KUNIHOKO KODAIRA AND COMPLEX MANIFOLDS

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Contents

It is my great pleasure to talk about the mathematics of the great master Professor Kunihiko Kodaira. He is one of the founders of the theory of complex manifolds. Many mathematicians in Japan regard him as an ideal mathematician and would like to imitate at least some part of his mathematics. I am not his direct student, but a student in broader sense and received a lot of influence. I also received a few but important advice from him. I apologize that I can touch only very tiny part of his mathematics due to the lack of knowledge.

1. chronology

I would like to start with a short chronology of Professor Kodaira. He was born in Tokyo on March 16 in 1915. He graduated from Mathematics Department of the University of Tokyo in 1938. Then he graduated from Physics Department of the University of Tokyo in 1941. I do not know the reason why he graduated from two departments.

He became an assistant professor at Tokyo University of Education in1942, where a great physicist Shinichiro Tomonaga was a professor. He became an assistant professor at Physics Department of the University of Tokyo in 1944. The Mathematics and Physics Departments of the University of Tokyo had to evacuate from Tokyo to Nagano prefecture in 1945 in order to escape from the bombing toward the end of the war. Nagano prefecture is the home country of Kodaira, and he helped to save precious books of the departments.

He received a PhD degree from the University of Tokyo by his thesis "Harmonic fields in Riemannian manifolds" under the supervision of Professor Shokichi Iyanaga in 1949. Then he was invited to Institute for Advanced Study in Princeton in the same year. He became an associate professor at Princeton University in 1952, received a Fields Medal in 1954, and became a full professor at Princeton University in 1955. He moved to Johns Hopkins University in 1962, and then to Stanford University in 1965. I do not know why he moved so many times.

In 1968 he returned to Japan and became a professor at the University of Tokyo. The timing was not ideal due to the student movement. There was no entrance examination at the University of Tokyo in 1969 though the entrance examination is a national event in Japan. He was elected to the dean of the Faculty of Science (served from 1971 to 1973) and was involved in university politics. He had to face student meetings. But he managed to raise brilliant students who were later called Kodaira school. He also delivered beautiful graduate lectures (I will talk about the lecture notes later).

He retired at the age of 60 according to the rule at that time, and become a professor at Gakushuin University in 1975. He retired from Gakushuin University at the age of 70 according to the rule of the university in 1985. He served as a chairman of ICM Kyoto in 1990. He was very successful in fund raising thanks to his long time friendship with business leaders. He passed away on June 26, 1997.

2. Kodaira's lecture notes

The following are notes of Professor Kodaira's graduate course lectures delivered when he was a professor at the University of Tokyo. These are published in a series "Seminar Notes of Mathematics Department of University of Tokyo" (東大数学教室セミナリー・ノート). The handwritten lecture notes in Japanese with drawings are taken by his students and reproduced by mimeograph. I studied essentials of algebraic and complex geometry from these notes when I was a student.

vol 19. 複素多様体と複素構造の変形 I. 諏訪立雄記. 1968, 99pp. *Complex Manifolds and Deformations of Complex Structures I.* notes taken by Tatsuo Suwa.

vol 20. 代数曲面論. 山島成穂記. 1968, 89pp. *Algebraic Surfaces.* notes taken by Naruho Yamashima.

vol 31. 複素多様体と複素構造の変形 II. 堀川穎二記. 1974, 303pp. *Complex Manifolds and Deformations of Complex Structures II.* notes taken by Eiji Horikawa.

vol 32. 複素解析曲面論. 赤尾和男記. 1974, 173pp. *Complex Analytic Surfaces.* notes taken by Kazuo Akao.

vol 34. Nevanlinna 理論. 酒井文雄記. 1974, 115pp. *Nevanlinna Theorey.* notes taken by Fumio Sakai. translated into English by Takeo Ohsawa.

3. MISCELLANEA

I would like to translate some of Kodaira's words which were impressive.

"There maybe is a sense of mathematics beside five basic senses. In order to develop this sense, one has to work hard, read repeatedly proofs of known theorems, take notes, think about them long hours, and try to apply them in examples, digest, decompose into pieces and reconstruct, like taking nutrition."

"A correct proof is natural and appears straight forward, but is not boring because there are ideas."

"I happened to find many theorems, but they seem to be already there. If I did not find them, then someone else would have found them."

There are two books of articles dedicated to Kodaira by his friends [12], [13]. These were sources of study at the time when there were not so many papers.

4. COMPLEX MANIFOLDS

The subject of his research was the complex manifolds. Kodaira said that he wanted to generalize results in Herman Weyl's book "The Concept of a Riemann Surface" to higher dimensions. He considered complex manifolds which are not necessarily complex algebraic manifolds. Since there are no ample divisors on them, an induction argument on the dimension by using hyperplane sections or restrictions to divisors is not always possible. He even considered non-Kähler manifolds. In this way, he established the classification of compact complex manifolds of dimension 2 which are not necessarily Kähler. He used the minimal model program which was known in dimension 2. Nowadays some results are extended to dimension 3 or higher in the case that the manifolds are algebraic or Kähler.

Kodaira used two different approaches to complex manifolds. The first method is the complex analysis. He solved elliptic PDE using norm estimates. The second method is the algebraic geometry. He started to use the sheaf cohomology theory, when it was new, to solve geometric problems such as counting invariants of manifolds. He also used algebraic method of taking quotients and coverings as well as topological cutting and pasting. He applied both analytic and algebraic methods to study geometry.

Kodaira wrote numerous articles which are published in "Kunihiko Kodaira Collected Works" in 3 volumes ([11]). Roughly speaking, there are three major topics; harmonic analysis, deformation theory, and compact complex surfaces.

