Witten Gauge Theory

Gauge theory for gauge field A with gauge group G on a 3-manifold M_3

$$Z_{CS} \sim \int_{\frac{\mathcal{A}}{\mathcal{G}}} \mu(A)\, e^{\,i\,\frac{k}{4\,\pi}\int_{M_3} Tr\left(A\,dA + \frac{2}{3}A^3\right)}$$

Chern-Simons-Witten Gauge Theory

G compact: A piece of 21st –century mathematics that fell into the 20th century.

Lots of exact results.
Deep Mathematics & Physics

Relation to RCFT: Early example of holography;
FQHE & anyons
MTCs; WRT invariants; Quantum Groups;….

In the wider universe of QFT:
Among the Precious & Few:

Chern-Simons-Witten Gauge Theory

G finite dimensional noncompact Lie Group:

Much less understood

Growing literature:

Anderen-Kashaev; Beem; Collier; Dimofte; Eberhardt; Gaiotto; Garoufalidis; Gu; Gukov; Lenells, Marino; Mikhaylov; Teschner; Pasquetti; Pei; Putrov; Terashima; Vafa; Yamazaki; Witten; Zagier; …..

1. New 3-manifold invariants

2. Shed light on hyperbolic geometry

3. 3d Quantum Gravity

4. Closely related to 6d (2,0) theory

Chern-Simons-Witten Gauge Theory & Knot Homology

Choose complex gauge group G

$\mathcal{X} = \{$ complex G-gauge fields on M_3 $\}$

Morse function: $W = k \int_{M_3} Tr\left(A\, dA + \frac{2}{3} A^3\right)$

2d (2,2) LG theory with target $\mathcal{X}$ Morse function W

Physical formulation of knot homologies.

Haydys-Witten/Kapustin-Witten PDE's:
Analysis very nontrivial. Progress by Cliff Taubes

Potentially New Knot Invariants

Conjecture [Dimofte,Gaiotto,Khan,Moore,Neitzke,Yan]:

$a.)$ The h.e. class of the A_∞−category of $\mathfrak{Br}\left(CSLG(M_3)\right)$ is a 3-manifold invariant

b.) The h.e. class of A_∞ - algebras $End\left(\mathfrak{B}(L)\right)$ are (new?) colored link invariants.

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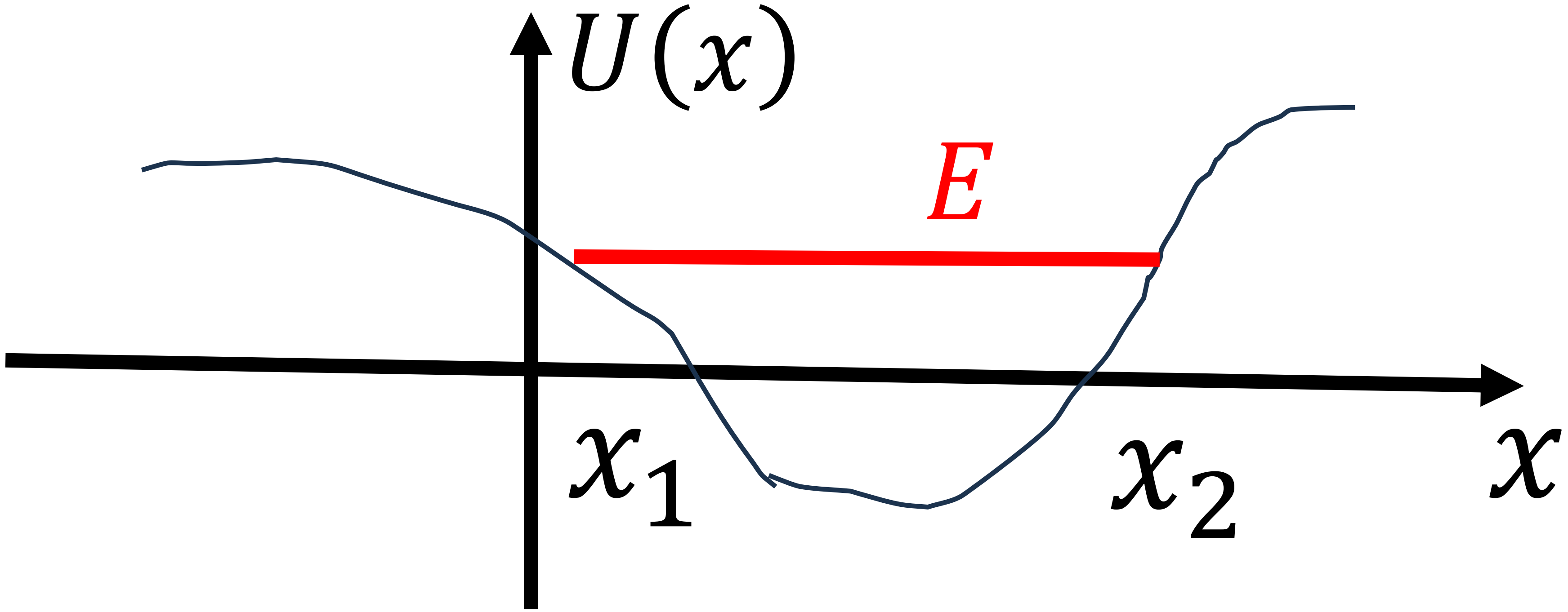
We’ve only told half the story about Stokes’ phenomenon

