Questions from first lecture

- uniqueness of Nash equilibrium

    —can be multiple equilibria as in

	B	F
B	2,1	0,0
F	0,0	1,2

    —in general need strong conditions to obtain uniqueness

    —but have "essential" uniqueness in two-player zero-sum games

- in proof of theorem that continuous game with compact strategy spaces has a Nash equilibrium looked at limit of NE in finite approximation

$$\lim_{t\to\infty}\left(p_1^t,\cdots,p_n^t\right)=\left(p_1^*,\cdots,p_n^*\right)$$

—does sequence converge?

—actually, sequence may not converge

—however, if it doesn't, can take convergent subsequence (which will exist)

- does there exist a NE in chess?
  - yes
  - but can say something stronger based on von Neumann's minimax theorem

Theorem (von Neumann): In finite, two-player zero-sum game,

- there exists (p_1^*, p_2^*) (minimax equilibrium) such that
  - $g_i(p_i, p_j^*) \leq g_i(p_i^*, p_j^*) \leq g_i(p_i^*, p_j)$ for all p_i, p_j

    saddle point property
  - $g_i(p_i^*, p_j^*) = \max_{p_i} \min_{p_j} g_i(p_i, p_j) = \min_{p_j} \max_{p_i} g_i(p_i, p_j)$
- if (p_1^{**}, p_2^{**}) also minimax equilibrium,

  so are (p_1^*, p_2^{**}) and (p_1^{**}, p_2^*)

  exchangeability

implication of Minimax theorem:

- in chess,
  - either both sides can guarantee (at least) draw
  - one side can guarantee win
  - strategies involve no randomization

Lecture 2: Mechanism Design

- imagine town that wants to adopt green energy (no carbon emissions)
- must decide among
  - solar
  - wind
  - nuclear
  - hydro
- suppose mayor wishes to adopt energy type that citizens want

- 3 citizens: Alice, Bob, Cal

either preferences are

θ_1

Alice	**Bob**	**Cal**
solar	nuclear	wind-solar
wind	wind	nuclear
hydro	hydro	hydro
nuclear	solar	

or

θ_2

Alice	**Bob**	**Cal**
nuclear	wind	wind-solar
solar	solar	nuclear
hydro	hydro	hydro
wind	nuclear	

θ_1

Alice	**Bob**	**Cal**
solar	nuclear	wind-solar
wind	wind	nuclear
hydro	hydro	hydro
nuclear	solar	

wind optimal

θ_2

Alice	**Bob**	**Cal**
nuclear	wind	wind-solar
solar	solar	nuclear
hydro	hydro	hydro
wind	nuclear	

solar optimal

- if θ_1 holds, mayor would want to pick *wind*
- if θ_2 holds, mayor would want to pick *solar*
- but suppose mayor doesn't *know* which of θ_1 or θ_2 actually holds

θ_1

Alice	**Bob**	**Cal**
solar	nuclear	wind-solar
wind	wind	nuclear
hydro	hydro	hydro
nuclear	solar	

wind optimal

θ_2

Alice	**Bob**	**Cal**
nuclear	wind	wind-solar
solar	solar	nuclear
hydro	hydro	hydro
wind	nuclear	

solar optimal

- mayor could *simply ask* Alice and Bob which state holds
- but not likely to get a straight answer
  - Alice prefers solar to wind in *both* states
    - so has incentive to say "θ_2" regardless of truth
  - Bob prefers wind to solar in both states
    - so has incentive to say "θ_1" regardless of truth
- so straightforward mechanism of asking citizens won't work

θ_1

Alice	**Bob**	**Cal**
solar	nuclear	wind-solar
wind	wind	nuclear
hydro	hydro	hydro
nuclear	solar	

wind optimal

θ_2

Alice	**Bob**	**Cal**
nuclear	wind	wind-solar
solar	solar	nuclear
hydro	hydro	hydro
wind	nuclear	

solar optimal

- Suppose instead mayor has Alice and Bob play following mechanism:

Bob

Alice		
	wind	hydro
	nuclear	solar

- Alice: can choose top row or bottom row
- Bob: can choose left column or right column
- if θ_1 holds,
  - Alice will prefer top row if Bob plays left column
  - Bob will always prefer left column
  - so (Alice plays top, Bob plays left) is (unique) Nash equilibrium
- mechanism induces optimal choice (wind) in state θ_1

θ_1

Alice	**Bob**	**Cal**
solar	nuclear	wind-solar
wind	wind	nuclear
hydro	hydro	hydro
nuclear	solar	

wind optimal

θ_2

Alice	**Bob**	**Cal**
nuclear	wind	wind-solar
solar	solar	nuclear
hydro	hydro	hydro
wind	nuclear	

solar optimal

- symmetrically, there is unique Nash Equilibrium (bottom right) in state θ_2 leading to optimal choice solar
- we have shown that mechanism *implements* optimal choice (always leads to optimal choice)

