Arithmetic Topology and Field Theory

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I. Knots and Primes (following Mazur, Manin, Mumford, ..)

Notation

F: an algebraic number field (usually totally imaginary)

$X := \mathrm{Spec}(\mathcal{O}_F)$

F_v: completion at $v \in X$

$X_v = \mathrm{Spec}(\mathcal{O}_{F_v})$

$k_v = \mathcal{O}_{F_v}/m_{F_v}$

$v = \mathrm{Spec}(k_v) \hookrightarrow X$, also $v \hookrightarrow X_v \longrightarrow X$

$T_v = \mathrm{Spec}(F_v) = X_v \setminus v$

B: finite set of points in X

$X^B := X \setminus B$

$T_B := \coprod_{v \in B} T_v$

$$\pi := \pi_1(X, b), \qquad \pi^B := \pi_1(X^B, b), \qquad \pi_v := \pi_1(T_v)$$

Analogies

$X := \mathrm{Spec}(\mathcal{O}_F) \sim$ 3-manifold

$v = \mathrm{Spec}(k_v) \hookrightarrow X \sim$ knot in 3-manifold

$X_v = \mathrm{Spec}(\mathcal{O}_{F_v}) \sim$ tubular neighbourhood of $v \sim$ solid torus

$T_v = \mathrm{Spec}(F_v) = X_v \setminus v \sim$ deleted tubular neighbourhood of $v \sim$ solid torus with interior removed=(hollow) torus

B: finite set of points in $X \sim$ collection of knots, i.e., link

$X^B := X \setminus B \sim$ 3-manifold with boundary

$T_B := \coprod_{v \in B} T_v \sim$ boundary of X_B

Goal: Explore these analogies from the viewpoint of (topological) quantum field theory.

II. Weil's Trichotomy

Analogy between Function Fields and Number Fields

Structural similarity:

$$F \sim k(C),$$

where F is an algebraic number field and C is a smooth projective curve over a finite field $k = \mathbb{F}_q$, $q = p^n$.

Also,

$$X^B \sim C^S := C \setminus S,$$

S finite set of closed points.

Analogy between Function Fields and Number Fields

Weil remarks that the analogy between F and $k(C)$ is

> *so strict and obvious that there is neither an argument nor a result in arithmetic that cannot be translated almost word for word to the function fields.*

Substantial consequences, e.g.

–Riemann hypothesis for varieties over finite fields;
–Langlands correspondence for function fields;
–The Fundamental Lemma;
–Weight monodromy conjecture for complete intersections.

Trichotomy (‘Rosetta Stone’)

Weil believed $k(C)$ to be an intermediate point in a bridge linking F and

$$\mathbb{C}(\Sigma),$$

the field of meromorphic functions on a compact smooth Riemann surface Σ:

$$F \sim k(C) \sim \mathbb{C}(\Sigma).$$

However, his sense of the the similarity between $k(C)$ and $\mathbb{C}(\Sigma)$ is expressed more cautiously:

> *The distance is not so large that a patient study would not teach us the art of passing from one to the other, and to profit in the study of the first from knowledge acquired about the second.*

Of course the analogy $k(X) \sim \mathbb{C}(\Sigma)$ is not quite right.

Trichotomy: Correction

A better analogy is

$$\bar{k}(C) \sim \mathbb{C}(\Sigma),$$

where $\bar{k}(C)$ is the field of rational functions on $\bar{C}$, the base-change of C to the algebraic closure $\bar{k}$ of k.

This is because of a comparison of (cohomological) *dimensions*.

Thus, we actually have two separate analogies

$$\bar{k}(C) \sim \mathbb{C}(\Sigma)$$

$$F \sim k(C)$$

How to extend these to trichotomies

$$? \sim \bar{k}(C) \sim \mathbb{C}(\Sigma)$$

$$F \sim k(C) \sim ?$$

Will focus today mostly on the second.

Trichotomy: Correction

Note that

$$\bar{C} \sim \Sigma,$$

an analogy of geometric objects and not just fields. Then we have

$$\begin{array}{ccc} \bar{C} & \hookrightarrow & C \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\bar{k}) & \longrightarrow & \mathrm{Spec}(k) \sim S^1 \end{array}$$

Trichotomy, Correction

We see that C itself is analogous to a fibered three manifold

$$\begin{array}{ccc} \Sigma & \hookrightarrow & M \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & S^1 \end{array}$$

with fibre Σ.

This is compatible with an analogy between $\mathrm{Spec}(\mathcal{O}_F)$ ($\sim C$) and a three-manifold.

