Automorphic representations in string amplitudes

Axel Kleinschmidt (MPI for Gravitational Physics, Potsdam)

Arithmetic Quantum Field Theory
CMSA, Harvard, 27 March 2024

Based on (partly ongoing) work with G. Bossard, P. Fleig, S. Friedberg, D. Gourevitch, H. Gustafsson, D. Persson, B. Pioline and S. Sahi

Summary

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Automorphic representations occur naturally in string scattering amplitudes

Fourier coefficients contain physical information (non-perturbative)

String theory suggests new types of automorphic representations

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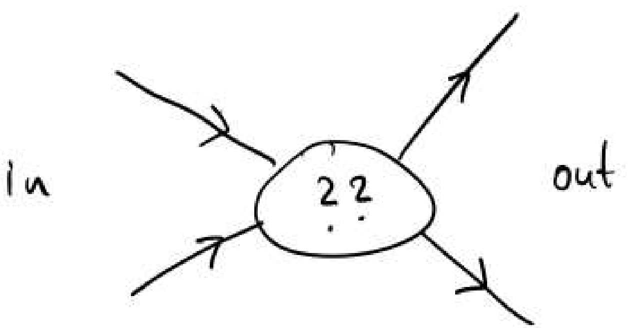
String theory suggests new types of automorphic representations

Many contributors over the last 20+ years:
A. Basu, E. D'Hoker, M. Green, M. Gutperle, S.D. Miller, N. Obers, J. Russo, S. Sethi, P. Vanhove, . . .

Scattering amplitudes

- determine probabilities for the outcomes of scattering experiments
- determined from fundamental interactions of a theory



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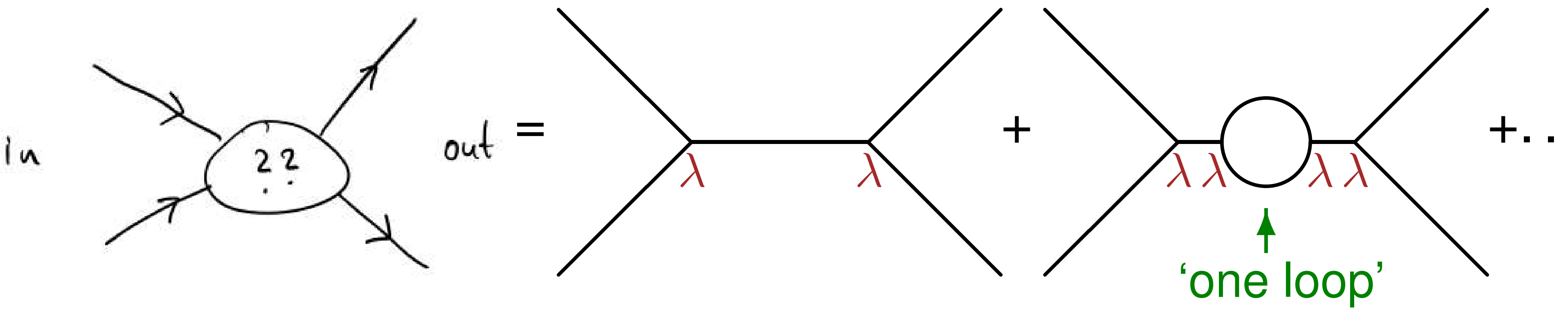
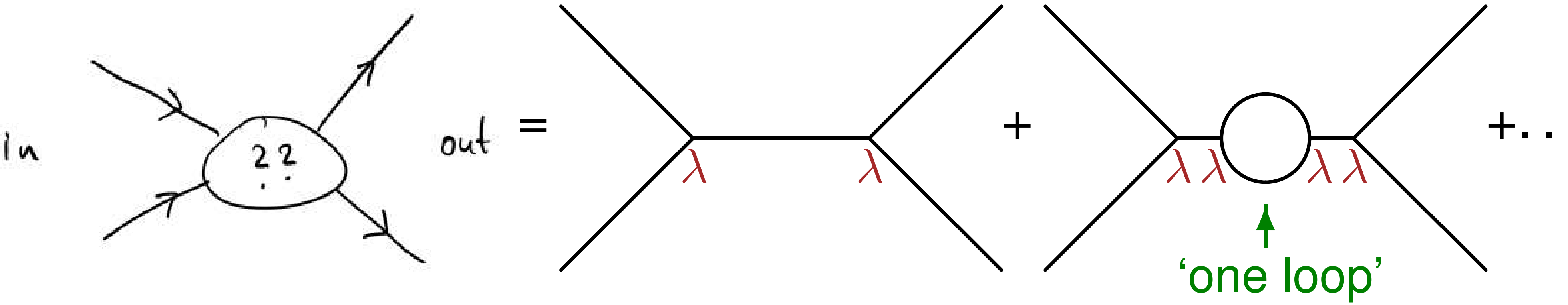
(Perturbative) textbook procedure with Feynman diagrams



λ denotes a 'small' coupling constant and each diagram corresponds to a specific (potentially high-dim'l) integral, typically divergent $\Rightarrow$ renormalization, regularization

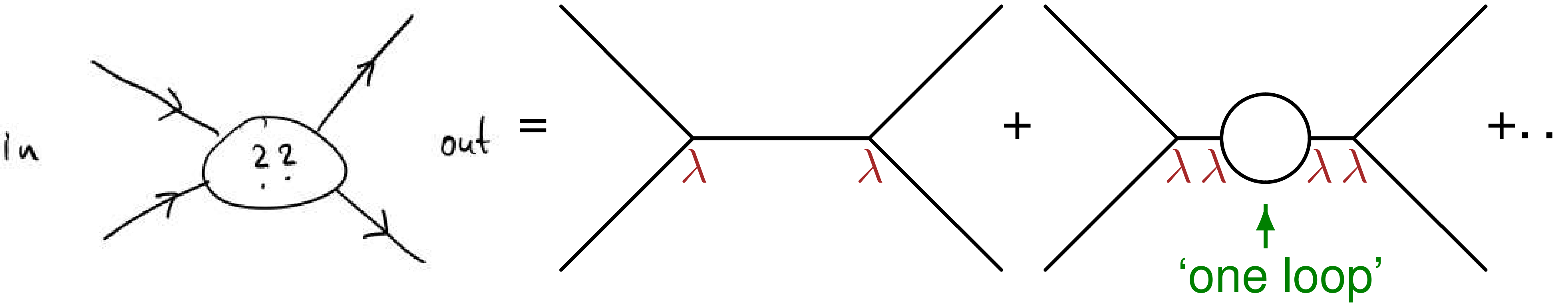
Scattering amplitude



Overall picture

$$\mathcal{A}(\{k_i, \epsilon_i\}; \{\lambda, ..\})$$

Scattering amplitude

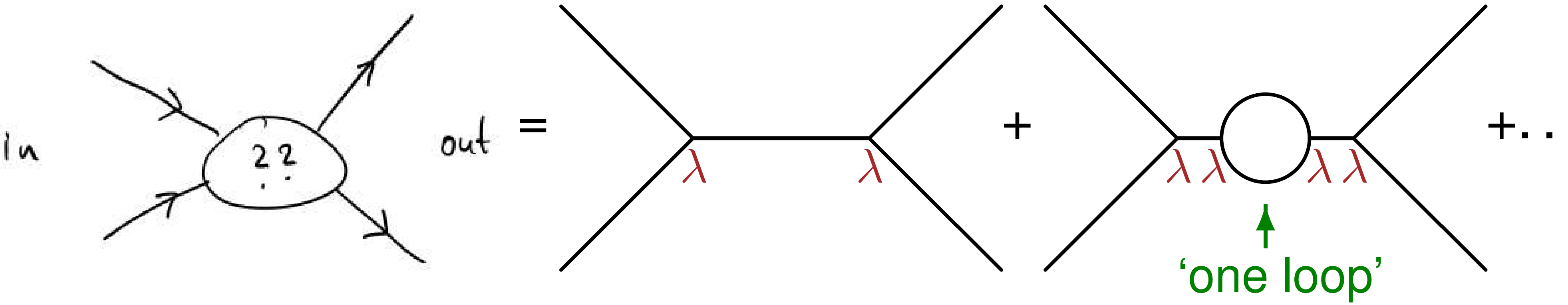


Overall picture 'theory data': coupling constants, ...

$$\mathcal{A}(\underbrace{\{k_i,\epsilon_i\}};\overbrace{\{\lambda,..\}})$$

'kinematic data' of scattering particles: momenta, polarizations, ...

