

John Conway, my thoughts after his passing

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The mathematician John Conway passed away on April 11, 2020. In this note, I am recounting some of my own experiences with him and his work on finite groups. Since I knew Conway for more than 50 years, It seems natural for me to write this article in the first person.

I met him in January 1970, at the Cambridge University DPMMS (Department of Pure Mathematics and Mathematical Statistics). I spent about five months there at the invitation of my advisor John Thompson, while working on my University of Chicago thesis on Schur multipliers of the finite simple groups.

Just before 1970, there had been a recent and dramatic increase in the number of known finite simple groups. (At that time, there was no sense that the classification of finite simple groups would be completed.) Conway was responsible for discovering several of these groups via his study of the 24-dimensional Leech lattice [1, 2]. He was justly proud. He determined the structure of the remarkably large isometry group of the Leech lattice, including its order $2^{22}3^95^47^211\cdot13\cdot23$. This group is usually notated Co_0 . Results included discoveries of some new sporadic simple groups (Co_1, Co_2, Co_3) and re-discoveries of then-recently discovered sporadic groups (those of Hall-Janko, Higman-Sims, McLaughlin, Suzuki). Built into the Leech lattice theory were the five Mathieu groups, discovered in the 19th century. A single context captured twelve sporadic simple groups! His results led to a tremendous increase in his status as a mathematician, and he soon received many lecture invitations, including one to the 1970 International Congress of Mathematicians in Nice.

Conway's office in DPMMS was near the lounge, which was full of puzzles and scratch paper. He was keen to discuss games and mathematics with whomever was available to listen. His Game of Life had attracted worldwide

interest. His quick wit was on display in frequent conversations in the lounge and throughout the building.

Around that time and later, Conway was attracted to “special phenomena” which he observed in a particular group and tried to generalize. Some of the things he looked into seemed like offbeat momentary interests, but they were clearly an expression of his broad curiosity and search for mathematical substance. One idea he had in the late 1960s involved what he called *trine groups*, groups generated by a conjugacy class of subgroups of order 3 so that any two generate a subgroup on a given short list. In his case, that list was something like $\mathbb{Z}_3 \times \mathbb{Z}_3$, Alt_4 , $SL(2, 3)$ and maybe also Alt_5 or $SL(2, 5)$. He collaborated with the computational mathematician John McKay. A partial list of trine groups was obtained; many were subgroups of the big Conway group, the isometry group of the Leech lattice. Such a short-lived and exploratory project could now be viewed in the context of the later theories of Fischer, Timmesfeld, Aschbacher, et al. about finite groups generated by a conjugacy class of elementary abelian subgroups so that any two generate a group on a restricted list.

Conway was aware of ongoing developments in the classification of finite simple groups and was ready to jump in if something caught his interest. Sometime in the spring of 1973, he and David Wales created a computer construction of the Rudvalis group, which happened, I believe, about a month after they heard evidence that such a group might exist. They built a 28-dimensional representation of the double cover of the Rudvalis group, written over the scalars $\mathbb{Z}[\sqrt{-1}]$ [5]. Conway was, however, not part of the team working systematically to classify finite simple groups.

In November 1973, Bernd Fischer and I independently found evidence for a very large sporadic group that later came to be known as the Monster. In brief, existence was proved [7] and uniqueness was proved later [10]. (We’ll soon say a bit more on this subject.) Around the early 1970s, there was an ongoing stream of conjectures about new finite simple groups that might exist, though at the time there was no clear idea as to how many, and when genuine finite simple groups would be generated by these searches.

I now introduce Simon Norton, a young math genius at Cambridge University. Conway recruited Norton in the early 1970s to work on finite simple groups as well as to pursue other interests. In the late 70s, John Conway and Simon Norton developed *Monstrous Moonshine* [4]. It involved a putative sporadic simple group, the Monster. (At this time, it was unclear as to whether existence of the Monster would ever be settled be-

cause of its size, about 8×10^{53}). Monstrous Moonshine theory was inspired by two ideas. The first was John McKay's surprising observation that 196884 (the first nontrivial coefficient of the elliptic modular function $j(z)$, for z in the upper half complex plane) equals $1+196883$. In more detail, $j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$, where $q = e^{2\pi iz}$. The number 196883 was expected to be the smallest degree of a nontrivial irreducible representation of the Monster. Secondly, John Thompson [12] proposed that there could be a graded space $V = \bigoplus_{n \geq -1} V_n$, where V_n is a finite dimensional module for the Monster, so that the formal series $\sum_{n \geq -1} \dim(V_n)q^n$ equals $j(z) - 744$ and that the series $\sum_{n \geq -1} \text{tr}(g |_{V_n})q^n$ could be interesting for all g in the Monster. They were indeed.

