Arithmetic Topology and Field Theory

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VI. L-functions

L-functions

L functions are topological invariants of $(X, \mathcal{F})$, where X is an arithmetic scheme and $\mathcal{F}$ is a locally constant sheaf of A-modules.

Usually $A \subset \mathbb{C}$, $A \subset \bar{\mathbb{Q}}_p$, or A is finite.

The L-function evaluated on $(X, \mathcal{F})$ is the canonical assignment

$$(X, \mathcal{F}) \mapsto L(X, \mathcal{F}) \in \det R\Gamma_c(X, \mathcal{F})^*$$

of an element in the determinant of cohomology of $\mathcal{F}$.

This assignment is subject to a collection of conditions, including some difficult functoriality that I will not spell out. However, I will focus on a few interesting conditions.

L-functions

I. (Multiplicativity)

Given an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

the canonical isomorphism

$$\det R\Gamma_c(X, \mathcal{F}_2) \simeq \det R\Gamma_c(X, \mathcal{F}_1) \otimes \det R\Gamma_c(X, \mathcal{F}_3)$$

takes $L(X, \mathcal{F}_2)$ to $L(X, \mathcal{F}_1) \otimes L(X, \mathcal{F}_3)$.

L-functions

Thus, for a triple

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z,$$

get

$$0 \longrightarrow j_! j^*(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow i_* i^*(\mathcal{F}) \longrightarrow 0$$

and hence

$$L(X, \mathcal{F}) = L(U, j^*\mathcal{F}) L(Z, i^*\mathcal{F}).$$

Doing this for Z an increasing union of points, formally get

$$L(X, \mathcal{F}) = \prod_{x \in X_0} L(x, \mathcal{F}).$$

L-functions

II. (Normalisation for finite fields)

For $X = \mathrm{Spec}(\mathbb{F}_q)$, a sheaf is equivalent to an A-module $\mathcal{F}$ with Fr_q-action. In this case, cohomology is computed via the exact sequence

$$0 \longrightarrow H^0(\mathcal{F}) \longrightarrow \mathcal{F} \xrightarrow{I - Fr_q} \mathcal{F} \longrightarrow H^1(\mathcal{F}) \longrightarrow 0,$$

from which one gets the isomorphism

$$\det(\mathcal{F})^* \otimes \det(\mathcal{F}) \simeq \det R\Gamma_c(\mathcal{F})^*.$$

Thus, the element $Id \in \det(\mathcal{F})^* \otimes \det(\mathcal{F})$, gives rise to a canonical element

$$L(\mathrm{Spec}(\mathbb{F}_q), \mathcal{F}) \in \det R\Gamma_c(\mathcal{F})^*,$$

which normalises the L-function over a finite field.

L-functions

In case $\mathcal{F}$ is *acyclic*, get another trivialisation

$$\det R\Gamma_c(\mathcal{F})^* \simeq A,$$

with respect to which the element $L(\mathrm{Spec}(\mathbb{F}_q), \mathcal{F})$ is identified with

$$\frac{1}{\det((I - Fr_q)|\mathcal{F})}.$$

Thus, for general X, we get the formal expression

$$L(X, \mathcal{F}) = \prod_{x \in X_0} \frac{1}{\det((I - Fr_x)|\mathcal{F}_x)}.$$

L-functions

III. Complex normalisation.

When $A \subset \mathbb{C}$, the assignment should be the natural (infinite) product when it converges, for example, when $\mathcal{F}$ has sufficiently negative weight.

This means the eigenvalues of Fr_x acting on $\mathcal{F}_x$ are of size $|N(x)|^c$ for a finite collection of exponents c sufficiently negative.

IV. p-adic normalisation.

In the p-adic case, one would like an interpolation property giving compatibility with the complex L-function.

L-functions

The variation of $L(X, \mathcal{F})$ with the sheaf $\mathcal{F}$ is the subject of the Hasse-Weil conjecture when $A \subset \mathbb{C}$ and the Main Conjecture of Iwasawa theory when $A \subset \bar{\mathbb{Q}}_p$.