5. HARMONIC ANALYSIS

Let (X, ω) be a compact Kähler manifold. Let Ω_X^p be the sheaf of holomorphic p-forms for $0 \le p \le \dim X$, and let $H^{p,q}(X) = H^q(X, \Omega_X^p)$. Then the Hodge decomposition theorem says that there is a direct sum decomposition

$$H^k(X, \mathbf{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

and each cohomology class in $H^{p,q}$ is represented uniquely by a harmonic (p,q)-form.

There is an exact sequence

$$0 \to \mathbf{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

given by the exponential function $z \mapsto e^{2\pi i z}$. For a holomorphic line bundle L on X which is given by a set of transition functions $h_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$, where $\{U_\alpha\}$ is an open covering, we define its first Chern class $c_1(L) \in H^2(X, \mathbf{Z})$ to be the image of the class $[L] \in H^1(X, \mathcal{O}_X^*)$ by the connecting homomorphism

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X).$$

We have $c_1(L) \in H^{1,1}(X)$ because it goes to zero in the third term. Thus it is represented by a harmonic real $(1,1)$-form $\phi = i \sum \phi_{ij} dz_i \wedge d\bar{z}_j$. L is said to be *positive* if (ϕ_{ij}) is a positive definite hermitian matrix at each point of X

Theorem 5.1 (Kodaira vanishing theorem 1953 [3])**.** *Let X be a compact Kähler manifold of dimension n and let L be a positive holomorphic line bundle on X. Then $H^p(X, L^{-1}) = 0$ for $p < n$.*

This important theorem is proved in a 5 page paper published in Proc. Nat. Acad. Sci. This is a full paper with the size of an announcement.

We note that L is assumed to be positive, and not ample. The ampleness is a posteriori proved by the next embedding theorem. The positivity is a differential geometric property while the ampleness an algebro-geometric one.

By the Serre duality theorem, the statement of the theorem is converted to $H^p(X, L \otimes K_X) = 0$ for $p > 0$, where $K_X = \Omega_X^n$ is the canoical line bundle ([32]). Raynaud proved that the Kodaira vanishing theorem does not hold over a field of positive characteristic when L is assumed to be ample ([30]).

The Kodaira vanishing theorem gives precise and sometimes optimal condition for the vanishing of cohomology compared to the following Serre vanishing theorem:

Theorem 5.2 (Serre vanishing theorem [33]). *Let X be a projective scheme and let L be an ample line bundle on X. Then for any coherent sheaf F on X, there exists an integer m_0 such that $H^p(X, F \otimes L^{\otimes m}) = 0$ for $p > 0$ and $m \geq m_0$.*

This theorem holds in a much more general situation, where X may have singularities and the base field is arbitrary, but not as precise as the Kodaira vanishing theorem.

The proof of the Kodaira vanishing theorem is an application of the harmonic analysis. The method is an explicit tensor calculus called Bochner's method. The holomorphic line bundle $F = L^{-1}$ has a resolution by C^∞ vector bundles defined by the operator $\bar{\partial}$ (Dolbeault sequence)

$$0 \to F \to C^\infty(\bar{\Omega}^0 \otimes F) \to \cdots \to C^\infty(\bar{\Omega}^n \otimes F) \to 0.$$

Any cohomology class in $H^p(X, F)$ has a unique representative of a harmonic $(0,p)$-form ϕ with coefficients in F:

$$\phi \in C^\infty(\bar{\Omega}^p_X \otimes F), \;\; \bar{\partial}\phi = \bar{\partial}^*\phi = 0.$$

Then a tensor calculus shows that $\phi = 0$ using an argument such as, if a real function $f \geq 0$ satisfies $I = \int_X f d\mu \leq 0$, then $f = 0$.

Theorem 5.3 (Kodaira embedding theorem 1954 [4]). *Let (X, ω) be a compact Kähler manifold. Assume that the cohomology class of the Kähler form $[\omega] \in H^2(X, \mathbf{C})$ belongs to the image of the natural map $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{C})$. Then X is a projective manifold, i.e. there is an embedding $X \subset \mathbf{P}^N$ to a projective space.*

The proof of the embedding theorem is a typical application of the vanishing theorem. We explain the outline.

First of all, we find a line bundle L on X with the given 1st Chern class $c_1(L) = [\omega]$ thanks to the exponential sequence (Lefschetz $(1,1)$-theorem):

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X).$$

We take a basis $s_0, \dots, s_N \in H^0(X, L^{\otimes m})$ for a large integer m. Then we define a map to a projective space $f : X \to \mathbf{P}^N$ by

$$f(x) = [s_0(x) : \cdots : s_N(x)].$$

We can prove that f is an embedding in the following way:

(0) We prove that there is no *base point*: for each $x \in X$, there exists i such that $s_i(x) \neq 0$, so that f is defined everywhere. The statement is equivalent to the surjectivity of the following natural homomorphism

$$H^0(X, L^{\otimes m}) \twoheadrightarrow H^0(L^{\otimes m} \otimes \mathcal{O}_x)$$

for all $x \in X$.

(1) We prove that f *separates points*:

$$H^0(X, L^{\otimes m}) \twoheadrightarrow H^0(L^{\otimes m} \otimes \mathcal{O}_x) \oplus H^0(L^{\otimes m} \otimes \mathcal{O}_y)$$

for all $x, y \in X$ with $x \neq y$.