We will examine this from the point of view of TQFT.

III. A few elements of (topological) quantum field theory

Quantum Field Theory

Typical ingredients of field theory of dimension d:

1. Manifold M of dimension d, the model for spacetime. For example, $\mathbb{R}^4$ with Minkowski or Euclidean metric.

2. Fibre bundle $F \longrightarrow M$. For example, $M \times N$, tensor bundles, principal bundles, bundle of connections on a principal bundle. The space F could be a stack in general, e.g., $M \times BG$.

3. $\mathcal{F}_M = \Gamma(M, F)$, space of fields. For example, vector fields, tensor fields, connections, maps to some other manifold. Bundles themselves.

Quantum Field Theory

4. A *theory* consists of a function

$$S : \mathcal{F}_M \longrightarrow \mathbb{C}$$

called the *action*, typically expressed as

$$S(\phi) = \int_M L(\phi(x), \nabla\phi(x), \nabla^2\phi(x), \ldots) dvol_M.$$

The function L is usually of the form

$$\langle D\phi(x), D\phi(x)\rangle + \text{ higher order terms}$$

for some linear differential operator D.

5. In classical field theory, one studies the space of classical states

$$\mathbb{S}_M \subset \mathcal{F}_M,$$

consisting of fields that satisfy the Euler-Lagrange equation for S describing the extrema of the function.

Quantum Field Theory

6. In a quantum field theory, one considers integrals like

$$\int_{\mathcal{F}_M} \exp(-\pi S(\phi)) dvol_{\mathcal{F}}$$

or

$$\int_{\mathcal{F}_M} g_1(\phi) g_2(\phi) \cdots g_k(\phi) \exp(-\pi S(\phi)) dvol_{\mathcal{F}},$$

where the $g_i(\phi)$ are usually local functions of ϕ, e.g.,

$$\phi \mapsto \phi(x), \frac{\partial}{\partial t}\phi(x).$$

Integrals like the first one are often viewed as invariants of the manifold M, once the theory is fixed and makes sense on any manifold.

Quantum Field Theory

For example, for electromagnetism on a compact Riemannian manifold with $H^1(M) = 0$, one might get

$$\int_{\mathcal{F}_M} \exp(-\pi S(\phi)) dvol_{\mathcal{F}} = \frac{1}{\sqrt{\det \Delta_1}},$$

where Δ_1 is the Laplacian on 1-forms.

Quantum Field Theory

When N is a manifold of dimension $d-1$, since one can consider the theory on

$$M = N \times [0,1],$$

there is also a vector space of initial conditions $Z(N)$ attached to N, approximately thought of as

$$Z(N) = L^2_{hol}(\mathbb{S}_{N\times[0,1]}, \mathbb{C}).$$

If M is a cobordism from N_1 to N_2, one should also get a linear transformation

$$Z(M) : Z(N_1) \longrightarrow Z(N_2),$$

thought of as an integral operator with kernel

$$K(\phi_1, \phi_2) := \int_{\phi|N_1=\phi_1, \phi|N_2=\phi_2} \exp(-\pi S(\phi)) dvol_{\mathcal{F}}$$

Quantum Field Theory

There is a monoidal property

$$Z(\phi) = \mathbb{C}.$$

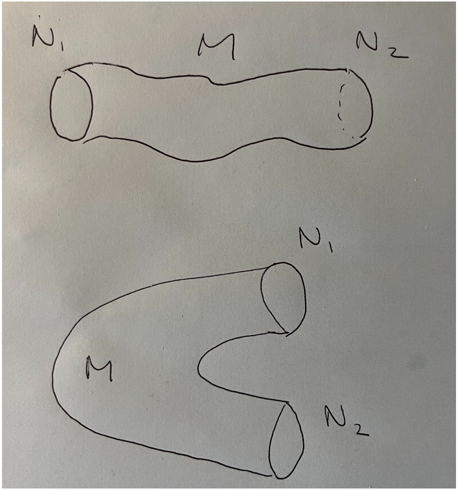
$$Z(N \coprod N') = Z(N) \otimes Z(N').$$

$$Z(-N) = Z(N)^*.$$

The operator associated to a cobordism can be compactly expressed as

$$Z(M) \in Z(N)$$

when $\partial M = N$.



N_1
M
N_2
N_1
M
N_2

Quantum Field Theory

The reason is that

$$\mathbb{S}_M \longrightarrow \mathbb{S}_N$$

is a Lagrangian, which gives rise to a vector

$$Z(M) \in Z(N).$$

In an extended field theory, get

$$Z(N_{d-k})$$

for a manifold of codimension k.