Stokes’ phenomenon plays and important role in the theory of Differential Equations

One-dimensional Schrödinger equation



$$\left(-\hbar^2 \frac{d^2}{dx^2} + U(x)\right)\psi(x) = E\,\psi(x)$$

Find L^2 −normalizable solutions. Only exist for special E

Zoom in on classical turning point x_2

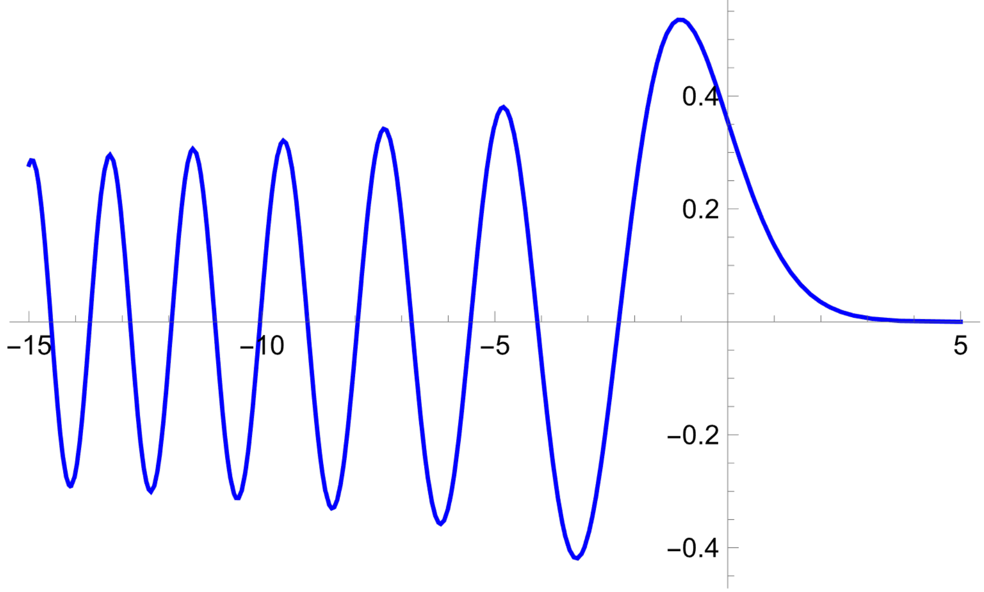
$$U(x) = E + U'(x_2)(x - x_2) + \mathcal{O}\left((x - x_2)^2\right)$$

$$z = \left(U'(x_0)\right)^{\frac{1}{3}} \hbar^{-\frac{2}{3}}(x - x_2)$$

$$\left(\frac{d^2}{dz^2} - z\right)\psi = 0$$

Only one ``small'' solution for $z \to +\infty$:

$$\psi(z) = const.\,Ai(z)$$



Startling prediction: There is a nonzero probability we'll all find ourselves stuffed into a small bottle of water.

