HOMOTOPY SPECTRA
AND DIOPHANTINE EQUATIONS

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To Xenia and Paolo, from Yuri and Matilde, with all our love and gratitude.

ABSTRACT. Arguably, the first bridge between vast, ancient, but disjoint domains of mathematical knowledge, – topology and number theory, – was built only during the last fifty years. This bridge is *the theory of spectra in the stable homotopy theory.*

In particular, it connects $\mathbf{Z}$, the initial object in the theory of commutative rings, with the sphere spectrum $\mathbf{S}$: see [Sc01] for one of versions of it. This connection poses the challenge: discover new information in number theory using the independently developed machinery of homotopy theory. (Notice that a passage in the reverse direction has already generated results about computability in homotopy theory: see [FMa20] and references therein.)

In this combined research/survey paper we suggest to apply homotopy spectra to the problem of distribution of rational points upon algebraic manifolds.

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CONTENTS

0. Introduction and summary

0.0. A brief history. For a long stretch of time in the history of mathematics, number theory and topology formed vast, but disjoint domains of mathematical knowledge. Emmanulel Peyre reminds us in [Pe19] that the Babylonian clay tablet Plimpton 322 (about 1800 BC) contained a list of integer solutions of the "Diophantine" equation $a^2 + b^2 = c^2$: archetypal theme of number theory, named after Diophantus of Alexandria (about 250 BC).

"Topology" was born much later, but arguably, its cousin – modern measure theory – goes back to Archimedes, author of *Psammites* ("Sand Reckoner"), who was approximately a contemporary of Diophantus.

In modern language, Archimedes measures the volume of observable universe by counting the number of small grains of sand necessary to fill this volume. Of course, many *qualitative* geometric models and *quantitative* estimates of the relevant distances known in a narrow small world of scientists of his times precede his calculations. Moreover, since the estimated numbers of grains of sands are quite large (about 10^{64}), Archimedes had to invent and describe a system of notation for large numbers going far outside the possibilities of any of the standard ancient systems.

The construction of the first bridge between number theory and topology was accomplished only about fifty years ago (around 1970): it is it is

the theory of spectra in stable homotopy theory

In this paper, after a brief survey of relevant setups in homotopy theory and number theory, we focus upon the ongoing research dedicated to the applications of spectra to the problems of distribution of rational/algebraic points on algebraic varieties.

0.1. Integers and finite sets. Below, we will briefly describe the contents of our paper appealing to the intuition of a reader who is already accustomed to *categorical* reasoning.

Intuitively, the set of non–negative integers $\mathbf{N}$ can be imagined as embodiment of *"sizes"* (formally, cardinalities) of all finite sets (including the empty set).

Already to compare sizes of two disjoint sets, that is to interpret the inequality $m \leq n$ in $\mathbf{N}$, one needs to look at the *category* $FSets$: if m is the size of M and n is the size of N, then $m \leq n$ means that in $FSets$ there is a *monomorphism* $M \to N$, or equivalently, an *epimorphism* $N \to M$.

A categorical interpretation of multiplication of integers uses *direct products of finite sets*. Associativity of multiplication is a reflection of the *monoidal structure* on $FSet$.

Addition in $\mathbf{N}$ already requires more sophisticated constructions in $FSet$: if two sets X, Y are disjoint, then the cardinality of $X \cup Y$ is the sum of cardinalities of X and Y, but far from all pairs (X, Y) are disjoint. So an additional formalism must be developed.

0.2. Integers and topological spaces. Much less popular is the background vision of a natural number $n \in \mathbf{N}$ as the *dimension of a manifold,* say the sphere S^n.

Here already the definition of *equality* of numbers requires quite sophisticated technicalities, if we want to lift it: it involves functorial definitions of classes of morphisms that we will declare *invertible ones* after passing to the category of categories. It is at this step that the idea of *homotopy*, homotopically equivalent maps, and homotopy classes of spaces and maps, acquires the key meaning.

While dimension is not a homotopy invariant, the natural numbers have another topological manifestation, in terms of certain categories, which we review below, describing the combinatorics of simplices.

But as soon as we accomplish this, addition and multiplication in $\mathbf{N}$ become liftable to an appropriate category and, after some more work, passage from $\mathbf{N}$ to $\mathbf{Z}$ becomes liftable as well.

0.3. From finite sets to topological spaces, and back. The critically important bridging of finite sets and topological spaces is furnished by the machinery of *simplicial sets*, in particular simplicial sets associated to *coverings*.

Roughly speaking, this passage replaces a finite set X by the simplex $\sigma(X)$ of which X is the set of vertices, and an admissible set theoretical map $X \to Y$ by its extension by convex combinations from $\sigma(X)$ to $\sigma(Y)$. Admissibility condition can be described in terms of the simplex category recalled below.

In the reverse direction, the passage from a *topological space with its covering* (say, by a family of open subsets) $\{U_a \,|a \in A\}$ to a simplicial set replaces each U_a by a vertex of a simplex, and associates simplices Δ_J with non–empty intersections $\cap_{i \in J} U_{a_i}$.

For a detailed treatment, see [GeMa03]. Here we will only recall the combinatorial setup (see [GeMa03] Section II.4(a) p. 58) with the following properties.

The simplex category Δ has objects the totally ordered sets $[n] := \{0, 1, \ldots, n\}$ and morphisms are nondecreasing maps $f : [n] \to [m]$.

The set of all morphisms is generated by two classes of maps: "i–th face" ∂_n^i and "i–th degeneration" σ_n^i. The i–th face is the increasing injection $[n-1] \to [n]$ not taking the value i, and σ_n^i is the nondecreasing surjection taking the value i twice.

All the relations between faces and degenerations are generated by the relations

$$\partial_{n+1}^j \partial_n^i = \partial_{n+1}^i \partial_n^{j-1} \quad \text{for} \quad i < j;$$

$$\sigma_{n-1}^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j+1} \quad \text{for} \quad i \le j;$$

$$\sigma_{n-1}^j \partial_n^i = \begin{cases} \partial_{n-1}^i \sigma_{n-2}^{j-1} & \text{for} \quad i < j; \\ id_{[n-1]} & \text{for} \quad i \in \{j, j+1\}; \\ \partial_{n-1}^{i-1} \sigma_{n-2}^j & \text{for} \quad i > j+1. \end{cases}$$

A *simplicial set* $X_\bullet$ (or simply X) can be then defined as a functor from the simplex category Δ to some category Set of sets.

Thus, a simplicial set is the structure consisting of a family of sets (X_n), $n = 0, 1, 2, \ldots$ and a family of maps $X(f) : X_n \to X_m$, corresponding to each nondecreasing map $f : [m] \to [n]$, such that $X(id_{[n]}) = id_{X_n}$, and $X(g \circ f) = X(f) \circ X(g)$.

Restricting ourselves by consideration of only $X(\partial_n^i)$ and $X(\sigma_n^i)$, and the respective relations, we get the convenient and widely used description of simplicial sets.

In order to pass from combinatorics to geometry, we must start with geometric simplices.

The *n–dimensional simplex* Δ_n is the topological space embedded into real affine $n+1$–dimensional space endowed with coordinate system:

$$\Delta_n := \left\{ (x_0, \ldots, x_n\} \in \mathbf{R}^{n+1} \cap [0,1]^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \right\}.$$

Starting with a simplicial set (X_n), we construct its *geometric realization* $|X|$. It is the set, endowed with a topology,

$$|X| := \cup_{n=0}^{\infty} (\Delta_n \times X_n)/R,$$

where R is the weakest equivalence relation identifying boundary subsimplices in $\Delta_n \times X_n$ and $\Delta_m \times X_m$ corresponding to nondecreasing maps $f : [m] \to [n]$: see [GeMa03], pp. 6–7. In particular, the boundary of Δ_{n+1} for $n \geq 2$ is the geometric realisation of the union of $n+1$ boundary $(n-1)$–dimensional simplices: this is a simplicial model of S^{n-1}.

One popular path in the opposite direction produces a simplicial set from a topological space $\mathcal{Y}$ endowed with a covering by open (or closed) sets (U_α), $\alpha \in A$.

Put for $n \geq 1$,

$$X_n := \{(\alpha_1, \dots, \alpha_n) \mid \cap_{i=1}^n U_{\alpha_i} \neq \emptyset\},$$

$$X(f)(\varphi) := \varphi \circ \Delta_f, \quad \text{where } f : [m] \to [n], \quad \Delta_f : \Delta_m \to \Delta_n.$$

The role of homotopy appearing at this point finds a beautiful expression in the following fact: *the geometric realisation $|X|$ is homotopically equivalent to $\mathcal{Y}$, if U is a locally finite covering, and all nonempty finite intersections $\cap_{i=1}^n U_{\alpha_i}$ are contractible.*

These constructions form the background for the much more sophisticated machinery of *homotopical spectra* sketched below in Sec. 1.

A note on terminology: in this paper we will adopt the somewhat unconventional terminology "homotopical spectra" or "homotopy spectra" to simply refer to "spectra as meant in the context of homotopy theory" (as opposed to all the other conflicting notions of "spectrum" that exist in mathematics). While we understand that this terminology is nonstandard, it is a convenient shorthand, especially when other notions of spectrum may also be present.

0.4. Structure of the paper. Homotopy theory methods have found some recent significant applications in arithmetic geometry, for instance in the work of Corwin–Schlank [CorSch20] on higher obstructions to the existence of rational points. We argue in this paper that homotopy theoretic methods, especially centered on the notions of assemblers and spectra, can also provide a novel viewpoint on various aspects of Manin's problem on the distribution of rational points of bounded height and some related ideas, including the case of the motivic height zeta functions studied by Chambert-Loir and Loeser [ChamLoe15], and Bilu [Bilu18]. The main new results in this paper consist of the construction of various relevant sieves and assemblers (Lemma 3.5.1 and Propositions 3.6.2 and 4.4.1, as well as Propositions 7.2, 7.5, 7.6, 7.7, and Theorem 7.4), and the proof that the motivic Fourier transform and the motivic Poisson summation formula, stated as relations in a Grothendieck ring of varieties with exponentials, lift to the categorical level in the form of covering families in an assembler (Theorems 7.12 and 7.14.1). We present various related questions concerning the categorification of aspects of classical and motivic height zeta function.

0.4.1 Plan of exposition. Section 1 is dedicated to a brief, but sufficiently formal presentation of homotopy-theoretic spectra. In particular, we would like to draw the attention of a reader to the category of Γ–spaces that has several different applications to the construction and study of new geometries: for the context of geometries "in characteristic 1", cf. [CoCons16], [MaMar18], [LMM19], and references therein. The main use of Γ-spaces in this paper will be in Sections 3.3 and 7.9.1.

Section 2 introduces an approach to distributions of rational/algebraic points on algebraic varieties based upon counting of points of bounded sizes. Here the central role is played by the notion of *heights*: ways to measure sizes of rational/algebraic points on algebraic varieties taking into account their positions in a projective space to which the variety is/can be embedded.

The pure categorical Section 3 is dedicated to the basic machinery of *assemblers* that we later use to bridge diophantine geometry and homotopical algebra, following [Za17a], [Za17b], [Za17c]. Its central intuitive notion is similar to one that lies in the background of the definition of functor K_0 of an abelian category: we replace each object of a category by the formal sum of its "irreducible" pieces, neglecting ways in which these pieces are assembled together.

The new results of Section 3 are contained in Sections 3.5 and 3.6, where we construct sieves (Lemma 3.5.1) and an assembler $\mathcal{C}_U$ (Proposition 3.6.2) with Grothendieck group $K_0(\mathcal{C}_U)$ that detects decompositions of U into a closed subvariety V and its open complement $W = U \smallsetminus V$ where $W \subset U$ satisfies a strict inequality $0 < \sigma(W, L_W) < \sigma(U, L_U)$ of the abscissas of convergence of the height zeta function with respect to a line bundle L_U. Arithmetic stratifications in the sense of [BaMa90] give examples of disjoint covering families in this assembler.

In Section 4 we introduce arithmetical/geometric environments that are more restricted, but better describable by homotopy means. In particular Section 4.2 recalls specific classes of varieties (such as the Fano complete intersections discussed in Section 4.2.3 and the generalized flag manifolds discussed in Section 4.2.4) for which asymptotic formulae are known. Section 4 is based mostly on [BaMa90] and [FrMaTsch89]. The only new results are in Section 4.4 (Proposition 4.4.1).

In Section 4.4 we show that the sieves and assemblers constructed in Section 3.5 and 3.6, based on conditions on the abscissa of convergence of the height zeta function can be generalized to other similar constructions, involving asymptotic conditions on the number of rational points of bounded height. We construct in Proposition 4.4.1 assemblers with associated Grothendieck group that detects splittings of U into a closed subvariety Y and its open complement, where Y is a strongly (or weakly) accumulating subvariety.

Section 5 considers possibilities to avoid point count on algebraic varieties, replacing it by a machinery of measurable sets in adelic spaces. This allows us to include consideration of Brauer–Manin obstructions and more intuitive notions of equidistribution of rational points. We pose the question of a possible formulation in the language of assemblers. Section 5 is mostly based on [Pe18] and [Sa20].

Section 6 is a survey of some very recent (2020) results of Corwin–Schlank [CorSch20], and we highlight the unexpected appearance of assemblers in the setup of varieties with empty sets of rational points, and obstructions for their existence.

Finally, in Section 7 we obtain categorifications of some parts of the recent work by Chambert–Loir and Loeser [ChamLoe15], dedicated to lifting arithmetic height zeta–functions (main generating functions for point counting) to motivic height zeta functions. We present the problem of how to lift these motivic height zeta functions to the categorical level of assemblers and homotopy-theoretic spectra. We discuss various aspects of this question, providing as initial result in this direction the construction of categorical lifts of the motivic Fourier transform and the motivic Poisson summation formula.

The first part of Section 7, from Section 7.1 to Proposition 7.7, introduces and extends various generalizations of the Grothendieck ring of varieties $K_0(\mathcal{V}_K)$. There are three generalizations to begin with that have been variously considered in previous literature:

(1) *from varieties to stacks* (see [BeDh07], [Ek09]): the Grothendieck ring of stacks $K_0(\mathcal{S}_K)$ is obtained as a localization $\mathcal{M}_K$ of the Grothendieck ring of varieties $K_0(\mathcal{V}_K)$, obtained by inverting the classes $[Gl_n]$;

(2) *from varieties to exponential sums* (see [CluLoe10], [ChamLoe15]): classes $[X]$ of varieties in $K_0(\mathcal{V}_K)$ are replaced by classes $[X, f]$ of varieties with a morphism $f : X \to \mathbf{A}^1$ in the Grothendieck ring with exponentials $KExp_K$, with relations designed to reflect the properties of exponential sums $\sum_{x\in X(\mathbf{F}_q)} \chi(f(x))$, for $\chi : \mathbf{F}_q \to \mathbf{C}^*$ a character.

(3) *from Grothendieck rings to assemblers* (see [Za17a], [Za17b], [Za17c]): the Grothendieck ring of varieties in $K_0(\mathcal{V}_K)$ is realized as $\pi_0 K(\mathcal{C}_{\mathcal{V}_K})$ of the spectrum $K(\mathcal{C}_{\mathcal{V}_K})$ associated to an assember $\mathcal{C}_{\mathcal{V}_K}$.

A common generalization of (1) and (2) is introduced in Section 7.3 where we obtain a relation between Grothendieck rings summarized by the diagram

$$\begin{tikzcd} & K_0(\mathcal{V}_K) \arrow[r] \arrow[dl] \arrow[d] & KExp_K \arrow[d] \arrow[dr] & \\ K_0(\mathcal{S}_K) & \mathcal{M}_K \arrow[l, "\simeq"'] \arrow[r] & Exp\mathcal{M}_K \arrow[r, "\simeq"] & K_0(Exp\mathcal{S}_K), \end{tikzcd}$$

where $Exp\mathcal{M}_K$ is the localization of $KExp_K$ at the classes $[GL_n, 0]$ and $K_0(Exp\mathcal{S}_K)$ is the Grothendieck ring of stacks with exponentials.

A common generalization of (2) and (3) is obtained in Theorem 7.4, where it is shown that there is a simplicial assembler $\mathcal{C}_K^{KExp} = \mathcal{C}/\Phi$, obtained as the cofiber of an endomorphism of an assembler, with $\pi_0 K(\mathcal{C}/\Phi) = KExp_K$. This generalization fits into a diagram

$$\begin{tikzcd} K_0(\mathcal{V}_K) \arrow[r] & KExp_K \\ K(\mathcal{C}_{\mathcal{V}_K}) \arrow[u] \arrow[r] & K(\mathcal{C}/\Phi), \end{tikzcd}$$

where the bottom line is a map of connective spectra and the upward maps are maps from the spectral to their π_0's.

The common simultaneous generalization of (1), (2), and (3) is then obtained in Proposition 7.5, with the construction of a simplicial assembler $\mathcal{C}_K^{KExp\mathcal{S}} = \mathcal{C}^{\mathcal{S}}/\Phi$ with $\pi_0 K(\mathcal{C}_K^{KExp\mathcal{S}}) = Exp\mathcal{M}_K = K_0(Exp\mathcal{S}_K)$, the Grothendieck ring of stacks with exponentials.

Propositions 7.6 and 7.7 describe the relative version of the previous construction, where one starts with the Grothendieck ring $K_0(\mathcal{V}_S)$ of varieties over a base scheme S and considers the corresponding generalizations of the other notions. In particular, Proposition 7.6 obtains the relative version of the generalization of (2) and (3) and Propositions 7.7 gives the relative form of the generalization of (1), (2), and (3).

The main reason for developing these common generalizations of the Grothendieck rings of varieties, stacks, and varieties with exponentials, with their associated assemblers, is the fact that the Grothendieck rings $KExp_S$ and $Exp\mathcal{M}_S$ (the S-relative versions of the Grothendieck ring with exponentials and its localization) are the settings where the motivic Fourier transform and the Hrushovski–Kazhdan motivic Poisson summation formula naturally live, and the corresponding assemblers $\mathcal{C}_S^{KExp}$ and $\mathcal{C}_S^{KExp\mathcal{S}}$ constructed in Propositions 7.6 and 7.7 are where their categorical lifts will take place (Theorem 7.12 and Theorem 7.14.1).

In Section 7.8 we discuss a natural class of motivic measures on the Grothendieck rings of varieties with exponentials, in the case over a finite field, given by the exponential sums, and the associated zeta function. We present in Proposition 7.8.1 an Euler product expansion generalizing the usual case of the Hasse–Weil zeta function.

In Section 7.9 and 7.10 we consider a different approach to the categorification of Grothendieck rings, through the construction of associated categories of motives through Nori's formalism. In Section 7.9 we recall the general functioning of Nori diagrams and Nori motives. As we recall in Section 7.10, a motivic category related to the Grothendieck ring of varieties with exponentials is provided by the category $MotExp(K)$ of exponential motives of Fresán and Jossen [FreJo20], which has a unique ring homomorphism

$$\chi : KExp_K \to K_0(MotExp(K)).$$

As we discuss in Section 7.9.1, the Nori approach also leads to the construction of a homotopy-theoretic spectrum, through the construction of a Γ-space associated to the Nori category and an associated category of summing functors [Carl05], [Se73]. The spectrum obtained in this way is a delooping of the infinite loop space associated to the K-theory of the Nori category.

In Section 7.11 we return to our main goal of Section 7, which is to investigate categorical aspects of the motivic height zeta functions, and we recall the setting and the main properties of the motivic Fourier transform, following [ChamLoe15].

In Theorem 7.12 we prove that the motivic Fourier transform lifts to a morphism of assemblers, and the identity in $KExp_V$ (with V a finite dimensional K-vector space) satisfied by the square of the Fourier transform is induced by a covering family in the assembler $\mathcal{C}_V^{KExp\mathcal{S}}$ constructed in Proposition 7.6.

In Section 7.13 we recall the Hrushovski–Kazhdan motivic Poisson summation formula, following [ChamLoe15] and [HruKaz09], and in Section 7.14 we present its categorification. In Theorem 7.14.1 we prove that the motivic Poisson summation formula, seen as a relation in $Exp\mathcal{M}_k$, is induced by a covering family in a corresponding assembler.

The remaining part of Section 7 introduces the question of a possible categorification of the motivic height zeta function itself, and discusses some more specific aspects of this question. The motivic height zeta function is recalled in Section 7.15, along with the role of motivic Fourier transform and Poisson summation formula in establishing its properties, following [ChamLoe15] and [Bilu18]. General methods for categorical liftings of zeta functions using Witt rings are discussed in Section 7.16, following [LMM19]. While these methods would apply to zeta functions associated to good (exponentiable) motivic measures on Grothendieck rings of varieties with exponentials, they cannot be applied to the motivic height zeta function itself. We propose, however, that the multivariable versions introduced in [Bilu18] on the basis of multivariable versions of the Kapranov motivic zeta function and motivic Euler product decomposition, may be more suitable for the purpose of a possible categorification via assemblers.

1. Homotopy spectra: a brief presentation

The notion of *spectra* in homotopy theory (which we refer to as "homotopy spectra") evolved through several stages. Below we will briefly describe two of them: *sequential spectra*, which can be considered as the first stage (fundamental definition), and Γ–*spaces*, the construction method which we will mostly use. The latter is a powerful method (introduced by Segal in [Se74]) for generating spectra from data consisting of categories with a zero-object and a categorical sum.

1.1. Sequential spectra (see [SpWh53]). In order to define them, we need to introduce the *smash product* $\wedge$ in the category of based (or pointed) sets. It can be defined by

$$(X, *_X) \wedge (Y, *_Y) = ((X \times Y)/(X \times *_Y) \cup (*_X \times Y), \bullet),$$

where the base point $\bullet$ in the smash product is the image of the union of two "coordinate axes" $(X \times *_Y) \cup (*_X \times Y)$ after the contraction of this union. Similarly, the smash product of pointed simplicial sets is the quotient of $X \times Y$ that collapses the pointed simplicial subset $(X \times *_Y) \cup (*_X \times Y)$.

A *sequential spectrum* E is a sequence of based simplicial sets E_n, $n = 0, 1, 2, \dots$ (see 0.3 above), and the structure maps

$$\sigma_n : \Sigma E_n := S^1 \wedge E_n \to E_{n+1}.$$

The *sphere (sequential) spectrum* $\mathbf{S}$ consists of simplicial sets $S^n := S^1 \wedge \dots \wedge S^1$ and identical σ_n's.