- let's look at mechanism design problem in general
- society = $\{1, \ldots, n\}$
  - individual $i = 1, \ldots, n$
- A = set of possible outcomes
  - possible public projects
  - possible allocations of goods to individuals
  - possible political candidates
- Θ = possible states of the world

  state θ = complete description of all relevant data

  e.g.
  - individuals' payoffs from outcomes in A
  - available resources
  - production technology

  thus, A should depend on θ $A(\theta)$
  - ignore this dependence
- payoff function $u_i : A \times \Theta \to \mathrm{R}$

$$u_i(a,\theta) = \text{individual } i\text{'s payoff from outcome } a \text{ in state } \theta$$

social choice rule

$$f : \Theta \twoheadrightarrow A \text{ (set-valued function; correspondence)}$$

$$f(\theta) \subseteq A$$

$f(\theta)$ consists of "optimal" ("best") outcomes in state θ

- in energy example,

$$f(\theta_1) = \text{wind and } f(\theta_2) = \text{solar}$$

- if mechanism designer *knows* θ, then achieving optimal outcome easy
  - just choose $a \in f(\theta)$

- if *doesn't* know θ, must proceed more indirectly

- mechanism

$$g : S_1 \times \cdots \times S_n \to A$$

$S_i =$ individual i's *strategy set,* with typical element $s_i \in S_i$

- Nash equilibrium for g in state θ

$$\left(s_1, \ldots, s_n\right) = \left(s_i, s_{-i}\right)$$

such that

$$u_i\left(g\left(s_i, s_{-i}\right), \theta\right) \geq u_i\left(g\left(s_i', s_{-i}\right), \theta\right) \text{ for all } s_i' \in S_i$$

  - n - tuple of strategies such that no individual gains from deviating unilaterally

$$NE_g(\theta) = \{a \in A \mid \text{there exists Nash equilibrium} (s_i, s_{-i}) \text{ for } g \text{ in state } \theta \text{ such that } a = g(s_i, s_{-i})\}$$

= Nash equilibrium outcomes of mechanism g in state θ

mechanism g *implements* SCR f if

$$NE_g(\theta) = f(\theta) \text{ for all } \theta$$

- i.e., whatever the state

predicted outcome = *desired* outcome

When is SCR f implementable?

- *monotonicity* is key
- f is monotonic provided that

  for all $\theta, \theta' \in \Theta$ and $a \in f(\theta)$,

  if, for all i and b,

  $$(*) \quad u_i(a,\theta) \geq u_i(b,\theta) \Rightarrow u_i(a,\theta') \geq u_i(b,\theta')$$

  then $a \in f(\theta')$

- suppose a is optimal in state θ
- now change payoff functions so that for any b, if individual i prefers a to b in state θ, still prefers a to b in state θ'
  - a doesn't "fall" vis à vis any b in going from θ to θ' (condition $(*)$)

  then a optimal in state θ'

θ_1

Alice	**Bob**	**Cal**
solar	nuclear	wind-solar
wind	wind	nuclear
hydro	hydro	hydro
nuclear	solar	

wind optimal

θ_2

Alice	**Bob**	**Cal**
nuclear	wind	wind-solar
solar	solar	nuclear
hydro	hydro	hydro
wind	nuclear	

solar optimal

- $f(\theta_1) = \text{wind}$ $\quad$ $f(\theta_2) = \text{solar}$
  - wind falls in Alice's ranking going from $\theta_1 \text{ to } \theta_2$
  - solar falls in Bob's ranking going from $\theta_2 \text{ to } \theta_1$
  - so f is monotonic

Suppose

θ_3

Alice	Bob	Cal
solar	wind	hydro-nuclear
hydro	nuclear	wind
nuclear	hydro	solar
wind	solar	

Suppose

θ_4

Alice	Bob	Cal
solar	nuclear	hydro-nuclear
hydro	wind	wind
nuclear	hydro	solar
wind	solar	

$\hat{f}(\theta_3) = \text{hydro}$ $\hat{f}(\theta_4) = \text{nuclear}$

- hydro does not fall in preferences in going from θ_3 to θ_4
- So $\hat{f}$ not monotonic

Theorem 1: If f implementable, then monotonic

Proof:

- suppose f implemented by g
- consider θ and $a \in f(\theta)$
  - then there exists $(s_1,\ldots,s_n)$ such that

$$g(s_1,\ldots,s_n) = g(s_i,s_{-i}) = a$$

(**) $u_i(g(s_i,s_{-i}),\theta) \geq u_i(g(s_i',s_{-i}),\theta)$ for all i, s_i'

- consider θ' such that

(*) $u_i(a,\theta) \geq u_i(b,\theta) \Rightarrow u_i(a,\theta') \geq u_i(b,\theta')$ for all i and b

- To show: $a \in f(\theta')$
- from (*) and (**),

$$u_i(g(s_i,s_{-i}),\theta') \geq u_i(g(s_i',s_{-i}),\theta') \text{ for all } i, s_i'$$

