Schubert Eisenstein series and Poisson summation for Schubert varieties

YoungJu Choie

POSTECH, KOREA

.

Arithmetic Quantum Field Theory

March 27 2024

What is it?

Schubert Eisenstein series are defined by restricting the summation in a degenerate Eisenstein series to a particular Schubert variety.

It occurred in the work of Eisenstein series, Kronecker limit formula and connection among Eisenstein series, multiple Dirichlet series and Kashiwara crystals , etc , in disguised form.

Bump (1984), Bump-Goldfeld (1984), Vinogradov and Takhtajan (1978), Brubaker-Bump- Friedberg (2011) etc

- In the case of GL_2 this is known case (only two cases)
- In the case of GL_3 over $\mathbb{Q}$ it is known that these Schubert Eisenstein series have meromorphic continuations in all parameters (Bump- C)
- and conjectured the same is true in general .
- We revisit this problem.

Recent Progress

- It turns out that one can relate this conjecture to the program of Braverman-Kazhdan aimed at establishing generalizations of the Poisson summation formula.
- We prove the Poisson summation formula for certain schemes closely related to Schubert varieties and use it to refine and establish the conjecture in many cases.
- These results are joint work with Jayce Getz.

set up

1. G : a split reductive group over a global field F (ex: $G = GL_n, F = \mathbb{Q}$)
2. B : a Borel subgroup of G (ex: B upper triangular)
3. T : a maximal torus of G
4. $W = N(T)/T$: Weyl group (ex: permutation)
5. $\hat{T}$: the maximal torus of the L-group $\hat{G}(\mathbb{C})$
6. χ_ν: a character on $T(F)\backslash T(\mathbb{A}_F)$ parametried by $\nu \in \hat{T}(\mathbb{C})$
7. $\mathbb{A}_F$: the adele ring of F
8. $f_\nu \in Ind_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\chi_\nu)$: an element of the corresponding induced representation, so that

$$f_\nu(bg) = (\delta^{\frac{1}{2}}\chi_\nu)(b)f_\nu(g), b \in B$$

1. From **Bruhat decomposition** $G = \cup_{w \in W} BwB$

$$X := B\backslash G = \cup_{w \in W} Y_w, \;\; Y_w := Im(BwB \to B\backslash G) \text{ Schubert cell}$$

2. **Schubert variety**

$$X_w := \overline{Y_w} = \cup_{u \leq w} Y_u \quad \text{with Bruhat order } " \leq "$$

3. **Schubert Eisenstein series**

$$E_w(g, f_\nu) = \sum_{\gamma \in X_w(F)} f_\nu(\gamma g)$$

Schubert Eisenstein Series

For each $w \in W$,

$$E_w(g, f_\nu) = \sum_{\gamma \in X_w(F)} f_\nu(\gamma g)$$

Remark.

1. *If $w \in W$ is the long Weyl group element w_0, $E_{w_0}(g, f_\nu)$ is the usual Eisenstein series, so automorphic in G !*
2. *In general $E_w(g, f_\nu)$ is no longer automorphic in G!*

Question

1. *What is this?*
2. *How to compute this ?*
3. *What is good for?*

Bott-Samelson Varieties

How to understand this?: we use

the relationship among Bott-Samelson varieties, Schubert varieties and Bott-Samelson Factorization

1. α_i : simple roots
   $\sigma_i \in W$: the corresponding simple reflections
2. $\mathfrak{w} = (\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_k})$: a reduced decomposition of $w \in W$, which is a product of simple reflections $w = \sigma_{i_1} \cdots \sigma_{i_k} \in W$
3. P_j : the minimal parabolic subgroup generated by B and the one-dimensional unipotent subgroup U_{α_j} tangent to $-\alpha_j$.
4. define a left action of B^k on $P_{i_1} \times \cdots P_{i_k}$ by
   $$(b_1, \cdots, b_k) \cdot (p_{i_1}, \cdots, p_{i_k}) = (b_1 p_{i_1} b_2^{-1}, b_2 p_{i_2} b_3^{-1}, \cdots, b_k p_{i_k}).$$
5. The quotient $B^k \backslash (P_{i_1} \times \cdots \times P_{i_k})$ is the Bott-Samelson variety $Z_{\mathfrak{w}}$