The smash product can be extended to the category of sequential spectra itself, however there it loses the commutativity property (it no longer forms a symmetric monoidal category). One can form a commutative and associative smash product only after formally inverting stable equivalences, that is, by passing to the homotopy category. We will not review the construction here, but we refer the reader to [EKMM97], [HoShiSmi00], [Sc12]. We just remark that, when one uses Γ-spaces as a construction method for spectra, as we discuss in the next subsection, the smash product has a very transparent description as shown in [Ly99].

1.2. Γ–spaces. Γ–spaces were introduced by Graham Segal in [Se74]. Here we reproduce arguments of Lydakis in [Ly99] and mostly keep his notation. Let Γ (Γ^{op} in [Ly99]) be the category of based finite sets $n^+ := \{0, 1, \dots, n\}$ (formerly called $[n]$), with base preserving maps as morphisms. So here we *do not restrict morphisms by nondecreasing maps.*

A Γ*–space* E is a functor from Γ to based simplicial sets sending 0^+ to the point.

Γ–spaces themselves are objects of the category denoted $\mathcal{GS}$ in [Ly99], morphisms in which are natural functors (we omit a precise description).

Following Section 2 of [Ly99], we will call based simplicial sets simply *spaces.* A space is called *discrete* if its simplicial set is constant.

1.3. Theorem. *One can define a functor $\wedge : \mathcal{GS} \times \mathcal{GS} \to \mathcal{GS}$ such that there is a canonical isomorphism of functors in three variables*

$$\mathcal{GS}(F \wedge F', F'') \cong \mathcal{GS}(F, \mathrm{Hom}\,(F', F''))$$

where we denote by $\mathcal{GS}(F, F')$ the set of morphisms $F \to F'$ in $\mathcal{GS}$.

The category of Γ–spaces endowed with smash–product $\wedge$ is a symmetric monoidal category.

Sketch of Proof. Consider the category of $\Gamma \times \Gamma$–spaces $\mathcal{GGS}$. Its objects are pointed functors $\Gamma \times \Gamma \to \mathcal{S}$. Define also the external smash product $F \overline{\wedge} F'$ of two Γ–spaces F, F' as the functor that sends (m^+, n^+) to $F(m^+) \wedge F'(n^+)$.

Then one can check that

$$\mathcal{GS}(F, \mathrm{Hom}\,(F', F'')) \cong \mathcal{GGS}(F \overline{\wedge} F', RF''),$$

where R is a functor $\mathcal{GS} \to \mathcal{GGS}$, determined by the map $\Gamma \times \Gamma \to \Gamma$ sending (m^+, n^+) to $m^+ \wedge n^+$, by setting $RF''(m^+, n^+) = F''(m^+ \wedge n^+)$.

The q-simplices of $\mathrm{Hom}\,(F', F'')(m^+)$ are given by $\mathcal{GS}(F \wedge (\Delta^q)^+, F'(m^+ \wedge))$, where $F'(m^+\wedge) : n^+ \mapsto F'(m^+ \wedge n^+)$.

After that one can prove that R has a left adjoint functor $L : \mathcal{GGS} \to \mathcal{GS}$. Namely, LF''' is the colimit of $F'''(i^+, j^+)$ over all morphisms $i^+ \wedge j^+ \to n^+$.

This is essentially the statement of Theorem 1.3.

For many more details, see [Ly99], especially Theorem 2.2.

1.4. Definition. *The sphere spectrum* $\mathbf{S}$ *is the unit object in the symmetric monoidal category of* Γ*–spaces.*

This is the functor $\mathbf{S}$ given by the inclusion of the category Γ of pointed sets in the category of pointed simplicial sets.

Here is a more detailed description of $\mathbf{S}$. For any n^+ we can define the representable Γ–space Γ^n by

$$\Gamma^n(m^+) := \mathcal{GS}(n^+, m^+).$$

This satisfies $\mathcal{GS}(\Gamma^n \wedge (\Delta^q)^+, F) = F(n^+)$. From this it follows that $\mathbf{S}$ is canonically isomorphic to Γ^1, and is in fact the sphere sequential spectrum defined in 1.1 above (see Definitions 2.5 and 2.7, Lemma 2.6 and Proposition 2.8 of [Ly99]).

In particular, this construction suggests to consider homotopical enrichments of arithmetics passing through

ring of integers $\mathbf{Z} \Longrightarrow$ *the sphere spectrum* $\mathbf{S}$

This is in particular justified by the fact that $\mathbf{S}$ is a ring spectrum and $\mathbf{Z}$, viewed as the Eilenberg-MacLane spectrum $H\mathbf{Z}$, is a ring spectrum over $\mathbf{S}$. This idea leads naturally to one of the approaches to $\mathbf{F}_1$ geometry (geometry "below Spec($\mathbf{Z}$)") as based on the sphere spectrum $\mathbf{S}$, as in [ToVa09].

This also poses a challenge: discover a new information in number theory using the independently-developed machinery of homotopy theory.

1.5. Γ–spaces: from categories to spectra. Our main reason for introducing the properties of Γ-spaces lies in the fact that these provide a very useful general method for constructing spectra.

A Γ-space F, which is a functor from the category Γ of finite pointed sets to that of pointed simplicial sets, extends to an endofunctor of the category of pointed simplicial sets, by defining $F(K)$ as the coend

$$F(K) = \int^{n^+ \in \Gamma} K_n \wedge F(n^+),$$

with K_n the n-th skeleton of K. There are natural assembly maps $K \wedge F(K') \to F(K \wedge K')$.

Thus, one obtains a spectrum $\mathbf{S}(F)$ associated to the Γ space F by applying this functor to spheres, $\mathbf{S}(F)_n := F(S^n)$, with structure maps $S^1 \wedge F(S^n) \to F(S^{n+1})$.

In particular, as originally described by Segal in [Se74], Γ-spaces can be used to associate a spectrum to a category $\mathcal{C}$ that has a categorical sum and a zero object, via the construction of a Γ-space $F_{\mathcal{C}}$ obtained in the following way (see [Carl05]).

Let (X, x_0) be a finite pointed set. Consider the category $P(X)$ with objects all the pointed subsets $A \subset X$ and morphisms given by pointed inclusions. A *summing functor* $\Phi : P(X) \to \mathcal{C}$ is a functor satisfying $\Phi(A \cup A') = \Phi(A) \oplus \Phi(A')$ for A, A' in $P(X)$ with $A \cap A' = \{x_0\}$, and $\Phi(\{x_0\}) = 0$. Let $\Sigma_{\mathcal{C}}(X)$ denote the category of summing functors with morphisms given by invertible natural transformations.

The reason for using natural isomorphisms instead of arbitrary natural transformations as morphisms in $\Sigma_{\mathcal{C}}(X)$ is in order to ensure that the resulting nerve (see below) has interesting topology: using all natural transformations would lead to a contractible space (the category of summing functors has an initial object, since $\mathcal{C}$ has a zero object).

Recall that the nerve $\mathcal{N}$ of a category $\mathcal{A}$ is the simplicial set $\mathcal{N}\mathcal{A}$ whose vertices are indexed by objects A of $\mathcal{A}$; 1–simplices by morphisms $A_1 \to A_2$ of $\mathcal{A}$; and generally, n–simplices by sequences of morphisms $A_1 \to \cdots \to A_{n+1}$. Faces and degenerations (see 0.3 above) are determined via composition of morphisms in a pretty obvious way.

The Γ–space $F_{\mathcal{C}}$ is then determined by assigning to a finite pointed set (X, x_0) the pointed simplicial set given by the nerve $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of $\Sigma_{\mathcal{C}}(X)$, with associated spectrum $\mathbf{S}(F_{\mathcal{C}})$.

The intuition behind this construction is the following: the category of summing functors provides a *delooping* of the infinite loop space given by (a completion of) the classifying space of $\mathcal{C}$, see [Carl05] for a more detailed discussion.

While this construction of spectra via Γ-spaces only gives rise to connective spectra, it is shown in [Tho95] that all connective spectra can be obtained in this way.

In this paper, the main applications of this general method for the construction of spectra via Γ-spaces will appear in Section 3.3 and in Section 7.9.1. The first is the application of Γ-spaces that occurs within Zakharevich's theory of assemblers (assembler categories), where the Γ-spaces method provides a key step in the construction of the corresponding spectrum. We will review this in Section 3.3 below and we will apply it then several times to different settings, in Sections 6 and then more explicitly in Proposition 7.2, Theorem 7.4, Propositions 7.5 and 7.6, Theorems 7.12 and 7.14.1. The second occurrence of Γ-spaces will be an application of the construction of spectra from categories, as recalled in this subsection, applied to Nori diagrams in Section 7.9.1.

2. Diophantine equations: distribution of rational points on algebraic varieties

2.1. Diophantine equations and heights. We will be studying here how fast the number of solutions of a system of equations can grow, if one first restricts the counting to solutions of *height* $\leq H$, and then lets H grow.

In order to define heights over general algebraic number fields, we need the following preparations.

Let K be a number field, $\Omega_K = \Omega_{K,f} \sqcup \Omega_{K,\infty}$ the set of its places v represented as the union of finite and infinite ones, and let K_v denote the respective completion of K.

For $v \in \Omega_{K,f}$, denote by $\mathcal{O}_v$, resp. m_v, the ring of integers of K_v, resp. its maximal ideal. For a uniformizer π_v of $\mathcal{O}_v$, $|\pi_v|_v^{-1}$ is the size of the residue field $\mathcal{O}_v/m_v$. The Haar measure dx_v on K_v is normalised in such a way that the measure of $\mathcal{O}_v$ becomes 1. Moreover, for an archimedean v, the Haar measure will be the usual Lebesgue measure, if v is real, and for complex v it will be induced by Lebesgue measure on $\mathbf{C}$, for which the unit square $[0,1]+[0,1]i$ has volume 2.

Let the map $|.|_v : K_v \to \mathbf{R}^*_{\geq 0}$ be defined by the condition $d(\lambda x)_v = |\lambda|_v dx_v$. In particular, $|.|_v$ is the usual absolute value for v real, and its square for v complex.

Then for any $\lambda \in K^*$ we have the following product formula: $\prod_v |\lambda|_v = 1$.

Let $\mathbf{P}^n$ be a projective space with a chosen system of homogeneous coordinates $(x_0 : x_1 : \ldots : x_{n-1} : x_n)$, $n \geq 1$. Then we can define *the exponential Weil height* of a point $p = (x_0(p) : \cdots : x_n(p)) \in \mathbf{P}^n(K)$ as

$$h(p) := \prod_{v \in \Omega_K} \max \{|x_0(p)|_v, \ldots, |x_n(p)|_v\}.$$

Because of the product formula, the height does not change if we replace coordinates $(x_0 : \cdots : x_n)$ by $(\lambda x_0 : \cdots : \lambda x_n)$, $\lambda \in K$.

Notational remark: it is more common in the literature to use a different notation $H(p)$ for the exponential height and $h(p)$ for the logarithmic height, but we will be only considering the exponential height and we will simply use $h(p)$ for all instances of height functions in this paper.

2.2. Height zeta functions. Let now (U, L_U) be a pair consisting of a projective variety U over K and an ample line bundle L_U on it. Let $V \subset U$ be a locally closed subvariety of U, also defined over K. Then we can define the height function $h_{L_V}(p)$ on $p \in V(K)$ using the same formula as above, but this time interpreting (x_i) as a basis of sections in $\Gamma(V, L_V)$.

If h'_{L_V} is another height, corresponding to a different choice of the basis of sections, then there exist two positive real constants C, C' such that for all x,

$$Ch_{L_V}(x) \leq h'_{L_V}(x) \leq C'h_{L_V}(x).$$

Now define the height zeta–function

$$Z(V, L_V, s) := \sum_{x \in V(K)} h_{L_V}(x)^{-s}.$$

2.3. Convergence boundaries. Denote by $\sigma(V, L_V) \in \mathbf{R}$ the greatest lower bound of the set of positive reals σ for which $Z(V, L_V, s)$ absolutely converges if $Re\, s \geq \sigma$.

We will call $\sigma(V, L_V)$ the respective *convergence boundary.*

Clearly, it is finite (because this is so for projective spaces), and non–negative whenever $V(K)$ is infinite.

For example, one has $\sigma(\mathbf{P}^n, \mathcal{O}(m)) = (n + 1)/m$ (Section 1.3 of [BaMa90]).

Intuitively, we may say that V contains "considerably less" K–points than U, if

$$\sigma(V, L_V) < \sigma(U, L_U).$$

and "approximately the same" amount of K–points, if

$$\sigma(V, L_V) = \sigma(U, L_U),$$

in the sense that the rate of growth of the counting of such points (weighted by height) is lower or the same.

2.3.1. Example: accumulating subvarieties. Let $V \subset U$ be a Zariski closed subvariety over K. As in Section 3.2.3 of [Cham10], we say that V is a *strongly accumulating subvariety* if the fraction

$$\frac{\mathrm{card}\left\{x \in V(K) \,\middle|\, h_{L_V}(x) \leq H\right\}}{\mathrm{card}\left\{x \in U(K) \,\middle|\, h_{L_U}(x) \leq H\right\}} \to 1$$

as $H \to \infty$, and a *weakly accumulating subvariety* if

$$\mathrm{liminf}_H \frac{\mathrm{card}\left\{x \in V(K) \,\middle|\, h_{L_V}(x) \leq H\right\}}{\mathrm{card}\left\{x \in U(K) \,\middle|\, h_{L_U}(x) \leq H\right\}} > 0.$$

Clearly, then $\sigma(V, L_V) = \sigma(U, L_U)$.

We will return to these notions of strongly and weakly accumulating subvarieties in Proposition 4.4.1, where we construct assemblers whose associated Grothendieck groups detects the presence of such subvarieties through its scissor-congruence relations.

We will now describe a categorical environment appropriate for describing various versions of accumulation and connecting distributions with spectra.

3. Rational points, sieves, and assemblers

3.1. Grothendieck topologies, sieves, and assemblers. (See [Za17a], [Za17b], [MaMar18]). Let $\mathcal{C}$ be a category with a unique initial object $\emptyset$. Two morphisms $f_1 : U_1 \to U$ and $f_2 : U_2 \to U$ are called *disjoint*, if $U_1 \times_U U_2$ exists and is $\emptyset$.

Notice that if $U_1 \times_U U_2$ exists, then $U_2 \times_U U_1$ also exists, and these two relative products are canonically isomorphic, through a unique isomorphism.

A *sieve* in $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ such that if $f : V \to U$ is a morphism in $\mathcal{C}$, and U is an object of $\mathcal{C}'$, then V is also an object of $\mathcal{C}'$.

Fixing an object U in $\mathcal{C}$, can apply this notion also to the category of morphisms $f : V \to U$, or in other words to the "overcategory" $\mathcal{C}/U$. Functors between the overcategories $f_* : \mathcal{C}/W \to \mathcal{C}/V$ induced by composition with $f : W \to V$ induce functors (in the opposite direction) between sieves in the respective overcategories. In fact, any functor $f : \mathcal{C} \to \mathcal{D}$ between two categories induces a pullback map on sieves: if $\mathcal{S} \subseteq \mathcal{D}$ is a sieve, then the full subcategory $f^{-1}(\mathcal{S})$ is a sieve in $\mathcal{C}$.

This notion is convenient in order to define a *Grothendieck topology* on $\mathcal{C}$: it is *a collection of sieves* $\mathcal{J}(U)$ in $\mathcal{C}/U$, one for each object U of $\mathcal{C}$, satisfying three axioms:

(i) Any morphism $f : W \to V$ lifts to the map $f^* : \mathcal{J}(V) \to \mathcal{J}(W)$.

(ii) The full overcategory $\mathcal{C}/U$ belongs to $\mathcal{J}(U)$ for any object of U of $\mathcal{C}$.

(iii) Let $\mathcal{S} \in \mathcal{J}(U)$ and $\mathcal{T}$ be a sieve in $\mathcal{C}/U$. If $f^*(\mathcal{T}) \in \mathcal{J}(V)$ for all $f : V \to U$ in $\mathcal{S}$, then $\mathcal{T} \in \mathcal{J}(U)$.

For any object U of a category with Grothendieck topology $\mathcal{C}$ (called also Grothendieck site) we can define the notion of *covering family*: it is a collection of morphisms $\{f_i : U_i \to U \,|\, i \in I\}$ such that the full subcategory of $\mathcal{C}/U$ containing all morphisms in $\mathcal{C}$ factoring through one of the f_i's belongs to the initial collection of sieves $\mathcal{J}(U)$.

A family $\{f_i : U_i \to U\}$ is disjoint if f_i and f_j are disjoint for all $i \neq j$. As will become clear in the result of [Za17a] that we recall in Theorem 3.4 below, heuristically one should think of disjoint covering families as the categorification of scissor-congruence relations.

3.2. Assemblers. An assembler (also referred to as assembler category) is a small category $\mathcal{C}$ with a Grothendieck topology with the following properties:
(a) $\mathcal{C}$ has an initial object $\emptyset$, for which the empty family is a covering family;
(b) all morphisms in $\mathcal{C}$ must be monomorphisms;

(c) any two disjoint finite covering families must admit a common refinement which is also a finite disjoint covering family.

Assemblers themselves form a category, in which a morphism is a functor continuous in the respective Grothenieck topologies, sending initial object to initial object, and disjoint morphisms to disjoint morphisms.

Let $\mathcal{C}$ be a Grothendieck site. Denote by $\mathcal{C}^\circ$ its full subcategory of non-initial objects.

If we have a family of assemblers $\{\mathcal{C}_x\}$ numbered by elements x of a set X, we will denote by $\bigvee_{x\in X} \mathcal{C}_x$ the category whose non-initial objects are $\bigsqcup_{x\in X} \mathrm{Ob}\mathcal{C}_x^\circ$ and to which one initial object is formally added.

A simplicial assembler is a functor from the simplex category to the category of assemblers (a simplicial object in the category of assemblers), see Definition 2.14 of [Za17a].

3.3. From assemblers to Γ–spaces and spectra. (See [Za17a], also reviewed in [MaMar18], Sec. 4.4). Starting with an assembler $\mathcal{C}$, we can construct the following category $\mathcal{W}(\mathcal{C})$:

(a) *An object* of $\mathcal{W}(\mathcal{C})$ is a map $I \to Ob\,(\mathcal{C})$ where I is a finite set, and the map lands in non-initial objects. We may write it as $\{A_i \,|\, i \in I\}, A_i \in Ob\,(\mathcal{C})$.

(b) *A morphism* $f : \{A_i\}_{i\in I} \to \{B_j\}_{j\in J}$ consists of a map of finite sets $f : I \to J$ and a family of morphisms $f_i : A_i \to B_{f(i)}$ such that $\{f_i : A_i \to B_j : i \in f^{-1}(j)\}$ for each $j \in J$ is a disjoint covering family.

3.3.1. Proposition. (See [Za17a], Prop. 2.11.) *(i) All morphisms in $\mathcal{W}(\mathcal{C})$ are monomorphims.*

(ii) If $\mathcal{C}$ has all pullbacks (i.e. is closed), then $\mathcal{W}(\mathcal{C})$ is closed as well.

(iii) Given a family of assemblers $\{\mathcal{C}_x \,|\, x \in X\}$ indexed by elements of a set X, denote by $\oplus\mathcal{W}(\mathcal{C}_x)$ the full subcategory of $\prod \mathcal{W}(\mathcal{C}_x)$ whose objects are families of objects of $\mathcal{C}_x$ for which all but finitely many of them are indexed by $\emptyset$. Consider the functor

$$P : \mathcal{W}(\bigvee_{x\in X} \mathcal{C}_x) \to \prod_{x\in X} \mathcal{W}(\mathcal{C}_x)$$

induced by the morphisms of assemblers $F_y : \vee_{x\in X}\, \mathcal{C}_x \to \mathcal{C}_y$ that map each $\mathcal{C}_x$ to the initial object if $x \neq y$ and identically to $\mathcal{C}_y$ if $x = y$.

Then P induces an equivalence of categories

$$\mathcal{W}(\bigvee_{x\in X}\mathcal{C}_x) \to \bigoplus_{x\in X}\mathcal{W}(\mathcal{C}_x).$$

Proof. The proof is as in [Za17a], Proposition 2.11. We recall it here for the reader's convenience. The main part of our argument is a detailed study of pullback squares (see the diagram in the proof of Proposition 2.11 of [Za17a]).

Start with a morphism $f: \{A_i\}_{i\in I} \to \{B_j\}_{j\in J}$, and two more morphisms

$$g, h: \ \{C_k\}_{k\in K} \to \{A_i\}_{i\in I}.$$

In order to show that f is a monomorphism we must check that if $fg = fh$, then $g = h$. Choose any $k \in K$, and look at the respective commutative square with vertices $C_k, A_{g(k)}, A_{h(k)}, B_{fg(k)}$. We must have $g(k) = h(k)$, because otherwise $f_{g(k)}$ and $f_{h(k)}$ are disjoint.

The remaining statements easily follow from these remarks. ■

3.3.2. Move to Γ–spaces, spectra, and K–theory. (Cf. Sec. 2.2 of [Za17a]). To move from here to Γ–spaces, start with the category Γ^0 whose objects are finite sets $n^+ := \{0, 1, 2, \dots, n\}$ with base point 0, and morphisms are all maps sending 0 to 0 as morphisms, as in Subsection 1.2 above.

If X is a pointed set, and $\mathcal{C}$ an assembler, we can construct the assembler $X\wedge\mathcal{C} := \bigvee_{x\in X^\circ}\mathcal{C}_x$, where $X^\circ := X \setminus \{*\}$.

For each n, we have a simplicial assembler $S^n\wedge\mathcal{C}$. Then $\mathcal{W}(S^n\wedge\mathcal{C})$ is a simplicial category, hence its nerve $\mathcal{N}\mathcal{W}(S^n \wedge \mathcal{C})$ is a bisimplicial set. This is then the n-th space of the associated spectrum, with structure maps induced by $X \wedge \mathcal{N}\mathcal{W}(S^n \wedge \mathcal{C}) \to \mathcal{N}\mathcal{W}(X \wedge \mathcal{C})$:

$$X \wedge \mathcal{N}\mathcal{W}(\mathcal{C}) \cong \bigvee_{X^\circ}\mathcal{N}\mathcal{W}(\mathcal{C}) \to \mathcal{N}(\bigoplus_{X^\circ}\mathcal{W}(\mathcal{C})) \equiv \mathcal{N}\mathcal{W}(X \wedge \mathcal{C}).$$

I. Zakharevich defines the symmetric spectrum $K(\mathcal{C})$ as spectrum of simplicial sets in which the k–th space is given by the diagonal of the bisimplicial set $[n] \to \mathcal{N}\mathcal{W}((S^k)_n \wedge \mathcal{C})$ and writes

$$K_i(\mathcal{C}) := \pi_i K(\mathcal{C})\,.$$

Note that this is not just a sequential spectrum as described in Section 1.1, but a symmetric spectrum, where the sequence of spaces is endowed with compatible actions of the symmetric group, see Definition 1.2.1 of [HoShiSmi00]. Symmetric spectra allow for a good notion of smash product but require more care when taking homotopy groups (we refer the reader to [HoShiSmi00] and [Sc12]).

