Homological algebra and moduli spaces in Topological Field theories

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1980's

Counting the number of solutions of non-linear PDE gives an interesting invariant.

Donaldson: Number of solution of ASD equation on 4 manifolds.

$$F_A + *F_A = 0$$ A : **a connection on** X^4

Gromov: Number of solution of non-linear Cauchy-Riemann equation from a Riemann surface to a symplectic manifold

$$\bar{\partial} u = 0$$ $u : \Sigma \to X$

Note

One needs auxiliary choices to define equation.

$$F_A + {}^*F_A = 0$$

A metric (or conformal structure) to define $*$ Hodge star.

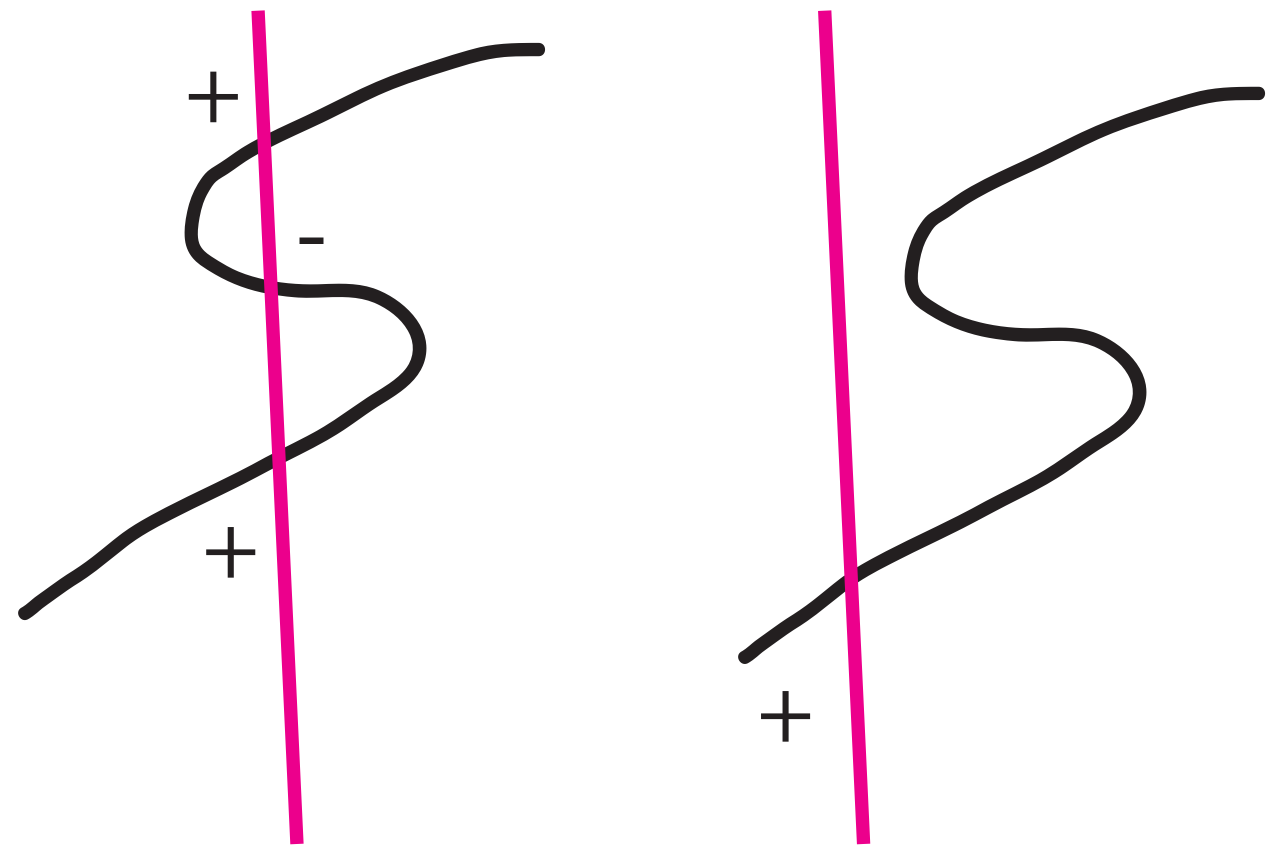
$$\overline{\partial} u = 0$$

An almost complex structure to define $\overline{\partial}$

Fact

The solution set depends auxiliary choices
However the number counted with sign is independent of such a choice.

Similar to the intersection number.



This is what topological field theory means.

The solution set depends on auxiliary choices.
However the number counted with sign is independent of such a choice.

Not all the solution space of non-linear PDE has this property.

○

ASD equation on 4 manifolds

non-linear Cauchy-Riemann equation

×

Yang-Mills equation on 4 manifolds

Harmonic map equation

The solution set depends on auxiliary choices.
However the number counted with sign is independent of such a choice.

