Arithmetic Topology and Field Theory

Minhyong Kim
ICMS/Maxwell Institute
Edinburgh

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VII. Arithmetic Actions

Arithmetic Actions

For technical reasons, we will assume throughout that F is a totally complex number fields.

Let R be a (sheaf of) p-adic Lie group(s) and $X = \mathrm{Spec}(\mathcal{O}_F)$

Would like to define arithmetic field theories via actions

$$S : \mathcal{C}(X, R) \longrightarrow K$$

as well as path integrals:

$$\int_{\rho \in \mathcal{C}(X,R)} \exp(-S(\rho)) d\rho$$

For example,

$$S : \mathcal{M}(X, R) = H^1(\pi_1(X), R) \longrightarrow K$$

$$\int_{\rho \in \mathcal{M}(X,R)} \exp(-S(\rho)) d\rho$$

Arithmetic Actions

Let μ_n be the n-th roots of 1. Then

$$H^3(X, \mu_n) = H^3(\mathrm{Spec}(\mathcal{O}_F), \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Follows from

$$H^3(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Recall

$$H^2(T_v, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

(Local class field theory.)

Arithmetic Actions

Global class field theory:

$$0 \longrightarrow H^2(X^B, \mathbb{G}_m) \longrightarrow \oplus_{v \in B} H^2(T_v, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where the last map is the sum.

This can be identified with

$$\longrightarrow H^2_c(X^B, \mathbb{G}_m) \longrightarrow H^2(X^B, \mathbb{G}_m) \longrightarrow \oplus_{v \in B} H^2(T_v, \mathbb{G}_m)$$
$$\longrightarrow H^3_c(X^B, \mathbb{G}_m) \longrightarrow 0$$

and

$$H^3_c(X^B, \mathbb{G}_m) \simeq H^3(X, \mathbb{G}_m).$$

Arithmetic Actions

Assume $\mu_n \subset F$. Then

$$H^3(X, \mathbb{Z}/n) \simeq H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

so we get a map

$$\mathrm{inv} : H^3(\pi_1(X), \mathbb{Z}/n) \longrightarrow H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Arithmetic Actions

On the moduli space

$$\mathcal{M}(X,R) = \mathrm{Hom}(\pi_1(X), R)//R,$$

of continuous representations of $\pi_1(X)$, a Chern-Simons functional is defined as follows.

The functional will depend on the choice of a cohomology class (a level)

$$c \in H^3(R, \mathbb{Z}/n).$$

Then

$$CS_c : \mathcal{M}(X,R) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is defined by

$$\rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \mathrm{inv}(\rho^*(c)).$$

Finite Arithmetic Chern-Simons Functionals

Example:

Let $R = \mathbb{Z}/n$. Then

$$\mathcal{M}(X, \mathbb{Z}/n) = Hom(\pi_1(X), \mathbb{Z}/n) = H^1_{et}(X, \mathbb{Z}/n).$$

Take $c \in H^3(R, \mathbb{Z}/n)$ to be given as

$$a \cup \delta a,$$

where $a \in H^1(R, \mathbb{Z}/n) = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/n)$ is the class coming from the identity map, while

$$\delta : H^1(R, \mathbb{Z}/n) \longrightarrow H^2(R, \mathbb{Z}/n)$$

is the Bockstein map coming from the extension

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Then

$$CS_{a\cup\delta a}(\rho) = \mathrm{inv}(\rho^*(a) \cup \rho^*(\delta a)).$$

Source of Examples

More general simple constructions come from extensions and characters. For example, a central extension

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow E \longrightarrow R \longrightarrow 0$$

gives a class $e \in H^2(R, \mathbb{Z}/n)$, which together with a character

$$\chi : R \longrightarrow \mathbb{Z}/n$$

then gives us

$$c = e \cup \chi \in H^3(R, \mathbb{Z}/n).$$

If particular, if $\rho : \pi \longrightarrow R$ admits a lifting to E, then $CS_c(\rho) = 0$.

A Small Application

[with Hee-Joong Chung, Dohyeong Kim, Jeehoon Park, Hwajong Yoo]

Let $d_1 = \prod_i p_i^*$, where $p^* = (-1)^{\frac{p-1}{2}} p$ for an odd prime p.

Let

$$\Delta(d_1, d_2) = \prod_i \left(\frac{d_2}{p_i} \right).$$

Proposition

If $\Delta(d_1, d_2) = -1$, then there is no number field

$$L \supset \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$$

such that $Gal(L/\mathbb{Q}) = Q_8$.

For example, $(d_1, d_2) = (13, 37), (13, 57), (17, 57)$.