Stokes' phenomenon has a manifestation in the theory of differential equations.

JWKB Approximation (1926)

(Invented by Liouville and Green in 1837.)

Ansatz: $$\psi(x) = e^{\frac{1}{\hbar}\int_{x_0}^{x} \lambda_x \, dx}$$

$$\lambda_x^2 = P(x) - \hbar \partial_x \lambda_x \qquad P(x) = U(x) - E$$

$$\lambda_x = \sqrt{P} - \frac{\hbar}{4}\frac{P'}{P} + \hbar^2 \sqrt{P}\,\frac{5(P')^2 - 4\,P\,P''}{32P^3} + \cdots$$

Two roots of $\sqrt{P}$ $\Rightarrow$ basis of (formal) solutions

1. Series in $\hbar$ is typically asymptotic

2. Borel resummation technique produces true functions with this asymptotic expansion, provided we continue $\hbar$ into the complex plane.

3. But such solutions will typically only exist in angular sectors in the complex $\hbar$ −plane. They are discontinuous across sectors:

A second version of Stokes' phenomenon.

(Relation to first: Solve Schrödinger via Feynman path integral.)

``Exact WKB method'': Balian, Delabaere, Ecalle, Parisi, Pham, Silverstone,Voros,….. [c. 1980]

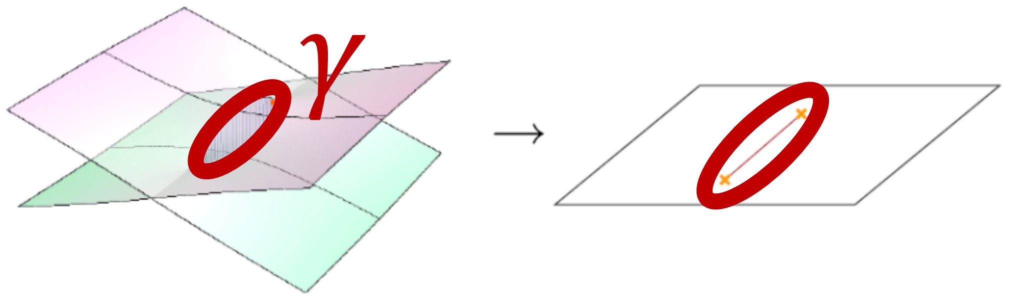
Exact JWKB starts by considering the ***geometry*** of the leading approximation:

$$\lambda_x^2 = P(x) - \hbar \partial_x \lambda_x \Rightarrow y^2 = P(x)$$

Equation for a Riemann surface Σ double-covering the x −plane, C

Exact JWKB is all about the geometry of flat connections over Σ , and C, and their relation to each other.

Bohr-Sommerfeld quantization:

Σ  C

$$\oint_\gamma \lambda = \left(n + \frac{1}{2}\right)\hbar \quad \Leftrightarrow \quad \exp\left(\frac{2\pi i}{\hbar}\oint_\gamma \lambda\right) = -1$$

Condition on the holonomy of a flat gauge field on Σ

Exact quantization is a condition on the holonomy of an $\hbar$ −modified flat gauge field on Σ

Geometrization of the quantization problem for Schrödinger operators

Amazingly, all this turns out to be closely connected to

1. Exact results in four-dimensional supersymmetric gauge theories.

2. Hitchin integrable systems

3. Hyperkahler geometry

Davide Gaiotto

Andy Neitzke

Geometrization Of Field Theoretic Phenomena

Class S & Spectral Networks: Geometrization of BPS States & RG Flow

Strong/Weak Coupling Dualities

AdS/CFT and Holography

Origin Of Gauge Theory

and many more …

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Physical mathematics is alive & well

We want, but don’t have, the perfect definition of Quantum Field Theory

Why is geometrization of field theoretic phenomena so widespread and so effective?


NOT
That's all Folks!