If $\mathcal{M}$ is a 'natural' moduli space of sheaves on X, then $\mathcal{M}$ carries a natural determinant line bundle

$$\mathrm{Det}^* \longrightarrow \mathcal{M}.$$

The conjectures propose that one can canonically construct an analytic section

$$L(X, \cdot) \in \Gamma(\mathcal{M}, \mathrm{Det}^*),$$

compatible with the conditions above.

L-functions

In particular, the Hasse-Weil conjecture says that one should start with any motivic sheaf $\mathcal{F}$ on $\mathrm{Spec}(\mathcal{O}_F[1/B])$, and then naturally define

$$L(\mathrm{Spec}(\mathcal{O}_F[1/B]), \mathcal{F} \otimes \| \cdot \|^s)$$

for any complex parameter s.

Meanwhile the main conjecture says something similar, that one should be able to define the p-adic analytic section

$$L(\mathrm{Spec}(\mathcal{O}_F[1/B]), \mathcal{F} \otimes \chi_p^s)$$

for all $s \in \mathbb{Z}_p^\times$, where χ_p is the p-adic cyclotomic character.

VII. A Review of Some Elementary Physics

Classical Systems

Space of all possible states: *symplectic manifold* $(\mathbb{S}, \omega)$, where $\omega \in \Omega^2(\mathbb{S})$ is non-degenerate and closed.

Examples:

$$\mathbb{R}^3 \times \mathbb{R}^3 = \{(p, q)\}$$

states of a single point particle in Euclidean space. Symplectic form

$$\sum_{i=1}^{3} dp_i \wedge dq_i = d(\sum_i p_i dq_i)$$

Generalises to

$$T^* X,$$

where X is a manifold: $\theta := \sum_i p_i dq_i$ is invariantly defined. Put $\omega = d\theta$.

Classical Systems

Examples (continued):

Smooth complex projective varieties. Symplectic form is associated to Kaehler metric pulled back from projective space:

$$\omega = g(J\cdot, \cdot)$$

In the C^∞-case, every symplectic structure locally looks like

$$\sum_{i=1}^{n} dp_i \wedge dq_i = d(\sum_i p_i dq_i)$$

Important class of examples are space of *fields*: Solution spaces of differential equations for sections of fibre bundles over a spacetime manifold.

Classical Systems

One motivation for symplectic structures: Hamilton's equations

$$\frac{\partial q}{\partial t} = \frac{\partial h}{\partial p}$$
$$\frac{\partial p}{\partial t} = -\frac{\partial h}{\partial q}$$

where $h(p, q)$ is a function representing energy, e.g.,

$$h(p, q) = \frac{p^2}{2m} + kq^2.$$

This can be written as the vector field X_h associated to dh:

$$\omega(X_h, \cdot) = dh.$$

The equation

$$\gamma'(t) = X_h(\gamma(t))$$

locally looks like Hamilton's equations. (Here as in the following, may be many sign errors.)

Quantum Systems

Space of states is a Hilbert space $\mathbb{H} = \{\psi\}$.

Role of functions played by self-adjoint operators

$$O : \mathbb{H} \longrightarrow \mathbb{H}.$$

Evaluating a function f at a point gets replaced by

$$\psi \mapsto \frac{\langle \psi, O\psi \rangle}{\langle \psi, \psi \rangle},$$

the expectation value of O in the state ψ.

Time evolution given by Schroedinger's equation:

$$\frac{d\psi}{dt} = \frac{1}{i\hbar} H\psi$$

for an operator H representing energy.

Quantum Systems

Quantisation refers to a process

$$(M, \omega, h) \longrightarrow (\mathbb{H}, H).$$

It should come with a process for converting functions to operators:

$$a \mapsto \hat{a}$$

so that physical quantities like energy, momentum, position have quantum mechanical expectation values. Energy is especially important in practice.