BF-theory

Have a function

$$H^1(X,V) \times H^1(X,D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

defined by

$$(a,b) \mapsto \mathrm{inv}(da \cup b)$$

For this, V is a finite n-torsion group scheme that admits a suitable Bockstein map

$$d : H^1(X,V) \longrightarrow H^2(X,V)$$

and $D(V)$ is the Cartier dual.

Variant:

$$H^1(X^B,V) \times H^1_c(X^B,D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

Remark on arithmetic differentials

The Bockstein map

$$d : H^1(X, \mathbb{Z}/n) \longrightarrow H^2(X, \mathbb{Z}/n)$$

is very much like a differential. In crystalline cohomology of varieties over perfect fields of positive characteristic, Bockstein maps on crystalline cohomology sheaves are used to construct the De Rham-Witt complex.

In general, whenever you have an extension

$$0 \longrightarrow V \longrightarrow E \longrightarrow V \longrightarrow 0,$$

there is a differential

$$H^1(X, V) \longrightarrow H^2(X, V)$$

that can be used to construct arithmetic functionals.

Arithmetic Path Integrals

[Joint work with H. Chung, D. Kim, G. Pappas, J. Park, H. Yoo]

Let $n = p$, a prime and assume the Bockstein map

$$d : H^1(X, \mathbb{Z}/p) \longrightarrow H^2(X, \mathbb{Z}/p)$$

is an isomorphism.

Then

$$\sum_{\rho \in H^1(X, \mathbb{Z}/p)} \exp[2\pi i CS(\rho)]$$

$$= \sqrt{|Cl_X[p]|} \left(\frac{\det(d)}{p} \right) i^{[\frac{(p-1)^2 \dim(Cl_X[p])}{4}]}.$$

Brief Interlude

Arithmetic duality

$$H^2_c(X^B, \mu_n) \simeq H^1(X^B, \mathbb{Z}/n)^*,$$

where the dual refers to $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$.

This follows from the isomorphism

$$H^3_c(X^B, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

The duality is essentially abelian class field theory. For example, when $B = \phi$, this becomes

$$H^2(X, \mu_n) \simeq H^1(X, \mathbb{Z}/n)^*.$$

Brief Interlude

The RHS is

$$\mathrm{Hom}(\pi_1(X), \mathbb{Z}/n)^* = \mathrm{Hom}(\pi_1(X)^{ab}, \mathbb{Z}/n)^* \simeq \pi_1(X)^{ab}/n.$$

The LHS fits into

$$H^1(X, \mathbb{G}_m) \xrightarrow{n} H^1(X, \mathbb{G}_m) \longrightarrow H^2(X, \mu_n) \longrightarrow H^2(X, \mathbb{G}_m)$$

But $H^2(X, \mathbb{G}_m) = 0$ (Mazur). So

$$H^2(X, \mu_n) \simeq Cl(X)/n.$$

The resulting isomorphism

$$Cl(X)/n \simeq \pi_1(X)^{ab}/n$$

is just the unramified reciprocity isomorphism mod n.

Brief Interlude

However, class field theory is:

Langlands reciprocity for $GL_1 \sim$ *electromagnetic duality.*

Thus,

arithmetic Poincare duality $\sim$ *electromagnetic duality.*

Does this seem right?

Arithmetic BF-theory: [Joint work with Magnus Carlson]

$$BF : H^1(X, \mu_n) \times H^1(X, \mathbb{Z}/n) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

$$(a, b) \mapsto \mathrm{inv}(da \cup b).$$

Proposition

For $n >> 0$,

$$\sum_{(a,b)\in H^1(X,\mu_n)\times H^1(X,\mathbb{Z}/n)} \exp(2\pi i BF(a, b))$$

$$= |Cl_X[n]||\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n|.$$

Compare with

$$\frac{L^{(r)}(Triv, 0)}{r!} = -|Cl_X||\|\det(\mathcal{O}_F^\times)\||$$

Arithmetic BF-theory

Similarly, if E is an elliptic curve with Neron model $\mathcal{E}$, then we have

$$0 \longrightarrow \mathcal{E}[n] \longrightarrow \mathcal{E}[n^2] \longrightarrow \mathcal{E}[n] \longrightarrow 0$$

for n coprime to the conductor and the orders of component groups of $\mathcal{E}$.