This should be thought of as the quantisation of $\mathbb{S}_{N_{d-k}}$ which has a $[k-1]$-shifted symplectic structure. Thus $Z(N_{d-k})$ is a $(k-1)$-category (Safronov).

When

$$\partial M_{d-k+1} = N_{d-k},$$

then

$$\mathbb{S}_{M_{d-k+1}} \longrightarrow \mathbb{S}_{N_{d-k}}$$

is a Lagrangian, and hence, gives an object

$$Z(M_{d-k+1}) \in Z(N_{d-k}).$$

Especially important is when $k = 2$, so that $Z(N_{d-k})$ is a category and $Z(M_{d-k+1})$ is an object in it. Importantly, get such an object from any

$$Lag \longrightarrow \mathbb{S}_{N_{d-k}},$$

not just $\mathbb{S}_{M_{d-k+1}}$.

An important case is that of a *boundary condition*.

Quantum Field Theory

Suppose $M = M_1 \cup_f M_2$, where

$$f : -\partial M_1 \simeq \partial M_2.$$

Then

$$Z(M) =< Z(M_1), f^* Z(M_2) > .$$

An important variant is when $M = N \times [0,1]$ and $f : N \simeq N$. Then

$$Z(M) = Tr(f | Z(N)).$$

IV. Return to Trichotomy

Return to Trichotomy: Application

Suppose there were a 3d arithmetic topological field theory.

Then it would assign to C a number $Z(C)$ and to $\bar{C}$ a finite-dimensional vector space $Z(\bar{C})$.

But

$$C \sim M \simeq (\Sigma \times [0,1])/f,$$

where $f : \Sigma \simeq \Sigma$ is a monodromy diffeomorphism.

The glueing formula implies that the isotopy class of f acts on $Z(\Sigma)$ and

$$Z((\Sigma \times [0,1])/f) = Tr(f^* | Z(\Sigma))$$

The analogy is that

$$C \sim (\bar{C} \times [0,1])/Fr_q.$$

Trichotomy: Application

Construction [joint with Akshay Venkatesh]:

Let Y be a lift of $\bar{C}$ to $W = W(\bar{k})$.

Let $L \longrightarrow J_Y$ a theta line bundle on the Jacobian of Y, giving a principal polarisation.

Let N be an odd prime such that $q \equiv 1 \mod N$.

Then

$$H = Z(\bar{C}) := \Gamma(J_Y, L^N) \otimes \mathbb{C}.$$

Trichotomy: Application

H is acted on by the finite Heisenberg group with centre μ_N:

$$\mathcal{H}_N = \mu_N \times J[N]$$

with group structure given by

$$(\lambda, a) \circ (\mu, b) = (\langle a, b \rangle^{1/2} \lambda\mu, a + b).$$

There is also an action of the finite symplectic group of $J[N]$ (Gurevich and Hadani).

Since C is defined over $\mathbb{F}_q$, the Frobenius Fr_q acts on $J[N]$ by symplectic transformations. So F_q acts on H. Then

$$Z(C) := Tr(F_q | H).$$

Trichotomy: Application

Formula:

Assume there is a Lagrangian subspace $M \subset J[N]$ such that $F_q(M) = M$. Then

$$Z(C) = \pm\sqrt{|Cl(C)[N]|}$$

Remark:

Gaitsgory, Rosenblyum, Raskin, ...study a $4d$ theory over finite fields.

Thus,

$$Z(\bar{C})$$

is a dualisable category.

They then take a categorical trace

$$Tr(Fr_q|Z(\bar{C}))$$

which is a vector space over $\bar{\mathbb{Q}}_\ell$. This is identified with a space of automorphic forms.

Trichotomy: Application

Proof of Formula:

H is the unique (up to almost unique isomorphism) irreducible representation of $\mathcal{H}_N$ with trivial central character.

Thus,

$$H \simeq C_{M^o} = \mathrm{Fun}(J[N]/M, \mathbb{C}),$$

where M^o denotes M with some fixed basis of $\wedge^{top} M$. Hadani and Gurevich show that there are canonical isomorphisms

$$T_{M^o,(M')^o} : C_{M^o} \simeq C_{(M')^o},$$

for any pair of oriented Lagrangians.