The construction above of the spectrum $K(\mathcal{C})$ can be seen as a case of the construction of spectra $\mathbf{S}(F)$ via Γ-spaces F that we recalled in Section 1.5. Indeed, the assignment

$$F : n^+ \mapsto F(n^+) = \mathcal{N}\mathcal{W}(n^+ \wedge \mathcal{C})$$

defines a Γ-space and $K(\mathcal{C})$ is the spectrum (called $\mathbf{S}(F)$ in Section 1.5) determined by this Γ-space. This fact is explicitly used in the proof of Theorem 2.13 in [Za17a], which we recall below, to identify generators and relations of $K_0(\mathcal{C})$ in terms of noninitial objects of $\mathcal{C}$ and morphisms of $\mathcal{W}(\mathcal{C})$, using the identification $K_0(\mathcal{C}) = \pi_0(\mathbf{S}(F))$, see also Section 4 of [Se74].

The following result (Theorem 2.13 in [Za17a]) furnishes the first justification of the intuition encoded in the word "assembler".

3.4. Theorem. *Let $\mathcal{C}$ be an assembler. Then $K_0(\mathcal{C})$ is canonically isomorphic to the abelian group generated by (isomorphism classes of) objects $[A]$ of $\mathcal{C}$ modulo the family of relations indexed by finite disjoint covering families $\{A_i \to A \,|\, i \in I\}$: each such family produces the relation*

$$[A] = \sum_{i \in I} [A_i].$$

Sketch of proof. The calculation of generators and relations for $K_0(\mathcal{C})$ can be reduced to making explicit the composition of functors

$$P^{-1} : \ \mathcal{W}(\mathcal{C}) \oplus \mathcal{W}(\mathcal{C}) \to \mathcal{W}(\mathcal{C} \wedge \mathcal{C})$$

and

$$\mu : \ \mathcal{W}(\mathcal{C} \vee \mathcal{C}) \to \mathcal{W}(\mathcal{C}).$$

Here P^{-1} sends a pair of objects $(\{A_i \,|\, i \in I\}, \{B_j \,|\, j \in J\})$ to the object $\{C_k \,|\, k \in I \cup J\}$, where for $k \in I$, resp. $k \in J$, we put $C_k = A_k$, resp. $C_k = B_j$: cf. Proposition 3.3 above. The map μ^{-1} is induced by the folding map of assemblers.

Finally, it remains to remark that relations in π_0 are generated by morphisms $\{A_i \to A \,|\, i \in I\} \to \{A\}$. ■

In most significant applications, the Grothendieck group $K_0(\mathcal{C})$ is also endowed with a ring structure. This is induced by a symmetric monoidal structure on the assembler $\mathcal{C}$, which in turn determines an E_∞-ring structure on the spectrum $K(\mathcal{C})$, see [Za17a].

For most of the applications to constructions of assemblers and associated spectra that we consider in this paper, we will be constructing some explicit disjoint covering families in an assembler $\mathcal{C}$. These covering families will provide categorical lifts of certain relations between classes in the corresponding Grothendieck ring $K_0(\mathcal{C})$. This will be the case, for example, in Theorems 7.12 and 7.14.1.

We now move to our main preoccupation here: constructing assemblers related to the distributions of rational points on algebraic varieties.

We will first of all define formally certain *sieves* via point distribution.

3.5. Categories $\mathcal{C}(U, L_U)$. Let U be a projective variety over K and L_U an ample rank 1 vector bundle on U over K.

By definition, objects of $\mathcal{C}(U, L_U)$ are locally closed subvarieties $V \subset U$ also defined over K, and morphisms are the structure embeddings $i_{V,U}$, or simply i_V: $V \to U$. Here we did not mention L explicitly, but it is natural to endow each V by $L_V := i_V^*(L_U)$.

Structure embeddings are compatible with these additional data so that we have in fact structure functors $\mathcal{C}(V, L_V) \to \mathcal{C}(U, L_U)$ which make of each $\mathcal{C}(V, L_V)$ a full subcategory of $\mathcal{C}(U, L_U)$ closed under precomposition, that is, *a sieve.*

We will call such $\mathcal{C}(V, L_V)$ *geometrical sieves*, and now introduce the *arithmetical sieves* $\mathcal{C}^{ar}(V, L_V)$ in the following way.

3.5.1. Lemma. *The family of those morphisms $i_{V,U}$ as above, together with their sources and targets, for which*

$$0 < \sigma(V, L_V) < \sigma(U, L_U),$$

forms a sieve in $\mathcal{C}(U, L_U)$ denoted $\mathcal{C}^{ar}(U, L_U)$.

Proof. If we have a two–step ladder of locally closed embeddings $W \subset V \subset U$ such that $0 < \sigma(V, L_V) < \sigma(U, L_U)$ and $0 < \sigma(W, L_W) < \sigma(V, L_V)$, then of course

$0 < \sigma(W, L_W) < \sigma(U, L_U)$, so that the composition of these embeddings is also a morphism in $\mathcal{C}^{ar}(U, L_U)$. ■

Notice, that if $V(K)$ is a finite set, then $\sigma(V, L_V) = 0$, but the converse is not true: $\sigma(V, L_V) = 0$ for any abelian variety V/K and for many other classes of V. A complete geometric description of this class of varieties seemingly is not known.

Note also that the arithmetic sieves $\mathcal{C}^{ar}(V, L_V)$ behave differently from the sieves $\mathcal{C}(V, L_V)$, in the sense that in general

$$\mathcal{C}^{ar}(U, L_U) \cap \mathcal{C}(V, L_V) \neq \mathcal{C}^{ar}(V, L_V).$$

3.6. Arithmetic assemblers. Using sieves $\mathcal{C}^{ar}(U, L_U)$, we can easily introduce the respective arithmetic assemblers $\mathcal{C}_U$ that we describe below.

We first need to identify a good class of morphisms to use for the construction of the assembler $\mathcal{C}_U$.

3.6.1. Lemma. *If $V \hookrightarrow U$ is a locally closed subvariety satisfying $0 < \sigma(V, L_V) < \sigma(U, L_U)$, then the complement $W = U \smallsetminus V$ satisfies $\sigma(W, L_W) = \sigma(U, L_U)$.*

Proof. It suffices to check this by considering height zeta functions as Dirichlet series, and normalize heights in such a way that, for $x \in U(K)$, the height $h(x)$ does not depend on other choices (e.g. does not depend on V and L_V, if $x \in V(K)$). Then the corresponding zeta functions satisfy $Z(U, s) = Z(V, s) + Z(U \smallsetminus V, s)$. If there is a real point s where one of the two summands in the right-hand-side diverges and another converges, then $Z(U, s)$ will diverge at this point, and if both summands converge at s then $Z(U, s)$ will also converge. ■

This shows that, if we want to have an assembler with non-trivial disjoint covering families, we cannot just use morphisms in $\mathcal{C}^{ar}(V, L_V)$. We can, however, relax this requirement by considering morphisms that are arbitrary finite compositions of open embeddings in $\mathcal{C}^{ar}(V, L_V)$ and arbitrary closed embeddings, where we do not impose the strict inequality between the abscissas of convergence of the respective height zeta functions.

3.6.2. Proposition. *Let $\mathcal{C}_U$ be the category whose objects are locally closed subvarieties V of U and whose morphisms are finite composites of open embeddings and closed embeddings, where the open embeddings are required to be in $\mathcal{C}^{ar}(V, L_V)$, namely $i_{W,V} : W \hookrightarrow V$ with W Zariski open in V and $0 < \sigma(W, L_W) < \sigma(V, L_V)$.*

The Grothendieck topology generated by the disjoint covering families of the form $\{W \hookrightarrow U, U \smallsetminus W \hookrightarrow U\}$*, where* $W \subset U$ *is open with* $0 < \sigma(W, L_W) < \sigma(U, L_U)$*. The category* $\mathcal{C}_U$ *is an assembler, and the associated* $K_0(\mathcal{C}_U)$ *group is generated by isomorphism classes* $[V]$ *of subvarieties of* U *with the relations* $[U] = [V] + [W]$ *where* $W = U \smallsetminus V$ *is open with* $0 < \sigma(W, L_W) < \sigma(U, L_U)$*.*

Proof. The empty scheme is the initial object. Both open and closed embeddings are monomorphisms. The category has pullbacks as closed embeddings are preserved under pullbacks and open embeddings $i_{W,V} : W \hookrightarrow V$ satisfying $0 < \sigma(W, L_W) < \sigma(V, L_V)$ are also preserved, since the intersection $W = W_1 \cap W_2$ of open subvarieties satisfying $0 < \sigma(W_i, L_{W_i}) < \sigma(V, L_V)$ will also satisfy the strict inequality $0 < \sigma(W, L_W) < \sigma(V, L_V)$. This ensures that any two finite disjoint covering families have a common refinement. The finite disjoint covering families are determined by sequences

$$U \supset V_0 \supset V_1 \supset V_2 \cdots \supset V_n$$

of Zariski open subvarieties with $0 < \sigma(V_{i+1}, L_{V_{i+1}}) < \sigma(V_i, L_{V_i})$, by taking $\{f_i : X_i \hookrightarrow U\}$ with $X_0 = U \smallsetminus V_0$ and $X_i = V_i \smallsetminus V_{i+1}$, with $V_{n+1} = \emptyset$. The Grothendieck group $K_0(\mathcal{C}_U)$ is then generated by the objects of $\mathcal{C}_U$ with the scissor-congruence relations determined by the disjoint covering families. This means that $K_0(\mathcal{C}_U)$ is generated by the $[V]$ for locally closed subvarieties $V \subset U$ with relations $[U] = [V] + [W]$ for $W = U \smallsetminus V$ open with $0 < \sigma(W, L_W) < \sigma(U, L_U)$. ■

3.6.3. Example: arithmetic stratifications. An arithmetic stratification of U with respect to the line bundle L_U, in the sense of [BaMa90], consists of a descending sequence of Zariski open subvarieties $U \supset V_0 \supset V_1 \supset V_2 \cdots$, where V_{i+1} is the maximal open subset of V_i with $0 < \sigma(V_{i+1}, L_{V_{i+1}}) < \sigma(V_i, L_{V_i})$. Different choices of the line bundle give rise to different arithmetic stratifications. An arithmetic stratification defines a disjoint covering family in the assembler $\mathcal{C}_U$ by taking $X_i = V_i \smallsetminus V_{i+1}$.

The remaining two sections sketch two diverging paths leading from distribution of K–rational points $U(K)$ in U to various further versions of the arithmetic sieve $\mathcal{C}^{ar}(U, L_U)$.

In Section 4 we look at the more narrow class of varieties, Fano varieties, for which more precise data about behaviour of some heights are known, or at least, conjectured.

In Section 5 (restricted to $K = \mathbf{Q}$), we consider rational points as a subset of *adelic* points, and try to go beyond heights by using new tools for studying the distribution of adelic points $U(A_{\mathbf{Q}})$ themselves.

4. Anticanonical heights and points count

4.1. Anticanonical heights: dimension one. Let (U, L_U) be as above a pair consisting of a variety and ample line bundle defined over K, $[K : \mathbf{Q}] < \infty$. Choose a exponential height function h_L, and set for $B \in \mathbf{R}_+$

$$N(U, L_U, B) := \mathrm{card} \left\{ x \in U(K) \,\middle|\, h_L(x) \le B \right\}.$$

In Section 3, we based the definition of an arithmetical sieve upon an intuitive idea that $i_V : V \to U$ belongs to this sieve, *if the number of K–points on U is "considerably less" that such number on V.* To make this idea precise, we used convergence boundaries.

Below, we will use a considerably more precise counting of points, in order to define subtler sieves on a more narrow class of varieties U, using *counting functions themselves* $N(U, L_U, B)$ in place of convergence boundaries.

Start with one–dimensional U.

If U is a smooth irreducible curve of genus g, *with nonempty set* $U(K)$, we have the following basic alternatives:

$$g = 0 : \quad U = \mathbf{P}^1, \quad L_U = -K_V, \quad N(U, L_U, B) \sim cB.$$

$$g = 1 : \quad U = \text{ an elliptic curve with Mordell–Weil rank } r, \quad N(U, L_U.B) \sim c\,(\log B)^{r/2}.$$

$$g > 1 : \quad N(U, L_U, B) = const, \text{ if } B \text{ is big enough.}$$

A survey of expected typical behaviours of multidimensional analogs can be found in the Introduction to [FrMaTsch89].

Below, our attention will be focussed upon *Fano varieties*, that is, varieties with ample anticanonical bundle $\omega_V^{-1} = -K_V$, as a wide generalisation of the one–dimensional case $g = 0$.

In the case of $\mathbf{P}^n$ (the simplest Fano variety), it was shown in [Scha79] that

$$N(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), B) \sim c'\, B^{n+1} \quad \text{and} \quad N(\mathbf{P}^n, -K_{\mathbf{P}^n}, B) \sim c\, B,$$

where $-K_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(n+1)$. The constant c' here is expressible in terms of basic invariants of the number field K: the class number, the value $\zeta_K(n+1)$ of the Dedekind zeta function, the number of real and complex embeddings of K, the regulator, see [Scha79]. As mentioned in Section 2.3 above, for the line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$, the abscissa of convergence is $\sigma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) = n+1$ (see Section 1.3 of [BaMa90] and Section 4.1.1 of [Cham10]).

4.2. Anticanonical heights for Fano varieties. The most precise conjectural asymptotic formula for Fano varieties (or Zariski open subsets of them) with dense $U(K)$ has the form

$$N(U, -K_U, B) = cB(\log B)^t + O(B(\log B)^{t-1}), \quad t := \mathrm{rk}\,\mathrm{Pic}\,U - 1,$$

with $\mathrm{Pic}\,U$ the Picard group.

Note that in the case of $\mathbf{P}^n$ the Picard group is $\mathrm{Pic}\,\mathbf{P}^n \simeq \mathbf{Z}$, by $m \mapsto \mathcal{O}_{\mathbf{P}^n}(m)$, so that $t = 0$, but the $(\log B)^t$ term already occurs in the case of products of projective spaces.

This conjectural asymptotic formula certainly is *wrong* for many subclasses of Fano varieties. On the other hand, it is

(i) *stable under direct products;*

(ii) *compatible with predictions of Hardy–Littlewood for complete intersections;*

(iii) *true for quotients of semisimple algebraic groups modulo parabolic subgroups.*

We reproduce below some of the arguments of [FrMaTSCH89], Sections 1–2, proving these statements.

4.2.1. Direct products. Let (U, L_U) and (V, L_V) be data as above. We must study the behaviour of $N(U \times V, L_{U\times V})$ where $L_{U\times V} := pr_U^*(L_U) \otimes pr_V^*(L_V)$.

In this context, we may restrict ourselves by consideration of exponential heights satisfying the exact equality

$$h_{L_{U\times V}}(x, y) = h_{L_U}(x) h_{L_V}(y).$$

This will be applied to the case of anticanonical heights.

We will now change notation without referring anymore to the specical properties of heights on Fano varieties etc, as in [FrMaTsch89].

Consider two infinite families of nondecreasing real numbers indexed by $1, 2, \dots :$ $\{\lambda_i\}$ and $\{\mu_j\}$. We allow each number to be repeated several times, so that they can have finite mulltiplicities. We can then form a new family: $\lambda\mu := \{\lambda_i\mu_j\}$ again ordered nondecreasingly. Put $N_\lambda(B) := \mathrm{card}\,\{i \,|\, \lambda_i < B\}$, and similarly for $N_\mu, N_{\lambda\mu}$.

Now "stability under the direct product" from 4.2 is a consequence of the following (see Proposition 2 of [FrMaTsch89]).

4.2.2. Lemma. *If*

$$N_\lambda(B) = c_\lambda B\log^s B + O(B\log^{s-1} B),$$

$$N_\mu(B) = c_\mu B\log^r B + O(B\log^{r-1} B),$$

then

$$N_{\lambda\mu}(B) = C\, c_\lambda c_\mu B\log^{r+s+1} B + O(B\log^{r+s} B),$$

where the constant $C = C(r,s)$ is the Euler beta–function.

Proof. Directly from the definitions and assumptions, we get

$$N_{\lambda\mu}(B) = \sum_{i=1}^{N_\lambda(B/\mu_i)} N_\mu(B/\lambda_i) = c_\mu \sum_{i=1}^{N_\lambda(B/\mu_i)} B/\lambda_i \log^r(B/\lambda_i)$$

$$+O(\sum_{i=1}^{N_\lambda(B/\mu_i)} B/\lambda_i \log^{r-1}(B/\lambda_i)) \tag{4.1}$$

Since the error term has the same structure as the main one, with r replaced by $r-1$, we can apply this formula inductively, and get the following expression for the main term

$$c_\lambda c_\mu B \sum_{j=1}^{N} a(j) B/j \log^r(B/j) \tag{4.2}$$

where

$$a(j) := \mathrm{card}\,\{i \,|\, \lambda_1 + j \le \lambda_i < \lambda_1 + j + 1\},\ N := [B/\mu_1 - \lambda_1] + 1.$$

So the main term of (4.1) can be rewritten as

$$c_\lambda c_\mu B \sum_{j=1}^{N} j \log^r j \int_j^{j+1} x^{-2} \log^s (B/x) dx.$$

After approximating the sum by the integral, we get the expected result. ■

4.2.3. Hardy–Littlewood method and complete intersections. Below we will sketch, following [Ig78], methodology and results of applications of the Hardy–Littlewood method in the setup of Fano complete intersections in projective spaces, explained in [FrMaTsch89]. Besides showing the "typical" asymptotic behaviour for some of them, they can serve as an example of study of distribution of rational points in *adelic spaces*. Later we will survey some recent developments in this direction, cf. Section 5 below.

Let A_K be the ring of adèles of a number field K and let X be an affine variety over K. The adèlic Hardy–Littlewood formula is the identity

$$\sum_{x \in X(K)} \varphi(x) = \int_{X(A_K)} \varphi(x)\omega(x) + R(\varphi),$$

where ω is a measure on $X(A_K)$ and φ is in a class of test functions on $X(A_K)$. The integral $P(\varphi) = \int_{X(A_K)} \varphi(x)\omega(x$ is the *singular series* and R the error term, see Section 1.3 of [FrMaTsch89].

Let $\mathbf{P}^n$ be a projective space with a fixed system of homogeneous coordinates $(x_0 : x_1 : \cdots : x_n)$ as in 2.1 above, K a base field of finite degree over $\mathbf{Q}$. Consider $m \geq 1$ forms $f_i \in K[x_0, \ldots, x_n]$ and put $d_i := \deg f_i$. Denote by V the projective variety over K which is the nonsingular complete intersection of hypersurfaces $f_i = 0$. Then its anticanonical line bundle is

$$-K_V = \omega_V^{-1} = \mathcal{O}_V(n + 1 - d_0 - \cdots - d_m).$$

Therefore V is a Fano variety iff $n + 1 > d_0 + \cdots + d_m$.

Let W be a cone over the projective complete intersection V, For some $\tau > 0$ and $\tilde{\tau}_v := \tau^{[K_v:\mathbf{Q}_v]/[K:\mathbf{Q}]}$, and for a test function φ on $W(A_K)$, set $\varphi_\tau(x) = \varphi(\tilde{\tau}^{-1}x)$. If $P(\varphi)$ converges then

$$P(\varphi_\tau) = \tau^{n+1-\sum_i d_i} P(\varphi_1),$$

as shown in Proposition 4 of [FrMaTsch89]. If φ_1 is the restriction to $W(A_K)$ of the characteristic function of a product of balls at the infinite places and the v-adic integers at finite places, then the sum $\sum_{x\in W_0(K)} \varphi_1(x)$, with $W_0 = W \smallsetminus \{0\}$, provides a counting of K-points of W_0, which in the case $K = \mathbf{Q}$ agrees with the counting of rational points of V with $\mathcal{O}(1)$-height bounded above by τ.

Finally, how summation over points may be approximated by integration over adelic spaces, as in the case of Proposition 4 of [FrMaTsch89] recalled above, is described in Section 5 below.

4.2.4. Generalised flag manifolds. The last class of Fano varieties we considered in [FrMaTsch89] consists of generalised flag manifolds $V = P\backslash G$, where G is a semisimple linear algebraic group, and P is a parabolic subgroup, both defined over K. For convenience, we assumed moreover that P contains a fixed minimal parabolic subgroup P_0. Cf. also [BlBrDeGa90].

Denote by $\pi : G \to V$ the canonical projection. Let $\mathcal{X}^*(P)$ be the group of characters of P defined over K. Each character $\chi \in \mathcal{X}^*(P)$ defines a line bundle on V that we will denote L_χ. Its local sections come from those local functions on G upon which the left multiplication by $p \in P$ lifts to the multiplication by $\chi(p)$.

The anticanonical line bundle is located among the L_χ. It is denoted $L_{-2\rho}$ in Section 2 of [FrMaTsch89].

Now comes the main part of the construction: explicit description of the anticanonical height. Let A_K be the adele ring of K. There exists a maximal compact subgroup $\mathcal{K} = \prod_v \mathcal{K}_v \subset A_K$ such that $G(A_K) = P_0(A_K)\mathcal{K}$, where P_0 is the fixed minimal parabolic subgroup. Over an open set W we write sections as

$$\Gamma(W, L_\chi) = \left\{ f \in \Gamma(\pi^{-1}(W), \mathcal{O}_G) \,\middle|\, f(pg) = \chi(p)f(g),\ \forall p \in P,\, g \in G \right\}.$$

If s is a local section of L_χ on a neighborhood W of a point $x \in V(K_v)$, we refer to f as above as the corresponding local function. Each bundle $L_\chi \otimes F_v$ on $V \otimes K_v$ is endowed by a canonical K_v–invariant v–adic norm: if s is a local section of L_χ with local function f, we choose $k \in K_v$ with $\pi(k) = x$ and put $|s|_v = |f(k)|_v$. We define the height function by

$$h_\chi = h_{L_\chi} := \prod_v |s|_v^{-1}.$$

Using this description, we can identify the anticanonical height zeta–function of V with one of the Eisenstein series from [La76], Appendix II. The analytic properties of these Eisenstein series are then used to establish the asymptotic formula for the height zeta function. We omit further details and we refer the reader to Section 2, Theorem 5 and subsequent corollary in [FrMaTsch89].