Physicist's interpretation:

Those numbers are integration of certain 'closed differential forms' on infinite dimensional space:

$$\int_{a \in \mathfrak{X}} \exp(F(a))\Phi(a)\mathfrak{D}a$$

by `super symmetry' it cancel out in most of the places and reduces to a integration of a finite dimensional space the critical point set of $F(a)$ that is the set of solution of the non-linear PDE.

1980's

Floer : Counting the number → Defining the groups

Donaldson: Number of solution of ASD equation on 4 manifolds X^4.

Floer : Defining the group, Floer homology (or instanton homology) of 3 manifolds M^3

Gromov: Number of solution of non-linear Cauchy-Riemann equation from a Riemann surface to a symplectic manifold

Floer : Defining the group, Floer homology using holomorphic map from disks (2 manifolds with boundary) or $\mathbb{R} \times S^1$

Floer : Defining the group, Floer homology (or instanton homology) of 3 manifolds M^3

Chain complex

Generator: A flat connection on M^3

Boundary operator: Count the number of the solution of ASD equation on $M^3 \times \mathbb{R}$.

Floer : Defining the group, Floer homology using holomorphic map from disks (2 manifolds with boundary)

(X^{2n}, ω) a symplectic manifold. (ω is a closed two form, $\omega^n =$ volume form)

$L_i^n \subset X^{2n}$ Lagrangian submanifold $\omega|_{L_i} = 0$

$HF(L_1, L_2)$ Lagrangian Floer homology

$HF(L_1, L_2)$ Lagrangian Floer homology

Chain complex $CF(L_1, L_2)$

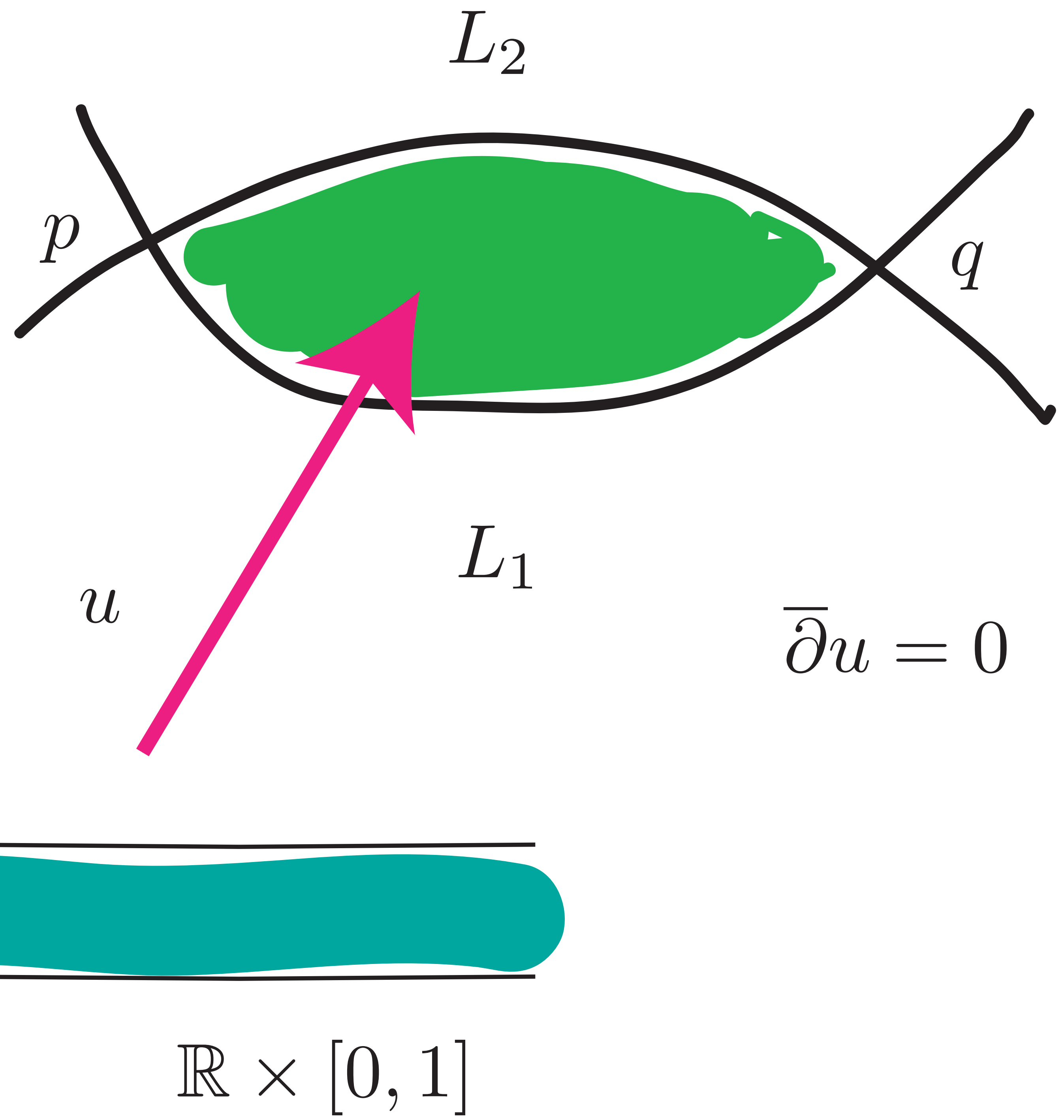
Generator: An intersection point $p \in L_1 \cap L_2$