- so $(s_1,\ldots,s_n)$ is a Nash equilibrium in state θ'
- so $g(s_1,\ldots,s_n) = a \in f(\theta')$ (by definition of an implementing mechanism)

- converse *not* true
  - there exist monotonic SCRs that can't be implemented
- but *almost* true

No Veto Power: f satisfies no veto power provided that, for all *i,*

if

$$u_j(a,\theta) \geq u_j(b,\theta) \text{ for all } b \text{ and } j \neq i$$

then

$$a \in f(\theta)$$

i.e., if all people except possibly *i* agree that *a* is their favorite outcome, then *i* cannot veto it

- f and $\hat{f}$ in energy example both satisfy no veto power

Theorem 2: If $n \geq 3$ and f is monotonic and satisfies no veto power, then f is implementable

Proof:

- Let $S_i = \Theta \times A \times \mathrm{N}$
  $s_i = (\theta_i, a_i, m_i)$

- individual i announces
  - state θ_i
  - outcome a_i
  - positive integer m_i

(A) If $(\theta_1, a_1) = \cdots = (\theta_n, a_n) = (\theta, a)$

and

$a \in f(\theta)$

- then $g(s_1, \ldots, s_n) = a$

(B) If $(\theta_j, a_j) = (\theta, a)$ for all $j \neq i$

and

$a \in f(\theta)$

- then

$$g(s_1, \ldots, s_n) = \begin{cases} a_i, \text{ if } u_i(a, \theta) \geq u_i(a_i, \theta) \\ \\ a, \text{ if } u_i(a_i, \theta) > u_i(a, \theta) \end{cases}$$

(requires $n \geq 3$)

(C) In all other cases

$$g(s_1, \ldots, s_n) = a_i,$$

where $m_i = \arg\max_j m_j$

Claim 1: If $a \in f(\theta)$ and

(i) $s_1 = \cdots = s_n = (\theta, a, 1)$,

then $(s_1, \ldots, s_n)$ is a Nash equilibrium in state θ and

(ii) $g(s_1, \ldots, s_n) = a$

Proof:

- From (A), (i) implies (ii)
- From (B), if i deviates from (s_i, s_{-i}), can't get anything better than a, so

  $(s_1, \ldots, s_n)$ is a Nash equilibrium in state θ

- Thus if $a \in f(\theta)$, there exists Nash equilibrium producing a

- Remains to show that if a is Nash equilibrium outcome in state θ,

  then $a \in f(\theta)$

Claim 2: If

(iii) $a \in f(\theta)$,

(iv) $(\theta_i, a_i) = (\theta, a)$ for all i,

and

(v) $(s_1, \ldots, s_n)$ is a Nash equilibrium in state θ',

then $a \in f(\theta')$

Proof: From (iii), (iv), and (A)

$$g(s_1, \ldots, s_n) = a$$

- suppose for some i and b

(vi) $u_i(a, \theta) \geq u_i(b, \theta)$

- From (B), (vi) implies that if $s_i' = (\theta, b, 1)$

(vii) $g(s_i', s_{-i}) = b$

- Now if $u_i(b, \theta') > u_i(a, \theta')$,

  then from (vii) $(s_1, \ldots, s_n)$ is not a Nash equilibrium in state θ', contradicitng (v). Hence (vi) implies

(viii) $u_i(a, \theta') \geq u_i(b, \theta')$

- From monotonicity, (iii), (vi), and (viii) imply
  $a \in f(\theta')$, as claimed

Claim 3: If

(ix) $(\theta_j, a_j) = (\theta, a)$ for all $j \neq i$

$(\theta_i, a_i) \neq (\theta, a)$,

and

(x) $(s_1, \ldots, s_n)$ is a Nash equilibrium in state θ',

then

$$g(s_1, \ldots, s_n) \in f(\theta')$$

Proof:

From (ix) and (C), each individual $j \neq i$ can get favorite

- outcome by choosing $s'_j = (\theta'_j, a'_j, m'_j)$ where

$\theta'_j \neq \theta$

a'_j is j's favorite outcome in state θ'

$m'_j > \max_{k \neq j} m_k$

because $(s_1, \ldots, s_n)$ is Nash equilibrium in state θ',

- Hence, $u_j(g(s_1, \ldots, s_n), \theta') \geq u_j(b, \theta')$ for all b
- Thus, from no veto power,

$$g(s_1, \ldots, s_n) \in f(\theta')$$

Claim 4: If $(s_1, \ldots, s_n)$ is a Nash equilibrium in state θ' for which (C) applies, then $g(s_1, \ldots, s_n) \in f(\theta')$

Proof: Since (C) applies,

- *all j* can deviate and get favorite alternative
- so from no veto power, $g(s_1, \ldots, s_n) \in f(\theta')$

So $NE_g(\theta) = f(\theta)$ for all θ, i.e., g implements f