This gives us a map

$$BF : H^1(X, \mathcal{E}[n]) \times H^1(X, \mathcal{E}[n]) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

as

$$(a, b) \longrightarrow \mathrm{inv}(da \cup b).$$

Arithmetic BF-theory

Proposition

For n as above,

$$\sum_{(a,b)\in H^1(X,\mathcal{E}[n])\times H^1(X,\mathcal{E}[n])} \exp(2\pi i BF(a,b))$$

$$= |Ш(A)[n]||E(F)/n|^2.$$

Compare

$$\frac{L^{(r)}(T_pE,0)}{r!} = (\prod_v c_v)|Ш_E|\|\det(E(F))\|^2$$

Chern-Simons Theory for Elliptic Curves

For $a \in H^1(X, \mathcal{E}[p])$, define

$$CS(a) := BF(a, a).$$

This is a mod p version of the p-adic height.

Local operators: Let $\ell \equiv 1 \mod p$ a prime of good reduction and $y \in \mathcal{E}(\mathbb{F}_\ell)$, define

$$O_{\ell,y} : H^1(X, \mathcal{E}[p]) \longrightarrow \mu_p$$

as

$$O_{\ell,y}(a) := \langle a \mod \ell, y \rangle$$
$$= (a(Fr_\ell), y),$$

where the last bracket is the Weil pairing.

Chern-Simons Theory for Elliptic Curves

$$\sum_{a \in H^1(X, \mathcal{E}[p])} O_{\ell_1, y_1}(a) O_{\ell_2, y_2}(a) \cdots O_{\ell_k, y_k}(a) \exp(2\pi i CS(a)) = ?$$

Partition Function of an Elliptic Curve

Can also consider a sum

$$\sum_{x \in E(F)} e^{-h(x)}$$

where h is the Neron-Tate height.

This is the value at one of the *height zeta function* of E:

$$\sum_{x \in E(F)} e^{-sh(x)},$$

which introduces a parameter analogous to inverse temperature.

Adelic variant

$$\int_{E(\mathbb{A}_F)} \prod_v e^{-sh_v(x_v)} \prod_v dx_v.$$

(cf. Candelas and de la Ossa)

VIII. Boundaries

Finite Arithmetic Chern-Simons Functionals with Boundaries

$X^B = \mathrm{Spec}(\mathcal{O}_F[1/B])$ for a finite set B of primes;

$\partial X^B = \coprod_{v \in B} \mathrm{Spec}(F_v)$.

$D_v := \mathrm{Spec}(\mathcal{O}_{F_v})$

$D_B = \coprod D_v$.

$$\pi_1(X^B) := \mathrm{Gal}(F_B^{un}/F), \quad \pi_v := \mathrm{Gal}(\bar{F}_v/F_v),$$

and fix a tuple of homomorphisms

$$i_S = (i_v : \pi_v \longrightarrow \pi_1(X^B))_{v \in B}$$

corresponding to embeddings $\bar{F} \hookrightarrow \bar{F}_v$.

Assume B contains all places dividing n.

Finite Arithmetic Chern-Simons Functionals with Boundaries

In addition to the global moduli space

$$\mathcal{M}(X^B, R) = \mathrm{Hom}(\pi_1(X^B), R)//R$$

we have the local moduli space

$$\mathcal{M}(\partial X^B, R) := \{\phi_B = (\phi_v)_{v \in B} \mid \phi_v : \pi_v \longrightarrow R\}//R$$

Thus, we get a localisation map

$$\mathrm{loc}_B = i_B^* : \mathcal{M}(X^B, R) \longrightarrow \mathcal{M}(\partial X^B, R)$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

Key cohomological facts:

$$H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

$H^i(\pi_v, \mathbb{Z}/n) = 0$ for $i > 2$.

There is a symplectic non-degenerate pairing

$$H^1(\pi_v, \mathbb{Z}/n) \times H^1(\pi_v, \mathbb{Z}/n) \longrightarrow H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

There is an exact sequence

$$0 \longrightarrow H^2(X^B, \mathbb{Z}/n) \longrightarrow \prod_{v \in B} H^2(\pi_v, \mathbb{Z}/n) \xrightarrow{\sum} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0.$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

Now $c \in Z^3(R, \mathbb{Z}/n)$ will denote a 3-cocycle.

For any $\phi_B = (\phi_v)$, each $\phi_v^*(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial. Thus,

$$\mathcal{T}_v := d^{-1}(\phi_v^*(c)) \in C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n)$$

is a torsor for $H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Hence,

$$\prod_{v \in B} \mathcal{T}_v$$

is a torsor for

$$\prod_{v \in B} H^2(\pi_v, \mathbb{Z}/n) \simeq \prod_{v \in B} \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

.

Finite Arithmetic Chern-Simons Functionals with Boundaries

We push this out using the sum map

$$\Sigma : \prod_{v \in B} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

to get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T}(\phi_B) := \Sigma_*(\prod_v d^{-1}(\phi_v)).$$

As ϕ_B varies, we get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T} \longrightarrow \mathcal{M}(\partial X^B, R)$$

over the local moduli space.