Boundary operator: Count the number of holomorphic map

$$u : \mathbb{R} \times [0,1] \to X$$

$$\partial[p] = \sum n_{p,q}[q]$$

$n_{p,q}$



Donaldson or Gromov-Witten invariant

The solution set depends on auxiliary choices
However the number counted with sign is independent of such a choice.

Floer homology

The chain complex $CF(L_1, L_2)$ $CF(M^3)$

depends on auxiliary choices.

However its homology (or chain homotopy type) is independe
of the choices.

Floer homology

homology (or chain homotopy type) is independe
of the choices.

As we go on, the ‘structure’ we obtain is more and more sophisticated and what we mean by ‘well-defined’ becomes more involved.

This is the typical place where ‘homological algebra’ appears in topological field theory.

Donaldson invariant and Floer homology

(Topological Field theory picture: late 1980's)

$$\partial X^4 = M^3$$

Z_X : (Donaldson) invariant is not a number

but is an element of $HF(M^3)$: Floer (instanton) homology.

$$\partial X_1 = M^3 = -\partial X_2 \qquad X = X_1 \cup_{M^3} X_2$$

$$Z_X = \langle Z_{X_1}, Z_{X_2} \rangle$$

inner product in Floer homology

Warning: This is an oversimplified picture.

Donaldson invariant is not actually a number but is a polynomial defined on homology group.

Donaldson invariant is not always well-defined. It is ill defined if intersection form on second cohomology is negative definite and has a ‘chamber structure and wall crossing’ if the intersection form has exactly one positive eigenvalue.

Somewhat similar to stable and unstable homotopy theory.

$\pi_{n+m}(S^m X)$ is independent of m is m is sufficiently large,
but for small m the behavior is harder (more interesting).

$X^4 \# m(S^2 \times S^2)$ differential topology is simple if m is large
gauge theory invariant is mostly trivial.

Warning: This is an oversimplified picture.

There is a axiomatic study of topological field theory, which are interesting.

However for the theory defined by non-linear PDE those axiom is satisfied only roughly.
Most of the interesting geometric application or non-trivially of such ‘topological field theory’ comes from the phenomenon in unstable range, where axiom is not literally satisfied.

Donaldson invariant and Floer (Instanton) homology

Topological Field theory picture: late 1980's.

Gromov-Witten invariant and Lagrangian Floer homology

The relation is not so simple as the gauge theory case,
but we now have an answer which I will explain later.

Categorification of invariants:

Started again in late 1980’s

G. Segal: Categorification of conformal field theory.

Witten's 3 manifold (and knot) invariant.

3 manifold X^3 + possibly a link $l = S^1 \cup \ldots \cup S^2$ in it.

$$Z(X,l) = \int_{a \in \mathfrak{B}} \exp(cs(a))\Phi_l(a)\mathfrak{D}a$$

path integral on the set $\mathfrak{B}$ of all connections

Witten invariant ?

Donaldson invariant Floer (Instanton) homology

Witten invariant Conformal block associated to 2 manifold

Donaldson invariant Floer (Instanton) homology

Including knot this picture gives a topological field theory description of Jones polynomial (Witten).

G. Segal: Categorification of conformal field theory.

G. Segal: Categorification of conformal field theory.

M^3	Witten invariant
Σ^2	Conformal field theory
S^1	Category

Axiomatic study of topological Field theory.

X^n	Number
X^{n-1}	Vector space or group
X^{n-2}	Category
X^{n-3}	2-Category

$\bullet \bullet \bullet$

This is a general idea and studied much. Started early 1990's.

How this works in the case of gauge theory ?

X^4 Donaldson invariant Z_X
Number

M^3 Floer homology $HF(M)$
Group

Σ^2 A category $\mathscr{F}(\Sigma^2)$

$$\partial X^4 = M^3$$

$Z_X \in HF(M)$

$$\partial M^3 = \Sigma^2$$

HF_M an object of $\mathscr{F}(\Sigma^2)$

Donaldson (1992)

$R(\Sigma)$ the moduli space of flat connections on Σ^2

symplectic manifold (Goldman)

Σ^2 A category $\mathscr{F}(\Sigma^2)$

$\mathscr{F}(\Sigma^2)$ is a category whose object is a Lagrangian submanifold of $R(\Sigma)$

$\partial M^3 = \Sigma^2$ HF_M an object of $\mathscr{F}(\Sigma^2)$

$HF_M = R(M)$ $R(M) \to R(\Sigma)$ restriction, gives Lagrangian immersion (generically).