Scattering amplitude



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$$\mathcal{A}(\{k_i,\epsilon_i\};\{\lambda,..\}) = \mathcal{A}^{\text{pert.}}(\{k_i,\epsilon_i\};\{\lambda,..\}) + \mathcal{A}^{\text{non-pert.}}(\{k_i,\epsilon_i\};\{\lambda,..\})$$

Above diagrams correspond to perturbative expansion of the amplitude, i.e. analytic in coupling λ.
Typically an asymptotic series and also non-perturbative terms needed, e.g. $\sim e^{-1/\lambda}$

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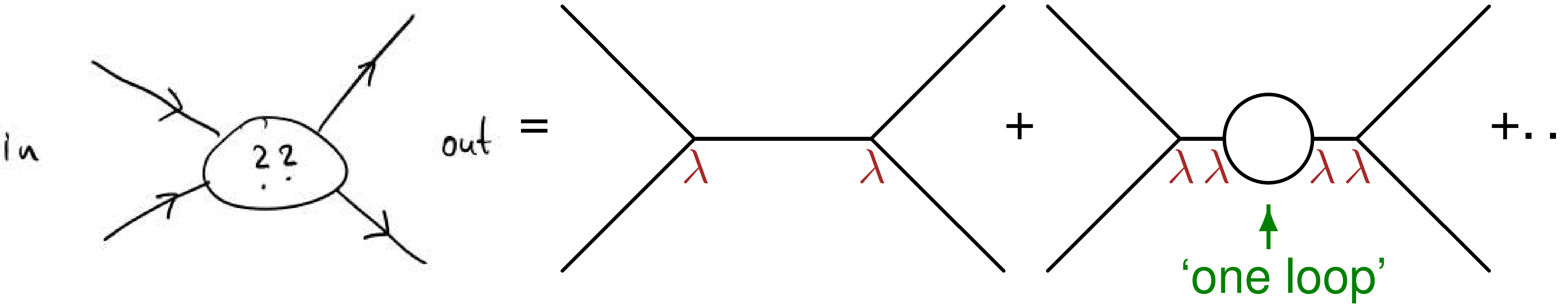

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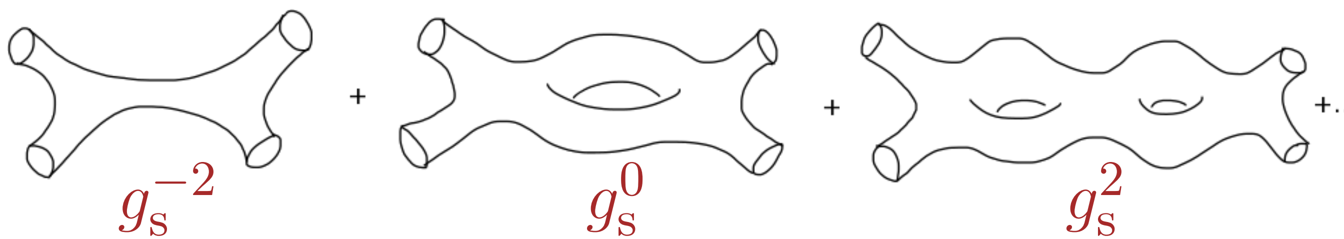
Exact structure of $\mathcal{A}$ of all its arguments typically intractable

Perturbative string theory

String theory replaces point-like particles by one-dimensional strings. Rather than ‘world-lines’ get world-sheets (two-dim’l surfaces) in space-time

World-sheet view of a scattering process: genus expansion



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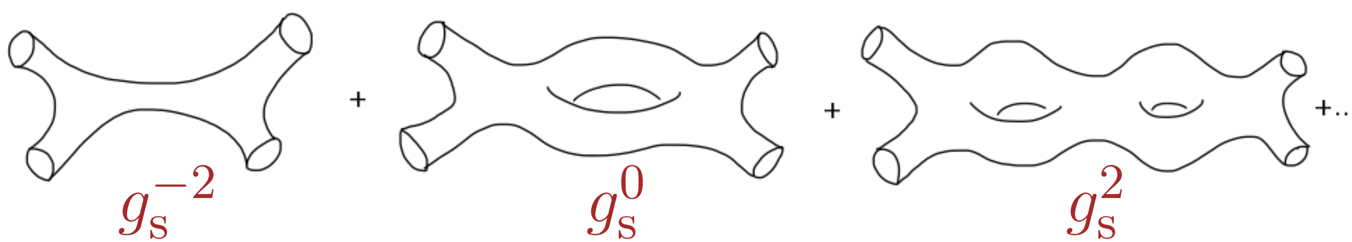

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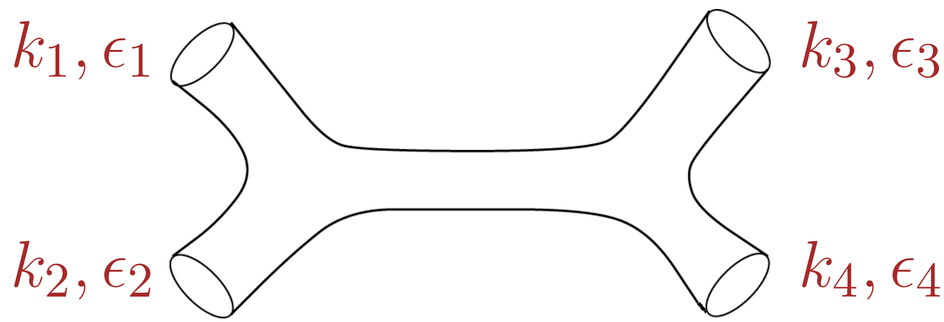
n-point amplitude at h loops: Integral over world-sheets of genus h with n punctures; weighted by $g_s^{2(h-1)}$.

Given by integrals over $\mathcal{M}_{h,n}$ moduli space

String theory scattering amplitudes

Example: Four-graviton scattering at tree level (flat target)



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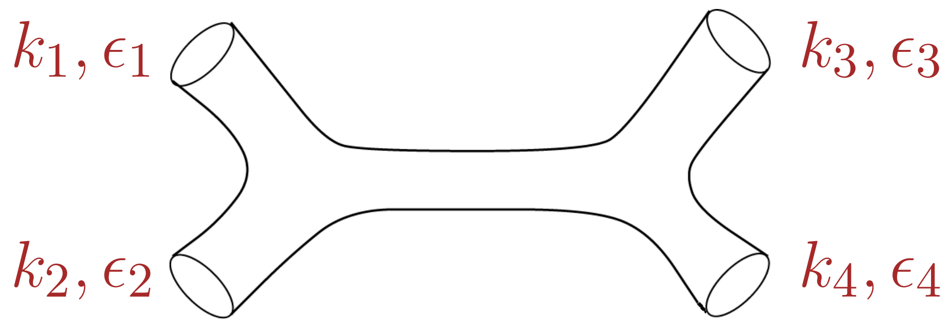

$$\mathcal{A}^{\text{tree}}(s,t,u) = g_{\text{s}}^{-2} \frac{(\alpha')^4}{stu} \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)} \mathcal{R}^4$$

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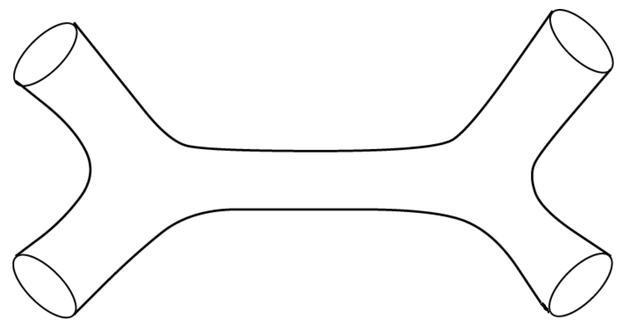

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Mandelstam variables

string coupling: tree level

absorbs polarisation tensors ϵ_i

$$s = -(k_1+k_2)^2, \;\; t = -(k_1+k_4)^2, \;\; u = -(k_1+k_3)^2$$

Four gravitons at tree-level

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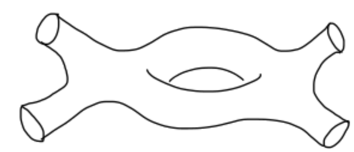

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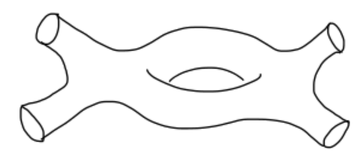

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For tree-level amplitude by expanding gamma function.

More generally, yields plethora of (novel) invariants on higher genus moduli spaces: Modular graph forms

[Basu, Broedel, Dorigoni, Doroudiani, Duke, Gerken, Green, Gürdoğan, D'Hoker, Kaderli, Kaidi, AK, Mafra, Pioline, Russo^2, Schlotterer, Vanhove, Verschinin, Zagier, Zerbini, ...]