4.3. Heights with respect to more general line bundles L_U on Fano varieties. In [BaMa90] it was suggested that only a slight generalisation of formula in 4.2 should be "typical" (in the sense of being the expected behavior when the previous formula fails, although in fact it is valid in a much more restricted set of cases):

$$N(U, L_U, B) \sim cB^{\beta}(\log B)^t, \quad t := \mathrm{rk}\,\mathrm{Pic}\,U - 1.$$

As was argued in [BaMa90], β should be defined by the relative positions of L_U and $-K_U$ in the cone of pseudo–effective divisors of U: see precise conjectures there.

Sh. Tanimoto in [Ta19] provided arguments proving various inequalities for these numbers related to the conjectures in [BaMa90].

Finally, the subtlest information about such asymptotics is given by several conjectures and proofs regarding exact value of the constant c.

4.4. From asymptotic formulas to sieves. If we restrict ourselves by those V for which we can define a *Grothendieck topology, whose objects satisfy strong asymptotic formulas discussed above*, or their weaker versions, as stated at the beginning of Section 4.2 and Section 4.3, then we can try to define sieves in it by some inequalities weakening that in Section 3.6, such as conditions given directly in terms of the number of points of bounded height, such as

$$N(V, L_V, B)/N(U, L_U, B) = o(1)$$

or in terms of asymptotics, such as, for $\beta(U, L_U) = \beta(V, L_V)$,

$$t(V, L_V) < t(U, L_U),$$

where $t(V, L_V)$ and $\beta(V, L_V)$ refer to the asymptotic formula mentioned at the beginning of Section 4.3.

Along the lines of Section 3.6, we can then construct other assemblers $\mathcal{C}_U$ with objects given by locally closed subvarieties V and morphisms given by inclusions $W \hookrightarrow V$, where the condition $0 < \sigma(W, L_W) < \sigma(V, L_V)$ is now replaced by one of the conditions listed here above.

Moreover, we can use conditions on the number of points of bounded height, such as those describing accumulating subvarieties in Example 2.3.1, to obtain an assembler whose Grothendieck group detects these subvarieties, as follows.

4.4.1. Proposition. *Let $\mathcal{C}^s_U$ and $\mathcal{C}^w_U$, be the categories where objects are, in both cases, the locally closed subvarieties $V \subset U$ and morphisms are finite compositions of open embeddings and closed embeddings, where, in the case of $\mathcal{C}^s_U$ the closed embeddings $i_{W,V} : W \hookrightarrow V$ satisfy*

$$\frac{N(W, L_W, B)}{N(V, L_V, B)} \longrightarrow 1, \quad \textit{for } B \to \infty,$$

while in the case of $\mathcal{C}^w_U$ the closed embeddings $i_{W,V} : W \hookrightarrow V$ satisfy

$$\operatorname{liminf}_B \frac{N(W, L_W, B)}{N(V, L_V, B)} > 0.$$

The categories $\mathcal{C}^s_U$ and $\mathcal{C}^w_U$ are assemblers, and the associated Grothendieck groups are generated by classes $[V]$ of subvarieties of U with the relations $[U] = [V]+[U \smallsetminus V]$ where V is a strongly or weakly accumulating subvariety, in the case of $\mathcal{C}^s_U$ or $\mathcal{C}^w_U$, respectively.

Proof. The argument is exactly the same as in Proposition 3.6.2. We just need to verify, as in Lemma 3.5.1, that the composition $W' \subset W \subset V$ of closed embeddings that satisfy one of the two conditions above still satisfies the same condition. This is clear both for the first case where one still has convergence to 1 of the product

$$\frac{N(W', L_{W'}, B)}{N(W, L_W, B)} \cdot \frac{N(W, L_W, B)}{N(V, L_V, B)}$$

and for the second case where, since all the terms are non-negative, one has

$$\operatorname{liminf}_{\mathrm{B}} \frac{N(W', L_{W'}, B)}{N(V, L_V, B)} \geq \operatorname{liminf}_B \frac{N(W', L_{W'}, B)}{N(W, L_W, B)} \cdot \operatorname{liminf}_B \frac{N(W, L_W, B)}{N(V, L_V, B)} > 0.$$

■

5. Sieves "beyond heights" ?

5.1. Thin sets and Tamagawa measures. Below we survey recent attempts to define the geometry of subsets of rational points of $V(K)$, containing "considerably less" points than V.

From our viewpoint, these definitions should also be tested on compatibility with our philosophy of sieves and assemblers.

Below, we adopt the framework of [Pe18], which was further studied by W. Sawin ([Sa20], under the additional restriction $K = \mathbf{Q}$). His paper starts with Conjecture 1.1, called "Modern formulation of Manin's conjecture". It involves both shifts from and extensions of the setup in our previous Section 4.

We first need to define the following objects/quantities, in order to state this modern formulation:

(i) *Summation* over points x of height $\leq B$ is replaced by the *averaging* of the measures δ_x of the same points embedded into the adelic space $V(A_{\mathbf{Q}})$.

(ii) The definition of the respective Tamagawa measure τ assumes that V is a geometrically integral smooth projective Fano variety, with Picard group of rank r. Let $\mathcal{V}$ be its proper integral model over $\mathbf{Z}$. The L-function $L(s, Pic\,V_{\overline{\mathbf{Q}}})$ defined by the Picard group and the respective local zetas L_v are zetas of lattices (in the sense that they are zeta functions associated to Artin (Galois) representations on lattices), see Construction 3.28 in [Pe18].

Then

$$\tau := (\lim_{s\to 1}(s-1)^r L(s, Pic\,V_{\overline{\mathbf{Q}}}))\prod_v L_v(s, Pic\,V_{\overline{\mathbf{Q}}})^{-1}\omega_v.$$

Here ω_v is defined by the natural measure on local non–archimedean points of V or archimedean volume form.

(iii) Finally, define the numbers $\alpha(V)$ and $\beta(V)$ by

$$\alpha(V) := r\,\mathrm{vol}\,\{y \in ((\mathrm{Pic}(V)\otimes\mathbf{R})^{eff})^\vee \,|\, K_V\cdot y \leq 1\},$$

and

$$\beta(V) := \mathrm{card}\,H^1(\mathrm{Gal}(\overline{\mathbf{Q}/\mathbf{Q}}), \mathrm{Pic}\,V_{\overline{\mathbf{Q}}}).$$

Then the modern formulation of the conjecture on the number of rational points of bounded height on a Fano variety, according to [Sa20] can be stated as follows.

Let $f : U \to V$ be a morphism of geometrically integral smooth projective varieties. Call it a *thin morphism*, if the induced map $U \to f(U)$ is generically finite of degree $\neq 1$.

The modern formulation of the Manin conjecture can then be stated in the following way (Conjecture 1.1 of [Sa20]).

5.2. Conjecture. *There exists a finite set of thin maps $f_i : Y_i \to X$, with W the complement of the union of the sets $f_i(Y_i(\mathbf{Q}))$, such that we have an exact formula for the weak limit of the form*

$$\lim_{B\to\infty} \frac{1}{B(\log B)^{r-1}} \sum_{\substack{x\in W(\mathbf{Q}) \\ H(x)<B}} \delta_x = \alpha(V)\beta(V)\tau^{Br},$$

where τ^{Br} is the restriction of the Tamagawa measure on the subset of $V(A_{\mathbf{Q}})$ on which the Brauer–Manin obstruction vanishes.

We will review the Brauer–Manin obstruction, along with other related material, in Section 6.1, where we also refine the general question proposed in this section.

For a class of varieties for which this conjecture is valid, we obtain interesting new possibilities for defining sieves. Thus, we can formulate this as a more general open ended question.

5.3. Question. *Is there a natural construction of sieves and assemblers based on subvarieties satisfying the conditions mentioned above? Such assemblers would be categorifications (with associated spectrifications) of these possible ways of expressing the notion of having asymptotically "fewer rational points than the ambient variety", without relying directly on the height zeta function. What information can be obtained from the homotopy invariants of the spectra of such assemblers?*

Regarding Brauer–Manin obstruction itself, see the monograph [CThSk19] and many references therein, in particular, [Sko95], [Sko09]. This obstruction cannot explain many cases of missing rational points even when the initial obstruction for them vanishes, and it was generalised and/or modified in different ways many times: see in particular [LeSeTa18], [Ta19], [DePi19]. It is worth mentioning, in particular, Skorobogatov's étale Brauer–Manin obstruction, [CoPaSk16], [Sko99], [Sko09].

The recent article [CorSch20] introduces interesting contexts that are very suitable for considering generalised obstructions from the viewpoint of sieves (see also

[LiXu15] and many other references in these papers). We dedicate the next Section 6 to a description of these contexts.

Finally, we should mention the related but not identical machinery for studying varieties with finite but non–empty sets of K–points. For this, we refer the reader to [Qu15].

6. Obstructions and sieves

6.1. "Invisible" varieties. Let V and U be two objects of a category with unique initial object $\emptyset$. Then we will say that V is *"U–invisible"*, if $\mathrm{Hom}(U, V) = \emptyset$.

The main setup we will have in mind is the category of K–schemes, where K is a number field. The Spec$K$-invisible varieties over K are exactly those with no K-points. Then if $U = \mathrm{Spec}\, L$, where L is a commutative K–algebra, in place of "U–invisibility" we will sometimes speak about "L–invisibility". Hopefully, this will not lead to a confusion. In particular, we will often speak about K–invisible varieties over K, and also A_K–invisible ones, where A_K is the ring of adèles.

K–invisible varieties are the objects of a category $\mathcal{X}_K$, whose morphisms are all morphisms of K–varieties (or a subfamily of them closed under composition.)

The local–global principle for a class of K–varieties X is the statement that $X(A_K) \neq \emptyset$ implies $X(K) \neq \emptyset$, where A_K is the ring of adèles of K. Thus, such a variety can be K–invisible only if it has no adèlic points. (Formerly this was called *the Hasse principle*).

The *Brauer–Manin obstruction* to the local–global principle is provided by the definition of a set $X(A_K)^{\mathrm{Br}}$ such that $X(K) \subseteq X(A_K)^{\mathrm{Br}} \subseteq X(A_K)$. One says that failure of the local–global principle for X *is explained by this obstruction,* if $X(A_K) \neq \emptyset$, but $X(A_K)^{\mathrm{Br}} = \emptyset$.

The first remarkable new result of [CorSch20] is the following theorem (Corollary 1.1 on p. 6 of [CorSch20]):

6.1.1. Theorem. *Assume that K is either totally real, or imaginary quadratic field. Then for any K–invisible K–variety V there exists its Zariski open covering $X = \cup_i U_i$ such that invisibility of each U_i is explained by the Brauer–Manin obstruction.*

The notions of "sieves", "assemblers" etc. do not appear explicitly in [CorSch20], and there are no references to the papers of I. Zakharevich there. However, Theorem 6.1.1. leaves no doubt that the setup of sieves is essential for the better understanding of their work.

Some details and explanations follow.

6.2. Functor Obstructions. (Sec. 8.1 of [Po17]). Corwin and Schlank work with definitions of obstructions, that were systematically developed in [Po17]. Denote by Sch_K, resp. Var_K, the category of K–schemes, resp. algebraic K–varieties. Let F be a contravariant functor $Sch_K \to Sets$ (presheaf of sets on the category of K–schemes).

If $V = \operatorname{Spec} L$ where L is a commutative K–algebra, we will write $V(L)$, resp. $F(L)$, in place of $V(\operatorname{Spec} L)$, resp. $F(\operatorname{Spec} L)$, so that with this notation, F becomes a *covariant* functor of its argument L.

6.2.1. Definition. *Let ω be a subfunctor of the functor $F : Var_K \to Sets$.*

We will say that ω is a generalised obstruction to the Hasse principle, if $V(K) \subseteq \omega(V(A_K))$ for all K–varieties V.

We will often write $V(A_K)^\omega$ in place of $\omega(V(A_K))$, as in the notation for the Brauer–Manin obstruction above.

From the definitions, one easily sees that if V is a proper variety, then

$$V(A_K) = V(\prod_v K_v) = \prod_v V(K_v).$$

Generally, we have only inclusion

$$V(A_K) \subseteq V(\prod_v K_v) = \prod_v V(K_v).$$

Let now F be a contravariant functor from Sch_K to $Sets$, and L a K–algebra.

In [Po17], pp. 227– 228, it is shown how to define for each $A \in F(V)$, and each K–variety V, the "evaluation" map

$$ev_A : V(L) \to F(L).$$

Namely, an L–point $x \in V(L)$ considered as morphism of K–schemes, induces a map $F(V) \to F(L)$. The image of x in $F(L)$ is denoted $ev_A(x)$, or simply $A(x)$.

From the definitions, it follows that we have a commutative diagram

$$\begin{tikzcd} V(K) \arrow[r] \arrow[d, "ev_A"] & V(A_K) \arrow[d, "ev_A"] \\ F(K) \arrow[r] & \prod_v F(K_v) \end{tikzcd}$$

The upper horizontal arrow is an embedding. Denote by $V(A_K)^A$, for $A \in F(V)$, the subset of $V(A_K)$ consisting of elements whose image in $F(A_K)$ lies in the image of $F(K)$.

Finally, put

$$V(A_K)^F := \cap_{A\in F(V)} V(A_K)^A.$$

Clearly, $V(K) \subseteq V(A_K)^F$.

We can similarly define the respective subsets in $V(\prod K_v)) = \prod V(K_v)$ and, using $\prod F(K_v)$, define $V(\prod K_v)^F$.

6.2.2. Definition. *If*

$$V(\prod K_v) \neq \prod V(K_v)^F := \prod F(K_v),$$

we will say that this inequality is explained by an F–obstruction for weak approximation.

In the proof of Theorem 6.1.1 above, Corwin and Schlank use also versions of these definitions involving adèles with restricted denominators. We will briefly sketch parts of their arguments below.

Using notations of Subsection 2.1 above, choose a subset of places $S \subset \Omega_K$, and denote (as in Section 1.1 of [CorSch20]) by $A_{K,S}$ the restricted product $A_{K,S} = \prod'_{v\in S} K_v$ where the components belong to $\mathcal{O}_v$ for all but finitely many $v \in S$. In particular, one considers the case where $S = \Omega_{K,f}$ is the set of finite places.

Generally, a triple (K, S, ω) as above is called *an obstruction datum* in Definition 2.3 of [CorSch20].

6.2.3. Definition. ([CorSch20], Def. 2.4). *A K–variety V satisfies VSA (very strong approximation principle) for (K, S, ω), if $V(K) = V(A_{K,S})^\omega$.*

As in Sec. 4.2 above, we can check that the family of K–varieties satisfying VSA for (K, S, ω) (or simply VSA–varieties) is closed with respect to:

(i) *direct products;*

(ii) *passage to locally closed subvarieties.*

6.3. Theorem. ([CorSch20], Theorem 4.3.) *Assume that there is an open subvariety of $\mathbf{P}^1_K$ satisfying VSA for (K, S, ω).*

Then for any K–variety V there exists a finite affine open cover $V = \cup_i U_i$ such that each U_i satisfies VSA for (K, S, ω).

Sketch of proof. Any closed point of V belongs to an open affine subvariety of V. Because of stability of VSA varieties with respect to direct products and locally closed embeddings, we can find an affine neighbourhood of each closed point satisfying VSA, and then to choose from all such neighbourhoods a finite open cover. ■

7. Assemblers and spectra for Grothendieck rings with exponentials

Starting with this section, we investigate the homotopy theoretic framework underlying another aspect of the height function and the associated height zeta function, in the framework of the *motivic height zeta function* introduced by Chambert-Loir and Loeser in [ChamLoe15]. In this section we focus on the relevant Grothendieck rings, namely the *Grothendieck rings with exponentials* defined in [ChamLoe15] and their localizations. We construct associated assembler categories and homotopy theoretic spectra, in the sense of [Za17a], [Za17b], [Za17c].

7.1. Localization of the Grothendieck ring. Consider the localization $\mathcal{M}_K$ of the Grothendieck ring of varieties $K_0(\mathcal{V}_K)$ obtained by inverting the Lefschetz motive $\mathbf{L} = [\mathbf{A}^1]$ and all $\mathbf{L}^n - 1$ for $n \geq 1$. This localization can be identified with the Grothendieck ring of algebraic stacks $K_0(\mathcal{S}_K)$ (see [BeDh07], [Ek09]) by expressing it as the localization with respect to the classes

$$[GL_n] = (\mathbf{L}^n - 1)(\mathbf{L}^n - \mathbf{L}) \cdots (\mathbf{L}^n - \mathbf{L}^{n-1}).$$

The Grothendieck ring of stacks $K_0(\mathcal{S}_K)$ is generated as a group by the isomorphism classes $[X]$ of algebraic stacks X of finite type over the field K with the property that all their automorphism group schemes are affine, modulo relations

$$[X] = [Y] + [U]$$

for a closed substack Y and its complement U, and

$$[E] = [X \times \mathbf{A}^n]$$

for a vector bundle $E \to X$ of rank n. The multiplication is given, as in the case of varieties, by the product

$$[X \times Y] = [X] \cdot [Y].$$

There is a natural map $K_0(\mathcal{V}_K) \to K_0(\mathcal{S}_K)$. The classes $[GL_n]$ are invertible in $K_0(\mathcal{S}_K)$. This can be seen by first observing that, if $X \to Y$ is a GL_n-torsor of algebraic stacks of finite type over K, then $[X] = [GL_n] \cdot [Y]$ in $K_0(\mathcal{S}_K)$. This fact follows from the relation $[E] = \mathbf{L}^n \cdot [S]$ for rank n vector bundles over S, repeatedly applied to $X = X_n \to X_{n-1} \to \cdots \to Y$ with X_k the stack of k linearly independent vectors in the vector bundle determined by X, and viewing each $X_k \to X_{k-1}$ as a complement of a vector subbundle in a vector bundle (see Proposition 1.1 of [Ek09]). The invertibility then follows by considering the GL_n-torsor $Spec(K) \to BGL_n$ that gives $1 = [GL_n] \cdot [BGL_n]$ in $K_0(\mathcal{S}_K)$.

This implies that the map $K_0(\mathcal{V}_K) \to K_0(\mathcal{S}_K)$ induces a map from the localization

$$\mathcal{M}_K \to K_0(\mathcal{S}_K).$$

In order to produce the inverse map, one assigns to the stack given by a global quotient $[X/GL_n]$, the class $[X]/[GL_n]$ in $\mathcal{M}_K$, where the resulting class is independent of the description of initial one as a global quotient, because for the general case one can stratify a stack by global quotients and add the resulting classes (see Theorem 1.2 of [Ek09]).

Observe also that the second relation $[E] = \mathbf{L}^n[Y]$ in $K_0(\mathcal{S}_K)$ for a rank n vector bundle $E \to Y$ is in fact equivalent to assuming the relation $[X] = [GL_n] \cdot [Y]$ for a GL_n-torsor $X \to Y$. One direction was explained above, while in the other direction it suffices to observe that one can realize a rank n vector bundle $E \to Y$ as $E = V \times_{GL_n} X$, where V is an n-dimensional vector space and $X \to Y$ is a GL_n-torsor. The quotient map

$$V \times X \to V \times_{GL_n} X$$

is also a GL_n-torsor, hence we have

$$[GL_n] \cdot [E] = [V] \cdot [X] = [V] \cdot [GL_n] \cdot [Y],$$

which gives $[E] = \mathbf{L}^n[Y]$ since the class $[GL_n]$ is invertible.

Notice, moreover, that it suffices to consider the class $\mathcal{S}^{\mathcal{Z}}$ of stacks where the automorphism group schemes in the class $\mathcal{Z}$ are connected finite type group schemes for which torsors over any finitely generated extension of K are trivial. Indeed, it is shown in Proposition 1.4 of [Ek09] that $K_0(\mathcal{S}_K) = K_0(\mathcal{S}_K^{\mathcal{Z}})$, and that for G a group scheme in $\mathcal{Z}$ any G-torsor $X \to Y$, with $X, Y \in \mathcal{V}_K$, has a stratification where the torsor is trivial on each stratum, so that $[X] = [G][Y] \in K_0(\mathcal{V}_K)$. If K is an algebraically closed field, the group schemes in $\mathcal{Z}$ are affine and with the property that $GL_n \to GL_n/G$ is a Zariski locally trivial fibration ([Ek09], Remark after Proposition 1.4).

We can define an assembler category underlying the localization $\mathcal{M}_K = K_0(\mathcal{S}_K)$ and the associated homotopy theoretic spectrum in the following way.

7.2. Proposition. *Let $\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}$ be the category whose objects are algebraic stacks X of finite type over the field K with automorphism group schemes in $\mathcal{Z}$, and whose morphisms are locally closed embeddings. A Grothendieck topology on $\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}$ is generated by the families*

$$\{Y \hookrightarrow X, U \hookrightarrow X\}$$

where $Y \hookrightarrow X$ a closed substack and U its complement. The category $\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}$ is an assembler, and the associated spectrum $K\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}$ satisfies the relation

$$\pi_0(K\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}) = K_0(\mathcal{S}_K^{\mathcal{Z}}).$$

Proof. The proof that this is an assembler category uses similar arguments as in [Za17a]: the empty set is the initial object; finite disjoint covering families are given by

$$X_i \hookrightarrow X$$

with

$$X_i = Y_i \smallsetminus Y_{i-1}$$

for a chain of embeddings

$$\emptyset = Y_0 \hookrightarrow Y_i \hookrightarrow \cdots \hookrightarrow Y_n = X.$$

Any two finite disjoint coverings have a common refinement because the category has pullbacks, as in [Za17a].

The spectrum $K\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}$ determined by the assembler $\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}$ has $\pi_0(K\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}})$ generated by the objects of $\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}$ with the scissor-congruence relations determined by the disjoint covering families. These include the relations $[X] = [Y] + [U]$ for closed substacks $Y \hookrightarrow X$ and their complements $U \hookrightarrow X$, as well as the relations $[X] = [GL_n][Y]$ for GL_n-torsors $X \to Y$ through the existence of a disjoint covering family determined by a stratification where the torsor is trivial on each stratum as discussed above. Thus we obtain that $\pi_0(K\mathcal{C}^{\mathcal{S}_K^{\mathcal{Z}}}) = K_0(\mathcal{S}_K^{\mathcal{Z}})$. ■

7.3. Grothendieck rings with exponentials. The *Grothendieck ring with exponentials* of [ChamLoe15] is designed to describe a motivic version of exponential sums of the form

$$\sum_{x \in X(\mathbf{F}_q)} \chi(f(x)),$$

where X a variety over a finite field, endowed with a function $f : X \to \mathbf{A}^1$ and a fixed non-trivial character $\chi : \mathbf{F}_q \to \mathbf{C}^*$.