Relationship between Bott-Samelson varieties and Schubert varieties

1. There is a morphism between Bott-Samelson varieties and Schubert variety

$$BS_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$$

induced by the multiplication map that sends

$$(p_{i_1}, \cdots, p_{i_k}) \to p_{i_1} \cdots p_{i_k}$$

2. This map is a surjective birational morphism.
3. Bott-Samelson varieties $Z_{\mathfrak{w}}$ are always nonsingular, so this gives a resolution of the singularities of X_w.

a Bott-Samelson factorization

1. If $BS_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ is an isomorphism:
   1. every $\gamma \in X_w$ can be written, uniquely,

$$\gamma = \iota_{\alpha_1}(\gamma_1) \cdots \iota_{\alpha_k}(\gamma_k),$$

$$\iota_{\alpha_i} : SL_2 \hookrightarrow G$$

   the embedding (Chevalley embedding) of SL_2 into G corresponding to a simple root α_i so that the image of ι_{α_i} lies in the Levi subgroup of P_i

   2. Using this factorization Schubert Eisenstein series can be written as building up the Schubert Eisenstein series by repeating SL_2 summations :

$$E_{\sigma_1 \cdots \sigma_k}(g, f_\nu) = \sum_{\gamma_k \in B_{SL(2)} \backslash SL(2,F)} E_{\sigma_1 \cdots \sigma_{k-1}}(\iota_{\alpha_k}(\gamma_k) g, f_\nu)$$

2. Even if $BS_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ is not an isomorphism, a modification of this method should be applicable

$G = GL_3(\mathbb{R})$

$G = GL_3, K = O(3), Z = Cent(G)$

1. (Iwasawa decomposition)

$$g = y_0 \begin{pmatrix} y_1 y_2 & x_1 y_2 & x_3 \\ & y_2 & x_2 \\ & & 1 \end{pmatrix} k_0 \in G/ZK, \quad y_0 \neq 0$$

2. with the Langlands parameters $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, take

$$(\delta^{\frac{1}{2}} \chi_\nu) \begin{pmatrix} y_1 y_2 & x_1 y_2 & x_3 \\ & y_2 & x_2 \\ & & 1 \end{pmatrix} = |y_1|^{2\nu_1 + \nu_2} |y_2|^{\nu_1 + 2\nu_2}$$

3. so

$$f_\nu(g) = |y_1|^{2\nu_1+\nu_2} |y_2|^{\nu_1+2\nu_2} f(k_0), f(k_0) = 1$$

4. $W = \langle \sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rangle$

$G = GL_3(\mathbb{R})$

1. (Bruhat decomposition)

$$G = \cup_{w \in W} BwB, \text{ with Borel subgroup } B$$

2.

$$\text{Flag variety } X := B \backslash G = \cup_{w \in W} Y_w,$$

$$\text{Schubert cell } Y_w := Im(BwB \to B \backslash G)$$

3. Schubert variety

$$X_w = \overline{Y_w} = \cup_{u \leq w} Y_u,$$

4. Study

$$E_w(g, f_\nu) = \sum_{\gamma \in X_w(\mathbb{Z})} f_\nu(\gamma g)$$

Schubert Eisenstein series

$$w \in \{id, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 = w_0\}$$

Using Bott-Samelson factorization we get :

1. for $w = \sigma_1$,

$$E_{\sigma_1}(g, f_{\nu_1,\nu_2}) = (y_1 y_2^2)^{\frac{1}{2}\nu_1+\nu_2} E\left(\frac{3\nu_1}{2}, \tau_1\right)$$

   with GL_2-Eisenstein series $E(s, \tau_1) = \sum_{c,d\in\mathbb{Z},(c,d)\neq(0,0)} \frac{{y_1}^s}{|c\tau_1 + d|^s}$