Low-energy expansion

For $\alpha' s \ll 1$, $\alpha' t \ll 1$, $\alpha' u \ll 1$ at tree level

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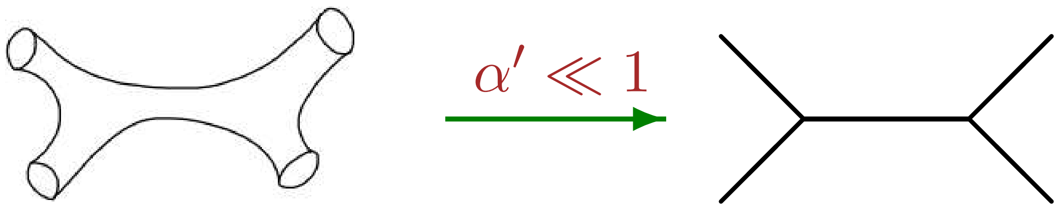

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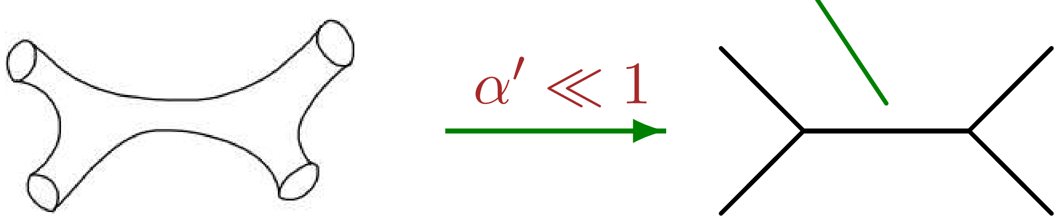

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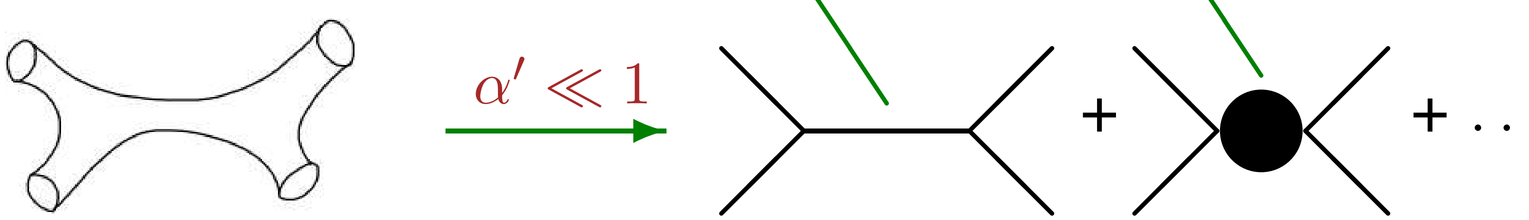

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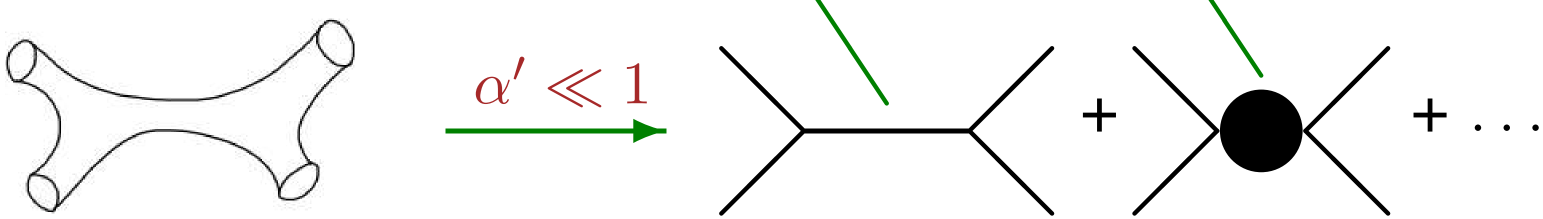


Generates an effective quantum field theory with new types of interactions

String theory scattering amplitudes

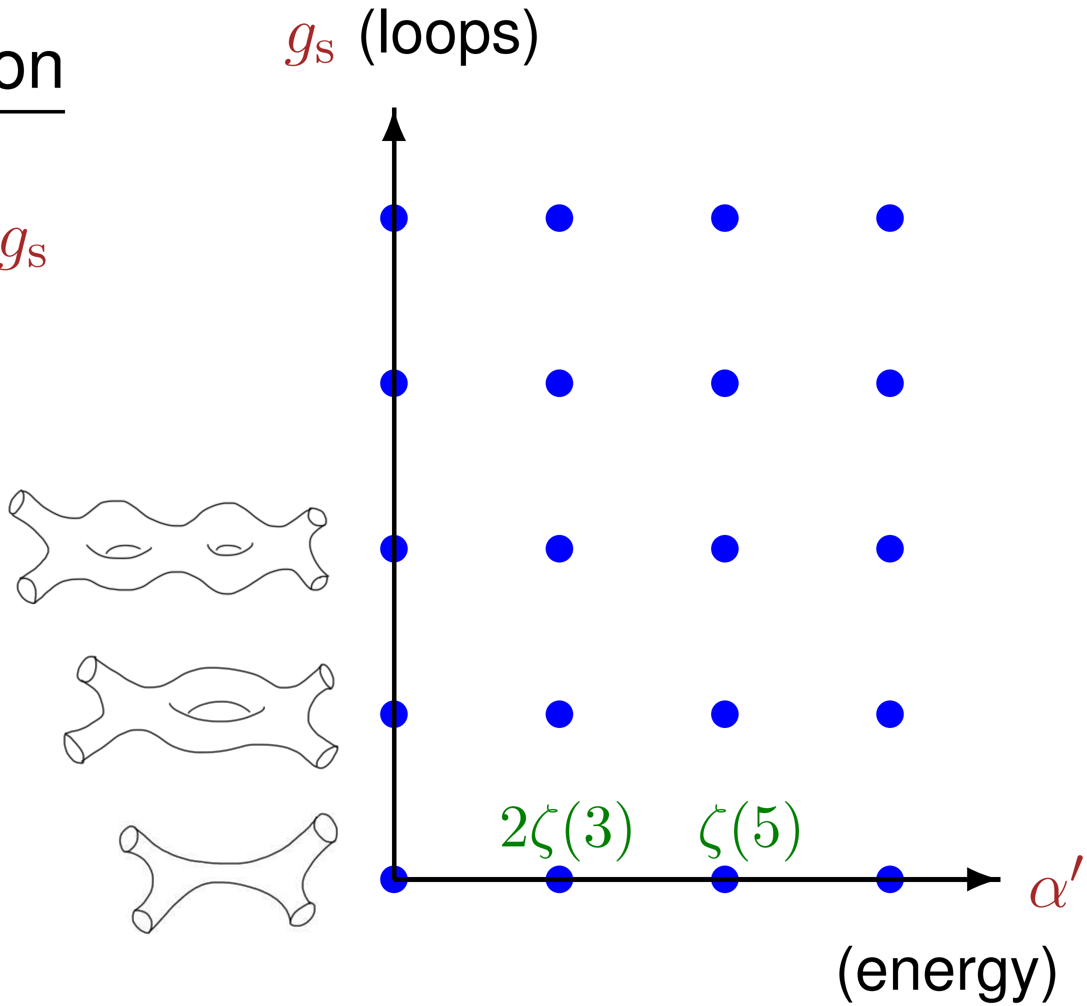
Scattering amplitudes of strings have a double expansion

- Perturbative loop expansion

  Diagram weighted by powers of string coupling g_s

- Low-energy expansion

  Energies involved in interaction measured in powers of string scale $\ell_s^2 = \alpha'$



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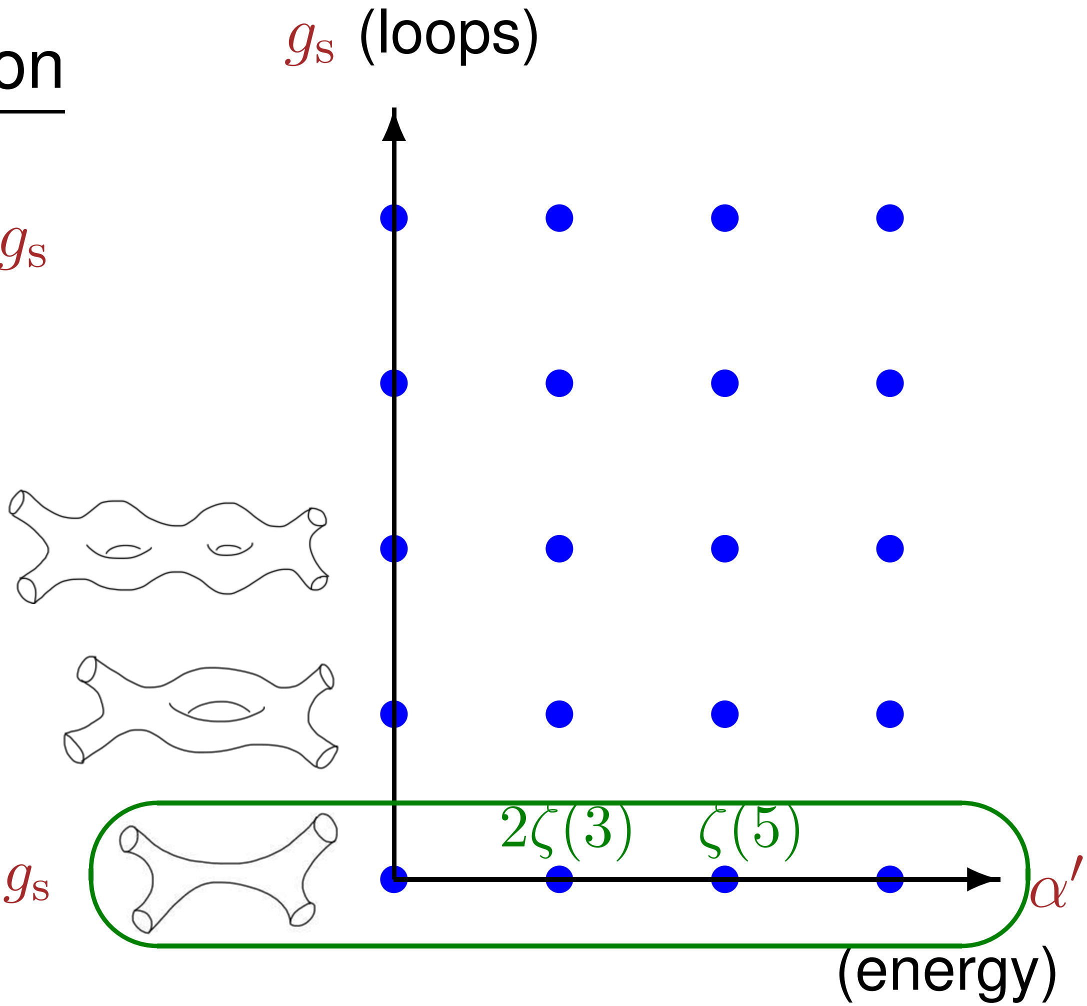