(2) We prove that f *separates infinitely near points*:

$$H^0(X, L^{\otimes m}) \twoheadrightarrow H^0(L^{\otimes m} \otimes \mathcal{O}_X/\mathfrak{m}_x^2)$$

for all $x \in X$.

We can prove the above surjectivities by using the vanishing theorem as follows. Let $g : Y \to X$ be the blowing up at x (resp. x, y) in the cases (0) and (2) (resp. (1)), and let $g^{-1}(x), g^{-1}(y) = E_x, E_y \cong \mathbf{P}^{n-1}$ be the exceptional divisors. Then we have exact sequences of sheaves

$$\begin{aligned}
&0 \to \mathcal{O}_Y(mg^*L - E_x) \to \mathcal{O}_Y(mg^*L) \to \mathcal{O}_{E_x} \to 0, \\
&0 \to \mathcal{O}_Y(mg^*L - E_x - E_y) \to \mathcal{O}_Y(mg^*L) \to \mathcal{O}_{E_x} \oplus \mathcal{O}_{E_y} \to 0, \\
&0 \to \mathcal{O}_Y(mg^*L - 2E_x) \to \mathcal{O}_Y(mg^*L) \to \mathcal{O}_{2E_x} \to 0,
\end{aligned}$$

where we use the additive notation of divisors instead of the multiplicative notation of sheaves, e.g., we have $\mathcal{O}_Y(D + D') \cong \mathcal{O}_Y(D) \otimes \mathcal{O}_Y(D')$. Then we have exact sequences

$$\begin{aligned}
&H^0(X, mg^*L) \to H^0(\mathcal{O}_{E_x}) \to H^1(Y, mg^*L - E_x), \\
&H^0(X, mg^*L) \to H^0(\mathcal{O}_{E_x}) \oplus H^0(\mathcal{O}_{E_y}) \to H^1(Y, mg^*L - E_x - E_y), \\
&H^0(X, mg^*L) \to H^0(\mathcal{O}_{2E_x}) \to H^1(Y, mg^*L - 2E_x).
\end{aligned}$$

Therefore it is sufficient to prove that the vanishings

$$\begin{aligned}
&H^1(Y, mg^*L - E_x) = 0, \\
&H^1(Y, mg^*L - E_x - E_y) = 0, \\
&H^1(Y, mg^*L - 2E_x) = 0,
\end{aligned}$$

hold, which follow from the positivity of divisors $g^*(mL - K_X) - nE_x$, $g^*(mL - K_X) - nE_x - nE_y$ and $g^*(mL - K_X) - (n+1)E_x$ for sufficiently large m, where $n = \dim X$.

The above argument is a prototype of applications of the vanishing theorem to the study of linear systems. Kodaira's argument is greatly generalized later to the study the minimal model program of higher dimensional algebraic varieties in characteristic zero ([20], [22]).

6. Deformation of complex structures

Let X be a compact complex manifold. A *deformation* of X is a proper smooth morphism $f : \mathcal{X} \to S$ of complex manifolds with a base point $s \in S$ and an isomorphism $f^{-1}(s) \cong X$. Each point $\tilde{x} \in \mathcal{X}$ has an open neighborhood with coordinates with respect to which f becomes a projection to a subspace. Then there is an open neighborhood $s \in S' \subset S$ such that the restriction $f|_{f^{-1}(S')}$ is differentially trivial; $f^{-1}(S') \sim S' \times X$ as differentiable manifolds. We consider only *small* deformations, i.e., the base space is considered as a germ of S at s, and $\mathcal{X} \to S$ can always be replaced by a restricted family $f^{-1}(S') \to S'$.

Kodaira and Spencer considered all possible small deformations of a given complex manifolds. They constructed deformations over a formal power series ring by method of infinitesimal calculus, and then proved the convergence of the formal deformations by global analysis of norm estimates.

Let M be the differentiable manifold which underlies X. Then X is regarded as a pair of M and a *complex structure* on M. The small deformations of X are realized by changing complex structures on M.

A complex structure is an *integrable almost complex structure* J, an action on the real tangent bundle T_M such that $J^2 = -1$. J induces a decomposition into eigenspaces of the complexfied tangent bundle

$$T_M \otimes \mathbf{C} = T_X \oplus \bar{T}_X$$

with the eigenvalues $i, -i$.

An almost complex structures J corresponds bijectively to the subspace $\bar{T}_X \subset T_M \otimes \mathbf{C}$, and in turn to the decomposition

$$d = \partial + \bar{\partial}$$

of the exterior derivation d of differential forms on M. T_X is determined as the complex conjugate of $\bar{T}_X$.

A differentiable function $h \in C^\infty(M)$ is holomorphic with respect to J if and only if it satisfies the *Cauchy-Riemann equation*

$$\bar{\partial} h = 0.$$

The almost complex structure J is said to be *integrable*, and thereby defines a complex structure, if there are sufficiently many holomorphic functions such that M becomes locally isomorphic to a polydisk.

A small deformation of the given almost complex structure J is determined by a differentiable form

$$\phi \in C^\infty(T_X \otimes \bar{T}^*_X) = C^\infty(Hom(\bar{T}_X, T_X)),$$

for which the complexified tangent bundle decomposes into

$$\bar{T}_\phi = \{v + \phi(v) \mid v \in \bar{T}_X\},$$

and its complex conjugate T_ϕ, so that we have a decomposition $d = \partial_\phi + \bar{\partial}_\phi$.

The almost complex structure J_ϕ thus defined is integrable if and only if the *Maurer-Cartan equation*

$$\bar{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0$$

is satisfied, where we note that

$$[\phi, \phi] \in Im([T_X, T_X] \otimes \bigwedge^2 \bar{T}^*_X \to T_X \otimes \bigwedge^2 \bar{T}^*_X)$$

with $\bar{T}_X^* = \bar{\Omega}^1$ and $\bigwedge^2 \bar{T}_X^* = \bar{\Omega}^2$.