Donaldson (1992)

$\mathscr{F}(\Sigma^2)$ is a category whose object is a Lagrangian submanifold of $R(\Sigma)$

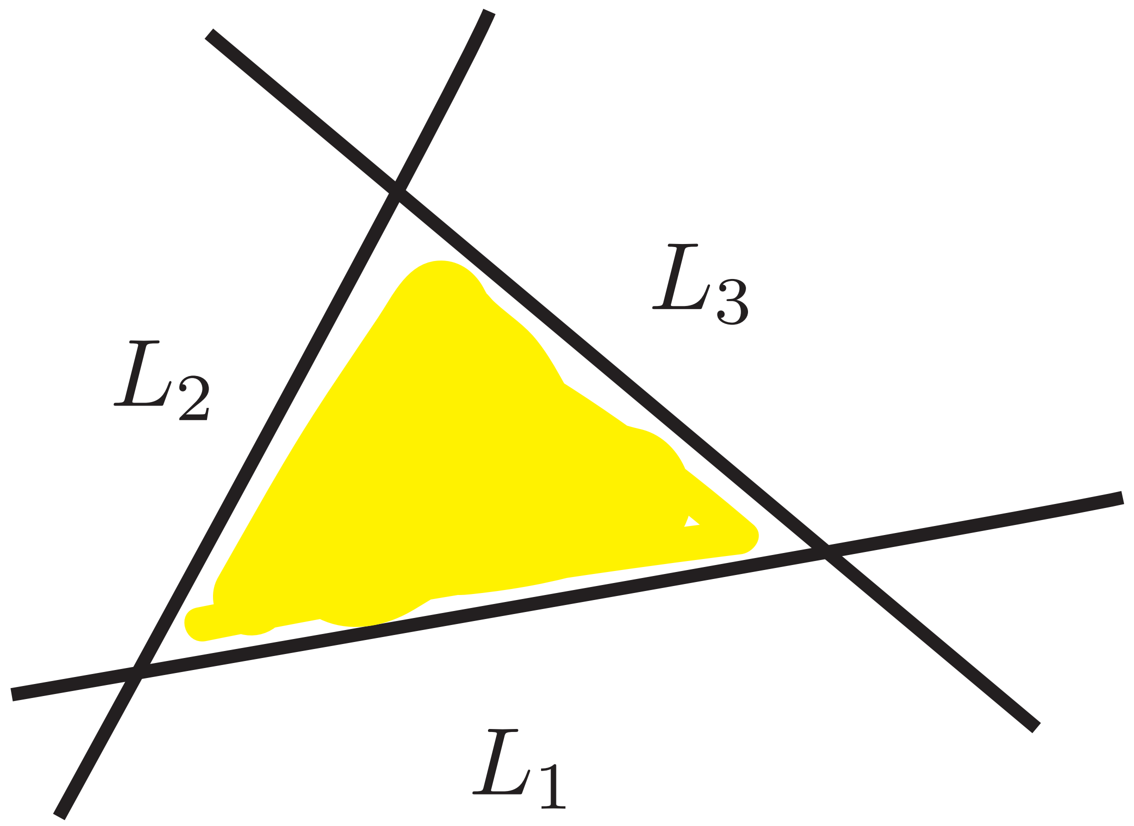
$L_i \subset R(\Sigma)$ Lagrangian subspace = object of $\mathscr{F}(\Sigma^2)$

Space of morphisms from L_1 to L_2 = $HF(L_1, L_2)$

(Lagrangian Floer homology)

composition of morphisms: count triangle

$$HF(L_1, L_2) \otimes HF(L_2, L_3) \to HF(L_1, L_3)$$



Donaldson (1992)

$\mathscr{F}(\Sigma^2)$ is a category whose object is a Lagrangian submanifold of $R(\Sigma)$

This proposal is closely related to a conjecture by Atiyah-Floer.