fixed order in g_s

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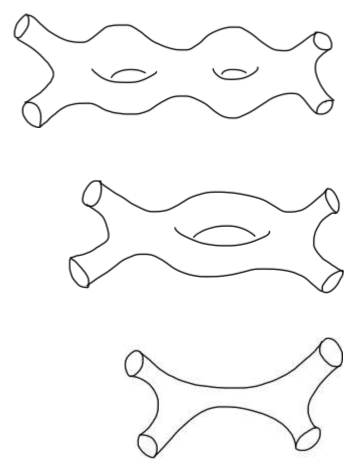

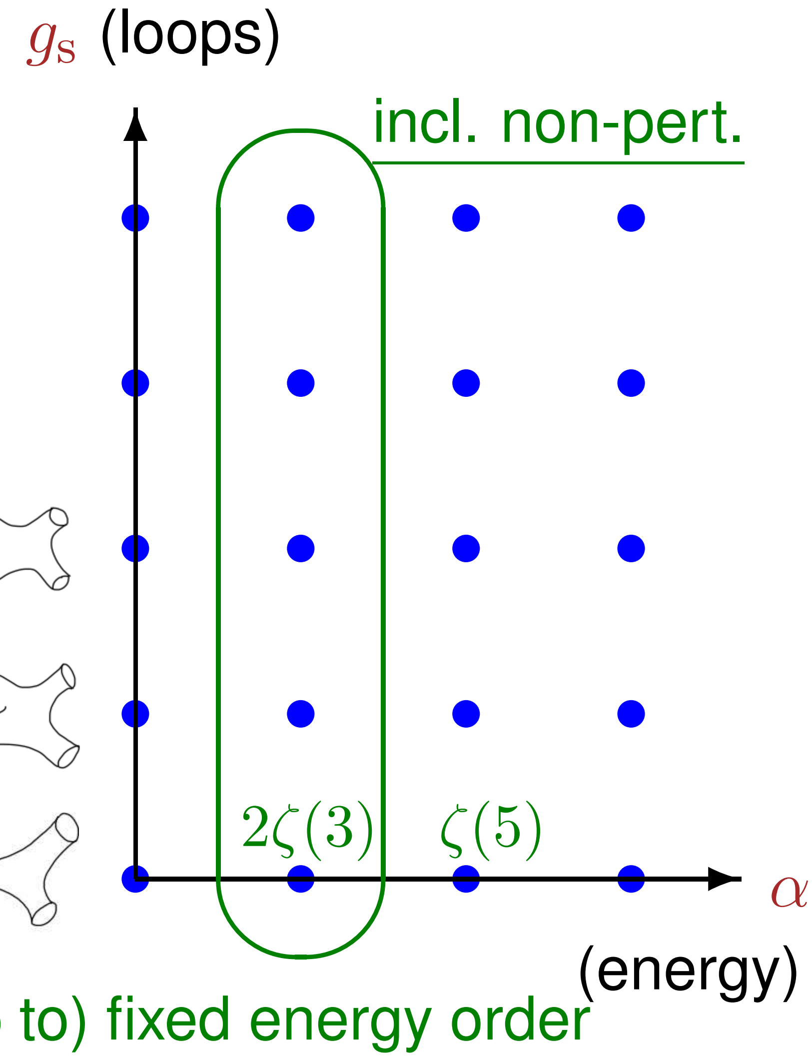

...sometimes fixed by (discrete) symmetries/automorphy!

Moduli and U-duality (I)

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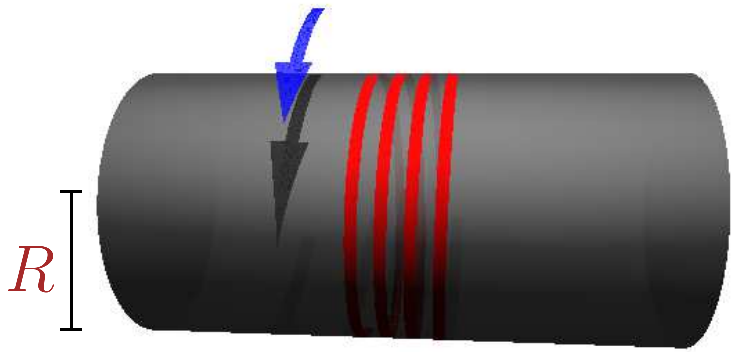

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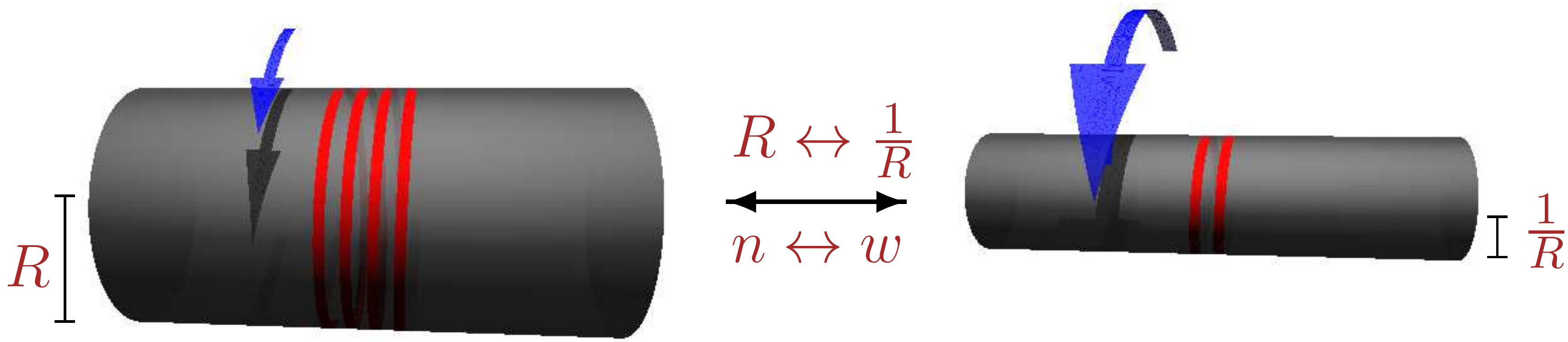

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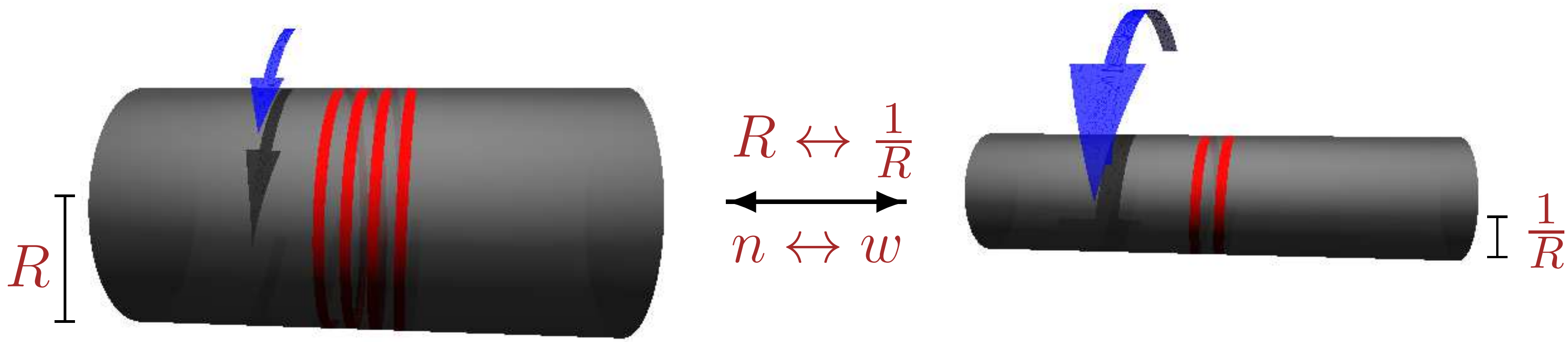