In this way we define an infinite dimensional variety $\mathcal{MC} = \{\phi \mid \bar{\partial}\phi + \frac{1}{2}[\phi,\phi] = 0\}$ inside an infinite dimensional vector space $C^\infty(Hom(\bar{T}_X, T_X))$. In order to obtain the *local moduli space*, or the *Kuranishi space*, $\mathcal{M}$ of small deformations of X, we need to divide $\mathcal{MC}$ by the group action of the connected component of the diffeomorphism group $\text{Diffeo}(M)^0$ in order to discard the redundancy: $\mathcal{M} = \mathcal{MC}/\text{Diffeo}(M)^0$. Let $u \in C^\infty(T_X)$ be a vector field on M, an element of the Lie algebra corresponding to $\text{Diffeo}(M)^0$. Then the action is given by a formula

$$\exp(u)(\phi) = \phi + \sum_{n \geq 0} \frac{\text{ad}(u)^n}{(n+1)!}([u,\phi] - \bar{\partial}u).$$

In particular, the tangent space of $\mathcal{M}$ is given as a quotient space

$$T_{\mathcal{M},0} = \{\phi \mid \bar{\partial}\phi = 0\}/\{\phi \mid \phi = \bar{\partial}u\} = H^1(X, T_X)$$

where $0 \in \mathcal{M}$ corresponds to the original complex structure X. In more modern language, small deformations of X are controlled by an infinite dimensional differential graded Lie algebra

$$\bigoplus_{p \geq 0} C^\infty(T_X \otimes \bigwedge^p \bar{T}_X^*).$$

We analyse the infinitesimal structure of the local moduli space $\mathcal{M}$, the base space of the *semi-universal deformation*, by using the Dolbeault cohomology groups $H^p(X, T_X)$. Namely the tangent space is given by $H^1(X, T_X)$, and the singularities are determined by the *obstruction space* $H^2(X, T_X)$. We note that the vector spaces $H^p(X, T_X)$ are finite dimensional.

In the algebraic language, the tangent space $T_{\mathcal{M},0}$ is equal to the set of maps $\text{Spec}(\mathbf{C}[t]/(t^2)) \to \mathcal{M}$ which induce $\text{Spec}(\mathbf{C}) \to 0 \in \mathcal{M}$. This is the set of all deformations of X over $\text{Spec}(\mathbf{C}[t]/(t^2))$ up to isomorphism. More generally, let $(R, \mathfrak{m})$ be an Artin local algebra over $\mathbf{C}$ with residue field $R/\mathfrak{m} \cong \mathbf{C}$. R is finite dimensional as a $\mathbf{C}$-vector space. An *infinitesimal deformation* over R is a deformation of X over $\text{Spec}(R)$, that is a smooth morphism $f : \mathcal{X} \to Spec(R)$ such that $\mathbf{C} \otimes_R \mathcal{X} \cong X$.

Let $p : (R', \mathfrak{m}') \to (R, \mathfrak{m})$ be a surjective homomorphism of Artin local algebras such that the ideal $I = \text{Ker}(p)$ satisfies $\mathfrak{m}'I = 0$. Given a deformation $f : \mathcal{X} \to Spec(R)$, there exists an extended deformation $f' : \mathcal{X}' \to Spec(R')$ such that $R \otimes_{R'} \mathcal{X}' \cong \mathcal{X}$ if and only if an *obstruction class* $\xi(f) \in H^2(X, T_X) \otimes I$ vanishes.

We will see how these cohomology groups appear using an open covering of the complex manifold X. Let $X = \bigcup U_i$ be an open covering by small polydisks U_i. Then the deformations of X are obtained by changing the gluing $U_i|_{U_{ij}} \cong U_j|_{U_{ij}}$ on $U_{ij} = U_i \cap U_j$. The change over $\text{Spec}(\mathbf{C}[t]/(t^2))$ is realized by a 1-cochain $\{v_{ij}\}$ with $v_{ij} \in H^0(U_{ij}, T_X)$ consisting of infinitesimal automorphisms of the U_{ij}. One needs a compatibility condition, i.e., a cocycle condition $v_{jk} - v_{ik} + v_{ij} = 0$ on the overlap $U_{ijk} = U_i \cap U_j \cap U_k$. As the coboundary of a 0-cochain $\{w_i\}$ consisting of infinitesimal automorphisms $w_i \in H^0(U_i, T_X)$ of the U_i, $v_{ij} = w_j|_{U_{ij}} - w_i|_{U_{ij}}$ represents a trivial deformation. Therefore the tangent space of the local moduli space is given by the cohomology group $H^1(X, T_X)$.