The theory of Conway and Norton was conceived just as two feelings were developing within the finite group theory community during the late 1970s. The first sentiment held that the classification of finite simple groups *might* be within reach since an end game had been envisioned. The second was that finite simple group theory had, for years, been getting increasingly deep and thereby (unintentionally) distancing itself from the rest of mathematics. Conway and Norton's Monstrous Moonshine theory asserted extensive and dramatic connections between a putative Monster and the theory of modular forms. Roughly speaking, it was based on their discovery of a near-bijective correspondence between conjugacy classes of the Monster and the known set of genus 0 function fields on the upper half complex plane. The series indicated in the previous paragraph played an important role. Suddenly, there were rich possibilities for studying connections between two important areas of mathematics. Mathematicians from outside finite group theory took interest. A few years later, the Monster was proved to exist (see below) [7]. Work on Monstrous Moonshine and related matters continues to this day.

In early 1980s, I announced construction of the Monster, a very large sporadic simple group, of order $2^{46}3^{20}5^97^611^213^317 \cdot 19 \cdot 23 \cdot 39 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$, about 8×10^{53} [7]. Its existence proof was one of the final steps in completing the classification of finite simple groups. Actually, I constructed both a commutative, nonassociative algebra of dimension $196883 = 47 \cdot 59 \cdot 71$ and a group of algebra automorphisms. The idea to use such an algebra was inspired by Norton's study of invariants of the Monster on a putative irreducible module R of dimension 196883. Norton conjectured that R was self-dual and had an invariant symmetric tensor of degree 3. This means that R would have the structure of a commutative algebra for which the Monster would act as algebra automorphisms. Conway and Norton dialogued regularly about

such an algebra and similar ones associated with smaller finite groups. To the best of my knowledge, no useful axioms for such algebras were ever made firm.

I studied many aspects of a possible algebra structure on a 196883 dimensional irreducible module. Defining the algebra structure on a vector space of dimension 196883 and suitable generators for a finite group with the right properties was quite hard. Given such an algebra structure on this irreducible module, there was a multi-parameter family of ways to define an algebra of dimension 196884 with identity and the same automorphism group. Eventually, a fairly natural choice arose. It involved promoting a 299-dimensional subalgebra of the original algebra to a 300-dimensional algebra isomorphic to the Jordan algebra of 24×24 symmetric matrices.

Many people examined my proof. Jacques Tits made many improvements and studies, both on the 196883-dimensional algebra and a determination of the automorphism group. (See for example, [13, 14] and other articles.) In 1984, Conway offered a more efficient construction of the Monster [3]. He used the Parker loop, which is a Moufang loop (a kind of nonassociative group) built from the binary Golay code of dimension 12 over the field of integers mod 2. It has 2^{13} elements. The loop enabled Conway to write down compact formulas for the algebra multiplication and show existence of an “extra automorphism,” a point that was particularly difficult in [7]. His formulas were compatible with my original formulas. Such a loop was not known to exist until the early 1980s, several years after the original construction of the Monster [6, 7]. For background on related loops, see [8, 9]. There is now a shorter existence proof of the Monster with relatively little computation using vertex algebra theory, which was done by Ching-Hung Lam and myself [11]. Uniqueness of the monster was finally proved in [10]. The symbol \mathbb{M} is now common notation for the Monster.

Conway was a popular lecturer in introductory math courses, in talks to math educators (such as the presentation he made at Canadian Math Society meeting on Victoria Island in 1990), and to students at all levels. He frequently gave vivid demonstrations with physical objects (pink rubber gloves, oversized playing cards, wooden puzzles), and liked to engage audience members (to be holders of ribbons for illustrating action of the braid group at a recent University of Michigan undergraduate colloquium). He loved sharing the pleasures of mathematics.

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