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Equivalent string theories! T-duality $SO(d-1, d-1, \mathbb{Z})$

Moduli and U-duality (II)

On $g_{\rm s}$ and (RR) axion χ action of $SL(2,\mathbb{Z})$ S-duality

$$\Omega = \chi + i g_{\rm s}^{-1} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \Omega = \frac{a\Omega + b}{c\Omega + d}$$

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All moduli g together form moduli space $\mathcal{M}$ [Hull Townsend]

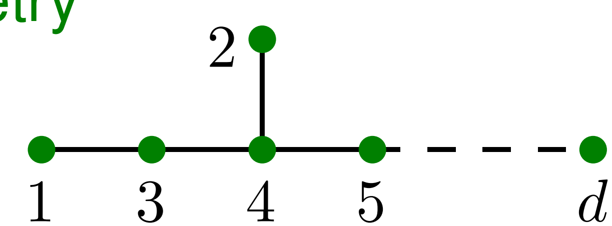
$$g \in \mathcal{M} = E_d(\mathbb{Z}) \backslash E_{d(d)}(\mathbb{R}) / K(E_d)$$

U-duality

Cremmer–Julia hidden symmetry (split real)

compact subgroup

strings on $\mathbb{R}^{1,10-d} \times T^{d-1}$



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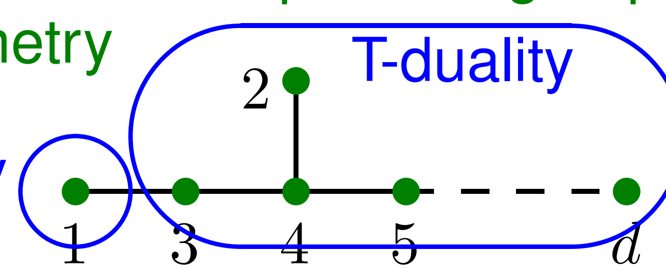
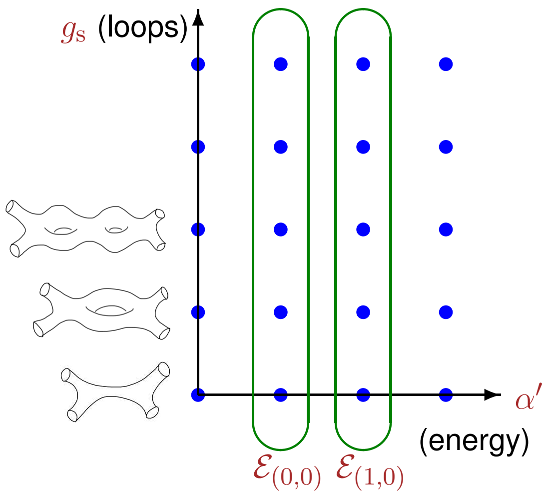


Coefficient functions in amplitude (I)

Expand the (analytic part of the) full scattering amplitude in energy direction

$$\mathcal{A}(s,t,u;g) = \mathcal{R}^4 \left(\frac{1}{stu} + \sum_{p,q\geq 0} \mathcal{E}_{(p,q)}(g)\sigma_2^p \sigma_3^q \right)$$



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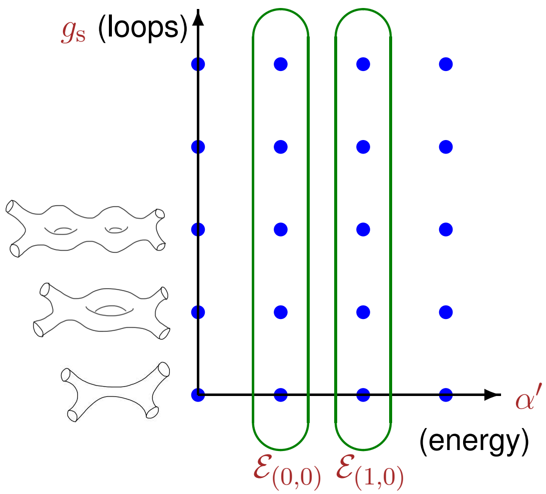

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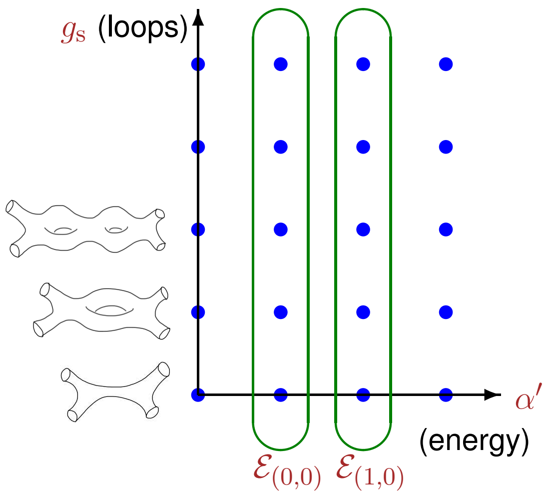

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$\Rightarrow$ Looking for (spherical) automorphic forms on E_d

Coefficient functions in amplitude (II)

A lot known for lowest $\mathcal{E}_{(p,q)}$ from supersymmetry and internal consistency [Green, Gutperle, Kiritsis, Miller, Obers, Pioline, Russo, Sethi, Vanhove, Waldron, ...]

R^4	$\mathcal{E}_{(0,0)}(g) = 2\zeta(3) E_{\alpha_1,3/2}(g)$
$D^4 R^4$	$\mathcal{E}_{(1,0)}(g) = \zeta(5) E_{\alpha_1,5/2}(g)$
$D^6 R^4$	$\mathcal{E}_{(0,1)}(g) =$ later

in terms of (maximal parabolic) Eisenstein series

$$E_{\alpha_1,s}(g) = \sum_{\gamma \in P_1(\mathbb{Z}) \backslash E_d(\mathbb{Z})} e^{\langle 2s\Lambda_1, H(\gamma g)\rangle}$$

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R^4	$\mathcal{E}_{(0,0)}(g) = 2\zeta(3)E_{\alpha_1,3/2}(g)$
D^4R^4	$\mathcal{E}_{(1,0)}(g) = \zeta(5)E_{\alpha_1,5/2}(g)$
D^6R^4	$\mathcal{E}_{(0,1)}(g) =$ later

in terms of (maximal parabolic) Eisenstein series

$$E_{\alpha_1,s}(g) = \sum_{\gamma\in P_1(\mathbb{Z})\backslash E_d(\mathbb{Z})} e^{\langle 2s\Lambda_1, H(\gamma g)\rangle}$$

fund. weight dual to $\alpha_1^\vee$

logarithm map $G \to \mathfrak{h}$ (CSA)

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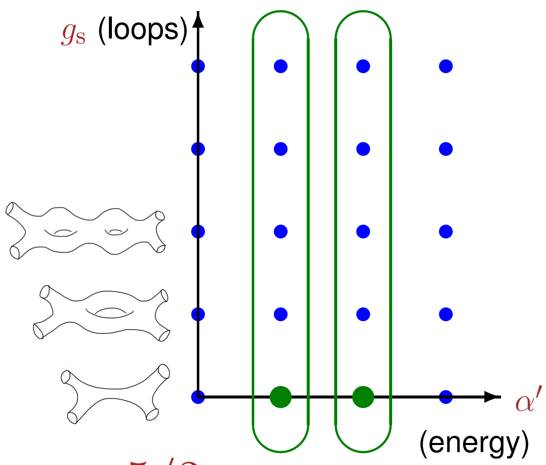

Consistency with tree-level results ($\gamma = 1$)

$$\mathcal{E}_{(0,0)}(g) = 2\zeta(3)g_s^{-3/2} + \ldots, \quad \mathcal{E}_{(1,0)}(g) = \zeta(5)g_s^{-5/2} + \ldots$$

Automorphic representations

Given an automorphic form η one can generate an automorphic representation by G right action:

$$\eta \mapsto \pi(g)\eta\,, \quad (\pi(g)\eta)\,(h) = \eta(hg)$$

Best done adelically. More properly form a $(\mathfrak{g}, K)$ module at archimedean places. Does not upset discrete invariance under $G(\mathbb{Q})$ left action: $\eta(g) = \eta(\gamma g)$

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$\Longrightarrow$ Link to representation theory

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$\Longrightarrow$ Anything special about the string theory cases?