2. Similarly for $w = \sigma_2$
3. for $w = \sigma_1\sigma_2$,

$$E_{\sigma_1\sigma_2}(g, f_{\nu_1,\nu_2}) = \sum_{\gamma_2\in B_{SL_2}(\mathbb{Z})\backslash SL_2(\mathbb{Z})} E_{\sigma_1}\left(\begin{pmatrix} 1 & \\ & \gamma_2 \end{pmatrix} g, f_{\nu_1,\nu_2}\right)$$

4. similarly for $w = \sigma_2\sigma_1$

These are the case when $BS_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ is an isomorphism

GL_3 (Bump-C)

1. Note two reduced words $\mathfrak{w} = (\sigma_1, \sigma_2, \sigma_1)$ or $\mathfrak{w} = (\sigma_2, \sigma_1, \sigma_2)$ representing long element $w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$
2. If $\mathfrak{w}$ is either of these, $BS_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ is not isomorphism
3. But since it is birational, it is a local isomorphism on the complement of a closed subvariety
4. $BS_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_{w_0}$ consists of discarding the auxiliary piece of data, which is a subvariety of X_{w_0}, where $BS_{\mathfrak{w}}$ has a fiber that consists of more than one point
5. still possible

$$E_{w_0}(g, f_{\nu_1,\nu_2}) = \sum_{\gamma_3 \in B_{SL_2}(\mathbb{Z})\backslash SL_2(\mathbb{Z})} (E_{\sigma_1\sigma_2} - E_{\sigma_1}) \left(\begin{pmatrix} \gamma_3 & \\ & 1 \end{pmatrix} g, f_{\nu_1,\nu_2} \right) + E_{\sigma_1}(g, f_{\nu_1,\nu_2})$$

Application a connection with Kronecker limit formula for totally real cubic field :

1. Expansion of Eisenstein series at $\nu_1 = \nu_2 = 0$:

$$E_{w_0}(g, f_{\nu_1,\nu_2}) = \frac{\kappa(g)}{\nu_1} + \cdots$$

2. $\kappa(g)$ appear in the first Kronecker limit formula of totally real cubic field K over $\mathbb{Q}$ (Bump-Goldfeld (1983))
   - $\mathfrak{a}$: an ideal class of K
   - associate $\mathfrak{a}$ to a compact torus of GL_3
   - $L_{\mathfrak{a}}$: the period of $\kappa(g)$ over this torus
   - $$\lim_{s\to 0}(\zeta(s,\mathfrak{a}) - \frac{\rho}{s}) = L_{\mathfrak{a}}$$

GL_3 case: (Bump-C)

1. $\kappa(g)$ can be expressed in terms of Schubert Eisenstein series

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{\sigma_2\sigma_1}(g;0,0) + E^{**}_{\sigma_1}(g;1,0)\right] + c_0$$

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{\sigma_1\sigma_2}(g;1,0) + \phi_{\sigma_2}(g)\right] + c_0'$$

2. As a result, at the point where the residue is taken the Schubert Eisenstein series can be promoted to the full GL_3 automorphicity in the sense of the following:
   write

$$E^*_{\sigma_1}(g;\nu_1,\nu_2) = \frac{\rho}{2\nu_1} + \phi_{\sigma_1}(g;\nu_2) + 0(\nu_1)$$

   Then $\phi_{\sigma_1}(g;\nu_2)$ is essentially GL_2-automorphic form

$G = GL_3$: (Bump-C)

Results (C-Bump) contains :

1. $E_w(g, f_\nu)$ on $GL_3, w \in W$, have **meromorphic continuation to all values of the parameters** $\nu = (\nu_1, \nu_2)$
2. $E_w(g, f_\nu)$ have some **functional equations**
3. a Whittaker coefficient of $E_{\sigma_1\sigma_2}$, whose p-part is related to the **Demazure character**