Finite Arithmetic Chern-Simons Functionals with Boundaries

Can use the map

$$\exp 2\pi i : \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow S^1.$$

to push $\mathcal{T}$ out to a unitary line bundle $\mathcal{U}$ over $\mathcal{M}(\partial X^B, R)$ and define

$$H_{CS}(B) := \Gamma(\mathcal{M}(\partial X^B), R), \mathcal{U})$$

This is the Hilbert space associated by finite arithmetic CS theory to B.
Should define

$$H_{CS}(X^B) \in H_{CS}(B).$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

If $\rho \in \mathcal{M}(X^B, R)$, because $H^3(\pi_1(X^B), \mathbb{Z}/n) = 0$, we can solve

$$d\beta = \rho^*(c) \in Z^3(\pi_1(X^B), \mathbb{Z}/n),$$

and put

$$\mathbb{CS}(\rho) = \Sigma_*(\mathrm{loc}_B(\beta)) \in \mathcal{T}_{\mathrm{loc}_B(\rho)}.$$

Lemma

$\mathbb{CS}(\rho)$ *is independent of the choice of* β*.*

This follows immediately from the reciprocity sequence

$$0 \longrightarrow H^2(\pi_1(X^B), \mathbb{Z}/n) \longrightarrow \prod_{v \in B} H^2(\pi_v, \mathbb{Z}/n) \xrightarrow{\Sigma} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0,$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

Exponentiating, we get

$$\exp(2\pi i \mathbb{CS}(\rho)) \in \mathcal{U}_{\mathrm{loc}_B(\rho)}$$

and

$$\int_{\{\rho \mid \mathrm{loc}_B(\rho)=\rho_B\}} \exp(2\pi i \mathbb{CS}(\rho)) \in \mathcal{U}_{\rho_B}.$$

As ρ_B varies get an element

$$\Psi_{CS}(X^B) \in H_{CS}(B).$$

Many analogues of topological formulas carry over, e.g., glueing formula.

Computing Chern-Simons: Decomposition Formula

We have the natural map

$$\pi^B \xrightarrow{q_B} \pi.$$

Thus, we get the map

$$\mathcal{M}(X, R) \longrightarrow \mathcal{M}(X^B, R)$$

$$\rho \mapsto \rho \circ q_B.$$

$$\mathbb{CS}(\rho \circ q_B) \in \mathcal{T}(r(\rho)).$$

On the other hand, for each $v \in B$, we get a composed representation

$$\rho_v^{un} : \pi_v^{un} \longrightarrow \pi \xrightarrow{\rho} R,$$

where $\pi_v^{un} \simeq \mathrm{Gal}(\bar{k_v}/k_v)$ is the unramified quotient of π_v.

Computing Chern-Simons: Decomposition Formula

By solving

$$d\beta_v = (\rho_v^{un})^*(c)$$

with

$$\beta_v(\rho_v^{un}) \in C^2(\pi_v^{un}, \mathbb{Z}/n)/B^2(\pi_v^{un}, \mathbb{Z}/n) \longrightarrow Z^3(\pi_v, \mathbb{Z}/n)$$

for each v, we get another element

$$\sum_v (\beta_v(\rho_v^{un})) \in \mathcal{T}(r(\rho)).$$

This is independent of the choice of β_v because

$$H^2(\pi_v^{un}, \mathbb{Z}/n) = 0.$$

Thus, we can take the difference

$$\mathbb{CS}(\rho \circ q_B) - \sum_v (\beta_v) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

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Computing Chern-Simons: Decomposition Formula

Theorem (w/ H. Chung, D. Kim, J. Park, and H. Yoo)

$$\mathbb{CS}(\rho) = \mathbb{CS}(\rho \circ q_S) - \sum_v (\beta_v(\rho_v^{un})).$$

This is an analogue of the *decomposition formula* in Chern-Simons theory, and gives us a way to compute the values.

Key Point:

> $\mathbb{CS}(\rho)$ *is the difference between a global ramified trivialisation and a local unramified trivialisation.*

Computing Chern-Simons: Decomposition Formula

Put

$$\Psi_{CS}(D_B)((\rho_v)_v) := \exp(2\pi i(\sum_v(\beta_v(\rho_v))))$$

if all the ρ_v are unramified. Otherwise,

$$\Psi_{CS}(D_B)((\rho_v)_v) = 0$$

Theorem (Hirano, J. Kim, Morishita)

$$CS(X) = \langle \Psi_{CS}(X^B), \Psi_{CS}(D_B) \rangle.$$

Chern-Simons Entanglement of Primes

[With Chung, Kim, Park, Yoo and inspired by Balasubramanian, Vijay; Fliss, Jackson R.; Leigh, Robert G.; Parrikar, Onkar: Multi-boundary entanglement in Chern-Simons theory and link invariants.]