We check that the *obstruction space* is $H^2(X, T_X)$. We consider a problem of extending a deformation $f : \mathcal{X} \to Spec(R)$ to $f' : \mathcal{X}' \to Spec(R')$. We have an open covering $\mathcal{X} = \bigcup \mathcal{U}_i$ such that $\mathcal{U}_i \cong U_i \times Spec(R)$. Let $g_{ij} : \mathcal{U}_j|_{U_{ij}} \cong \mathcal{U}_i|_{U_{ij}}$ be the gluing of the $\mathcal{U}_i$. They satisfy the cocycle condition $g_{ij}g_{jk}g_{ki}|_{U_{ijk}} = \text{Id}$ on U_{ijk}. Let $g'_{ij} : \mathcal{U}'_j|_{U_{ij}} \cong \mathcal{U}'_i|_{U_{ij}}$ be any gluing of the $\mathcal{U}'_i = U_i \times Spec(R')$ extending g_{ij}. We ask whether they satisfy the cocycle condition. Since g_{ij} satisfies the cocycle condition, we can write $g'_{ij}g'_{jk}g'_{ki}|_{U_{ijk}} = v_{ijk} \in H^0(U_{ijk}, I \otimes T_X)$. The 2-cochain $\{v_{ijk}\}$ is closed by construction. If it is a coboundary $v_{ijk} = w_{jk} - w_{ik} + w_{ij}$, then we can modify g'_{ij} by w_{ij} so that $\bigcup \mathcal{U}'_i$ glue to yield $\mathcal{X}'$.

Let us take a basis $\{\phi_{1,1}, \dots \phi_{1,m}\} \in H^1(X, T_X)$, and let $\{t_1, \dots, t_m\} \in H^1(X, T_X)^*$ be the dual basis which are regarded as formal variables. Then we have a universal first order deformation given by $\phi_1 = \sum \phi_{1,i} t_i$ over $\mathbf{C}[t_1, \dots, t_n]/(t_1, \dots, t_n)^2$. Let us assume that the obstruction space vanishes: $H^2(X, T_X) = 0$. Then we can develop a *formal deformation*: $\phi = \phi_1 + \phi_2 + \phi_3 + \dots$ with $\deg_t(\phi_i) = i$ by solving Maurer-Cartan equation inductively:

$$\bar{\partial}\phi_i + \frac{1}{2}\sum_{j=1}^{i-1}[\phi_j, \phi_{i-j}] = 0.$$

By a priori estimates of norms $\|\phi\| \le C\|\psi\|$ for an equation $\bar{\partial}\phi = \psi$, the convergence of the formal series is proved:

Theorem 6.1 (Kodaira-Nirenberg-Spencer 1958 [5])**.** *Let X be a compact complex manifold. Assume that $H^2(X, T_X) = 0$. Then there exists a deformation of X over an open polydisc in the affine space $H^1(X, T_X)$ which contain all small deformations.*

This deformation is called a *semi-universal* deformation or a *Kuranishi family* because Kuranishi extended the theorem to the case where $H^2(X, T_X)$ does not vanish [23]. The Kuranishi space, the local moduli space, becomes a germ of a possibly singular analytic subvariety of $H^1(X, T_X)$.

Next we consider deformations of a submanifold $X \subset V$ of codimension 1 in a fixed ambient manifold V. Let $N_{X/V}$ be the normal bundle. Then the tangent space of the local moduli space is equal to $H^0(X, N_{X/V})$ and the obstruction space is $H^1(X, N_{X/V})$.

We have an exact sequence

$$0 \to \mathcal{O}_V \to \mathcal{O}_V(X) \to N_{X/V} \to 0.$$

$X \subset V$ is said to be *semi-regular* if the homomorphism $H^1(V, \mathcal{O}_V(X)) \to H^1(X, N_{X/V})$ is a zero map. The following theorem shows that a weaker condition than the vanishing of the whole obstruction space $H^1(X, N_{X/V})$ is sufficient for the unobstructedness of deformations:

Theorem 6.2 (*completeness of characteristic system*, Kodaira-Spencer 1959 [7]). *Let $X \subset V$ be a codimension 1 submanifold. Assume that X is semi-regular. Then there exists a semi-universal family of submanifolds $f : \mathcal{X} \subset V \times S \to S$ such that $S \subset H^0(X, N_{X/V})$ is an open neighborhood of the origin.*

This is a generalization of a result of Severi in the case $\dim X = 1$. This is the reason of a strange name of the theorem. Kodaira asked the following question:

Question 6.3. *In many interesting examples like complex tori, there are no obstructions for deformations even if the obstruction spaces do not vanish $H^2(X, T_X) \neq 0$. Why?*

There is now a theory of *unobstructed deformations.* For example, Bogomolov-Tian-Todorov theorem states that there are no obstructions for deformations of Calabi-Yau manifolds in a broader sense, i.e, $K_X \sim 0$ ([16], [35], [36], see also [29], [21]).

Kodaira also asked the following question:

Question 6.4. *Does any compact Kähler manifold have a small deformation which is a projective manifold?*

He obtained a positive answer in the case of dimension 2 as a consequence of his classification ([8], [9]). On the other hand, Voisin gave a negative answer in higher dimension $\dim X \geq 4$ ([37]). It is proved to be negative even in a birational sense ([38]); there exists compact Kähler manifold in each even dimension ≥ 8 without a bimeromorphic model which is deformation equivalent to a projective manifold. But Lin proved that the answer is positive in dimension 3 ([26]); any compact Kähler manifold of dimension 3 has a small deformation which is a projective manifold.

7. COMPACT COMPLEX SURFACES

Kodaira applied the general theory of compact complex manifolds to the investigation of compact complex surfaces, 2-dimensional complex manifolds. He said that "a beautiful general theory is not interesting if there are no concrete applications".

A map between compact complex manifolds $f : X \dashrightarrow Y$ is said to be a *bimeromorphic map* (or sometimes called a *birational map*) if there exists Zariski open dense subsets $U \subset X$ and $V \subset Y$ such that f induces an isomorphism $f^o : U \cong V$, and that the graph of f^o in $U \times V$ extends to a subvariety of $X \times Y$. We note that f is not necessarily defined on the whole manifold X. If it is defined everywhere and the pull-backs of holomorphic functions are holomorphic, then it is called a *morphism*.