Fourier expansion

Simplest case $G(\mathbb{R}) = SL(2,\mathbb{R})$, $\Omega \in SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/K$
$\Omega = \chi + i g_{\mathrm{s}}^{-1}$

$$\mathcal{E}_{(0,0)} = 2\zeta(3) E_{3/2}(\Omega) = 2\zeta(3) g_{\mathrm{s}}^{-3/2} + 4\zeta(2) g_{\mathrm{s}}^{1/2} + 2\pi \sum_{m\neq 0} \sqrt{|m|}\sigma_{-2}(m) e^{-2\pi|m|g_{\mathrm{s}}^{-1} + 2\pi i m\chi}\left(1 + O(g_{\mathrm{s}}^{-1})\right)$$

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$$+ 2\pi \sum_{m\neq 0} \sqrt{|m|}\sigma_{-2}(m) \underbrace{e^{-2\pi|m|g_s^{-1}+2\pi i m\chi}}_{\text{non-perturbative D(-1)-instanton/brane}} \left(1 + O(g_s^{-1})\right)$$

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Remarks

- Constant terms: perturbative calculation (agree and predict correctly)
- Fourier coefficients: non-perturbative effects
- Instanton labelled by charge $m \neq 0 \leftrightarrow$ nilpotent $\in \mathfrak{g}^*$

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Story gets more interesting for higher rank groups...

Fourier coefficients

Consider a unipotent subgroup $U(\mathbb{Q}) \subset G(\mathbb{Q})$ and define the Fourier coefficient/unipotent period

$$F_{\psi_U}(\eta, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \eta(ug)\overline{\psi_U(u)}du$$

for $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to \mathbb{C}^\times$ a unitary character. Equivalently:

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Wave-front set (of an automorphic form):
Collection of nilpotent orbits supporting $F_{\psi_U} \neq 0$.

Discussed for instance in [Mœglin Waldspurger] [Matumoto] (locally) and [Jiang, Liu Savin] (globally), using Whittaker pairs in [Gomez, Gourevitch Sahi]

String theory expectations

Coefficient functions [Green, Russo Vanhove] [Green, Miller Vanhove] [Pioline]

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Small automorphic representations! [Ginzburg Rallis, Soudry] [Miller Sahi] [Ciubotaru Trapa]

Which Fourier coefficients to compute?

In string theory on torus T^{d-1} typically interested in U coming from max. parabolic subgroups $P = LU$. Levi subalgebras for example of the form

- $\mathfrak{so}(d-1, d-1) \oplus \mathfrak{gl}(1)$: cusp $g_\mathrm{s} \to 0$ and D-instantons
- $\mathfrak{gl}(d)$: cusp $\mathrm{vol}(T^{d-1}) \to \infty$ and M-instantons
- $\mathfrak{e}_{d-1}$: cusp where one radius $R \to \infty$ and black holes

Also interested in explicit form of Fourier coefficients

Fourier and Whittaker coefficients

Compared to Fourier coefficients F_{ψ_U} more known for Whittaker coefficients

$$W_{\psi_N}(\eta, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \eta(ng)\overline{\psi_N(n)}dn$$

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Have reduction algorithm for relating various coefficients of arbitrary automorphic forms [Gourevitch, Gustafsson AK, Persson, Sahi] $\rightarrow$ more forms

Reduction formula

Eisenstein series defined by choice of weight $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho, H(\gamma g) \rangle}$$

(ρ: Weyl vector)

Character ψ_N for E_d defined by $(m_1, .., m_d) \in \mathbb{Q}^d$ (‘charges’)

Degenerate ψ_N: some m_i vanish. Non-zero ones select subgroup $G' \subset E_d$ such that $\psi_N\big|_{N'}$ is non-degenerate

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Proposition [Hashizume] [Fleig, AK Persson]

Intertwiner= $\prod_{\substack{\alpha>0 \\ w_c^{-1}\alpha<0}} \frac{\zeta^\star(\lambda\cdot\alpha)}{\zeta^\star(\lambda\cdot\alpha+1)}$

$$W^G_{\psi_N}(\lambda, 1) = \sum_{w_c w'_{\text{long}} \in W/W'} M(w_c^{-1}, \lambda) W^{G'}_{\psi_{N'}}(w_c^{-1}\lambda, 1)$$

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can vanish for special λ
small representations!

Reduction formula (II)

Using reduction formula can show that for E_d [Gustafsson AK, Persson]

- for η_{min}: $W_{\psi_N} \neq 0$ only if a single $m_i \neq 0$ (type A_1), $G' = SL(2)$ and a single term in sum (Eulerian)
- for η_{ntm}: $W_{\psi_N} \neq 0$ if a single $m_i \neq 0$ (non-Eulerian) or two disconnected $m_i \neq 0$ (type $2A_1$, Eulerian)

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What about Fourier coefficients for other unipotents U?

Something like Piatetski-Shapiro–Shalika formula for η or F_{ψ_U} in terms of W_{ψ_N}?

Relating coefficients

Algorithm in [Gourevitch, Gustafsson AK, Persson, Sahi], building on [Miller Sahi] [Gomez, Sahi Gourevitch]

Representative (simplified) results

- For η_{min}, U unipotent radical of maximal parabolic and $\psi_U \neq 0$ only on root space defining maximal parabolic:

$$F_{\psi_U}(\eta_{\text{min}}, g) = W_{\psi_N}(\eta_{\text{min}}, g)$$

extend ψ_U trivially from U to N

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- For η_{ntm} and ψ_U of rank two, can find $\gamma \in G(\mathbb{Q})$ such that $\psi_N = \mathrm{Ad}^*_\gamma \psi_U$ on two orthogonal simple root spaces $(2A_1)$

$$F_{\psi_U}(\eta_{\mathrm{ntm}}, g) = \int_{V_\gamma(\mathbb{A})} W_{\psi_N}(\eta_{\mathrm{ntm}}, v\gamma g)dv$$

Lie $V_\gamma = \mathfrak{g}_{>1}^{\gamma S \gamma^{-1}} \cap \bar{\mathfrak{b}}$, where S defines U by e-val ≥ 2

Relating coefficients (II)

General structure of relations

$$F_\psi \text{ or } \eta = \sum \int W$$

replace by
'Levi distinguished'
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'linearly determined by' or 'spanned by' [Hundley Sayag]

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Idea of proof:

1. Initial coefficient F_ψ through Whittaker pair (S,ψ). Defines unipotent integration domain $N_{S,\psi}$
2. 'Fill up' integration domain by deforming Whittaker pair to another (H,ψ) via $H = S + t(H-S)$ for $0 \le t \le 1$
3. At 'critical points' t_* can obtain (i) root exchanges [Ginzburg Rallis, Soudry], (ii) additional Fourier expansion. Might have to conjugate ψ for standard Borel. Discard all terms outside wave-front set of η

Example: ntm of $SL(4)$

One-parameter family of next-to-minimal representations for $SL(4)$ of GK-Dim=4, spherical vector is max. parabolic Eisenstein series $E_{\alpha_2,s}$ for P_2.

Want $F_{\psi_U} = \int_{V_\gamma} W_{\psi_N}(v\gamma)dv$ (Eulerian!) with $U = U_2$ unipotent of middle parabolic and ntm character ψ_U of form

$$P_2 = \left(\begin{array}{cc|cc} * & * & U & U \\ * & * & U & U \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right), \quad \psi_U = \left(\begin{array}{cc|cc} 0 & 0 & m & 0 \\ 0 & 0 & 0 & n \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\gamma = w_2} \left(\begin{array}{cc|cc} 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{array}\right) = \psi_N$$

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$$p<\infty: \qquad F_{p,\psi_U}(1) = (1-p^{2s})(1-p^{2s-1}) \sum_{d|\Gamma} (\det\Gamma)^{2-2s} d^{2s-1} \sigma_{2s-2}\left(\frac{\det\Gamma}{d^2}\right)$$

$$p=\infty: \quad F_{\infty,\psi_U}(1) = \frac{4\pi^{2s-1/2}}{\Gamma(s)\Gamma(s-1/2)} |mn|^{s-1} \int_{\mathbb{R}} K_{s-1}(2\pi|m|\sqrt{1+u^2}) K_{s-1}(2\pi|n|\sqrt{1+u^2}) du$$

(Agrees with direct calculation.)