In fact, we will see in Section 7.8 that this heuristics can be made precise, in the sense that assigning such an exponential sum to a class in the Grothendieck ring with exponentials defines a ring homomorphism.

In the formulation of [ChamLoe15], the Grothendieck ring with exponentials $KExp_K$ is defined as follows.

7.3.1. Definition. *The Grothendieck ring with exponentials $KExp_K$ is generated by isomorphism classes of pairs (X, f) of a K-variety X and a morphism $f : X \to \mathbf{A}^1$, where two such pairs (X_1, f_1) and (X_2, f_2) are isomorphic if there is an isomorphism $u : X_1 \to X_2$ of K-varieties such that $f_1 = f_2 \circ u$. The relations are given by*

$$[X, f] = [Y, f|_Y] + [U, f|_U],$$

for a closed subvariety $Y \hookrightarrow X$ and its open complement U, and

$$[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}] = 0$$

where $\pi_{\mathbf{A}^1} : X \times \mathbf{A}^1 \to \mathbf{A}^1$ is the projection on the second factor. The ring structure is given by the product

$$[X_1, f_1] \cdot [X_2, f_2] = [X_1 \times X_2, f_1 \circ \pi_{X_1} + f_2 \circ \pi_{X_2}]$$

where $f_1 \circ \pi_{X_1} + f_2 \circ \pi_{X_2} : (x_1, x_2) \mapsto f_1(x_1) + f_2(x_2)$.

The relation $[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}] = 0$ corresponds to the fact that, for an exponential sum as above, one has $\sum_{a \in \mathbf{F}_q} \chi(a) = 0$.

There is an embedding of the ordinary Grothendieck ring of varieties $K_0(\mathcal{V}_K)$ in the Grothendieck ring with exponentials $KExp_K$ that maps $[X] \mapsto [X, 0]$. This map is compatible with localizations and it induces an embedding of the localization $\mathcal{M}_K$ in the corresponding localization $Exp\mathcal{M}_K$ of $KExp_K$.

Given the identification $\mathcal{M}_K \simeq K_0(\mathcal{S}_K)$ of the localization $\mathcal{M}_K$ of the Grothendieck ring of varieties and the Grothendieck ring of algebraic stacks, one can similarly obtain a description of the corresponding localization $Exp\mathcal{M}_K$ of $KExp_K$ of the Grothendieck ring of varieties with exponentials in terms of a *Grothendieck ring of algebraic stacks with exponentials.*

7.3.2. Definition. *The Grothendieck ring of algebraic stacks with exponentials* $K_0(Exp\mathcal{S}_K)$ *is generated by the isomoprhism classes* $[X, f]$ *of pairs* (X, f) *of an algebraic stack* X *and a morphism* $f : X \to \mathbf{A}^1$*, where an isomorphism* $(X_1, f_1) \simeq (X_2, f_2)$ *is given by an isomorphism* $u : X_1 \to X_2$ *with* $f_2 \circ u = f_1$*. The relations in* $K_0(Exp\mathcal{S}_K)$ *are given by* $[X, f] = [Y, f|_Y] + [U, f|_U]$ *for a closed substack* Y *and its open complement;* $[E, f] = \mathbf{L}^n \cdot [S, f_S]$ *for* $\pi : E \to S$ *a vector bundle and* $f : E \to \mathbf{A}^1$ *with* $f_S \circ \pi = f$*, with* $\mathbf{L}^n = [\mathbf{A}^n, 0]$*; and with the further relation* $[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}] = 0$*. The ring structure is defined as in* $KExp_K$*.*

The argument of [Ek09] recalled above, for the identification of $K_0(\mathcal{S}_K)$ with the localization $\mathcal{M}_K$ of $K_0(\mathcal{V}_K)$ extends to the case with exponentials.

7.3.3. Lemma. *The natural morphism* $KExp_K \to K_0(Exp\mathcal{S}_K)$ *factors through the localization* $Exp\mathcal{M}_K$ *and induces an isomorphism* $Exp\mathcal{M}_K \simeq K_0(Exp\mathcal{S}_K)$*.*

Proof. As in the original argument of [Ek09], the vector bundle relation implies that, if $X \to Y$ is a GL_n-torsor, endowed with compatible maps f_X, f_Y to $\mathbf{A}^1$, then $[X, f_X] = [GL_n] \cdot [Y, f_Y]$. The invertibility of $[GL_n] = [GL_n, 0]$ follows from its invertibility in $K_0(\mathcal{S}_K)$. This implies that the homomorphism $KExp_K \to K_0(Exp\mathcal{S}_K)$ factors through a homomorphism $Exp\mathcal{M}_K \to K_0(Exp\mathcal{S}_K)$. To construct the inverse map we assign to a class $[X/GL_n, f]$, where X/GL_n is a global quotient, the class $[GL_n]^{-1} \cdot [X, \tilde{f}]$ where $\tilde{f} = f \circ \pi$ for the quotient map $\pi : X \to X/GL_n$. This map extends an arbitrary $[X, f]$ by stratifying the stack X by global quotients with the restriction of the morphism f to the strata. ∎

We can then consider assembler and homotopy theoretic spectra associated to these Grothendieck rings. This can be done using the setting of [Za17a] for sub-assemblers and cofiber sequences.

7.4. Theorem. *A simplicial assembler $\mathcal{C}_K^{KExp}$ with $\pi_0 K(\mathcal{C}_K^{KExp}) = KExp_K$, the Grothendieck ring with exponentials, is obtained as the cofiber of the morphism of assemblers $\Phi : \mathcal{C} \to \mathcal{C}$, where the objects of $\mathcal{C}$ are pairs (X, f) of a K-variety X and a morphism $f : X \to \mathbf{A}^1$, and morphisms are locally closed embeddings of subvarieties with compatible morphisms to $\mathbf{A}^1$. Finally,*

$$\Phi(X, f) := (X \times \mathbf{A}^1, f \circ \pi_X + \pi_{\mathbf{A}^1}).$$

Proof. Endow the category $\mathcal{C}$ with the Grothendieck topology generated by the families

$$\{(Y, f|_Y) \hookrightarrow (X, f), (U, f|_U) \hookrightarrow (X, f)\}$$

where $Y \hookrightarrow X$ is a closed subvariety, and U its open complement. The category $\mathcal{C}$ is an assembler, since the category has pullbacks, hence finite disjoint families have a common refinement, and morphisms are compositions of embeddings and therefore monomorphisms. The associated spectrum $K(\mathcal{C})$ has $\pi_0 K(\mathcal{C})$ generated by the isomorphism classes $[X, f]$ with relations

$$[X, f] = [Y, f|_Y] + [U, f|_U].$$

Consider then the endofunctor $\Phi : \mathcal{C} \to \mathcal{C}$ that assigns to an object (X, f) the object

$$\Phi(X, f) = (X \times \mathbf{A}^1, f \circ \pi_X + \pi_{\mathbf{A}^1})$$

for $\pi_X, \pi_{\mathbf{A}^1}$ the projections onto the two factors of $X \times \mathbf{A}^1$. This functor is a morphism of assemblers and the induced map on $\pi_0 K(\mathcal{C})$ is given by multiplication by the class $[\mathbf{A}^1, id]$. Note that this is not the multiplication by the Lefschetz motive, as in this setting $\mathbf{L} = [\mathbf{A}^1, 0]$. We introduce the notation $\mathbf{Y} := [\mathbf{A}^1, id]$ and we write the map on $\pi_0 K(\mathcal{C})$ as $\cdot \mathbf{Y}$.

By the localization theorem (Theorem C of [Za17a]), the morphism $\Phi : \mathcal{C} \to \mathcal{C}$ of assemblers (seen as a morphism between constant simplicial assemblers) has an associated simplicial assembler $\mathcal{C}/\Phi$ with a morphism $\iota : \mathcal{C} \to \mathcal{C}/\Phi$ that gives a cofiber sequence

$$K(\mathcal{C}) \xrightarrow{K(\Phi)} K(\mathcal{C}) \xrightarrow{K(\iota)} K(\mathcal{C}/\Phi) .$$

There is an associated exact sequence

$$\pi_1 K(\mathcal{C}) \longrightarrow \pi_1 K(\mathcal{C}/\Phi) \longrightarrow \pi_0 K(\mathcal{C}) \xrightarrow{\cdot \mathbf{Y}} \pi_0 K(\mathcal{C}) \longrightarrow \pi_0 K(\mathcal{C}/\Phi)\ .$$

Thus, the cokernel $\pi_0 K(\mathcal{C}/\Phi)$ of the map

$$\pi_0 K(\mathcal{C}) \to \pi_0 K(\mathcal{C})$$

mapping

$$[X, f] \mapsto [X, f] \cdot \mathbf{Y}$$

can be identified with the Grothendieck ring generated by the classes $[X, f]$ and the relations

$$[X, f] = [Y, f|_Y] + [U, f|_U]$$

and

$$[X \times \mathbf{A}^1, \pi_{\mathbf{A}^1}] = 0.$$

Note that $\pi_0 K(\mathcal{C}/\Phi)$ has in fact all relations of the form $[X \times \mathbf{A}^1, f \circ \pi_X + \pi_{\mathbf{A}^1}]$, but these are all zero because $[\mathbf{A}^1, \pi_{\mathbf{A}^1}] = 0$ and the product is well defined. ∎

In a similar way, we can treat the localization $Exp\mathcal{M}_K$ of the Grothendieck ring with exponentials $KExp_K$, by describing it in terms of algebraic stacks with exponentials as in Lemma 7.3.3.

The following statement can be proved in the same way, as Theorem 7.4.

7.5. Proposition. *Let $\mathcal{C}_K^{KExp\mathcal{S}}$ denote the cofiber of the morphism of assemblers*

$$\Phi : \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathcal{S}}$$

where $\mathcal{C}^{\mathcal{S}}$ has objects given by pairs (X, f) of an algebraic stack X and a morphism $f : X \to \mathbf{A}^1$, and morphisms given by locally closed embeddings of substacks with compatible maps to $\mathbf{A}^1$, and

$$\Phi(X, f) = (X \times \mathbf{A}^1, f \circ \pi_X + \pi_{\mathbf{A}^1}).$$

The localization $Exp\mathcal{M}_K$ of the Grothendieck ring with exponentials $KExp_K$ is identified with

$$Exp\mathcal{M}_K = \pi_0 K(\mathcal{C}_K^{KExp\mathcal{S}}).$$

One can also consider the relative version of the Grothendieck ring with exponentials and its localization, as in Section 1.1.5 of [ChamLoe15].

7.5.1. Definition. *Let S be a K-variety. Consider pairs (X, f) where X is an S-variety (a variety endowed with a morphism $u : X \to S$) and $f : X \to \mathbf{A}^1$ is a morphism. The relative Grothendieck ring with exponentials $KExp_S$ is defined as in Definition 7.3.1, generated by isomorphism classes $[X, f]_S$ with the relations*

$$[X, f]_S = [Y, f|_Y]_S + [U, f|_U]_S$$

for $Y \hookrightarrow X$ a closed embedding of S-varieties (with morphism $u|_Y : Y \to S$) and

$$[X, f]_S \cdot [\mathbf{A}^1, \pi_{\mathbf{A}^1}]_S = 0.$$

A morphism $\varphi : S \to T$ induces a ring homomorphism

$$\varphi^* : KExp_T \to KExp_S$$

given by

$$\varphi^*[X, f]_T = [X \times_T S, f \circ \pi_X]_S.$$

There is an embedding

$$K_0(\mathcal{V}_S) \to KExp_S$$

given by

$$[X]_S \mapsto [X, 0]_S,$$

with the localizations satisfying

$$\mathcal{M}_S = \mathcal{M}_K \otimes_{K_0(\mathcal{V}_K)} K_0(\mathcal{V}_S), \qquad Exp\mathcal{M}_S = Exp\mathcal{M}_K \otimes_{KExp_K} KExp_S.$$

If we interpret Grothendieck rings with exponentials as abstract motivic functions, the relative case corresponds to a motivic generalization of functions that for finite base fields take the form

$$\Psi : S(\mathbf{F}_q) \to \mathbf{C}, \qquad \Psi(s) = \sum_{x \in X_s(\mathbf{F}_q)} \chi(f(x)).$$

Here X is a variety over S with fibers X_s over $s \in S$, and χ is a fixed character $\chi : \mathbf{F}_q \to \mathbf{C}^*$.

We then have the following statements whose proof is analogous to the previous theorem.

7.6. Proposition. *Let $\mathcal{C}_S^{KExp}$ denote the cofiber of the morphism of assemblers*

$$\Phi : \mathcal{C}_S \to \mathcal{C}_S.$$

Here objects of $\mathcal{C}_S$ are triples (X, u, f), where X an S–variety, $u : X \to S$ its structure morphism, and $f : X \to \mathbf{A}^1$. Morphisms between such objects are locally closed embeddings of S–subvarieties compatible with the maps to $\mathbf{A}^1$. Finally, $\Phi(X, u, f)$ is the S–variety obtained by taking a product with $\mathbf{A}^1$, endowed with the morphism

$$f \circ \pi_X^* + \pi_{\mathbf{A}^1}$$

to $\mathbf{A}^1$. Then

$$\pi_0 K(\mathcal{C}_S^{KExp}) = KExp_S.$$

7.7. Proposition. *The case of the localization $Exp\mathcal{M}_S$ can be treated similarly in terms of an assembler $\mathcal{C}_S^{\mathcal{S}}$ with objects given by algebraic stacks with morphisms to S and to $\mathbf{A}^1$ and the cofiber $\mathcal{C}_S^{KExp\mathcal{S}}$ of the morphism*

$$\Phi : \mathcal{C}_S^{\mathcal{S}} \to \mathcal{C}_S^{\mathcal{S}}$$

that multiplies a stack X by $\mathbf{A}^1$ with the morphism

$$f \circ \pi_X^* + \pi_{\mathbf{A}^1}$$

to $\mathbf{A}^1$, so that

$$\pi_0 K(\mathcal{C}_S^{KExp\mathcal{S}}) = Exp\mathcal{M}_S.$$

7.8. Grothendieck ring with exponentials and zeta functions. The Hasse-Weil zeta function of a variety X over a finite field $\mathbf{F}_q$,

$$\zeta_X^{HW}(t) = \exp\left(\sum_{m \geq 1} \frac{\# X(\mathbf{F}_{q^m})}{m} t^m\right)$$

has an Euler product expression of the form of a product over closed points of X,

$$\zeta_X^{HW}(t) = \prod_{x \in X} (1 - t^{\deg(x)})^{-1}.$$

One can see this by observing that the latter expression can be written in the form

$$\prod_{x \in X} (1 - t^{\deg(x)})^{-1} = \prod_{r \geq 1} (1 - t^r)^{-a_r},$$

where

$$a_r := \#\{x \in X \mid [k(x) : \mathbf{F}_q] = r\}$$

with $k(x)$ the residue field at the point x with $deg(x) = [k(x) : \mathbf{F}_q]$, and that the counting

$$N_m(X) := \#X(\mathbf{F}_{q^m})$$

is given by

$$N_m(X) = \sum_{r|m} r \cdot a_r.$$

One can also reformulate the expression of the Hasse-Weil zeta function as a product over points in the form of sum over effective zero-cycles of a given degree, by expanding the product,

$$\zeta_X^{HW}(t) = \sum_{n \geq 0} \#\{\text{effective 0-cycles of degree n on X}\} t^n.$$

The latter expression motivated the introduction of Kapranov's motivic zeta function [Kap00],

$$Z_X(t) = \sum_{n \geq 0} [Sym^n X]\, t^n,$$

where $[Sym^n X] \in K_0(\mathcal{V})$ are the classes in the Grothendieck ring of varieties of the symmetric products $Sym^n X = X^n/\Sigma_n$ of X. The symmetric product $Sym^n X$ indeed parameterizes the effective zero-cycles of degree n on X.

In general, by a motivic measure we mean here a ring homomorphism

$$\mu : K_0(\mathcal{V}) \to R$$

from the Grothendieck ring of varieties to a commutative ring R.

As above, consider Kapranov's motivic zeta function

$$Z_X(t) = \sum_{n \geq 0} [Sym^n X] \, t^n,$$

with $Sym^n X$ the symmetric products. For a motivic measure $\mu : K_0(\mathcal{V}) \to R$ one obtains an associated zeta function by taking

$$\zeta_\mu(X, t) := \sum_{n=0}^{\infty} \mu(Sym^n X) \, t^n.$$

Consider the case of varieties over a finite field $K = \mathbf{F}_q$, and the Grothendieck ring with exponentials $KExp_K$. The choice of a character $\chi : \mathbf{F}_q \to \mathbf{C}^*$ determines a motivic measure, namely a ring homomorphicm $\mu_\chi : KExp_K \to \mathbf{C}$ by

$$\mu_\chi[X, f] = \sum_{x \in X(\mathbf{F}_q)} \chi(f(x)).$$

This is indeed a ring homomorphism as

$$\mu_\chi([Y, f|_Y] + [U, f|_U]) = \mu_\chi[Y, f|Y] + \mu_\chi[U, f|_U]$$

and

$$\mu_\chi([X_1, f_1] \cdot [X_2, f_2]) = \mu_\chi([X_1 \times X_2, f_1 \circ \pi_{X_1} + f_2 \circ \pi_{X_2}) =$$

$$\sum_{(x_1, x_2) \in X_1(\mathbf{F}_q) \times X_2(\mathbf{F}_q)} \chi(f_1(x_1)) \chi(f_2(x_2)) = \mu([X_1, f_1] \cdot \mu[X_2, f_2]).$$

By precomposition with the embedding $K_0(\mathcal{V}_K) \hookrightarrow KExp_K$ given by $[X] \mapsto [X, 0]$, the homomorphims μ_χ induces a ring homomorphism $\mu : K_0(\mathcal{V}_K) \to \mathbf{C}$, that is, a motivic measure in the usual sense. One has $\mu_\chi[X, 0] = \#X(\mathbf{F}_q)$, hence this induced motivic measure is independent of the character χ, takes values in $\mathbf{Z}$, and is just given by the usual counting function for varieties over finite fields, whose associated zeta function $\zeta_\mu(t)$ is the Hasse–Weil zeta function.

In this setting one considers the Kapranov motivic zeta function in $KExp_K$ and the zeta function ζ_{μ_χ} is obtained by composing it with the given motivic measure. Namely, given a class $[X,f] \in KExp_K$, one considers the symmetric products

$$Sym^n[X,f] := [Sym^n X, f^{(n)}],$$

where the morphism $f^{(n)} : Sym^n(X) \to \mathbf{A}^1$ is given by

$$f^{(n)}[x_1, \dots, x_n] = f(x_1) + \cdots + f(x_n),$$

where $[x_1, \dots, x_n]$ is the class in $Sym^n(X) = X^n/\Sigma_n$ of $(x_1, \dots, x_n) \in X^n$.

Note that this shows, in particular, that there is a unique way of interpreting the term $\chi(f(x))$ for a character $\chi : \mathbf{F}_q \to \mathbf{C}^*$ when the point x has degree $r > 1$, namely as χ evaluated at the trace of $f(x)$. This follows from the description of such x as an $\mathbf{F}_q$-point of $Sym^n(X)$.

One can then define an analog in $KExp_K[[t]]$ of the Kapranov motivic zeta function as

$$Z_{(X,f)}(t) = \sum_{n \geq 0} [Sym^n X, f^{(n)}]\, t^n,$$

and zeta functions associated to motivic measures $\mu : KExp_K \to R$, with values in a commutative ring R with values in $R[[t]]$ defined by $\zeta_\mu : KExp_K \to R[[t]]$,

$$\zeta_\mu((X,f),t) := \sum_{n \geq 0} \mu[Sym^n X, f^{(n)}]\, t^n.$$

In particular, for a choice of a character $\chi : \mathbf{F}_q \to \mathbf{C}^*$, we can consider the zeta function associated to the corresponding motivic measure $\mu_\chi : KExp_{\mathbf{F}_q} \to \mathbf{C}$,

$$\zeta_{\mu_\chi}((X,f),t) = \sum_{n \geq 0} \mu_\chi[Sym^n X, f^{(n)}]\, t^n.$$

$$= \sum_{n \geq 0} N_\chi(Sym^n X, f^{(n)})\, t^n,$$

This zeta function is a generalization of the Hasse–Weil zeta function to which it restricts in the case $f = 0$.

For a given pair (X, f) of an $\mathbf{F}_q$-variety and a morphism $f : X \to \mathbf{A}^1$, and a character $\chi : \mathbf{F}_q \to \mathbf{C}^*$, consider for $r \in \mathbf{N}$ and $\alpha \in \mathbf{C}$ the sets

$$X_{\alpha,r} := \{x \in X \mid \deg(x) = r \text{ and } \chi(f(x)) = \alpha\},$$

with $k(x)$ the residue field and $\deg(x) = [k(x) : \mathbf{F}_q]$. Let $a_{\alpha,r} := \# X_{\alpha,r}$.

7.8.1. Proposition. *For (X, f) of an $\mathbf{F}_q$-variety and a morphism $f : X \to \mathbf{A}^1$ and $\chi : \mathbf{F}_q \to \mathbf{C}^*$ a character as above, the zeta function $\zeta_{\mu_\chi}((X, f), t)$ has an Euler product expansion of the form*

$$\zeta_{\mu_\chi}((X, f), t) = \prod_{\alpha} \prod_{r \geq 1} (1 - \alpha t^r)^{-a_{\alpha,r}},$$

and can be written equivalently in the form

$$\zeta_{\mu_\chi}((X, f), t) = \exp\left(\sum_{m \geq 1} N_{\chi,m}(X, f) \, \frac{t^m}{m} \right),$$

with coefficients

$$N_{\chi,m}(X, f) = \sum_{\alpha} \sum_{r | m} r \, a_{\alpha,r} \, \alpha^{\frac{m}{r}}.$$

Proof.