$$M^3 = H_g \cup_{\Sigma_g} H'_g$$

H_g, H'_g handle bodies

$$HF(M^3) = HF(R(H_g), R(H'_g))$$

Instanton Floer homology

Lagrangian Floer homology

$$HF(M^3) = HF(R(H_g), R(H'_g))$$

Instanton Floer homology

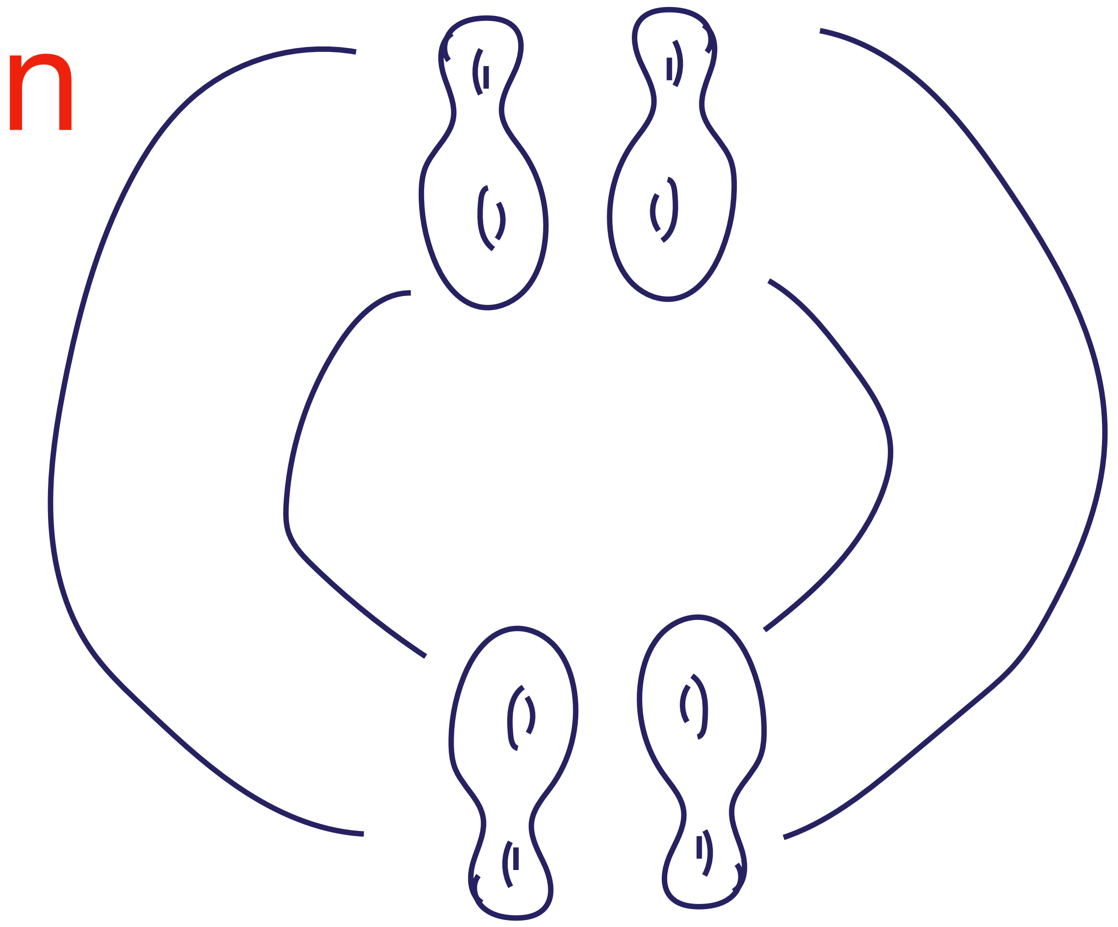
Lagrangian Floer homology

Many attempts to solve Atiyah-Floer conjecture in early 1990's.

Salamon, Yoshida, Lee-Li …..

One variant was proved by Dostoglou-Salamon

$$M^3 = (\Sigma_g \times [0,1]) \cup_{\Sigma_g \sqcup \Sigma_g} (\Sigma_g \times [0,1])$$



Donaldson (1992)

$\mathscr{F}(\Sigma^2)$ is a category whose object is a Lagrangian submanifold of $R(\Sigma)$

This is mostly or in principle correct formulation, but

$\Sigma = S^2 \qquad R(\Sigma) = pt$

$\partial M^3 = S^2 \quad R(M) \to R(\Sigma)$ finite to one map.

$R(M) = R(M^+) \quad M^+ = M \cup D^3$

$R(M)$ immersed Lagrangian submanifold (of a 0 dim. symplectic manifold)
remember the set of Flat connections (generator)
but not boundary operator.

$\partial M^3 = S^2$ $R(M) \to R(\Sigma)$ finite to one map.

$R(M) = R(M^+)$ $M^+ = M \cup D^3$

$R(M)$ immersed Lagrangian submanifold (of a 0 dim. symplectic manifold)
remember the set of Flat connections (generator)
but not boundary operator.

So to obtain a correct relative invariant HF_M

it is slightly different from an object of $\mathscr{F}(\Sigma^2)$

Category theory

Yoneda's Lemma

category of functors

Embedding of categories $\mathcal{C} \to \mathcal{FUNK}(\mathcal{C}^{\mathrm{op}}, Sets)$

$$c \mapsto (c' \mapsto \mathcal{C}(c', c))$$

A functor $\mathcal{C}^{\mathrm{op}} \to Sets$

maybe regarded as an enhancement of an object of $\mathcal{C}$

The Yoneda embedding maybe regarded as a 'categorificaiton' of the embedding: $C^\infty(M) \to$ distribution on M.