   This is a generalization of the **Casselman-Shalika formula** that express the Whittaker coefficients of Eisenstein series in terms of characters of irreducible representations of the L-group
4. more $\cdots$, $\cdots$

Conjecture for general G

Conjecture

(Bump-C (2014))

- *The Schubert Eisenstein series always have* **meromorphic continuation** *to all values of the parameters*
- *Although they will not have the full group of functional equations that Eisenstein series has, they should have* **some functional equations**
- *linear combinations of Schubert Eisenstein series can be entire, that is, have no poles in the parameters.*
- *It may be possible to associate a Whittaker function with E_w. This would be an Euler product whose p-part may be expressed in terms of* **Demazure characters.**
- *more* $\cdots, \cdots,$

Recent Progress

Instead of using Bott-Samalson factorization
it turns out that one can relate this conjecture to the program of Braverman-Kazhdan aimed at establishing generalizations of the Poisson summation formula.

From now on this is a joint work with Jayce Getz

Another approach

Remark.

1. *Braverman and Kazhdan introduced the conjectures, now called "Poisson summation conjecture" ,*
   generalizing the Fourier transform and the Poisson summation formula
2. *Their conjectures may imply that quite general Langlands L-functions have meromorphic continuations and functional equations*
3. *Roughly it is about the existence of a nice "Schwartz space " over a spherical variety :*
   - *a spherical variety X for a reductive group G over F is G-variety with an open dense Borel -orbit in X.*
   - *For instance Flag varieties, symmetric spaces are spherical*

Poisson Summation conjecture (rough form)

Braveman-Kazhdan , Ngô , Lafforgue, Sakellaridis, etc :

Conjecture

G: a reductive group

- *Assume X is a G- spherical variety with smooth locus X^{sm}.*
- *Then there should be a Schwartz space $\mathcal{S}(X(\mathbb{A}_F)) < C^\infty(X^{\mathrm{sm}}(\mathbb{A}_F))$ and a map (Fourier transform)*

$$\mathcal{F}_X : \mathcal{S}(X(\mathbb{A}_F)) \longrightarrow \mathcal{S}(X(\mathbb{A}_F))$$

- *satisfying a certain "twisted-equivariance property" under $G(\mathbb{A}_F)$ such that for $f \in \mathcal{S}(X(\mathbb{A}_F))$ satisfying certain local conditions*
-

$$\sum_{x \in X^{\mathrm{sm}}(F)} f(x) = \sum_{x \in X^{\mathrm{sm}}(F)} \mathcal{F}_X(f)(x).$$

refered as " **Poisson summation conjecture"**

known cases

1. The only case that is completely understood is that of a vector space
2. Braverman-Kazhdan space :
3. Triples of quadratic spaces : Getz-Liu, Getz-Hsu
4. generalized Schubert varieties : C-Getz
5. more...

Note that the above cases are not necessarily spherical

Braverman-Kazhdan space

- $P < G$: a parabolic subgroup
- $P = MN$ Levi-decomposition
- $P^{der} = [P, P], M^{ab} = M/M^{der}$
- $X_P^0 := P^{der} \backslash G$: **Braverman-Kazhdan space**
- there is a left and right action of $M^{ab} \times G$ on X_P^0

$$X_P^0 \times M^{ab} \times G \to X_P^0$$
$$(x, m, g) \to m^{-1} x g$$

- $X_P = \overline{X_P^0}$: affine closure

Plücker embedding

How to study this space?
Use Plücker embedding

$$Pl_P : X_P \to V_P := \prod_{\beta \in \Delta_P} V_\beta$$

where V_β is an irreducible representation of G with of highest weight dual to coroot $\beta^\vee$