This is used to define the action of the symplectic group: Given $g \in Sp(J[N])$,

$$C_{M^o} \simeq^{\circ g^{-1}} C_{gM^o} \simeq^{T_{(gM^o),M^o}} C_{M^o}.$$

Trichotomy: Application

Proof of Formula (continued):

When $gM = M$, then $T_{g(M^o),M^o} = \pm 1$. Thus,

$$Tr(Fr_q|H) = \pm Tr(Fr_q|C_M) = \pm Tr(Fr_q|\mathrm{Fun}(M', \mathbb{C})),$$

where $M' \subset J[N]$ is a complementary subspace.

Easy to see that

$$Tr(Fr_q|\mathrm{Fun}(M', \mathbb{C})) = |(M')^{Fr_q}|.$$

Via duality given by the Weil pairing

$$|(M')^{Fr_q}| = |M^{Fr_q}|,$$

so that

$$|(M')^{Fr_q}| = \sqrt{|(M \times M')^{Fr_q}|} = |J[N]^{Fr_q}| = |Cl(X)[N]|.$$

V. Plan

Analogy Reminder

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$$\pi = \pi_1(X), \qquad \pi^B = \pi_1(X^R), \qquad \pi_v = \pi_1(T_v).$$

Improved Analogy

Suitable moduli space of sheaves on $X^B \sim$ Suitable moduli space of sheaves on $C^B \sim$ space of fields on 3d spacetime.

For example,

$$\mathcal{M} = \mathrm{Hom}(\pi, R) /\!\!/ R \quad \text{or} \quad H^1(X, R)$$

for a p-adic Lie group R (e.g., $R = G(\mathbb{Z}_p)$ for a reductive group G) or for a sheaf R (e.g., p-adic rep of π).

In the first instance, a pair (v, V), where $v \in X$ and V is a $\mathbb{Q}_p$-representation of R defines a function

$$\mathcal{M} \longrightarrow \mathbb{Q}_p$$

$$\rho \mapsto \mathrm{Tr}(\rho(Fr_v)|V).$$

In short, the analogy between decorated subschemes and extended operators in QFT is an improvement.

Improved Analogy

What about actions and path integrals?

Arithmetic Chern-Simons

On suitable

$$\mathcal{M} = \mathrm{Hom}(\pi, R) /\!\!/ R,$$

define

$$CS : \mathcal{M} \longrightarrow K.$$

On

$$\mathcal{M}_B^{\mathrm{loc}} = \prod_{v \in B} \mathrm{Hom}(\pi_v, R)$$

define line bundle L and space

$$Z(T_B) := \Gamma(\mathcal{M}_B^{\mathrm{loc}}, L).$$

To X^B, associate

$$Z(X^B) \in Z(T_B).$$

Field Theories and L-functions

In old paper, thought this construction should be related to L-functions, which are also canonical trivialisations of determinant lines.

Not quite right. Should have something like line bundle

$$L^{glob} \longrightarrow \mathcal{M}_B^{glob} = \mathrm{Hom}(\pi^B, R) /\!\!/ R$$

and section

$$\mathcal{L} \in \Gamma(\mathcal{M}_B^{glob}, L^{glob}).$$

That is, need 4d theory. However, this seems to require a bounding arithmetic 4-manifold.

However, Ben-Zvi, Sakellaridis, and Venkatesh are pointing out that such vectors also come from *boundary conditions*.

More generally, suffices to have a Lagrangian

$$Lag \longrightarrow \mathcal{M}_B^{glob}$$

Field Theories and L-functions

But $\mathcal{M}_B^{glob}$ is typically not symplectic.

There should rather be a conic n-shifted symplectic moduli space $\mathbb{S}$ of arithmetic sheaves whose sheared quantisation is a suitable $n+2$-category

$$Q(\mathbb{S}).$$

A conic Lagrangian

$$Lag \longrightarrow \mathbb{S}$$

should then give an object

$$Z(Lag) \in Q(\mathbb{S})$$

Might try to construct L-function as a trace.

Field Theories and L-functions

For example, might consider

$$H^1(X_\infty^{(p)}, Lie(R)^*(1)/R)$$

for

$$X_\infty^{(p)} = \mathrm{Spec}(\mathbb{Z}[\mu_{p^\infty}][1/p]).$$

For a graded hyperspherical Hamiltonian R-space M, get conic Lagrangian

$$H^1(X^{(p)}, M/R) \longrightarrow H^1(X_\infty^{(p)}, Lie(R)^*(1)/R)$$

?

Variant, work on $X^{(p)}$ with Λ-adic sheaves, where

$$\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q})]] \simeq \mathbb{Z}_p[[T]].$$