Conjecture (F 1993)

$\partial M^3 = \Sigma^2$ HF_M is a functor from $\mathscr{F}(\Sigma^2)$

Note morphisms of $\mathscr{F}(\Sigma^2)$ is a group or a chain complex.

So analogue of Yoneda embedding is not

$$\mathcal{F}(\Sigma^2) \to \mathcal{FUNK}(\mathcal{F}(\Sigma^2), Sets)$$

but is

$$\mathcal{F}(\Sigma^2) \to \mathcal{FUNK}(\mathcal{F}(\Sigma^2), ch)$$

ch is the category of chain complexes.

For Yoneda embedding

$$\mathcal{F}(\Sigma^2) \to \mathcal{FUNK}(\mathcal{F}(\Sigma^2), ch)$$

to work, we need more homological algebra.

$\mathscr{F}(\Sigma^2)$ the space of morphisms is not Floer homology group $HF(L_1, L_2)$
but a chain complex $CF(L_1, L_2)$ which defines Floer homology.
Composition of morphism is associative only up to homotopy .....

A_∞ category

we associate a A_∞ category $\mathscr{F}(Y)$ to a symplectic manifold Y.

Conjecture (F 1993)

$\partial M^3 = \Sigma^2$ HF_M is an A infinity functor from

$\mathscr{F}(R(\Sigma^2))$ to *ch.*

Such a functor nowadays is called A infinity module.

$\partial M^3 = \Sigma^2$

More explicitly A infinity functor $\mathscr{F}(R(\Sigma^2)) \to ch$

associate a chain complex $CF(M; L)$ to a Lagrangian submanifold $L \subset R(\Sigma)$ which is functorial with respect to *L.*

An idea to obtain $CF(M; L)$ is studying ASD equation on $M \times \mathbb{R}$ with boundary condition given via L on $\partial M \times \mathbb{R} = \Sigma \times \mathbb{R}$

Studying ASD equation on

$M \times \mathbb{R}$ with boundary condition given via L on $\partial M \times \mathbb{R} = \Sigma \times \mathbb{R}$

This is a difficult boundary valued problem to study.

I was working on it in the second half of 1990's but that research was not completed that time.

In the first half of 2000's Salamon-Wehrheim gave a rigorous construction of $HF(M;L)$

However in their construction $L \mapsto HF(M;L)$ is not yet functorial.

In Heegard Floer theory (something which is isomorphic to Seiberg-Witten Floer theory but studying them with minimal use of non-linear PDE)

For $\partial M^3 = \Sigma^2$

an A infinity module is associated to certain DGA associated to Σ^2

by R. Lipshitz, P. Ozsváth, D. Thurston. in second half of 2000's.

With A. Daemi and partially with M. Lipyanski
I am on the way writing the 'functorial'
construction of $L \mapsto HF(M;L)$
in the situation when $R(\Sigma)$ space of flat connection
has no singularity.

(2017 - ???)

Let me go back A_∞ category $\mathscr{F}(Y)$ associated to a symplectic manifold Y.

Lagrangian Floer homology is cumbersome object to define and study.

Naive expectation to Lagrangian Floer homology.

1) $L_1, L_2 \subset Y$ ➡ $HF(L_1, L_2)$ defined

2) $\operatorname{rank} HF(L_1, L_2) \le \# L_1 \cap L_2$

3) It is invariant via Hamiltonian deformation

$$HF(L_1, L_2) \cong HF(\varphi(L_1), \varphi'(L_2))$$

4) $L_1 = L_2 = L$ ➡ $HF(L, L) = H(L)$

1) $L_1, L_2 \subset Y$ → $HF(L_1, L_2)$ defined

2) $\text{rank}\ HF(L_1, L_2) \leq \# L_1 \cap L_2$

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4) $L_1 = L_2 = L$ → $HF(L, L) = H(L)$

Actually this is oversimplification and in general not true.

$L_1, L_2 \subset \mathbb{R}^n$ 1)2)3) is true

↓

$HF(L_1, L_2) = 0$ → 4) does not hold.

$L_1, L_2 \subset \mathbb{R}^{2n}$ 1)2)3) is true

$$HF(L_1, L_2) = 0$$

So is Lagrangian Floer theory trivial on $\mathbb{R}^{2n}$?

No there are applications in such a case.

(eg. Classification of irreducible 3 manifolds which is a Lagrangian submanifold of $\mathbb{R}^6$. (Irie))

Situation is somewhat similar to:
Donaldson invariant is not well defined if intersection form is negative definite but ASD equation has important application in that case.