An important example of a birational map is a blow-up of Y along a smooth subvariety. The inverse of a blow-up is a blow-down. A curve C on a surface X is said to be a (-1)*-curve* if it is isomorphic to projective line $C \cong \mathbf{P}^1$ and the normal bundle has degree -1, i.e., $N_{C/X} \cong \mathcal{O}_{\mathbf{P}^1}(-1)$. The *Castelnuovo contraction* theorem states that, for a (-1)-curve C, there exists a birational morphism $f : X \to Y$ such that $f(C)$ point and $X \setminus C \cong Y \setminus f(C)$, so that f is a blow-up of $f(C)$. Any birational morphism $f : X \to Y$ in dimension 2 is decomposed into a succession of blow-downs [40]:

$$X = X_0 \to X_1 \to \cdots \to X_m = Y.$$

For any birational map $f : X \dashrightarrow Y$, there is decomposition into two birational morphisms $X \leftarrow Z \to Y$ for some intermediate surface Z. Therefore f is decomposed into blow-ups and then blow-downs:

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_m \to X_{m+1} \to \dots X_{m+n} = Y.$$

A surface is said to be *minimal* if it does not contain a (-1)-curve.

For arbitrary dimensional compact complex manifolds, any birational map is decomposed into a combination of blow-ups and blow-downs under the additional assumption that they are projective:

Theorem 7.1 (weak factorization theorem [14] [39])**.** *Let $f : X \dashrightarrow Y$ be a birational map between complex projective manifolds. Then f is decomposed into blow-ups and blow-downs:*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m = Y$$

where each map is either a blow-up or a blow-down with smooth centers.

A *birational invariant* is a invariant $I(X)$ of a compact complex manifold such that $I(X) = I(Y)$ if X and Y are birational. It is considered to be more fundamental than a general invariant. For example, some Hodge numbers $H^{p,0}(X) = \dim H^0(X, \Omega_X^p)$ and $H^{0,p}(X) = \dim H^p(X, \mathcal{O}_X)$ are birational invariants, while Betti numbers $B_i(X) = \dim H^i(X, \mathbf{Q})$ are not in general. The fundamental group $\pi_1(X)$ is also a birational invariant. The m-genus $P_m(X) := \dim H^0(X, mK_X)$ for a positive integer m is a birational invariant, where $K_X = \Omega_X^{\dim X}$ is the canonical line bundle with an additive notation. The growth rate $\kappa(X)$ of the pluri-genera is called the *Kodaira dimension*. It can take values among $-\infty, 0, 1, 2, \dots, \dim X$, and we have estimates $P_m(X) \sim m^{\kappa(X)}$ for sufficiently large and divisible integer m.

Kodaira considered the following problem:

Problem 7.2. *Classify all compact complex surfaces up to birational equivalence using birational invariants.*

For complex *algebraic* surfaces, there were already a theory by Italian algebraic geometers such as Enriques, Castelnuovo and so on.

One of the main methods is to consider linear systems of divisors and to calculate their intersection numbers. A *divisor* $D = \sum d_i D_i$ on X is a finite formal combination of codimension 1 subvarieties D_i with integer coefficients d_i. It is said to be *effective* $D \geq 0$ if all coefficients d_i are non-negative, when we assume that $D_i \neq D_j$ for $i \neq j$. Two divisors are said to be *linearly equivalent* $D \sim D'$ if the associated line bundles are isomorphic $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. A holomorphic non-zero section $s \in H^0(X, \mathcal{O}_X(D))$ gives an effective divisor $D' = \mathrm{div}(s)$ such that $D' \sim D$. A *linear system* is a set of effective divisors $|\Lambda| = (\Lambda \setminus \{0\})/\mathbf{C}^*$ corresponding to a linear subspace $\Lambda \subset H^0(X, \mathcal{O}_X(D))$. A complete linear system $|D| = (H^0(X, \mathcal{O}_X(D)) \setminus \{0\})/\mathbf{C}^*$ corresponds to the whole space. For example, if we have a non-vanishing $H^0(X, \mathcal{O}_X(D) \neq 0$, then we obtain a linear system of curves on X which gives a clue to the structure of X.

For a basis $\{s_0, \dots, s_N\} \in \Lambda$, we can construct a rational map $\Phi_\Lambda : X \dashrightarrow \mathbf{P}^N$ defined by $\Phi_\Lambda(x) = [s_0(x) : \dots : s_N(x)]$. Φ_Λ is a morphism if Λ is *base point free*, i.e., for each $x \in X$, there exists $s \in \Lambda$ such that $s(x) \neq 0$.

In order to calculate the space of holomorphic sections $H^0(X, \mathcal{O}_X(D))$, Kodaira proved a Riemann-Roch theorem, which is used together with two other theorems:

Theorem 7.3. *(1)* Riemann-Roch theorem *1951* [1]*:*

$$\chi(X, \mathcal{O}_X(D)) := \sum_{i=0}^{2} (-1)^i \dim H^i(X, \mathcal{O}_X(D)) = \frac{1}{2} D(D - K_X) + \chi(X, \mathcal{O}_X).$$

(2) Noether's formula

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X)).$$

(3) Serre duality

$$H^i(X, \mathcal{O}_X(D)) \cong H^{2-i}(X, \mathcal{O}_X(K_X - D))^*.$$

We note that $c_1(X)^2 = (K_X^2)$ and $c_2 = \sum_{i=0}^4 (-1)^i B_i$. The calculation of intersection numbers gives $\chi(X, \mathcal{O}_X(D))$. We can see the geometric meaning of $\dim H^0(X, \mathcal{O}_X(D))$ and $\dim H^2(X, \mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(K_X - D))$, but $\dim H^1(X, \mathcal{O}_X(D))$ is harder to understand. It is easier if the vanishing theorem holds.