Note that, if we identify points $[x_1, \dots, x_n] \in Sym^n X$ with effective divisors $D = x_1 + \cdots + x_n$ on X of degree n, then we can write

$$\mu_\chi[Sym^n X, f^{(n)}] = \sum_{D} \chi(f^{(n)}(D)),$$

where

$$\chi(f^{(n)}(D)) = \prod_{i=1}^{n} \chi(f(x_i)).$$

Thus we have

$$\zeta_{\mu_\chi}((X, f), t) = \sum_{n} \sum_{D \,:\, deg(D) = n} \chi(f^{(n)}(D)) \, t^n.$$

Let $X_{\alpha,r}$ be the level sets defined as in the statement above. We also denote by $X_\alpha = \{x \in X \,|\, \chi(f(x)) = \alpha\} = \cup_r X_{\alpha,r}$. Instead of considering the integer numbers $a_r = \#X_r$ we now consider $a_{\alpha,r} := \#X_{\alpha,r}$.

Observe then that we have

$$\sum_n \sum_{D\,:\,deg(D)=n} \chi(f^{(n)}(D))\, t^n = \sum_n \sum_{D=x_1+\cdots+x_n} \prod_{i=1}^n \chi(f(x_i)) t^{\deg(x_i)}$$

$$= \prod_\alpha \prod_{x\in X_\alpha} (1 + \alpha t^{\deg(x)} + \alpha^2 t^{2\deg(x)} + \cdots) =$$

$$= \prod_\alpha \prod_{r\geq 1} \prod_{x\in X_{\alpha,r}} (1-\alpha t^r)^{-1} = \prod_\alpha \prod_{r\geq 1} (1-\alpha t^r)^{-a_{\alpha,r}}.$$

This gives the Euler product expansion of $\zeta_{\mu_\chi}((X,f),t)$.

Moreover, we have

$$\log \zeta_{\mu_\chi}((X,f),t) = -\sum_\alpha \sum_{r\geq 1} a_{\alpha,r} \log(1-\alpha t^r)$$

$$= \sum_\alpha \sum_{r\geq 1} a_{\alpha,r} \sum_\ell \alpha^\ell t^{r\ell}$$

$$= \sum_{m\geq 1} \sum_{r|m} \sum_\alpha r\, a_{\alpha,r}\, \alpha^{m/r}\, \frac{t^m}{m},$$

where the sum over alpha ranges over the non-empty level sets $X_{\alpha,r} \neq \emptyset$. Consider then the sequence $N_{\chi,m} = N_{\chi,m}(X,f)$, for $m \geq 1$, defined as

$$N_{\chi,m} := \sum_\alpha \sum_{r|m} r\, a_{\alpha,r}\, \alpha^{m/r}.$$

We have

$$\log \zeta_{\mu_\chi}((X,f),t) = \sum_{m\geq 1} N_{\chi,m}(X,f)\, \frac{t^m}{m}.$$

■

Motivic Euler product expansions were considered, for instance, in [Bour09]. Our main focus in Proposition 7.8.1 above is on introducing a version for the case with exponentials.

In the next two subsections we discuss another categorification of the Grothendieck ring with exponentials, which instead of using the formalism of assemblers is based on Nori's diagrams and Nori's categories.

7.9. Nori's Tannakian formalism. We have presented in Theorem 7.4 and Propositions 7.5, 7.6, and 7.7 a categorification and spectrification of the Grothendieck ring with exponentials based on the categorical formalism of assemblers and on the associated spectra. There is, however, another possible approach to categorifying the Grothendieck ring with exponentials, via an appropriate category of motives. We discuss this other approach in this and the following subsection, where we show that the appropriate category of motives is provided by the exponential motives of Fresán and Jossen, [FreJo20], constructed through the general Tannakian formalism of Nori motives, which we review in this subsection. Thus, assemblers and Nori diagrams can be viewed as two complementary paths to the categorification of Grothendieck rings, one that leads naturally to the homotopy-theoretic world and the other to the motivic. The possible interactions between homotopy-theoretic and motivic settings appear to be most promising for future developments.

We recall here briefly Nori's formalism, constructing Tannakian categories associated to diagrams and their representations, and the application of this formalism to the construction of the category of Nori motives. As we recall below, a Nori diagram is like a quiver and the Nori formalism makes it possible to construct from representations of Nori diagrams in categories Mod_R of modules an abelian (and under suitable circumstances Tannakian) category that satisfied a universal property with respect to such representations. We recall here the main steps of this construction. A main reference for the material we review in this subsection is the book [HuMu-St17]. What we review here will be useful in the next subsection, where we describe another way of categorifying the Grothendieck ring of varieties with exponentials in terms of a category of exponential motives, due to Fresán and Jossen [FreJo20].

A category of Nori diagrams is defined as follows (Definition 7.1.1 of [HuMu-St17]). *A diagram* D consists of a family $V(D)$ of vertices and a family $E(D)$ of edges, with a boundary map $\partial : E(D) \to V(D) \times V(D)$, where $\partial(e) = (\partial_{out}(e), \partial_{in}(e))$ means source and target of the oriented edge. *A morphism* $D_1 \to D_2$ of diagrams consists of two maps $V(D_1) \to V(D_2)$ and $E(D_1) \to E(D_2)$ compatible with orientations and boundaries. *A diagram with identities* is a diagram D where for each $v \in V(D)$ there is a unique oriented edge id_v with $\partial(id_v) = (v, v)$. In the case of diagrams with identities, morphisms are require to map identity edges to identity edges.

To a category $\mathcal{C}$ one can associate the diagram $D(\mathcal{C})$ with $V(D(\mathcal{C})) = Obj(\mathcal{C})$ and $E(D(\mathcal{C})) = Hom_{\mathcal{C}}$. *A representation of a diagram D in a category $\mathcal{C}$* is a morphism of diagrams $T : D \to D(\mathcal{C})$. One considers representations in categories Mod_R of modules over a commutative ring R, and in particular representations in categories of vector spaces.

Given a diagram D and a representation T in Mod_R, for some commutative ring R, one defines the ring $End(T)$ as

$$End(T) = \left\{ (\phi_v) \in \prod_{v \in V(D)} End_R(T(v)) \,\middle|\, \phi_{\partial_{out}(e)} \circ T(e) = T(e) \circ \phi_{\partial_{in}(e)}, \ \forall e \in E(D) \right\}.$$

Nori's *diagram category* $\mathcal{C}(D, T)$ is then obtained in the following way. If the diagram D is finite then $\mathcal{C}(D, T)$ is the category $Mod_{End(T)}$ of finitely generated R–modules with an R–linear action of $End(T)$. If the diagram D is infinite, one considers all finite subdiagrams D_F and constructs the corresponding categories $\mathcal{C}(D_F, T|_{D_F})$. The category $\mathcal{C}(D, T)$ has as objects the union of all the objects of the $\mathcal{C}(D_F, T|_{D_F})$ for all finite $D_F \subset D$. An inclusion $D_F \subset D'_F$ determines a morphism $End(T|_{D'_F}) \to End(T|_{D_F})$ by projecting the product $\prod_{v \in V(D'_F)} End_R(T|_{D'_F}(v))$ onto $\prod_{v \in V(D_F)} End_R(T|_{D_F}(v))$. This morphism induces a functor from $Mod_{End(T|_{D_F})}$ to $Mod_{End(T|_{D'_F})}$. Morphisms in $\mathcal{C}(D, T)$ are then defined as colimits of morphisms in $\mathcal{C}(D_F, T|_{D_F})$ under these extensions. The category $\mathcal{C}(D, T)$ constructed in this way is R–linear abelian with an R–linear faithful exact forgetful functor $f_T : \mathcal{C}(D, T) \to Mod_R$. The representation $T : D \to Mod_R$ factors as $T = f_T \circ \tilde{T}$ with a representation $\tilde{T} : D \to \mathcal{C}(D, T)$. We refer the reader to [HuMu-St17] pp. 140–144 for more details.

Diagram categories satisfy the following universal property:

Any representation $F : D \to A$ where A an R–linear abelian category with an R–linear faithful exact functor $f : A \to Mod_R$ factors through a faithful exact functor $L(F) : \mathcal{C}(D, T) \to A$, where $T = f \circ F$, compatibly with the decomposition $T = f_T \circ \tilde{T}$. (See [HuMu-St17], pp. 140–141).

Tensor structures on diagram categories can be obtained through the notion of graded diagrams with a commutative product with unit, in the sense of Definition 8.1.3 of [HuMu-St17]. Here are some details.

A graded diagram D is a diagram endowed with a map $\deg : V(D) \to \mathbf{Z}/2\mathbf{Z}$ extended to $\deg : E(D) \to \mathbf{Z}/2\mathbf{Z}$ by $\deg(e) = \deg(s(e)) - \deg(t(e))$. The product $D \times D$ is the diagram with vertices the pairs $(v, w) \in V(D) \times V(D')$ and edges of the form (e, id) or (id, e'). *A product structure* on D is a map of graded diagrams (a degree preserving map of directed graphs) $D \times D \to D$ together with a choice of edges

$$\alpha_{v,w} : v \times w \to w \times v, \quad \forall v, w \in V(D)$$

$$\beta_{v,w,u} : v \times (w \times u) \to (v \times w) \times u,$$

$$\beta'_{v,w,u} : (v \times w) \times u \to v \times (w \times u), .$$

for all $v, w, u \in V(D)$. A unit is a vertex $\mathbf{1}$ with $\deg(\mathbf{1}) = 0$ and edges $u_v : v \to \mathbf{1} \times v$ for all $v \in V(D)$.

One can then consider those representations of D that are compatible with the grading and with the commutative product above, where the compatibility is expressed as the existence of isomorphisms (see Definition 8.1.3 of [HuMu-St17])

$$\tau_{v,w} : T(v \times w) \xrightarrow{\simeq} T(v) \otimes T(w)$$

for all $v, w \in V(D)$, with the following properties:

$$T(v) \otimes T(w) \xrightarrow{\tau_{v,w}^{-1}} T(v \times w) \xrightarrow{T(\alpha_{v,w})} T(w \times v) \xrightarrow{\tau_{w,v}} T(w) \otimes T(v)$$

is equal to multiplication by $(-1)^{\deg(v)\deg(w)}$; the β–maps satisfy $T(\beta_{v,w,u})^{-1} = T(\beta'_{v,w,u})$, and moreover

$$\tau_{v,w'} \circ T(1, e) = (id \otimes T(e)) \circ \tau_{v,w} : T(v \times w) \to T(v) \otimes T(w'),$$

$$\tau_{v',w} \circ T(e, 1) = (T(e) \otimes id) \circ \tau_{v,w} : T(v \times w) \to T(v') \otimes T(w),$$

$$\begin{tikzcd} T(v \times (w \times u)) \arrow[r, "T(\beta_{v,w,u})"] \arrow[d, "\tau\circ\tau"] & T((v \times w) \times u) \arrow[d, "\tau\circ\tau"] \\ T(v) \otimes (T(w) \otimes T(u)) \arrow[r, "\simeq"] & (T(v) \otimes T(w)) \otimes T(u)) \end{tikzcd}$$

and similarly for the inverse $T(\beta'_{v,w,u})$.

In Theorem 7.1.12 of [HuMu-St17] it is shown that if the representation T is valued in the subcategory of Mod_R of finite projective modules, and R is a Dedekind domain, the Nori diagram category $\mathcal{C}(D,T)$ is equivalent to the category of finitely generated comodules over the coalgebra $\mathcal{A}(D,T)$ given by the colimit

$$\mathcal{A}(D,T) = colim_{D_F} End(T|_{D_F})^\vee$$

over finite sub-diagrams D_F.

In Sec. 7.5.1 of [HuMu-St17] it is ahown, that if R is a Dedekind domain, then for the R–algebra $E = End(T|_{D_F})$ with D_F a finite diagram, the R–dual $E^\vee = Hom_R(E,R)$ has the property that the canonical map $E^\vee \otimes_R E^\vee \to Hom(E,E^\vee) \simeq (E \otimes_R E)^\vee$ is an isomorphism. If an E-module is finitely generated projective as an R–module then the same is true for comodule $E^\vee$. The coalgebra $\mathcal{A}(D,T)$ also carries an algebra structure induced by the monoidal structure of $\mathcal{C}(D,T)$ discussed in Sections 7.1.4 and 8.1 of [HuMu-St17], so that $\mathcal{A}(D,T)$ determines a pro–algebraic monoid scheme $Spec(\mathcal{A}(D,T))$ (see Section 7.1.4 of [HuMu-St17]). This is the general form of Nori's Tannakian formalism.

More specifically, for K a subfield of $\mathbf{C}$, the category of Nori motives (which we here call Nori *classical* motives, not to be mixed with *exponential* motives we will review in the next subsection) is constructed by considering the Nori diagram of *effective pairs* (see [HuMu-St17], pp. 207–208) where vertices are of the form (X,Y,n) with X a K–variety, $Y \subseteq X$ a closed embedding, and n an integer, and non–identity edges are of the following types:

(a) Let (X,Y) and (X',Y') be two pairs of closed embeddings. Every morphism $f : X \to X'$ such that $f(Y) \subset Y'$ produces functoriality edges f^* (or rather (f^*,n)) going from (X',Y',n) to (X,Y,n).

(b) Let $(Z \subset Y \subset X)$ be a pair of closed embeddings. Then it defines coboundary edges ∂ from (Y,Z,n) to $(X,Y,n+1)$.

A representation of the diagram $D(Pairs^{eff})$ obtained in this way in the category $Mod_{\mathbf{Z}}$ (or in the category $Vect_{\mathbf{Q}}$) is given by relative singular cohomology $H^n(X(\mathbf{C}),Y(\mathbf{C}),\mathbf{Z})$ (respectively, $H^n(X(\mathbf{C}),Y(\mathbf{C}),\mathbf{Q})$). The category of effective Nori motives is given by $\mathcal{C}(D(Pairs^{eff}),H^*)$ and the category of Nori motives is obtained as the localization at $(\mathbf{G}_m,\{1\},1)$. A tensor structure is obtained as discussed above, after restricting to a subcategory of *good pairs*, see Sections 8.1 and 9.3 of [HuMu-St17]. One obtains a Tannakian category of classical Nori motives.

7.9.1. Spectra from Nori diagrams. The categorification through assemblers discussed in the previous sections leads to the construction of an associated spectrum, obtained through a Γ-space, as discussed in Section 3.3.2. It is then natural to ask whether the categorical construction outlined here above can also have an associated homotopy-theoretic spectrum.

As recalled in Section 1.5, the formalism of Γ-spaces provides a general method for the construction of spectra from categories, through the nerve of the associated category of summing functors.

Given a Nori category $\mathcal{C}(D,T)$, obtained as above from a Nori diagram and a representation T, let $F_{\mathcal{C}(D,T)}$ be the associated Γ-space that maps a finite pointed set X to the pointed simplicial set given by the nerve $\mathcal{N}(\Sigma_{\mathcal{C}(D,T)}(X))$ of the category $\Sigma_{\mathcal{C}(D,T)}(X)$ of summing functors $\Phi : P(X) \to \mathcal{C}(D,T)$, with the notation of Section 1.5. This Γ-space determines then a spectrum $\mathbf{S}(F_{\mathcal{C}(D,T)})$, obtained by promoting the functor $F_{\mathcal{C}(D,T)}$ to an endofunctor of the category of pointed simplicial sets and applying it to spheres, see Section 1.5.

Since the category $\mathcal{C}(D,T)$ is abelian, its higher algebraic K-theory groups [Qui73] are the homotopy groups of an infinite loop space $K(\mathcal{C}(D,T))$. The spectrum $\mathbf{S}(F_{\mathcal{C}(D,T)})$ constructed as above provides a delooping of this infinite loop space, in the sense discussed in [Carl05].

7.10. Exponential motives. As mentioned in the previous section, another possible way to categorify the notion of Grothendieck ring with exponentials is through a category of "motives with exponentials". A general treatment of such "exponential motives" was presented by Fresán and Jossen in [FreJo20], where the relations between the resulting category of exponential motives, the Grothendieck ring of varieties with exponentials of [ChamLoe15], the formalism of Nori motives [HuMu-St17], and the "exponential periods" of Kontsevich–Zagier [KoZa01] is also explained.

We assume here that K is a subfield of $\mathbf{C}$. The category $MotExp(K)$ of exponential motives over a field K constructed in [FreJo20] is based on Nori diagrams where the vertices are tuples (X, Y, f, n, i) with X a variety over K and $Y \hookrightarrow X$ a closed subvariety, together with a morphism $f : X \to \mathbf{A}^1$, with the restriction $f|_Y : Y \to \mathbf{A}^1$, and integers n, i, respectively referred to as degree and twist; the edges are of the following types:

(1) a morphism of varieties $h : X \to X'$ with $h(Y) \subseteq Y'$ and $f' \circ h = f$ determines an edge $h^* : (X', Y', f', n, i) \to (X, Y, f, n, i)$;

(2) a pair of closed immersions $Z \subseteq Y \subseteq X$ determines an edge

$$\partial : (Y, Z, f|_Y, n-1, i) \to (X, Y, f, n, i);$$

(3) edges of the form

$$(X \times \mathbf{G}_m, (Y \times \mathbf{G}_m) \cup (X \times \{1\}), f \boxtimes 0, n+1, i+1) \to (X, Y, f, n, i)$$

A representation of these Nori diagrams in the category of vector spaces $Vect_{\mathbf{Q}}$ is given in [FreJo20] in terms of *rapid decay cohomology*

$$H^n(X, Y, f)(i)$$

where (i) denotes the Tate twist given by the tensor product with tensor powers of $H^1(\mathbf{G}_m, \mathbf{Q})$ and the rapid decay cohomology $H^n(X, Y, f)$.

Rapid decay cohomology is constructed in the following way. Let X be a complex variety with a closed subvariety $Y \subseteq X$ and a regular function $f : X \to \mathbf{A}^1$. For a real number $r \in \mathbf{R}$, denote by $S_r = \{z \in \mathbf{C} \mid Re(z) \geq r\}$ the closed half-plane with boundary the vertical line $Re(z) = r$. One then defines

$$H_n(X, Y, f) = \varinjlim_{r \to \infty} H_n(X, Y \cup f^{-1}(S_r)), \quad H^n(X, Y, f) = \varprojlim_{r \to \infty} H^n(X, Y \cup f^{-1}(S_r)),$$

as limits in $Vect_{\mathbf{Q}}$, with respect to the system of inclusions $f^{-1}(S_t) \subseteq f^{-1}(S_r)$ for $t \geq r$. This cohomology theory originates in the study of differential equations with irregular singularities, [DeMaRa07].

The reader should also look back at the definition of convergence boundaries in Sec. 2.3.

A property of rapid decay cohomology that is directly relevant to exponential motives and the Grothendieck ring of varieties with exponentials is the fact that one has

$$H^n(X \times \mathbf{A}^1, Y \times \mathbf{A}^1, \pi_{\mathbf{A}^1}) = 0,$$

which suggests that this is indeed the right cohomology theory that provides a realization compatible with the relations in the Grothendieck ring of varieties with exponentials that we discussed above.

A more general construction of rapid decay cohomology is given in [FreJo20] in terms of perverse sheaves. For our purposes we only review the more elementary definition above, and we refer the reader to [FreJo20] for the more general treatment.

The category $MotExp(K)$ of exponential motives is the abelian $\mathbf{Q}$–linear category obtained by applying the Nori formalism to these diagrams and representations. A tensor product on the category of exponential motives is also introduced in [FreJo20] following an analogous construction for Nori motives recalled in the previous subsection, and it is shown that each object admits a dual. The general Nori formalism then shows that the resulting category $MotExp(K)$ of exponential motives is Tannakian, as discussed in the previous subsection (see [HuMu-St17] for more details).

The relations between the category $MotExp(K)$ of exponential motives and other motivic theories discussed in [FreJo20] that are more directly relevant for us here can be summarized as follows. There is a fully faithful exact functor from the category of classical Nori motives to exponential motives, which gives rise to a morphism (in the reverse direction) of affine group schemes, between the respective motivic Galois groups, which is faithfully flat. Moreover, it is shown in Section 5.4 of [FreJo20] that there is a unique ring homomorphism

$$\chi : KExp_K \to K_0(MotExp(K))$$

such that, for a pair (X, f) of a K-variety and a morphism $f : X \to \mathbf{A}^1$

$$\chi[X, f] = \sum_n (-1)^n [H^n_c(X, f)],$$

where the bracket notation indicates that we view the $H^n_c(X, f)$ on the right-hand-side as elements in $K_0(MotExp(K))$.

Exponential periods arise from the pairing of de Rham cohomology and rapid decay homology

$$H^n_{dR}(X, f) \otimes H_n(X, f) \to \mathbf{C}.$$

Rapid decay homology describes cycles that are possibly non–compact but unbounded in directions where $Re(f) \to \infty$, so that the exponential periods are given by the pairing

$$\int_\gamma e^{-f}\omega = \lim_{r\to\infty} \int_{\gamma_r} e^{-f}\omega$$

where γ is a direct limit of compact cycles γ_r for $r \in \mathbf{R}$ with $\partial\gamma_r \subset f^{-1}(S_r)$.

An example of a number that is not expected to be a period of a classical motive but that is an exponential period (for $f(x) = x^2$) is

$$\sqrt{\pi} = \int_{\mathbf{R}} e^{-x^2} dx.$$

Exponential motives provide the motivic framework for exponential periods.

7.11. Motivic Fourier transform. Let K denote an algebraically closed field of characteristic zero, and V a finite–dimensional linear space over K. A motivic Fourier transform on the Grothendieck ring $KExp_V$ of varieties with exponentials over V, can be defined as follows (see Section 7.1 of [CluLoe10], Section 1.2 of [ChamLoe15] and Section 7.12 and Definition 2.2 of [Wy17]):

$$\mathcal{F} : KExp_V \to KExp_{V^\vee}$$

$$\mathcal{F}([X, f]_V) := [X \times V^\vee, f \circ \pi_X + \langle u \circ \pi_X, \pi_{V^\vee} \rangle]_{V^\vee},$$

where $V^\vee = Hom_K(V, K)$ is the dual linear space, $\langle \cdot, \cdot \rangle : V \times V^\vee \to K$ the natural pairing, and $u : X \to V$ is the structure morphism of X as a V–variety. This motivic Fourier transform satisfies the relation

$$\mathcal{F} \circ \mathcal{F}[X, f]_V = \mathbf{L}^{\dim V} \cdot j^*[X, f]_V,$$

where $j^* : KExp_V \to KExp_V$ is the pullback induced by the map $j : V \to V$ given by multiplication by -1, see [CluLoe10].

We consider here again the assemblers $\mathcal{C}_S^{KExp}$ of Proposition 7.6 and the associated homotopy–theoretic spectra $K(\mathcal{C}_S^{KExp})$ underlying the relative Grothendieck ring $KExp_S$ of varieties with exponentials over a base scheme S, as we discussed in Theorem 7.4 and Propositions 7.5 and 7.6.