FOOO (F-Oh-Ohta-Ono)

1) ~~$L_1, L_2 \subset Y \rightarrow HF(L_1, L_2)$~~ defined

$L \subset Y \rightarrow \mathscr{M}(L) \subset H^{\mathrm{odd}}(L)$

$b_i \in \mathscr{M}(L_i)$ $HF((L_1, b_1), (L_2, b_2))$ defined

Note $\mathscr{M}(L)$ can be empty.

4) $L_1 = L_2 = L \rightarrow$ ~~$HF(L, L) = H(L)$~~

$\rightarrow$ Spetre sequence

$H(L) \to HF(L, L)$

FOOO (F-Oh-Ohta-Ono)

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$L \subset Y \;\Rightarrow\; \mathscr{M}(L) \subset H^{\mathrm{odd}}(L)$

$b_i \in \mathscr{M}(L_i)$ $\quad HF((L_1, b_1), (L_2, b_2))$ defined

Note $\mathscr{M}(L)$ can be empty.

4) $L_1 = L_2 = L \;\Rightarrow\;$ ~~$HF(L, L) = H(L)$~~

$\Rightarrow$ Spetre sequence

$$H(L) \to HF(L, L)$$

FOOO (F-Oh-Ohta-Ono)

$HF((L_1, b_1), (L_2, b_2))$ has extra parameter

$b_i \in \mathscr{M}(L_i)$ bounding cochain.

This is related to homological algebra
= deformation theory of associative
or A infinity algebra.

This story is generalized by Akaho-Joyce so that it include immersed Lagrangians.

$$\mathscr{M}(L) \subset H^{\mathrm{odd}}(L) \qquad \text{for embedded Lagrangians}$$

If L is immersed, then

$$\mathscr{M}(L) \subset H^{\mathrm{odd}}(L) \oplus \text{ two extra generators for each self intersection points.}$$

Actually existence of this extra parameter makes Floer theory more applicable.

Example: $\mathbb{C}P^2$ T^2 acts on it.

$HF(L,L)$ L is a T^2 orbit

is mostly 0 but is nonzero for unique L

$\mathbb{C}P^2\# - \mathbb{C}P^2$ blow up. T^2 acts on it.

$HF(L,L)$ L is a T^2 orbit is always 0

But there exist two L such that

$HF((L,b),(L,b)) \neq 0$ for some $b \in \mathscr{M}(L)$

At this stage (2017) Lagrangian Floer theory gives a 2-functor

{Category of all compact symplectic manifolds}

$$\Downarrow$$

{2-category of all A infinity categories}

(FOOO,Akaho-Joyce,Wehrheim-Woodwards-Mau,F)

{Category of all compact symplectic manifolds}

{2-category of all A infinity categories}

Morphism from M_1 to M_2

is (L, b): $L \subset -M_1 \times M_2$

$b \in \mathscr{M}(L)$

We need homological algebra much to define and study such 2-category.

Going back to gauge theory.

$$\partial M^3 = \Sigma^2$$

So to obtain a correct relative invariant HF_M

we need more information than the Lagrangian submanifold

$R(M)$ the space of Flat connections on M.

Actually the extra information we need is bounding cochain of $R(M)$

Ex.

$$\partial M^3 = T^2 \sqcup T^2$$

consider SO(3) bundle on M which is nontrivial on T^2

$R(T^2 \sqcup T^2)$ is one point.

$R(M) \to R(T^2 \sqcup T^2)$ finite to one map.

$R(M)$ immersed Lagrangian submanifold (in a trivial way).

$b \in \mathscr{M}(R(M))$ is exactly the boundary operator of instanton Floer homology homology of $M \cup_{\partial M} (T^2 \times [0,1])$

Relation to the formulation

$$HF_M : \mathscr{F}(R(\Sigma)) \to \mathrm{ch}$$

$$HF_M(L) = HF(M; L)$$

$\partial M = \Sigma \qquad L \subset R(\Sigma)$

If we enhance $\mathscr{F}(R(\Sigma))$ so that immersed Lagrangian is its object

then $HF_M : \mathscr{F}(R(\Sigma)) \to \mathrm{ch}$ is representable by $(R(M), b_M)$.

Let me go back to the question.

Donaldson invariant and Floer (Instanton) homology

Topological Field theory picture: late 1980's.

Gromov-Witten invariant and Lagrangian Floer homology ?

Functoriality of Gromov-Witten theory

Quantum cohomology $HQ(X)$

Deformation of cohomology ring of a symplectic manifolds using the count of holomorphic sphere.

(It is the same as group with usual cohomology but ring structure is different.)

Note $X \mapsto HQ(X)$ is not functorial.

Note $X \mapsto HQ(X)$ is not functorial.

$X \to Y$ map $H^*(Y) \to H^*(X)$

ring homomorphism.

however $HQ^*(Y) \to HQ^*(X)$

is not a ring homomorphism.