7.1. **rationality criterion.** A compact complex surface X is said to be *rational* if there is a birational map $f : X \dashrightarrow \mathbf{P}^2$. A numerical criterion for the rationality was a great achievement of the Italian school of algebraic geometers. Kodaira gave a simpler proof using sheaf cohomology.

Theorem 7.4 (*Castelnuovo criterion* [9])**.** *Let X be an algebraic surface. Assume that*

$$P_2(X) := \dim H^0(X, 2K_X) = 0, \quad q(X) := \dim H^0(X, \mathcal{O}_X) = 0.$$

Then X is a rational surface.

The proof uses the theory of the *adjoint system* $|D + K_X|$ for a divisor D. The point is to prove that the *adjunction terminates*: $|D + mK_X| = \emptyset$ for any D if m is sufficiently large.

In the later development to higher dimensional varieties, we considered $|m(H + rK_X)|$ for an ample divisor H and for all $r > 0$. The numerical thresholds such as $t = \sup\{r \in \mathbf{R} \mid H + rK_X \text{ ample}\} \in \mathbf{Q}$ played important roles. There is now a theory of rationally connected manifolds, but the rationality criterion is still an open problem in dimension ≥ 3.

7.2. **K3 surfaces.** A compact complex surface X is called a *K3 surface* if the canonical bundle is trivial $K_X \sim 0$ (i.e., $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$) and the irregularity vanishes $q := \dim H^1(X, \mathcal{O}_X) = 0$. The name K3 came from three great mathematicians Kummer, Kähler and Kodaira according to Andre Weil.

For an algebraic K3 surface, there is an ample divisor L. The pair (X, L) is said to be a polarized K3 surface. The self intersection number $d = (L^2)$ is called the *degree*. It is an even positive integer. For each even positive integer d, there exists a 19-dimensional deformation family of polarized K3 surfaces of degree d. Therefore there are infinitely many families of algebraic K3 surfaces whose constructions are different for each degree. For example, K3 surfaces of degree 2 are double covers of $\mathbf{P}^2$ ramified along a curve of degree 6, those of degree 4 are hypersurfaces in $\mathbf{P}^3$ of degree 4, those of degree 6 are complete intersections of degrees 2 and 3 in $\mathbf{P}^4$, and those of degree 8 are complete intersections of degrees $2, 2, 2$ in $\mathbf{P}^5$.

But Kodaira realized that there is only one 20-dimensional family of K3 surfaces, all connected by deformations (Kodaira 1966 [9]). For example, the 19-dimensional family of quartic surfaces deforms out from $\mathbf{P}^3$. These general K3 surfaces have no equations because they are not algebraic, but their existence is guaranteed by the deformation theory. Indeed we have

$$\dim H^1(X, T_X) \cong \dim H^1(X, \Omega^1_X) = 20, \;\; H^2(X, T_X) = 0.$$

For a polarized K3 surface (X, L), the deformations preserving the polarization is 19-dimensional, because they are parametrized by a subspace $[L]^\perp \subset H^1(X, \Omega^1_X)$ of dimension 19.

An important example of a K3 surface is a *Kummer surface*, which is a desingularization of a quotient variety $A/\langle i \rangle$ of an abelian surface A by an involution $i : x \mapsto -x$. Kodaira showed that Kummer surfaces are dense in the moduli space of K3 surfaces. The advantage of Kummer surfaces is that they are easier to handle. The study of K3 surfaces became later one of the main topics in algebraic geometry around 1980. The whole moduli space is given by the period domain. The *global Torelli theorem* states that the period mapping is bijective.

Kodaira asked the following question:

Question 7.5. *A compact complex surface admits a Kähler metric if and only if its first Betti number is even.*

In 1974 his student Miyaoka gave an affirmative answer in the case of elliptic surfaces in his master's thesis of only 4 pages ([27]). Finally Siu gave a positive answer to the above question in 1983 ([34]).

The higher dimensional analogue of K3 surfaces are Calabi-Yau manifolds. In dimension 3, there are more than thousands known families of Calabi-Yau 3-folds which have different topological types. There are too many Calabi-Yau 3-folds, but Reid's fantasy [31] claims that their might be only one family of 3-folds whose generic members are non-Kähler such that all the Calabi-Yau 3-folds appear after degenerations and desingularizations at the boundaries of the moduli space.

7.3. **elliptic surfaces.** A compact complex surface X is said to be an *elliptic surface* if there exists a morphism $f : X \to C$ to a curve whose general fibers E are elliptic curves.

Kodaira classified all possible fibers $f^{-1}(t)$ assuming that f is *relatively minimal*, i.e., there are no (-1)-curves in fibers.

I_0: a regular fiber.

$\mathrm{I}_b = \tilde{A}_{b-1}$ for $b \geq 1$: a rational curve with one node if $b = 1$, cycle of b smooth rational curves $C_1 + C_2 + \cdots + C_b$ for $b \geq 2$, i.e., $C_i \cong \mathbf{P}^1$ and $(C_i^2) = -2$, $(C_i, C_j) = 1$ if $i = j \pm 1 \mod b$, and $(C_i, C_j) = 0$ if $i \neq j, j \pm 1 \mod b$.

$m\mathrm{I}_b$ for $b \geq 0$ and $m \geq 2$: multiple fiber of type I_b with multiplicity m, i.e, $m(C_1 + \cdots + C_b)$ as a divisor.