Example: $(\mathbb{R} \times \mathbb{R}, \omega)$, $h = \frac{p^2}{2m} + kq^2$ is quantised via

$$\mathbb{H} = L^2(\mathbb{R}, \mathbb{C}),$$

$q \longrightarrow$ multiplication operator;
$p \longrightarrow \frac{-id}{dq}$;

$$h \longrightarrow H = \frac{1}{2m}\nabla^2 + kq^2.$$

Quantum Systems

The prototype is $(\mathbb{R}^n \times \mathbb{R}^n = \{(q, p)\}, \omega)$, which quantises to $L^2(\mathbb{R}^n)$, with q_i going to the multiplication operator and p_i to $i\frac{\partial}{\partial q_i}$.

Quantise many h of the form

$$h = \frac{p^2}{2m} + V(q),$$

where $V(q)$ is interpreted as a potential energy.

However, this does *not* extend to

$$f \mapsto \hat{f}.$$

Works for linear and quadratic functions.

Quantum Systems

Two other constructions:

1. Can replace $L^2(\mathbb{R}^n)$ by $L^2(L)$, where $L \subset \mathbb{R}^n \times \mathbb{R}^n$ is any Lagrangian subspace.
Given such an L, almost canonical isomorphism

$$\mathcal{F}_L : L^2(L) \simeq L^2(\mathbb{R}^n)$$

2. Instead can use $L^2_{hol}(\mathbb{C}^n, \mu)$, where μ is a Gaussian measure.

This is naturally thought of as a completion of

$$\mathrm{Sym}(\mathbb{C}^n)$$

with respect to

$$\int |f(z)|^2 \exp(-|z|^2) dz d\bar{z}.$$

Then z_i acts by multiplication while $\bar{z}_i$ acts by d/dz_i with

$$z_i = p_i + iq_i, \bar{z}_i = p_i - iq_i.$$

Quantum Systems

Several advantages, including the fact that quantum states can be evaluated at a point on the classical state space.

This extends to the idea of Kaehler geometric quantisation: Given the symplectic $(\mathbb{S}, \omega)$, put on it a Kaehler structure. Construct a holomorphic line bundle $\mathcal{L} \longrightarrow \mathbb{S}$ with connection such that $c_1(\mathcal{L}, \nabla) = \omega$.

Note that $1 \in L^2_{hol}(\mathbb{C}^n, \mu)$ spans the unique line killed by the $\bar{z}_i$. Sometimes has the interpretation of a *vacuum* state. Thus, there is such a line spanned by $v_0 \in L^2(\mathbb{R}^n)$.

In fact, for any Lagrangian subspace L, there is a line spanned by $v_L \in L^2(\mathbb{R}^n)$: Write L locally as $q_1 = q_2 = \cdots = q_n = 0$ for a system (q_j, p_j) of symplectic coordinates.

Then v_L is the vector annihilated by

$$\frac{\partial}{\partial q_j} - i \frac{\partial}{\partial p_j}$$

Quantum Systems

This is believed to work quite generally: When $(\mathbb{S}, \omega)$ is quantised to $\mathbb{H}$, there should be something like a cycle map

$$L \mapsto v_L \in \mathbb{H}$$

from Lagrangian submanifolds to lines in $\mathbb{H}$.

General idea: Write L locally as $q_1 = q_2 = \cdots = q_n = 0$ for a system (q_j, p_j) of symplectic coordinates.
Then v_L is the vector annihilated by

$$\frac{\partial}{\partial q_j} - i\frac{\partial}{\partial p_j}$$

Clearly difficult to make sense of this.

Quantum Systems

When the system consists of *fields*, e.g., functions ϕ on spacetime, try to quantise coordinates like $\phi(x), \dot{\phi}(x)$.

So papers on quantum field theory are full of expressions like

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle$$

the expectation value of this composite of operators in the vacuum state.

But very difficult to define the Hamiltonian $\longrightarrow$ perturbation theory, renormalisation, etc. This allows people to compute scattering amplitudes.