7.12. Theorem. *The motivic Fourier transform $\mathcal{F}$ lifts to an morphism of assemblers $\mathcal{F} : \mathcal{C}_V^{KExp} \to \mathcal{C}_{V^\vee}^{KExp}$ and induces a morphism of the associated spectra. The second iterate $\mathcal{F} \circ \mathcal{F}$ of the Fourier transform determines a covering family*

$$\{(\mathbf{A}^d, 0) \times (Z, f|_Z) \hookrightarrow \mathcal{F} \circ \mathcal{F}(X, f), (Z^c \times V^\vee, h|_{Z^c}) \hookrightarrow \mathcal{F} \circ \mathcal{F}(X, f)\}$$

in the assembler $\mathcal{C}_V^{KExp}$, where $Z = \{(x, v) \in X \times V \mid u(x) + v = 0\}$ with $u : X \to V$ is the structure morphism of X as a V–variety. The term $(Z^c \times V^\vee, h|_{Z^c})$ in the *range of the endofunctor* $\Phi : \mathcal{C}_V^{KExp} \to \mathcal{C}_V^{KExp}$ *of the assembler* $\mathcal{C}_V^{KExp}$. *This family implements the relation*

$$\mathcal{F} \circ \mathcal{F}[X, f]_V = \mathbf{L}^{\dim V} \cdot j^*[X, f]_V$$

in $KExp_V = \pi_0 K(\mathcal{C}_V^{KExp}/\Phi)$.

Proof. The argument is similar to Section 2 of [Wy17]. For $(X, f)_V$ an object in $\mathcal{C}_V^{KExp}$, the second iterate of the Fourier transform is given by

$$\mathcal{F} \circ \mathcal{F}(X, f) = (X \times V \times V^\vee, f \circ \pi_X + \langle u \circ \pi_X + \pi_V, \pi_{V^\vee} \rangle),$$

where $u : X \to V$ is the structure morphism of X as a V–variety.

Consider the variety

$$Z = \{(x, v) \in X \times V \mid u(x) + v = 0\}.$$

The structure morphism of Z as a V–variety given by the second projection $\pi_V : Z \to V$ fits in the following commutative diagram, with $j(v) = -v$:

$$\begin{tikzcd} Z \arrow[r, "\pi_X"] \arrow[d, "\pi_V"] & X \arrow[d, "u"] \\ V \arrow[r, "j"] & V \end{tikzcd}$$

Consider the embeddings

$$(Z \times V^\vee, f|_Z \circ \pi_Z) \hookrightarrow (X \times V \times V^\vee, h) \hookleftarrow (Z^c \times V^\vee, h|_{Z^c}),$$

where Z^c is the complement of Z in $X \times V$ and $h : X \times V \times V^\vee \to K$ is given by

$$h(x, v, v^\vee) = \langle u(x) + v, v^\vee \rangle + f(x).$$

We have

$$(Z \times V^\vee, f|_Z \circ \pi_Z) = (\mathbf{A}^d, 0) \times (Z, f|_Z).$$

On the other hand, for the object $(Z^c \times V^\vee, h|_{Z^c})$ of the assembler $\mathcal{C}_V^{KExp}$ we can see that, for any $(x, v) \in Z^c$, the object

$$(\{(x, v)\} \times V^\vee, h_{(x,v)})$$

with

$$h_{(x,v)}(v^\vee) = f(x) + \langle u(x) + v, v^\vee \rangle$$

is in the range of endofunctor $\Phi : \mathcal{C}_V^{KExp} \to \mathcal{C}_V^{KExp}$ of the assembler $\mathcal{C}_V^{KExp}$. This in fact follows from the following general observation.

If W is a finite dimensional K–vector space and $\lambda : W \to K$ is a linear map, then the pair (W, λ) is either $(\mathbf{A}^d, 0)$ when λ is trivial, or a product

$$(W' \times \mathbf{A}^1, \lambda \circ \pi_W + \pi_{\mathbf{A}^1}),$$

where the $\mathbf{A}^1$ factor is spanned by a vector w_0 in W such that $\lambda(w_0) = 1$ in K, so that writing $w = w' + tv_0$ with $w' \in W'$ gives $\lambda(w) = \lambda(w') + t$. Thus, for λ nontrivial, such (W, λ) is in the range of the endofunctor $\Phi : \mathcal{C}_K^{KExp} \to \mathcal{C}_K^{KExp}$ of the assembler $\mathcal{C}_K^{KExp}$. We can then apply this to the case where (W, λ) is given by

$$(\{(x, v)\} \times V^\vee, h_{(x,v)})$$

for a fixed (x, v). This shows that the map $\varphi : Z^c \to Obj(\mathcal{C}_K^{KExp})$ with

$$\varphi(x, v) = (\{(x, v)\} \times V^\vee, h_{(x,v)})$$

has image contained in the range of Φ.

This implies the existence of some object $(W_{(x,v)}, h_{(x,v)}|_W)$ in $\mathcal{C}_K^{KExp}$, obtained as in the general observation above, such that

$$\varphi(x, v) = (W_{(x,v)}, h_{(x,v)}|_W) \times (\mathbf{A}^1, id),$$

for a subspace $W_{(x,v)} \subset V^\vee$. More precisely,

$$Z^c \times V^\vee = \cup_{(x,v)\in Z^c} W_{(x,v)} \times \mathbf{A}^1$$

with compatible morphisms to $\mathbf{A}^1$, and the decomposition given by identifying $\mathbf{A}^1$ with the span of a vector $w^\vee_{(x,v)} \in V^\vee$ such that

$$f(x) + \langle u(x) + v, w^\vee_{(x,v)} \rangle = 1$$

in $\mathbf{A}^1$. The bundle W over Z^c constructed in this way is locally trivial, hence by Noetherian descent induction we then have a finite decomposition $\{Z_i\}$ with locally closed $Z_i \subset Z^c$ such that

$$(Z_i \times V^\vee, h|_{Z_i \times V^\vee})$$

is isomorphic to a product

$$\xi : Z_i \times V^\vee \xrightarrow{\simeq} Z_i \times W \times \mathbf{A}^1$$

with, for $g = h \circ \xi$,

$$(Z_i \times W \times \mathbf{A}^1, g \circ \pi_{Z_i \times W} + \pi_{\mathbf{A}^1}).$$

Thus $(Z^c \times V^\vee, h|_{Z^c})$ is itself in the range of the endofunctor $\Phi : \mathcal{C}_V^{KExp} \to \mathcal{C}_V^{KExp}$ of the assembler $\mathcal{C}_V^{KExp}$.

This induces the relation

$$\mathcal{F} \circ \mathcal{F}[X, f]_V = \mathbf{L}^{\dim V} \cdot j^*[X, f]_V$$

in $KExp_V = \pi_0 K(\mathcal{C}_V^{KExp}/\Phi)$. ■

7.13. Motivic Bruhat–Schwartz functions and Poisson summation. In this subsection we review the Hrushovski–Kazhdan motivic Poisson summation formula, following [ChamLoe15] and [HruKaz09].

As in Section 2.1 above, with slightly changed notations, consider a global field F be a global field and denote by A_F its adèles group. For a place v of F, let F_v denote the completion of F at v. The space $\mathcal{S}(F_v)$ of Bruhat–Schwartz functions on F_v consists of rapidly decaying complex valued functions when v is archimedean and of locally constant and compactly supported complex valued functions when v is archimedean. Bruhat–Schwartz functions $\varphi \in \mathcal{S}(A_F)$ are linear combinations of complex valued functions of the form $\prod_v \varphi_v$, where v ranges over the places of F and $\varphi_v \in \mathcal{S}(F_v)$, with the property that $\varphi_v = 1_{\mathcal{O}_v}$, the characteristic function of the ring of integers $\mathcal{O}_v \subset F_v$ at all but finitely many of the non–archimedean places.

Since F is a discrete cocompact subgroup of the adèles group A_F, the Poisson summation formula gives

$$\sum_{x \in F} \varphi(x) = \frac{1}{\mu(A_F/F)} \sum_{y \in F} \hat{\varphi}(y),$$

where $\hat{\varphi}$ is the Fourier transform of φ:

$$\hat{\varphi}(y) := \int_{A_F} \varphi(x)\chi(xy)d\mu(x)$$

for a nontrivial character $\chi : A_F \to \mathbf{C}^*$ and the Haar measure μ. This classical fact has a motivic counterpart (for function fields), due to Hrushovski and Kazhdan ([HruKaz09]), where Bruhat–Schwartz functions are replaced by elements of a relative Grothendieck ring with exponentials, and motivic Fourier transform is used.

We will focus here on the case of function fields F for a curve C over a finite field, so there are only non–archimedean places corresponding to the points of the curve. As discussed in Section 5.2 of [Bilu18], Section 1.2 of [ChamLoe15], Section 4 of [HruKaz09], the Bruhat–Schwartz functions $\varphi \in \mathcal{S}(F_v)$ can be organized according to their level, labelled by two integers (M, N), respectively measuring "support" and "invariance". Namely, for a function $\varphi \in \mathcal{S}(F_v)$ there are integers $M, N \geq 0$ such that $\varphi \equiv 0$ outside of $t^{-M}\mathcal{O}_v$ and f is invariant modulo the subgroup $t^N\mathcal{O}_v$, where t is a uniformizer. Thus, φ descends to a function on the quotient $t^{-M}\mathcal{O}_v/t^N\mathcal{O}_v$, identified with an $M+N$ dimensional vector space $\mathbf{A}^{M+N}_{k_v}$ over the residue field k_v. This vector space is denoted by $\mathbf{A}^{(M,N)}_{k_v}$, to keep track of the levels.

The motivic version of Bruhat–Schwartz functions of level (M, N) on a nonarchimedean local field with residue field k are then defined (see Section 5.2 of [Bilu18], Section 1.2 of [ChamLoe15], Section 4 of [HruKaz09]) as elements in the relative localized Grothendieck ring with exponentials $Exp\mathcal{M}_{\mathbf{A}^{(M,N)}_k}$.

This definition is motivated by regarding, as we discussed earlier, classes in the Grothendieck ring with exponentials $KExp_{\mathbf{A}^{(M,N)}_k}$ as motivic versions of functions $\Psi_{[X,f],\chi} : \mathbf{A}^{(M,N)}(k) \to \mathbf{C}$ of the form

$$\varphi(s) := \Psi_{[X,f],\chi}(s) = \sum_{x \in X_s(k)} \chi(f(x)).$$

Consider the canonical morphism $\pi : \mathbf{A}^{(M,N)}_k \to Spec(k)$ and the induced morphism of additive groups $\pi_! : Exp\mathcal{M}_{\mathbf{A}^{(M,N)}_k} \to Exp\mathcal{M}_k$ (summation along the fibers)

$$\pi_! : [X, f]_{\mathbf{A}^{(M,N)}_k} \mapsto [X, f] = [X, f]_k.$$

Note that in this case there is only one fiber since the base is $Spec(k)$.

One then defines, for an element $\varphi = [X, f]_{\mathbf{A}_k^{(M,N)}} \in Exp\mathcal{M}_{\mathbf{A}_k^{(M,N)}}$,

$$\int \varphi(s) := \mathbf{L}^{-N} \pi_! \varphi \in Exp\mathcal{M}_k.$$

The dependence on the level (M, N) is regulated by the behaviour under the inclusions $\iota : \mathbf{A}_k^{N-M} \hookrightarrow \mathbf{A}_k^{N-(M-1)}$:

$$\iota : (x_M, \dots, x_{N-1}) \mapsto (0, x_M, \dots, x_{N-1})$$

and projections $\rho : \mathbf{A}_k^{(N+1)-M} \to \mathbf{A}_k^{(N-M}$:

$$\rho : (x_M, \dots, x_N) \mapsto (x_M, \dots, x_{N-1}),$$

with $\iota^* \iota_! = Id$ and $\rho_! \rho^* = \mathbf{L} \cdot$. The motivic Bruhat–Schwartz functions of all levels for a non–archimedean local field with residue field k are then defined as

$$\mathcal{S}^{mot} := \varinjlim_{M, \iota_!} \varinjlim_{N, \rho^*} Exp\mathcal{M}_{\mathbf{A}_k^{(M,N)}}.$$

For a smooth projective curve C over k, with $F = k(C)$, and for a place $v \in C(k)$ with residue field k_v, let $Res_{k_v/k}$ denote the restriction of scalars functor, satisfying

$$Res_{k_s/k}(\mathbf{A}_{k_v}^m) = \mathbf{A}_k^{m\,[k_v:k]}.$$

The adèlic version is then defined as

$$\mathcal{S}^{mot}(A_F) := \varinjlim_{B \subset C(k)} \mathcal{S}^{mot}(\prod_{v \in B} F_v),$$

where the limit is taken over all finite subsets B of the set of points $C(k)$ and $\mathcal{S}^{mot}(\prod_{v \in B} F_v)$ is obtained in the following way. Given a finite set B of places, and given levels M_v, N_v, define

$$\mathbf{A}_k^{(M_B, N_B)} := \prod_{v \in B} Res_{k_s/k}(\mathbf{A}_{k_v}^{(M_v, N_v)}) = \prod_{v \in B} \mathbf{A}_k^{(M_v[k_v:k], N_v[k_v:k])},$$

and consider the Grothendieck ring with exponentials $Exp\mathcal{M}_{\mathbf{A}_k^{(M_B,N_B)}}$. One then sets

$$\mathcal{S}^{mot}(\prod_{v\in B} F_v) := \varinjlim_{M_B,\iota_!} \varinjlim_{N_B,\rho^*} Exp\mathcal{M}_{\mathbf{A}_k^{(M_B,N_B)}}.$$

For $B' \supset B$, taking products with the unit elements $1_{\mathcal{O}_v}$ for $v \in B' \smallsetminus B$ gives the maps

$$\mathcal{S}^{mot}(\prod_{v\in B} F_v) \to \mathcal{S}^{mot}(\prod_{v\in B'} F_v)$$

that determine the directed system computing $\mathcal{S}^{mot}(A_F)$.

The motivic Fourier transform discussed in the previous subsection adapts to this setting of motivic adèlic Bruhat–Schwartz functions. One proceeds as follows (see Section 1.3 of [ChamLoe15] and Section 5 of [HruKaz09]). Consider a nontrivial k–linear map $r_v : F_v \to k$ of conductor c, meaning the smallest integer c such that r vanishes on $t^c\mathcal{O}_v$. Such linear maps can be obtained geometrically from the residue $res_v : \Omega_{F_v/k} \to k$ at the point $v \in C(k)$, of a chosen meromorphic differential form $\omega \in \Omega_{F_v/k}$ by setting $r_\omega : x \mapsto res_v(x\omega)$, with c the order of pole of ω at v. Such a k-linear map $r_v : F_v \to k$ induces linear morphisms $r^{(M,N)} : \mathbf{A}_k^{(M,N)} \to \mathbf{A}_k^1$, for $N \geq c$ (see Section 1.2.6 of [ChamLoe15]). The multiplication in F_v induces morphisms

$$\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \to \mathbf{A}_k^{(M+M',N'')}, \quad N'' = \min\{M'+N, M+N'\}.$$

For $N'' \geq c$, composing with $r^{(M+M',N'')}$ gives a morphism

$$\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \to \mathbf{A}_k^1,$$

which we write as

$$(v, v') \mapsto r(vv').$$

Then, for $\varphi = [X, f]_{\mathbf{A}_k^{(M,N)}}$ in $Exp\mathcal{M}_{\mathbf{A}_k^{(M,N)}}$, one defines the motivic Fourier transform as

$$\mathcal{F}_\omega\varphi := \mathbf{L}^{-N}[X \times_{\mathbf{A}_k^{(M,N)}} \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}, f\circ\pi_X + \langle u\circ\pi_1, \pi_2\rangle_\omega]_{\mathbf{A}_k^{(M',N')}},$$

with $(M', N') = (c-N, c-M)$, and where

$$X \times_{\mathbf{A}_k^{(M,N)}} \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}$$

is the fibered product over the structure morphism $u : X \to \mathbf{A}_k^{(M,N)}$ and the first projection of $\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}$, with the structure of $\mathbf{A}_k^{(M',N')}$–variety given by the projection onto the $\mathbf{A}_k^{(M',N')}$ factor. The morphism to $\mathbf{A}_k^1$ is given by

$$f(x) + \langle v, v' \rangle_\omega,$$

for $x \in X$ and $(v, v') \in \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')}$, with

$$\langle v, v' \rangle_\omega := r_\omega(vv').$$

With the integral notation discussed above, the local motivic Fourier transform is also written in the form

$$\mathcal{F}_{\omega_v} \varphi = \int \varphi(x) e(xy) dx,$$

where $e(xy)$ stands for the motivic Fourier kernel described explicitly here above.

As discussed earlier, this motivic Fourier transform satisfies a Fourier inversion formula, which in this case takes the form

$$\mathcal{F}_\omega \circ \mathcal{F}_\omega [X, f]_{\mathbf{A}_k^{(M,N)}} = \mathbf{L}^{-c} j^* [X, f]_{\mathbf{A}_k^{(M,N)}},$$

where j^* is the pullback on $KExp_{\mathbf{A}_k^{(M,N)}}$ of the map on $\mathbf{A}_k^{(M,N)}$ given by multiplication by -1.

The motivic Fourier transform defined as above extends from the local to the global case. Given a finite set $B \subset C(k)$ of places of the global function field $F = k(C)$, consider an adèlic motivic Bruhat–Schwartz function in $\varphi \in \mathcal{S}^{mot}(\prod_{v \in B} F_v)$, defined as discussed above.

Let $\omega \in \Omega_{F/k}$ be a meromorphic differential form, and $D = \sum_v c_v\, v$ its divisor, with $\deg(D) = \chi(C) = 2 - 2g$. This defines linear maps $r_v = r_{\omega_v} : F_v \to k$ by $r_v(x) = res_v(x\omega)$, the residue at $v \in C(k)$.

For $\varphi = [X, f]_{\mathbf{A}_k^{(M_B,N_B)}}$ in $Exp\mathcal{M}_{\mathbf{A}_k^{(M_B,N_B)}}$, the local motivic Fourier transform extends to this semi–local case, and we can write it in the form

$$\mathcal{F}_\omega \varphi = \mathbf{L}^{-\sum_v N_v [k_v : k]} [X \times_{\mathbf{A}_k^{(M_B,N_B)}} \mathbf{A}_k^{(M_B,N_B)} \times \mathbf{A}_k^{(M'_B,N'_B)}, f \circ \pi_X + \langle u \circ \pi_1, \pi_2 \rangle_\omega]_{\mathbf{A}_k^{(M'_B,N'_B)}},$$

with the Fourier inversion formula

$$\mathcal{F}_\omega \circ \mathcal{F}_\omega = \mathbf{L}^{-\sum_v [k_v:k] c_v} j^*.$$

By considering finite sets of places B that include the zeros and poles of ω, one can further extend this from the semi–local to the global adèlic case, with Fourier inversion formula

$$\mathcal{F}_\omega \circ \mathcal{F}_\omega = \mathbf{L}^{-\chi(C)} j^*,$$

with $\chi(C) = 2 - 2g$ for $g = g(C)$ the genus, see Section 1.3 of [ChamLoe15] for more details.

The motivic analog of summing a Bruhat–Schwartz function over the discrete $F \subset A_F$ is defined in the following way. For a divisor D on C, let $\mathcal{L}(D)$ denote the set of nontrivial rational functions h with $div(h) + D \geq 0$ together with 0, so that it forms a finite dimensional k-vector space, $\mathcal{L}(D) = H^0(C, \mathcal{O}(D))$. Given a finite set $B \subset C(k)$ and a class $\varphi = [X, f]_{\mathbf{A}_k^{(M_B,N_B)}}$, one defines

$$\sum_{x \in F} \varphi(x) := [\mathcal{L}(D) \times_{\mathbf{A}_k^{(M_B,N_B)}} X, f \circ \pi_2] \in Exp\mathcal{M}_k,$$

where D is the divisor $D = -\sum_v M_v\, v$, and the fiber product is taken over the structure morphism $u : X \to \mathbf{A}_k^{(M_B,N_B)}$ of X as a $\mathbf{A}_k^{(M_B,N_B)}$–variety and the morphism $\alpha : \mathcal{L}(D) \to \mathbf{A}_k^{(M_B,N_B)}$ with components determined by mapping $\mathcal{L}(D)$ into $t^{M_v}\mathcal{O}_v$ via the inclusions $F \hookrightarrow F_v$.

With this summation notation understood, the Hrushovski–Kazhdan motivic Poisson summation formula takes the form

$$\sum_{x \in F} \varphi(x) = \mathbf{L}^{1-g} \sum_{y \in F} \mathcal{F}\varphi(y).$$

7.14. Categorical aspects of Poisson summation. We now return to our point of view based on the assembler category $\mathcal{C}_k^{KExp\mathcal{S}}$ and the associated spectrum $K(\mathcal{C}_k^{KExp\mathcal{S}})$ underlying the Grothendieck ring $Exp\mathcal{M}_k = \pi_0 K(\mathcal{C}_k^{KExp\mathcal{S}})$, where the Hrushovski–Kazhdan motivic Poisson summation formula takes place. Our goal here is to rephrase the Poisson summation at the level of objects and morphisms in the category $\mathcal{C}_k^{KExp\mathcal{S}}$ discussed in Proposition 7.5 above.