More auto. functions from strings

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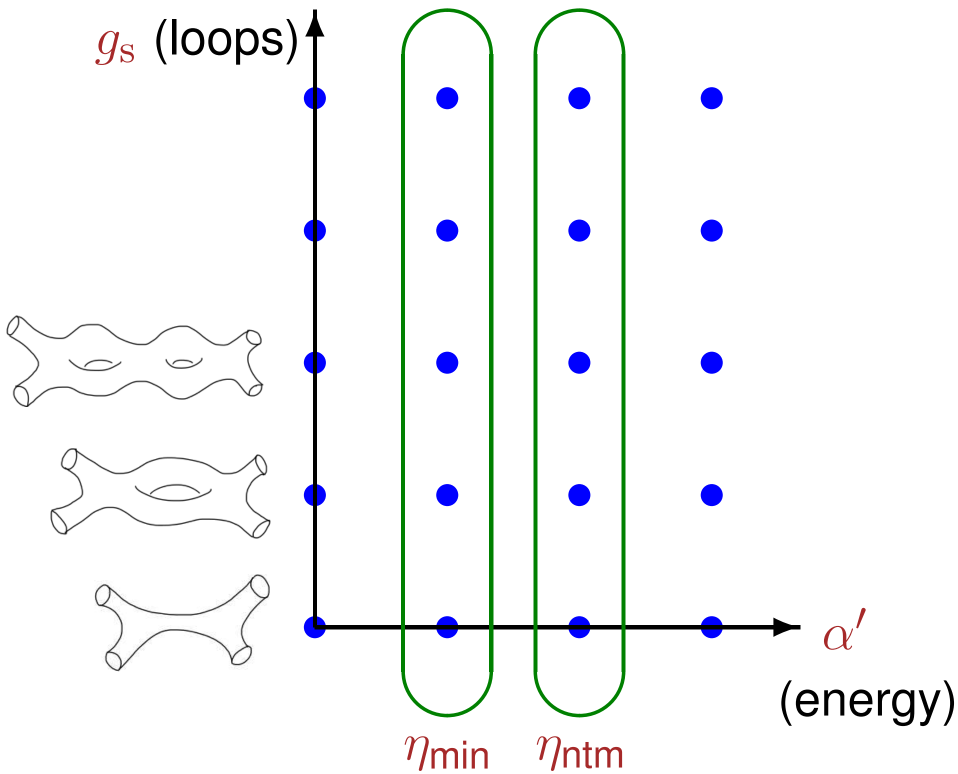

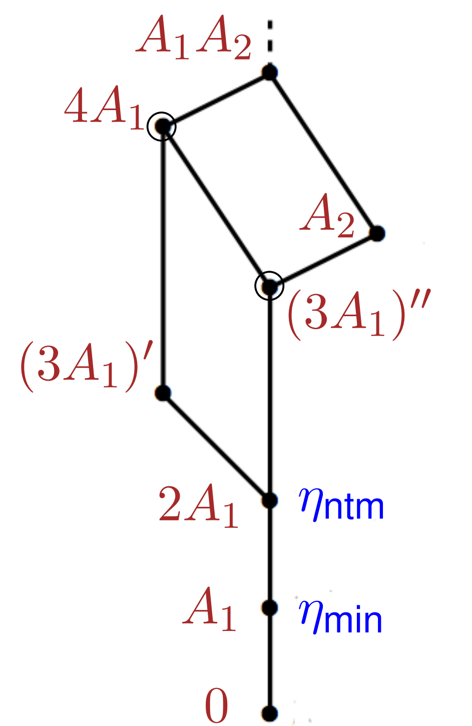

Hasse diagram for $E_7(\mathbb{R})$

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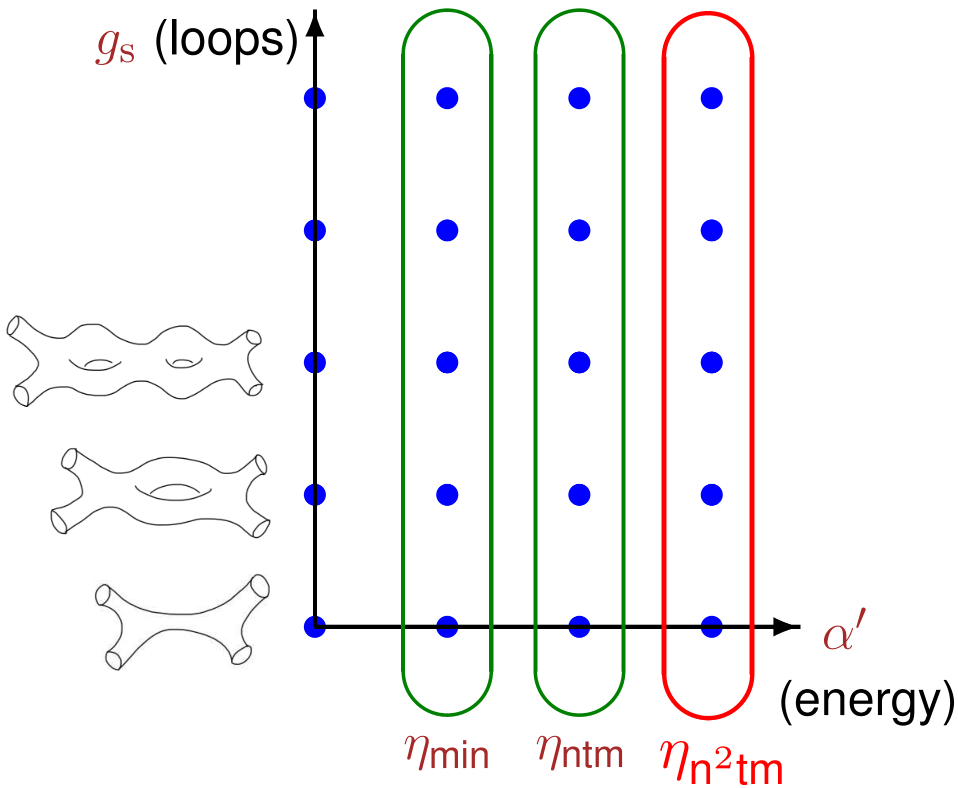

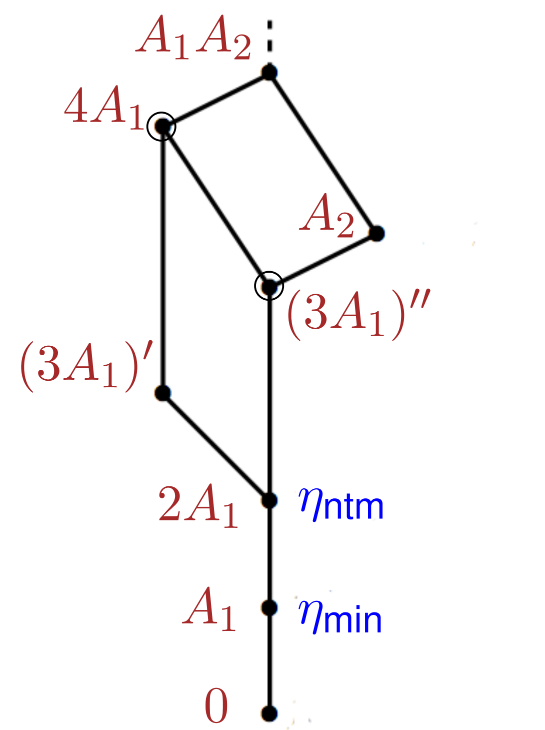

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Why not continue to higher orders in α', i.e. $\eta_{\text{n}^2\text{tm}}$?

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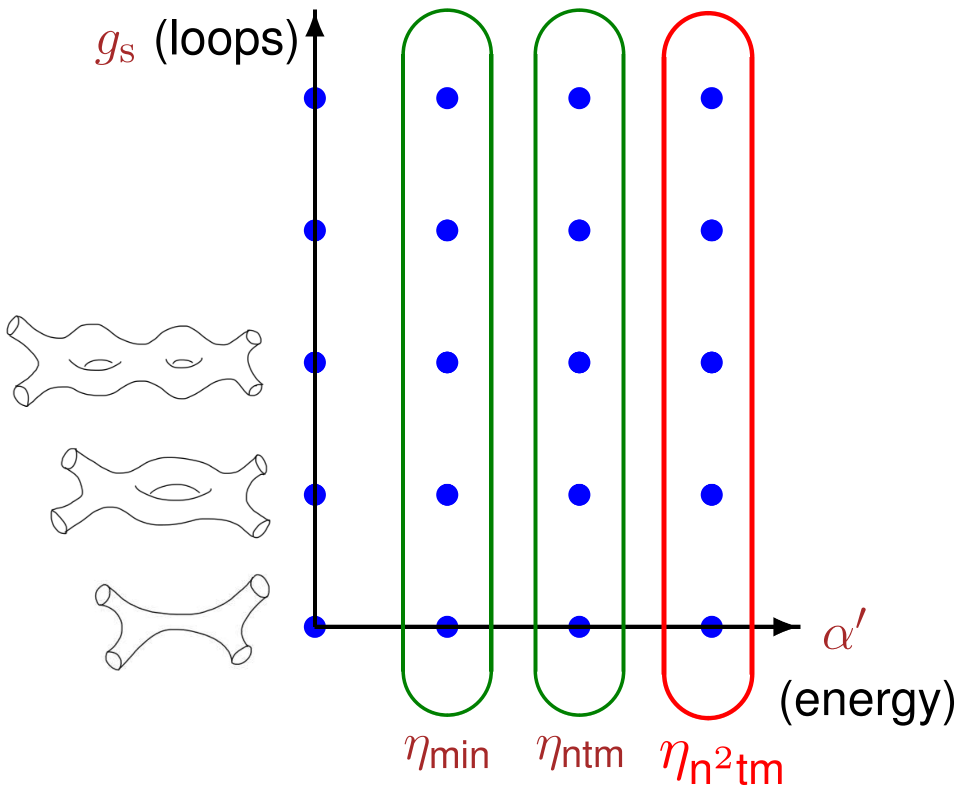