This embedding is a closed immersion

Plücker embedding

Example

$G = SL_3, P = B$

$$Pl_P : X_p \to V_P = \prod V_\beta$$

- Take a suitable ordering simple root β_1, β_2 of T in B
- V_{β_1}, V_{β_2} are standard representations $\mathbb{G}_a{}^3, \wedge^2 \mathbb{G}_a{}^3$
  Choose $(0,0,1)$ and $(0,1,0) \wedge (0,0,1)$ as the highest weight vectors
- Then
  1. $$Pl_P(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = (c, b \wedge c)$$
  2. $Pl_P(X_P) =$ the cone C points in F-algebra R are given by
  $$C(R) = \{(v_1, v_2) \in R^3 \times \wedge^2 R^3 : v_1 \wedge v_2 = 0\}$$

Recipe : action and Pairing

1. an embedding

$$Pl_P : X_P \to V_P := \prod_{\beta \in \Delta_P} V_\beta \quad \text{is } M^{ab}\text{-equivariant :}$$

$$Pl_P(m^{-1}g) = \omega_p(m) Pl_P(g)$$

with an isomorphism

$$\omega_P : M^{ab} \to \mathbb{G}_m$$

2. **a paring** between X_P and X_{P^*}
   - Take an opposite parabolic P^* of P
   - $V_P^\vee$ a representation of G dual to V_P
   - There is an embedding $Pl_{P^*} : X_{P^*} \to V_P^\vee$
   - Since there is a paring $<,>: V_P \times V_P^\vee \to \mathbb{G}_a$, it induces

$$\textbf{a paring} \quad <,>_{P|P^*}: X_P \times X_{P^*} \to V_P \times V_P^\vee \to \mathbb{G}_a$$

1. there is an **Intertwining operator**

$$\mathcal{R}_{P|P^*} : \mathcal{C}^{\infty}(X_P(F)) \to \mathcal{C}^0(X_{P^*}(F))$$

given by

$$\mathcal{R}_{P|P^*}(f)(y) := \int_{N_{P^*}(F)} f(ny)dn$$

2. **Schwartz space** $\mathcal{S}(X_P)$ is defined with the condition that Mellin transformation of f and $\mathcal{R}_{P|P^*}(f)$ have poles no worse than some product of Γ-factors

$$\mathcal{S}(X_P) \subset \mathcal{C}^{\infty}(X_P)$$

Fourier transform

1. Braverman-Kazhdan defined an explicit unitary isomorphism

$$\mathcal{F}_{P|P^*} : L^2(X_P) \to L^2(X_{P^*}),$$

commuting with actions of group $M^{ab} \times G$,
by re-normalizing the intertwining operator $\mathcal{R}_{P|P^*}$

Fourier transform

1. Getz-Hsu-Leslie (2023) gave an explicit formula: $\mathcal{F}_{P|P^*}$ is an isomorphism

$$\mathcal{F}_{P|P^*} : \mathcal{S}(X_P) \to \mathcal{S}(X_{P^*})$$

with formula

$$\mathcal{F}_{P|P^*} = \mu_P^{aug} \circ \mathcal{F}_{P|P^*}^{geo}$$

where

$$\mathcal{F}_{P|P^*}^{geo}(f)(y^*) = \int_{X_P^0} f(y)\psi(< y, y^* >_{P|P^*})dy$$

and an augmented operator

$$\mu_P^{aug} : \mathcal{S}(X_{P^*}) \to \mathcal{C}^\infty(X_{P^*}),$$

$$f \to \int_{M^{ab}} \psi(w_p(m))|w_p(m)|^{s+1}\delta_{P^*}^{\frac{\lambda}{2}}(m)f(m^{-\lambda}x)dm$$

$\lambda \in \mathbb{Z}$ and $s \in \mathbb{C}$ depends on $M^\vee$ and $N^\vee$

pursuing Schubert varieties and the Bott-Samelson resolution to study higher rank analogues we observe:

1. (observation)
   left side of $X_w = \overline{BwB}$ often fixed by larger group then B

$$X_{\sigma_1\sigma_2} = \overline{B\sigma_1\sigma_2 B} = P_{2,1}\sigma_1\sigma_2 P_{1,2},$$
$$P_{2,1} := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ & & h \end{pmatrix} \in SL_3 \right\}, \quad P_{1,2} := \left\{ \begin{pmatrix} a & b & c \\ & e & f \\ & g & h \end{pmatrix} \in SL_3 \right\}$$

2. Instead of pursuing Schubert varieties and the Bott-Samelson resolution to study higher rank analogues of the left hand side of the above equation, we generalize and study the right hand side directly.