On the other hand, Lagrangian Floer theory is functorial:

Lagrangian Floer theory gives a 2-functor

{Category of all compact symplectic manifolds}

↓

{2-category of all A infinity categories}

Gromov-Witten invariant and Lagrangian Floer homology ?

There exists an closed-open map (Kontsevich, Seidel, Albers, FOOO, Biran-Cornea ...)

$$\mathfrak{q} : HQ(X) \to HH^*(\mathfrak{F}(X))$$

quantum cohomology

Hochschild cohomology

this is a ring homomorphism and is expected to be an isomorphism for, say, smooth projective algebraic variety.

$$\mathfrak{C} \to HH^*(\mathfrak{C})$$

A infinity category → Hochschild cohomology

is neither contravariant nor covariant.

However:

$$\mathfrak{C} \to HH_*(\mathfrak{C})$$

A infinity category → Hochschild homology

is covariant.

There exists an open-closed map

$$\mathfrak{p} : HH_*(\mathfrak{F}(X)) \to H_*(X)$$

that is Poincare dual to closed-open map.

Conjecture Morphism from M_1 to M_2 (L, b): $L \subset -M_1 \times M_2$

$b \in \mathscr{M}(L)$

then the next diagram commutes.

$$\begin{array}{ccc} HH_*(\mathfrak{F}(M_1)) & \longrightarrow & HH_*(\mathfrak{F}(M_2)) \\ \downarrow \mathfrak{p} & & \downarrow \mathfrak{p} \\ H_*(M_1) & \longrightarrow & H_*(M_2) \end{array}$$

If yes does it mean something to the functoriality of GW invariant ?

Foundation:

Proof of all these results are based on the study of various moduli spaces of PDE’s.

Physicist's interpretation:

Those numbers are integration of certain 'closed differential forms' on infinite dimensional space:

$$\int_{a\in\mathfrak{X}} \exp(F(a))\Phi(a)\mathfrak{D}a \quad = \int_{\mathcal{M}} \Phi(a)da$$

In good case it reduces
to an integration on a finite dimensional space the set of solutions $\mathcal{M}$
of a non-linear PDE.

$$\int_{a\in\mathfrak{X}} \exp(F(a))\Phi(a)\mathfrak{D}a \quad = \int_{\mathcal{M}} \Phi(a)da$$

$\mathcal{M}$ is a finite dimensional space but can be much singular.

In our story, those number itself is not well defined but
only complicated system of such numbers are
well-defined in certain complicated sense. (homological algebra).

$$\int_{a\in\mathfrak{X}} \exp(F(a))\Phi(a)\mathfrak{D}a \quad = \int_{\mathcal{M}} \Phi(a)da$$

Classical approach (< 1996),

perturb the space $\mathcal{M}$
by certain explicit geometric parameter
(eg. metric, almost complex structure ….)
so that $\mathcal{M}$ becomes smooth.

In our story where problem becomes more and more cumber some it is becoming harder and harder to work out.

$$\int_{a\in\mathfrak{X}} \exp(F(a))\Phi(a)\mathfrak{D}a \quad = \int_{\mathcal{M}} \Phi(a)da$$

Virtual technique ($\geq$1996) (F-Ono, Tian-Li-Lu, Ruan, Siebert, ….)

1) Define some notion of 'singular' spaces in C^∞ category.

2) Prove that $\mathcal{M}$ is an example of such space.

3) Develop certain 'cohomology theory' or 'integration' on such 'singular' spaces and justify the right hand side.

Virtual technique ($\geq$1996)

There are 4 kinds of variants of Virtual technique being studied now.

Kuranishi structure	d-manifold	Polyfold	implicite atlas
F-Oh-Ohta-Ono	Joyce	Hofer ….	Pardon
manifold theory	scheme stack	functional analysis	algebraic topology

Virtual technique ($\geq$1996)

There are 4 kinds of variant of Virtual technique being studied now.

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It is becoming clearer that all 4 versions work for the purpose of studying moduli space of (pseudo) holomorphic curves (Gromov-Witten-Floer theory).

Virtual technique (≥1996)

How about gauge theory ?

Singularity of the moduli space of gauge theory is more singular than that of pseudo-holomorphic curve.

Ex:

$\mathcal{M}_d(X)$ compactified moduli space of ASD connection with instanton number d on X.

It contains a point where there is a d bubles and what is left is a trivial connection.

This singularity is not contained in spaces studied by virtual technique.

F - A.Daemi (in progress)

$$\int_{a \in \mathfrak{X}} \exp(F(a))\Phi(a)\mathfrak{D}a \quad = \int_{\mathcal{M}} \Phi(a) da$$

in case $\Phi(a) \equiv 1$ and $\dim \mathcal{M} = 0$

we can justify RHS by virtual technique in the gauge theory case.