When $B = \{p_1, p_2\}$, then

$$\Psi_{CS}(X_B) \in H_{CS}(B) \simeq H_{CS}(p_1) \otimes H_{CS}(p_2).$$

We can define the CS entanglement entropy of primes:

$$Ent_{CS}(p_1, p_2) := -Tr(Tr_{H(p_1)}(\Psi_{CS}(X_B)) \log Tr_{H(p_1)}(\Psi_{CS}(X_B))).$$

Chern-Simons Entanglement of Primes

Let $R = \mathbb{Z}/p$. Recall the localisation maps

$$\mathrm{loc}_{p_i} : H^1(X^B, \mathbb{Z}/p) \longrightarrow H^1(T_{p_i}, \mathbb{Z}/p).$$

Theorem

$$Ent_{CS}(p_1, p_2) = (dim H^1(X^B, \mathbb{Z}/p) - dim(Ker(loc_{p_1}) + Ker(loc_{p_2})) \log p.$$

IX. P-adic L-functions

Should be interesting, but hasn't led to much yet.

p-adic L-functions

For each $j \in \{1, 2, \ldots, p-1\}$ odd there is a unique power series

$$Z_j(T) \in \mathbb{Z}_p[[T]]$$

such that

$$Z_j((1+p)^n - 1) = (1 - p^{-n})\zeta(n)$$

for all $n < 0$, $n \equiv j \mod p-1$.

Allows the interpolation of the negative odd values of $\zeta(s)$ to p-adic analytic functions

$$L_j(s) := Z_j((1+p)^s - 1).$$

p-adic L-functions and path integrals

[Joint work with Magnus Carlson, Hee-Joong Chung, Dohyeong Kim, Jeehoon Park, and Hwajong Yoo.]

Let

$$X_B = \mathrm{Spec}(\mathbb{Z}[\mu_{p^n}][1/(\zeta_{p^n} - 1)])$$

and define the space of fields as

$$\mathcal{F}^m := H^1\left(X_B, \mu_{p^m}\right) \times H^1_c\left(X_B, \mathbb{Z}/p^m\mathbb{Z}\right).$$

p-adic L-functions and path integrals

There is a natural action of

$$G = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$$

on the space of fields $\mathcal{F}^m$, and we let

$$G' \subset G$$

be the unique subgroup of G of order $p-1$.

Since $p-1$ is not divisible by p, G' acts semi-simply on $\mathcal{F}^m$.

Define

$$\mathcal{F}_k^m := H^1\left(X_B, \mu_{p^m}\right)_{\omega^k} \times H_c^1\left(X_B, \mathbb{Z}/p^m\mathbb{Z}\right)_{\omega^{-k}}.$$

p-adic L-functions and path integrals

Further,

$$\mathcal{F}_k = H^1(X_B, \mathbb{Z}_p(1))_{\omega^k} \times H^1_c(X_B, \mathbb{Q}_p/\mathbb{Z}_p)_{\omega^{-k}}$$

$$= \varprojlim H^1\left(X_B, \mu_{p^m}\right)_{\omega^k} \times \varinjlim H^1_c\left(X_B, \mathbb{Z}/p^m\mathbb{Z}\right)_{\omega^{-k}}$$

and

$$\int_{\mathcal{F}_k} \exp(2\pi i BF(a,b)) dadb := \lim_{m\to\infty} \sum_{(a,b)\in \mathcal{F}_k^m} \exp\left(2\pi i BF(a,b)\right)$$

p-adic L-functions and path integrals

Theorem

Let $k \neq 1$ be odd. We have

$$\int_{\mathcal{F}_k} \exp(2\pi i BF(a,b)) da db = |\prod_{j=0}^{p^n-1} Z_{1-k}(\exp(2\pi i j/p^n) - 1)^{-1}|_p$$

Essentially just a repackaging of the main conjecture of Mazur and Wiles together with some generalities on arithmetic duality.

p-adic L-functions and path integrals

Remark

1. This is a partial unification of Mazur's paper 'Notes on the Alexander polynomial' and Witten's Jones polynomial paper.

2. It would be far more interesting to get

$$\prod_{j=0}^{p^n-1} Z_{1-k}(\exp(2\pi ij/p^n)-1)^{-1}$$