II: a rational curve with one cusp.

III: two smooth rational curves tangent at one point.

IV: three smooth rational curves meeting at one point.

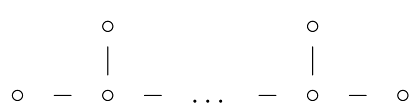
$\mathrm{I}_b^* = \tilde{D}_{b+4}$ for $b \geq 0$: tree of $b + 5$ smooth rational curves with the dual graph below.

$\mathrm{II}^* = \tilde{E}_8$: tree of 9 smooth rational curves with the dual graph below.

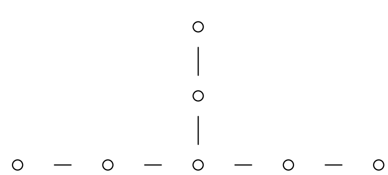
$\mathrm{III}^* = \tilde{E}_7$: tree of 8 smooth rational curves with the dual graph below.

$\mathrm{IV}^* = \tilde{E}_6$: tree of 7 smooth rational curves with the dual graph below.

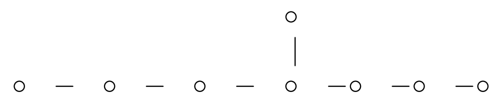
$\tilde{D}_n$:



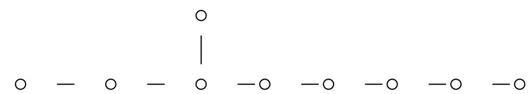
$\tilde{E}_6$:



$\tilde{E}_7$:



$\tilde{E}_8$:



In the above diagram, the vertexes correspond to smooth rational curves $\mathbf{P}^1$, and the edges to transversal intersections. The extended Dynkin diagrams of types $\tilde{A}, \tilde{D}, \tilde{E}$ appear in the classification of singular fibers. They have one more vertexes than A, D, E type diagrams. The Roman numerals II, III, IV correspond to topological Euler numbers. They are also related to eigenvalues of the local monodromies around the singular fibers which are reflected in the canonical bundle formula below. The eigenvalues for types II*, III*, IV* are the complex conjugates.

Kodaira proved the important *canonical bundle formula*:

$$K_X = f^*(K_C + D) + \sum_{F_i = m_i I_b} \frac{m_i - 1}{m_i} F_i$$

where D is a divisor on C such that

$$\deg(D) = \frac{1}{12}\left(\sum n_i + \sum b_i\right)$$

where the number n_i for a singular F_i is determined as

$$n_i = 0, 2, 3, 4, 6, 12 - 2, 12 - 3, 12 - 4$$

for types mI_{b_i}, II,III,IV,$\mathrm{I}^*_{b_i}$,II*,III*,IV*. The eigenvalue of the *local monodromy* on $H^1(E, \mathbf{Z})$ around the singular fiber F_i is equal to $e^{2\pi n_i/12}$.

The denominator 12 appears in many places:

(1) It is the denominator in Noether's formula.

(2) It is the degree of the automorphic function $\Delta = g_2^3 - 27g_3^2$, the *j-function*, where b_i is the order of its pole.

(3) It is the least common multiple of the orders of finite order automorphisms of elliptic curves.

Kodaira gave detailed description as well as the construction of elliptic surfaces by:

(1) j-function of fibers E.

(2) monodromy of $H^1(E, \mathbf{Z})$.

(3) torsor construction under the action of the group E.

(4) *logarithmic transformation* for multiple fibers.

In this way, one can say that *elliptic surfaces are now well understood.*

7.4. VII_0 **surfaces.** Kodaira classified all minimal compact complex surfaces into seven classes ($_0$ means minimal):

(I_0): b_1 even, $p_g := \dim H^0(X, K_X) = 0$. (algebraic surface)
(II_0): $b_1 = 0$, $p_g = 1$, $K_X \sim 0$. K3 surface.
(III_0): $b_1 = 4$, $p_g = 1$, $K_X \sim 0$. complex torus.
(IV_0): b_1 even, $p_g > 0$, $K_X \not\sim 0$, $(K_X^2) = 0$. (elliptic surface)
(V_0): b_1 even, $p_g > 0$, $(K_X^2) > 0$. (general type)
(VI_0): b_1 odd, $p_g > 0$, $(K_X^2) = 0$. (elliptic surface)
(VII_0): $b_1 = 1$, $p_g = 0$. (???)

The last class VII_0 was (and is) mysterious even for Kodaira. He considered only a special case. A compact complex surface is called a *Hopf surface* if its universal covering is isomorphic to $\mathbf{C}^2 \setminus \{0\}$.

An example of a Hopf surface is given by

$$X = (\mathbf{C}^2 \setminus \{0\})/\mathbf{Z}, \quad (z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$$

with $0 < |\alpha_i| < 1$. It is an elliptic surface if and only if $\alpha_1^i = \alpha_2^j$ for some positive integers i, j. Indeed the fibration is given by

$$f = z_1^i / z_2^j : X \to \mathbf{P}^1.$$

Otherwise, there are only 2 curves on X: $\{(z_1, 0)\}/\mathbf{Z}$ and $\{(0, z_2)\}/\mathbf{Z}$.

Theorem 7.6 (Kodaira 1966 [10]). *If X is homeomorphic to $S^1 \times S^3$, then X is a Hopf surface.*

An almost complex structure exists only on S^2 and S^6 among spheres. So a related question is whether there is a complex structure on S^6.

His students Inoue, Kato and Nakamura continued investigation of VII_0 surfaces [18], [19], [28], also [15], [17], [24], [25].

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