Classical and Quantum Systems: Correction

All of this is old fashioned!

It is restricted to a situation where

$$M = N \times \mathbb{R}.$$

Conventionally, the space $\mathbb{S}$ of classical states is often described alternatively as 'space of solutions to the equations of motion' or 'space of initial conditions'. The former is more intrinsic in that it doesn't rely on a splitting of spacetime. It is the latter that gives rise to the symplectic structure.

The convenient mental model for the space of quantum states, $L^2_{hol}(\mathbb{S}, \mathcal{L})$, applies to the case of a product manifold. That is, the symplectic manifold and Hilbert space are intrinsic to N, not M.

In recent years, becoming important to study arbitrary spacetimes, as well as 'initial conditions,' i.e., 'boundary conditions' in all dimensions, giving rise to *shifted symplectic manifolds* and higher algebraic structures.

Higher Quantum Systems

When $\mathbb{S}$ is a [1]-shifted symplectic manifold (or a graded [-1]-shifted s.m.), then

$$\mathbb{H}(\mathbb{S})$$

should be constructed as a suitable category of sheaves.

If $f : Lag \longrightarrow \mathbb{S}$ is Lagrangian, then get an object like

$$f_*(D) \in \mathbb{H}(\mathbb{S}),$$

which gives a construction of an L-sheaf. This is what happens in [BZSV], where $\mathbb{S}$ is the space of fields in codim 2 for something like 4D BF theory. This appears to be a more natural process than the construction of L-functions.

Raises the question of setting up this formalism for over rings of algebraic integers, where spaces of arithmetic sheaves $\sim$ spaces of fields

VII. Arithmetic Actions

Arithmetic Actions

For technical reasons, we will assume throughout that F is a totally complex number fields.

Let R be a (sheaf of) p-adic Lie group(s) and $X = \mathrm{Spec}(\mathcal{O}_F)$

Would like to define arithmetic field theories via actions

$$S : \mathcal{C}(X, R) \longrightarrow K$$

as well as path integrals:

$$\int_{\rho \in \mathcal{C}(X,R)} \exp(-S(\rho)) d\rho$$

For example,

$$S : \mathcal{M}(X, R) = H^1(\pi_1(X), R) \longrightarrow K$$

$$\int_{\rho \in \mathcal{M}(X,R)} \exp(-S(\rho)) d\rho$$

Arithmetic Actions

Let μ_n be the n-th roots of 1. Then

$$H^3(X, \mu_n) = H^3(\mathrm{Spec}(\mathcal{O}_F), \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Follows from

$$H^3(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Recall

$$H^2(T_v, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

(Local class field theory.)

Arithmetic Actions

Global class field theory:

$$0 \longrightarrow H^2(X^B, \mathbb{G}_m) \longrightarrow \oplus_{v \in B} H^2(T_v, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where the last map is the sum.

This can be identified with

$$\longrightarrow H^2_c(X^B, \mathbb{G}_m) \longrightarrow H^2(X^B, \mathbb{G}_m) \longrightarrow \oplus_{v \in B} H^2(T_v, \mathbb{G}_m)$$
$$\longrightarrow H^3_c(X^B, \mathbb{G}_m) \longrightarrow 0$$

and

$$H^3_c(X^B, \mathbb{G}_m) \simeq H^3(X, \mathbb{G}_m).$$

Arithmetic Actions

Assume $\mu_n \subset F$. Then

$$H^3(X, \mathbb{Z}/n) \simeq H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

so we get a map

$$\mathrm{inv} : H^3(\pi_1(X), \mathbb{Z}/n) \longrightarrow H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Arithmetic Actions

On the moduli space

$$\mathcal{M}(X,R) = \mathrm{Hom}(\pi_1(X), R)//R,$$

of continuous representations of $\pi_1(X)$, a Chern-Simons functional is defined as follows.