Generalized Schubert cells (C-Getz)

1. a pair of parabolic subgroups $P < P' \leq G$ such that P is maximal in P'
2. $Y \subset G$: any variety stable under left multiplication by P' (ex: $Y = X_w = \overline{BwB}, Y = \overline{P\gamma H}, H \leq G$)
3. $Y_P := Im(Y \to X_P^0) = P^{der} \backslash Y$
4. **Schwartz Space**

$$\mathcal{S}(Y_P(\mathbb{A}_F)) \subset C^\infty(X_P(\mathbb{A}_F))$$

5. $\mathcal{S}(Y_P(\mathbb{A}_F))$ is preserved under the left action of M^{ab} so we get **Mellin transforms**

$$\mathcal{S}(Y_P(\mathbb{A}_F)) \to Ind_P^G(\chi_\nu)|_{Y_P}$$

$$f \to f_\nu(y) = \int_{M^{ab}(F)} \delta_P^{\frac{1}{2}}(m)\chi_\nu(\omega_P(m))f(m^{-1}y)dm$$

f_ν converges absolutely for $Re(\nu_0)$ large

Result (C- Getz (2022))

Define the following generalized Schubert Eisenstein series:

for $f \in \mathcal{S}(Y_P(\mathbb{A}_F)) \to f_\nu \in Ind_P^G(\chi_\nu)$

$$E_{Y_P}(g, f_\nu) := \sum_{\gamma \in M^{ab}(F) \backslash Y_P(F)} f_\nu(\gamma g)$$

Theorem

(C-Getz (2022))

- *Let $f \in \mathcal{S}(Y_P(\mathbb{A}_F))$ and $f_\nu \in Ind_P^G(\chi_\nu)$, $\nu = (\nu_0, \cdots, \nu_k)$*
- *Fix $\nu_1, \cdots, \nu_k$ such that $\mathrm{Re}(\nu_i)$ is sufficiently large for $1 \le i \le k$.*
- *Then $E_{Y_P}(g, f_\nu)$ and $E_{Y_{P^*}}(g, \mathcal{F}_{P|P^*}(f)^*_\nu)$ are meromorphic in ν_0.*
- *Moreover one has*

$$E_{Y_P}(g, f_\nu) = E_{Y_{P^*}}(g, \mathcal{F}_{P|P^*}(f)^*_\nu).$$

set-up:

- Take $w \in G$,
- P': the stabilizer of $\overline{PwB}$ under the left action of G
- we may consider the family of generalized Schubert Eisenstein series

$$E_{P^{der}\backslash \overline{PwB}}(g, f_\nu)$$

Then Theorem provides, when $P < P'$ is maximal,

REMARK.

1. *one functional equation of* $E_{P^{der}\backslash \overline{PwB}}(g, f_\nu)$ *with respect* ν_0
2. *the meromorphic continuation of* $E_{P^{der}\backslash \overline{PwB}}(g, f_\nu)$ *in* ν_0
3. *a linear combinations of these generalized Schubert Eisenstein series is entire in* ν_0 *for fixed* $\nu_1, \cdots, \nu_k$.

PROOF

1. To prove Theorem we prove Poisson conjecture for Y_P
2. we don't know if $Y_P = P^{der} \backslash Y$, Y is left invariant under P', is spherical , but it is true for many cases)
3. we proved meromorphic continuation for one parameter ν_0 although it should hold for all ν_j

Thank you very much!