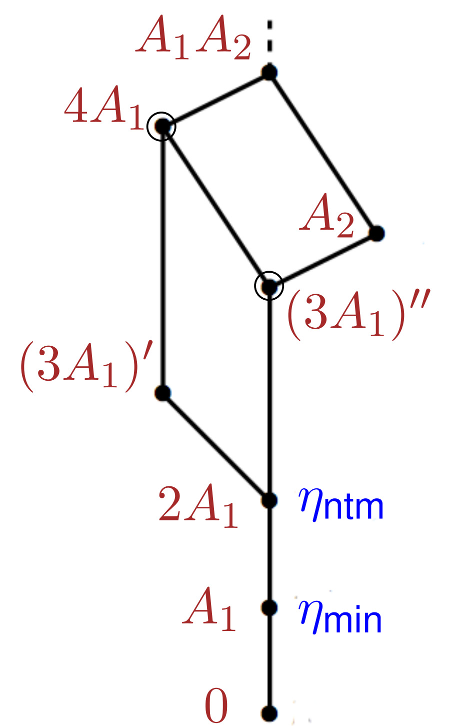

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What identifies η_{min} and η_{ntm} in string theory is Fourier support $\leftrightarrow$ differential equations (=annihilator ideal)

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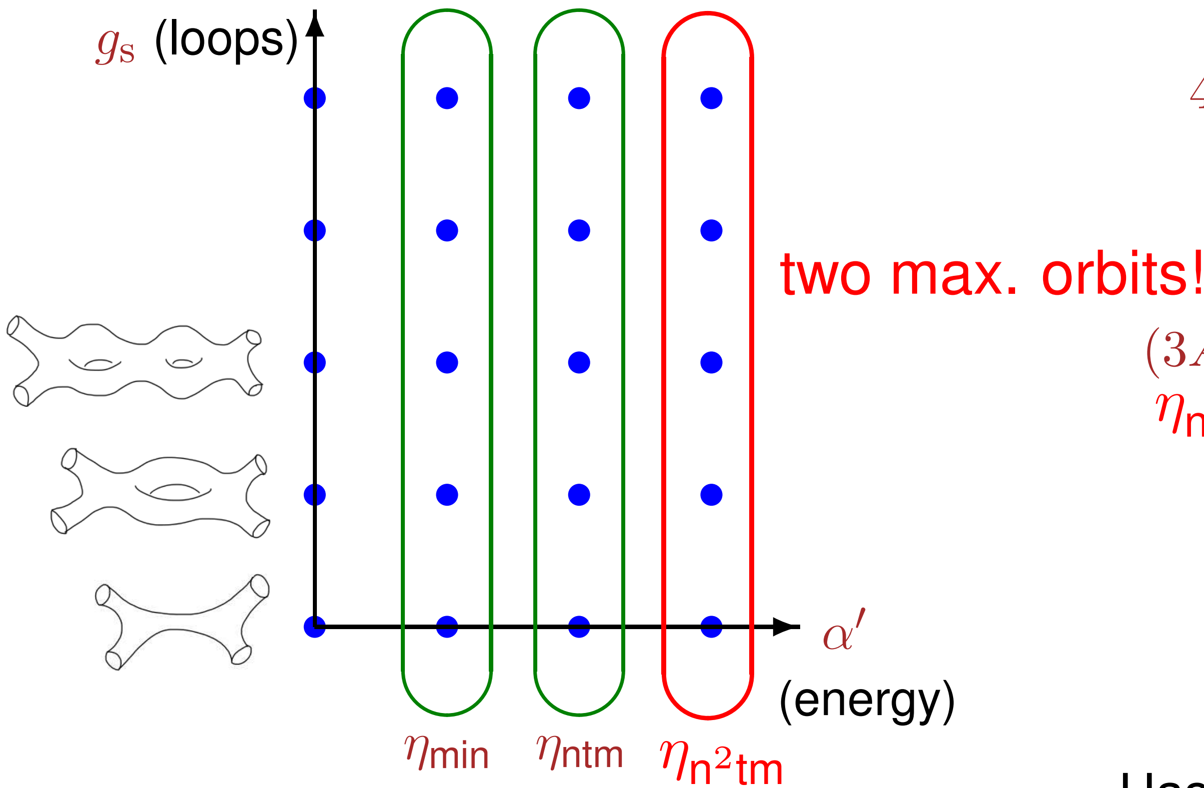

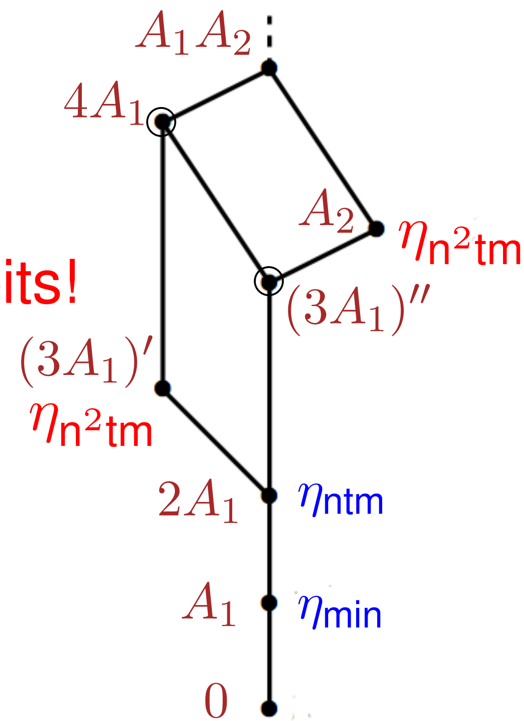

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Exemplary inhomogeneous differential equation for $SL(2)$

[Green Vanhove]

$$(\Delta - 12)\, \eta_{\mathsf{n^2tm}} = -\eta_{\mathsf{min}}^2 \qquad (*)$$

$SL(2)$ invariant Laplacian on UHP

Can be solved using Poincaré series, Fourier series or spectral methods [Green, Miller Vanhove] [Ahlén AK] [Dorigoni AK] [Klinger-Logan]

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Gen. of (*) to E_d [Bossard Verschinin] [Pioline]. Solution [Bossard, AK] [Bossard,AK Pioline]

$$\eta_{\mathsf{n^2tm}}(g) = \frac{4\pi}{3}\int_{\mathbb{R}^3_+}\frac{d^3\Omega_2}{(\det\Omega_2)^{\frac{7-d}{2}}}\varphi(\Omega_2)\sideset{}{'}\sum_{\substack{\Gamma_1,\Gamma_2\in\Lambda\\ \Gamma_i\times\Gamma_j=0}} e^{-\pi\Omega_2^{ij}G(\Gamma_i,\Gamma_j)} + \eta_{\mathsf{hom.}}(g)$$

More auto. functions from strings (II)

Exemplary inhomogeneous differential equation for $SL(2)$

[Green Vanhove]

$$(\Delta - 12)\,\eta_{\text{n}^2\text{tm}} = -\eta^2_{\text{min}} \qquad (*)$$

$SL(2)$ invariant Laplacian on UHP

Can be solved using Poincaré series, Fourier series or spectral methods [Green, Miller Vanhove] [Ahlén AK] [Dorigoni AK] [Klinger-Logan]

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g-dep. bil. form on lattice Λ in E_d-rep

$$\Omega_2^{ij} = \begin{pmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{pmatrix}$$

$\varphi(\Omega_2) = L_1+L_2+L_3-5\frac{L_1L_2L_3}{\det\Omega_2}$

$\sim$ Kawazumi–Zhang inv.

More auto. functions from strings (III)

Fourier expansion only partially analysed [Bossard, AK Pioline]

Note: $\eta_{\mathrm{n}^2\mathrm{tm}}$ not $\mathfrak{Z}$-finite due to inhomogeneity in equation.
New types of automorphic functions!

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Remark: Very similar (and more involved 'higher-depth') equations arise naturally for $SL(2,\mathbb{Z})$ modular graph forms. Connection to iterated Eisenstein integrals and multiple zeta values there [Broedel, Brown, Dupont, Enriquez, Gerken, AK, Matthes, Mizera, Panzer, Schlotterer, Stieberger, Taylor, Zagier, Zerbini,...] Also show up for integrated $\mathcal{N}=4$ super Yang–Mills correlators [Chester, Green Pufu, Wang, Wen] [Dorigoni Green, Wen] [Fedosova Klinger-Logan]

More auto. functions from strings (IV)

Possible picture (in progress [Bossard, Friedberg Gourevitch, AK, Persson]):

More general notion of $\mathcal{U}(\mathfrak{g})$-module associated with inhomogeneous equations of the form

$$(\Delta - \lambda)\, F = S$$

giving rise to an exact sequence

$$0 \to \mathcal{U}(\mathfrak{g})S \to \mathcal{U}(\mathfrak{g})F \to \mathcal{U}(\mathfrak{g})E_P \to 0$$

for some parabolic Eisenstein series E_P that descends to interesting consequences for Fourier coefficients (that grow exponentially per black hole counting).

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Thank you for your attention!

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