The functional will depend on the choice of a cohomology class (a level)

$$c \in H^3(R, \mathbb{Z}/n).$$

Then

$$CS_c : \mathcal{M}(X,R) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is defined by

$$\rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \mathrm{inv}(\rho^*(c)).$$

Finite Arithmetic Chern-Simons Functionals

Example:

Let $R = \mathbb{Z}/n$. Then

$$\mathcal{M}(X, \mathbb{Z}/n) = Hom(\pi_1(X), \mathbb{Z}/n) = H^1_{et}(X, \mathbb{Z}/n).$$

Take $c \in H^3(R, \mathbb{Z}/n)$ to be given as

$$a \cup \delta a,$$

where $a \in H^1(R, \mathbb{Z}/n) = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/n)$ is the class coming from the identity map, while

$$\delta : H^1(R, \mathbb{Z}/n) \longrightarrow H^2(R, \mathbb{Z}/n)$$

is the Bockstein map coming from the extension

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Then

$$CS_{a\cup\delta a}(\rho) = \mathrm{inv}(\rho^*(a) \cup \rho^*(\delta a)).$$

Source of Examples

More general simple constructions come from extensions and characters. For example, a central extension

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow E \longrightarrow R \longrightarrow 0$$

gives a class $e \in H^2(R, \mathbb{Z}/n)$, which together with a character

$$\chi : R \longrightarrow \mathbb{Z}/n$$

then gives us

$$c = e \cup \chi \in H^3(R, \mathbb{Z}/n).$$

If particular, if $\rho : \pi \longrightarrow R$ admits a lifting to E, then $CS_c(\rho) = 0$.

A Small Application

[with Hee-Joong Chung, Dohyeong Kim, Jeehoon Park, Hwajong Yoo]

Let $d_1 = \prod_i p_i^*$, where $p^* = (-1)^{\frac{p-1}{2}} p$ for an odd prime p.

Let

$$\Delta(d_1, d_2) = \prod_i \left(\frac{d_2}{p_i} \right).$$

Proposition

If $\Delta(d_1, d_2) = -1$, *then there is no number field*

$$L \supset \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$$

such that $Gal(L/\mathbb{Q}) = Q_8$.

For example, $(d_1, d_2) = (13, 37), (13, 57), (17, 57)$.

BF-theory

Have a function

$$H^1(X, V) \times H^1(X, D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

defined by

$$(a, b) \mapsto \mathrm{inv}(da \cup b)$$

For this, V is a finite n-torsion group scheme that admits a suitable Bockstein map

$$d : H^1(X, V) \longrightarrow H^2(X, V)$$

and $D(V)$ is the Cartier dual.

Variant:

$$H^1(X^B, V) \times H^1_c(X^B, D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

Remark on arithmetic differentials

The Bockstein map

$$d : H^1(X, \mathbb{Z}/n) \longrightarrow H^2(X, \mathbb{Z}/n)$$

is very much like a differential. In crystalline cohomology of varieties over perfect fields of positive characteristic, Bockstein maps on crystalline cohomology sheaves are used to construct the De Rham-Witt complex.

In general, whenever you have an extension

$$0 \longrightarrow V \longrightarrow E \longrightarrow V \longrightarrow 0,$$

there is a differential

$$H^1(X, V) \longrightarrow H^2(X, V)$$

that can be used to construct arithmetic functionals.

Arithmetic Path Integrals

[Joint work with H. Chung, D. Kim, G. Pappas, J. Park, H. Yoo]

Let $n = p$, a prime and assume the Bockstein map

$$d : H^1(X, \mathbb{Z}/p) \longrightarrow H^2(X, \mathbb{Z}/p)$$

is an isomorphism.

Then

$$\sum_{\rho \in H^1(X, \mathbb{Z}/p)} \exp[2\pi i CS(\rho)]$$

$$= \sqrt{|Cl_X[p]|} \left(\frac{\det(d)}{p} \right) i^{\left[\frac{(p-1)^2 \dim(Cl_X[p])}{4}\right]}.$$