The identity in the Grothendieck ring with exponentials $Exp\mathcal{M}_k$

$$\sum_{x\in F}\varphi(x)=\mathbf{L}^{1-g}\sum_{y\in F}\mathcal{F}\varphi(y)$$

given by Poisson summation can be viewed as the following statement. Consider the classes

$$\sum_{x\in F}\varphi(x)=[\mathcal{L}(D)\times_{\mathbf{A}_k^{(M_B,N_B)}}X, f\circ\pi_X],$$

where $D=-\sum_v M_v v$ and

$$\sum_{y\in F}\mathcal{F}\varphi(y)=$$

$$\mathbf{L}^{-\sum_v N_v[k_v:k]}[\mathcal{L}(D')\times_{\mathbf{A}_k^{(M'_B,N'_B)}}(X\times_{\mathbf{A}_k^{(M_B,N_B)}}\mathbf{A}_k^{(M_B,N_B)}\times\mathbf{A}_k^{(M'_B,N'_B)}), f\circ\pi_X+\langle u\circ\pi_1,\pi_2\rangle_\omega],$$

with $D'=-\sum_v M'_v v=\sum_v N_v v-\sum_v c_v v$. We view these as classes in the Grothendieck ring $\pi_0 K(\mathcal{C}^{\mathcal{S}})$ of the assembler $\mathcal{C}^{\mathcal{S}}$ of algebraic stacks with exponentials (see Proposition 7.5). The Poisson summation formula means that these classes satisfy

$$\sum_{y\in F}\mathcal{F}\varphi(y)-\mathbf{L}^{g-1}\sum_{x\in F}\varphi(x)\,\in Range(\mathbf{Y}\cdot),$$

where $\mathbf{Y}\cdot:\pi_0K(\mathcal{C}^{\mathcal{S}})\to\pi_0K(\mathcal{C}^{\mathcal{S}})$, given by multiplication by $\mathbf{Y}=[\mathbf{A}^1,id]$ is the morphism induced by the endofunctor $\Phi:\mathcal{C}^{\mathcal{S}}\to\mathcal{C}^{\mathcal{S}}$ of Proposition 7.5 with

$$Exp\mathcal{M}_k=\pi_0(\mathcal{C}_k^{KExp\mathcal{S}})=Coker(\mathbf{Y}\cdot)$$

with $\mathcal{C}_k^{KExp\mathcal{S}}=\mathcal{C}^{\mathcal{S}}/\Phi$ the cofiber of the morphism of assemblers.

As shown in Section 1.3.2 of [ChamLoe15], it suffices to consider the case where the motivic adèlic function φ is a *simple function*, that is, the motivic analog of a product of characteristic functions of balls for ordinary Bruhat–Schwartz functions. These are defined by assigning a finite set $B\subset C(k)$ of points, an element $a=(a_v)_{v\in B}$ in $\prod_{v\in B}F_v$ and levels (M_v,N_v) with $ord(a_v)\geq M_v$ for all $v\in B$, with $\mathbf{A}_k^{(M_B,N_B)}$ as above. A simple function is a class of the form $[Spec(k),0]_{\mathbf{A}_k^{(M_B,N_B)}}$, where the morphism $u_a:Spec(k)\to\mathbf{A}_k^{(M_B,N_B)}$ assigns the t_v-expansion of a_v for each $v\in B$. This is regarded as the motivic counterpart of the characteristic function of the product of balls with centers a_v and radii N_v in F_v. More general motivic adèlic functions can be identified with families of simple functions $\varphi_{a(z)}$ parameterized by $z \in Z$, where Z is a k–variety.

The following result shows that the Poisson summation formula (viewed as a relation in the ring $Exp\mathcal{M}_k$) comes from a relation at the categorical level, described in terms of a covering family in the assembler $\mathcal{C}_k^{\mathcal{S}}$.

7.14.1. Theorem. *For a motivic adèlic Bruhat–Schwartz function*

$$\varphi = [X, f]_{\mathbf{A}_k^{(M_B,N_B)}} \in Exp\mathcal{M}_{\mathbf{A}_k^{(M_B,N_B)}},$$

denote by $\mathcal{Q}_\varphi$, resp. $\mathcal{F}\mathcal{Q}_\varphi$, the objects in the category $\mathcal{C}_k^{\mathcal{S}}$ of Proposition 7.5 given by

$$\mathcal{Q}_\varphi := (\mathcal{L}(D) \times_{\mathbf{A}_k^{(M_B,N_B)}} X, f \circ \pi_X) \ \in Obj(\mathcal{C}_k^{\mathcal{S}}),$$

$$\mathcal{F}\mathcal{Q}_\varphi := (\mathcal{L}(D') \times_{\mathbf{A}_k^{(M'_B,N'_B)}} X \times_{\mathbf{A}_k^{(M_B,N_B)}} \mathbf{A}_k^{(M_B,N_B)} \times \mathbf{A}_k^{(M'_B,N'_B)}, f \circ \pi_X + \langle u \circ \pi_1, \pi_2 \rangle_\omega).$$

Then there is a covering family in $\mathcal{C}_k^{\mathcal{S}}$

$$((\mathbf{A}^{\deg(D)+g-1}, 0) \times \mathcal{Q}_{\varphi,1} \hookrightarrow \mathcal{F}\mathcal{Q}_\varphi, \mathcal{F}\mathcal{Q}_{\varphi,2} \hookrightarrow \mathcal{F}\mathcal{Q}_\varphi),$$

where $\mathcal{Q}_{\varphi,1}$ is a family of simple functions with $a(z) \in \mathcal{L}(div(\omega) + D)^\perp$ (the orthogonal with respect to the Serre duality pairing), and $\mathcal{F}\mathcal{Q}_{\varphi,2}$ is in the range of the functor $\Phi : \mathcal{C}_k^{\mathcal{S}} \to \mathcal{C}_k^{\mathcal{S}}$ of Proposition 7.5. This covering family lifts to the level of the assembler $\mathcal{C}_k^{\mathcal{S}}$ the motivic Poisson summation formula in $Exp\mathcal{M}_k$.

Proof. The definition of $\sum_{x \in F} \varphi(x)$ can be equivalently described as the composition of the pushforward to $Exp\mathcal{M}_k$ of the class in $Exp\mathcal{M}_{\mathcal{L}(D)}$ given by the pullback of the class φ in $Exp\mathcal{M}_{\mathbf{A}_k^{(M_B,N_B)}}$, along the morphism $\mathcal{L}(D) \hookrightarrow \mathbf{A}_k^{(M_B,N_B)}$.

Using this description, we can refer to the Theorem 1.3.10 of [ChamLoe15] which shows that if φ a simple function, the summation $\sum_{y \in F} \mathcal{F}\varphi(y)$ is zero unless $ord_v(y_v) + ord_v(D) \geq 0$. Moreover, if $ord_v(y_v) + ord_v(D) \geq 0$, then he computation reduces to the case of a linear function as morphism to $\mathbf{A}^1$, where (as we recalled in the proof of Theorem 7.12 above) the resulting class in $\pi_0 K(\mathcal{C}_k^{\mathcal{S}})$ is in the range of multiplication by $\mathbf{Y} = [\mathbf{A}^1, id]$ if the linear map is nonzero and is a power of $\mathbf{L}$ if it is zero.

This vanishing condition is satisfied when the element $a = (a_v)$, that determines the simple function φ, belongs to the orthogonal of $\mathcal{L}(div(\omega) + D)$ with respect to the Serre duality pairing, and in that case the resulting class is equal to $\mathbf{L}^{-\deg(D)+\dim \mathcal{L}(div(\omega)+D)}$. Thus, for the simple function $\varphi = \varphi_a$ the summation of the motivic Fourier transforms, seen as a class in $\pi_0 K(\mathcal{C}_k^{\mathcal{S}})$ is given by

$$\sum_{y \in F} \mathcal{F}\varphi(y) = \mathbf{L}^{-\deg(D)+\dim \mathcal{L}(div(\omega)+D)}$$

when $a = (a_v)_{v \in B} \in \mathcal{L}(div(\omega) + D)^{\perp}$, and is in the range of multiplication by $\mathbf{Y} = [\mathbf{A}^1, id]$ otherwise.

By the same argument, one can show that the left–hand–side $\sum_{x \in F} \varphi(x)$ belongs to the range of multiplication by $\mathbf{Y} = [\mathbf{A}^1, id]$ unless $a = (a_v)_{v \in B} \in \mathcal{L}(div(\omega)+D)^{\perp}$, and equal to $\mathbf{L}^{\dim \mathcal{L}(-D)}$ in that case. The Riemann–Roch formula for curves

$$\dim \mathcal{L}(-D) = \dim \mathcal{L}(div(\omega) + D) - \deg(D) + 1 - g$$

then shows, that the respective classes satisfy

$$\sum_{y \in F} \mathcal{F}\varphi(y) - \mathbf{L}^{g-1} \sum_{x \in F} \varphi(x) \in Range(\mathbf{Y}\cdot).$$

This argument extends from the case of simple functions to the more general case by identifying general motivic adèlic functions with families of simple functions over a parameterising k–variety Z.

In this case, one starts with a decomposition $Z = Z_1 \sqcup Z_2$, where points $z \in Z_1$ have corresponding simple functions $\varphi_{a(z)}$ with $a(z) \in \mathcal{L}(div(\omega) + D)^{\perp}$. We denote by $Z_2 = Z \smallsetminus Z_1$ the complementary set where this condition is not satisfied. The corresponding class $\sum_{y \in F} \mathcal{F}\varphi(y)$ then decomposes into a Z_1–part, that can be identified as above with the corresponding Z_1–part of $\mathbf{L}^{\deg(D)+g-1} \sum_x \varphi(x)$, and a Z_2–part belonging to $Range(\mathbf{Y}\cdot)$.

Let now X be a $\mathbf{A}_k^{(M_B,N_B)}$–variety, endowed with a morphism $f : X \to \mathbf{A}^1$, whose class $\varphi = [X, f]_{\mathbf{A}_k^{(M_B,N_B)}}$ belongs to $Exp\mathcal{M}_{\mathbf{A}_k^{(M_B,N_B)}}$. Consider the objects $\mathcal{Q}_\varphi$ and $\mathcal{F}\mathcal{Q}_\varphi$ of $\mathcal{C}_k^{\mathcal{S}}$ as in the statement, which correspond respectively to classes

$$[\mathcal{Q}_\varphi] = \sum_{x \in F} \varphi(x)$$

and

$$[\mathcal{F}\mathcal{Q}_\varphi] = \mathbf{L}^{\deg(D)} \sum_{y\in F} \mathcal{F}\varphi(y)$$

in $\pi_0 K(\mathcal{C}_k^{\mathcal{S}})$.

In particular, for a simple function $\varphi = \varphi_a = [Spec(k), 0]_{\mathbf{A}_k^{(M_B,N_B)}}$ we write simply $\mathcal{Q}_a$ and $\mathcal{F}\mathcal{Q}_a$ for the corresponding objects in $\mathcal{C}_k^{\mathcal{S}}$ as above. The situation described above for the Grothendieck classes corresponds to decompositions $[\mathcal{Q}_\varphi] = [\mathcal{Q}_{\varphi,1}] + [\mathcal{Q}_{\varphi,2}]$ and $[\mathcal{F}\mathcal{Q}_\varphi] = [\mathcal{F}\mathcal{Q}_{\varphi,1}] + [\mathcal{F}\mathcal{Q}_{\varphi,2}]$, where the first term corresponds to the subset where the $\mathcal{Q}_{a(z)}$ and $\mathcal{F}\mathcal{Q}_{a(z)}$ have $a(z) \in \mathcal{L}(div(\omega) + D)^\perp$, and the second is the complementary case as above. The argument described above for the classes can then be rephrased, as we did in Theorem 7.12 above, as the statement that, the object $\mathcal{F}\mathcal{Q}_\varphi$ in $\mathcal{C}_k^{\mathcal{S}}$ has a disjoint covering family

$$(\mathcal{F}\mathcal{Q}_{\varphi,1} \hookrightarrow \mathcal{F}\mathcal{Q}_\varphi, \mathcal{F}\mathcal{Q}_{\varphi,2} \hookrightarrow \mathcal{F}\mathcal{Q}_\varphi)$$

where one can identify $\mathcal{F}\mathcal{Q}_{\varphi,1}$ with $(\mathbf{A}^{\deg(D)+g-1}, 0) \times \mathcal{Q}_{\varphi,1}$ and where $\mathcal{F}\mathcal{Q}_{\varphi,2}$ is in the range of the functor $\Phi : \mathcal{C}_k^{\mathcal{S}} \to \mathcal{C}_k^{\mathcal{S}}$. $\blacksquare$

A generalisation of the Hrushovski–Kazhdan motivic Poisson summation formula was later obtained by Bilu in [Bilu18]. This more general motivic Poisson summation applies also to the case of motivic test functions that are infinite products. This requires a different new technique based on motivic Euler products, see Chapters 3 and 5 of [Bilu18].

7.14.2. Question. *Is there a categorification of Bilu's motivic Euler products and of the resulting motivic Poisson summation formula, in an appropriate assembler category, as discussed here, or in the category of exponential Nori motives recalled in Section 7.9?*

7.15. Motivic height zeta function After reviewing the Grothendieck ring with exponential and the motivic Poisson summation, and discussing some of their categorical properties in terms of assembler categories, we return to the main theme of height functions. In this subsection we summarize and compare the results of [ChamTsch02], [ChamTsch12] on the height zeta function of equivariant compactifications of vector groups, and the motivic counterpart of [ChamLoe15] based on a motivic version of the height zeta function and the motivic Poisson summation that we recalled in the previous subsections. We refer the reader to [ChamLoe15] and to the introductory chapter of [Bilu18] for a more detailed overview of this material.

As already discussed in Section 2 of this paper, given a projective Fano variety (or almost Fano as in [Pe01]) X over a number field F, endowed with an ample line bundle L, one has an associated height function $h = h_L : X(F) \to \mathbf{R}_+$. For $V \subset X$ and $B > 0$, the asymptotic behavior of

$$N_{V,h}(B) = \#\{x \in V(F) \,|\, h(x) \leq B\}$$

when $V(F)$ is infinite and for $B \to \infty$, is expected to be of the form

$$N_{V,h}(B) \sim C\, B^a (\log B)^{b-1},$$

for real numbers $C > 0$, $a > 0$ and for a half–integer $b \in \frac{1}{2}\mathbf{Z}$, with $b \geq 1$, where these parameters should carry a geometric interpretation. As we already discussed in Section 4, in the case of the height function associated to the anticanonical line bundle, the expected asymptotics has $a = 1$, $b = rkPic(X)$ and C expressible in terms of volumes of adelic spaces and a cohomological factor (see [Pe95], [BatTsch98]).

The investigation of the asymptotic behavior of the counting function $N_{V,h}(B)$ can be reformulated in terms of questions about the associated height zeta function

$$\zeta_{V,h}(s) = \sum_{x \in V(F)} h(x)^{-s}.$$

In particular, one can ask whether there is a dense open subvariety $V \subset X$ such that $\zeta_{V,h}(s)$ is absolutely convergent for $Re(s) > 1$ with meromorphic continuation to some strip $Re(s) > 1 - \delta$ and a unique pole of order $r = rk\, Pic(X)$ at $s = 1$.

The case of toric varieties was treated in [BatTsch98] using Poisson summation, and this approach was generalized to varieties with an action of an algebraic group with an open dense orbit (equivariant compactifications of algebraic groups). The case of vector groups, that is, equivariant compactifications of $\mathbf{G}_a^n$, was studied in [ChamTsch02] and [ChamTsch12] by considering the height zeta function

$$\zeta_{G,h}(s) = \sum_{x \in G(F)} h(x)^{-s}$$

and using Poisson summation for $G(F)$ as a discrete subgroup of $G(A_F)$, leading to the identity

$$\sum_{x \in G(F)} h(x)^{-s} = \sum_{y \in G(F)} \mathcal{F}(h^{-s})(y).$$

The Fourier transform $\mathcal{F}(h^{-s})(y)$ is then written as a product of local factors

$$\mathcal{F}(h^{-s})(y) = \prod_v \mathcal{F}(h_v^{-s})(y_v).$$

This can be expressed in terms of local Igusa zeta functions, which makes it then possible to show the existence of meromorphic continuation and identify the location and order of the main pole.

The formulation of Manin's problem on the asymptotic behavior of the number of rational points of bounded height in terms of the height zeta functions extends to the case of function fields. For a smooth projective curve C over a finite field $k = \mathbf{F}_q$, with function field $F = k(C)$, and an almost Fano variety X over F for which the set $X(F)$ is Zariski dense in X, one considers as above the height function h associated to the anti-canonical line bundle and the height zeta function $\zeta_{V,h}(t)$ with $t = q^{-s}$, and with $V \subset X$ an open dense subset. Then the above question (absolute convergence in the disk $|t| < q^{-1}$, meromorphic continuation in some disk $|t| < q^{-1+\delta}$, unique pole of order $r = rkPic(X)$ at q^{-1}) can also be formulated in this setting.

In this function field setting, given a model $\mathcal{X}$ of X over C with $u : \mathcal{X} \to C$, the rational points in $V(F)$ correspond to sections $\sigma : C \to \mathcal{X}$ of u with $\sigma(\eta_C) \in V(F)$ for the generic point η_C, and the height with respect to a line bundle L is given by $h(\sigma) = q^{\deg \sigma^* L}$. This makes it possible to rewrite the height zeta function in the form

$$\zeta_{V,h}(s) = \sum_{x \in V(F)} h(x)^{-s} = \sum_{d \geq 0} N_{V,L,d}\, q^{-ds},$$

where

$$N_{V,L,d} := \#\{\sigma : C \to \mathcal{X} \mid \sigma(\eta_C) \in V(F),\ \deg \sigma^* L = d\}.$$

This more geometric formulation then suggests the existence of a motivic version of the same problem, where the counting function $N_{V,L,d}$ is replaced by a class in the Grothendieck ring of varieties. This is the approach introduced by Chambert–Loir and Loeser in [ChamLoe15]. In this reformulation, one needs to show that the sections $\sigma : C \to \mathcal{X}$ satisfying the conditions $\sigma(\eta_C) \in V(F)$ and $\deg \sigma^* L = d$ form a moduli space $M_{V,L,d}$ that is a quasi-projective k–scheme, so that it makes sense to consider the Grothendieck classes $[M_{V,L,d}] \in K_0(\mathcal{V}_k)$ and one can form a motivic height zeta function

$$Z_{V,L}(T) := \sum_{d \in \mathbf{Z}} [M_{V,L,d}]\, T^d,$$

seen as a formal series in $K_0(\mathcal{V}_k)[[T]][T^{-1}]$. The analog of the height problem in this motivic setting becomes showing that the series $Z_{V,L}(T)$, viewed as an element in $\mathcal{M}_k[[T]][T^{-1}]$ with $\mathcal{M}_k$ the localization of the Grothendieck ring, has the property that

$$(1-\mathbf{L}^a T^a)^b Z_{V,L}(T) = P(T)$$

where $P(T)$ is an element in the subring of $\mathcal{M}_k[[T]][T^{-1}]$ generated by the inverses of the polynomials $1-\mathbf{L}^\alpha T^\beta$ for $\beta > \alpha \geq 0$, with value $P(\mathbf{L}^{-1})$ at $T=\mathbf{L}^{-1}$ given by a nontrivial effective class.

This problem can be formulated more generally for the data of a smooth projective curve C over a field k and a variety X with a line bundle L and an open subset $V \subset X$, with structure morphism $u : X \to C$, for which the moduli space $M_{V,L,d}$ of sections exists as a constructible set so that the Grothendieck class $[M_{V,L,d}]$ can be considered.

This motivic problem was solved affirmatively by Chambert–Loir and Loeser in [ChamLoe15] in the case of equivariant compactifications of $\mathbf{G}_a^n$. The argument is conceptually similar to the original case of [ChamTsch02] and [ChamTsch12] for the ordinary height zeta function. The Hrushovski–Kazhdan motivic Poisson summation formula replaces the ordinary Poisson summation formula, by writing the motivic height zeta function in the form

$$Z_{V,L}(T) = \sum_{x\in G(F)} \prod_{v\in B} (\sum_{m\in\mathbf{Z}} \varphi_{v,m}(x) T^m),$$

where the $\varphi_{v,m}(x)$ are motivic Bruhat–Schwartz functions as recalled in the previous subsections, vanishing for $m << 0$. Applying termwise in T the Hrushovski–Kazhdan motivic Poisson summation formula, we get

$$Z_{V,L}(T) = \mathbf{L}^{(1-g)n} \sum_{y\in G(F)} \hat{Z}_{V,L}(Y,y),$$

where

$$\hat{Z}_{V,L}(T,y) = \prod_v \hat{Z}_{V,L,v}(T,y), \quad \text{with} \quad \hat{Z}_{V,L,v}(T,y) = \sum_m \mathcal{F}_v \varphi_{v,m}(y)\, T^m.$$

Here the Hrushovski–Kazhdan motivic Poisson summation formula is used in the multidimensional version

$$\sum_{x\in F^n} \varphi(x) = \mathbf{L}^{(1-g)n} \sum_{y\in F^n} \mathcal{F}\varphi(y),$$

which is proved as in the case $n = 1$ we recalled in the previous subsections (see Theorem 1.3.10 of [ChamLoe15]). The classes $[M_{V,L,d}]$ in the coefficients of $Z_{V,L}(T)$ are classes in $K_0(\mathcal{V}_k)$. After considering them as classes in the localization $\mathcal{M}_k$, and using the embedding $\mathcal{M}_k \hookrightarrow Exp\mathcal{M}_k$, the Hrushovski–Kazhdan motivic Poisson summation formula applies and the resulting terms $\mathcal{F}_v\varphi_{v,m}(y)$ give classes in the Grothendieck ring with exponentials $Exp\mathcal{M}_k$.

The $y = 0$ term is identified as the leading term responsible for the main pole. This term $\hat{Z}_v(T, 0)$ is a motivic version of the local Igusa zeta function and is understood in terms of motivic integration. This method is used in [ChamLoe15] to show both the rationality and identify the leading pole of the motivic height zeta function.

7.16. The lifting problem for zeta functions. In the context of categorical structures underlying Grothendieck rings, one can one can ask, which particular zeta functions of arithmetic origin may be lifted to the categorical level. In the case of zeta functions associated to exponentiable motivic measures that satisfy conditions of rationality and factorization, a lift to the categorification of the Witt ring was constructed in [LMM19]. We review this construction here. We then outline the question of a possible categorification for the motivic height zeta function.

7.16.1. Witt rings and exponentiable motivic measures. Let R be an associative and commutative ring. Let End_R be the category of endomorphisms of projective R–modules of finite rank. The objects of this category are pairs (E, f), where $f \in End_R(E)$. With the direct sum and the tensor product defined componentwise on the objects, the Grothendieck group $K_0(End_R)$ also acquires a commutative ring structure. Let $K_0(R)$ be the ideal generated by the pairs of the form $(E, f = 0)$. Then one defines

$$W_0(R) = K_0(End_R)/K_0(R).$$

The ring $W_0(R)$ embeds as a dense subring of the big Witt ring $W(R)$ via the map

$$L : (E, f) \mapsto \det(1 - t\, M(f))^{-1},$$

where $M(f)$ is the matrix representing $f \in End_R(E)$, and $\det(1 - t\, M(f))^{-1}$ is viewed as an element in $\Lambda(R) = 1 + tR[[t]]$. The subring $W_0(R) \hookrightarrow W(R)$ consists of the rational Witt vectors

$$W_0(R) = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} \mid a_i, b_i \in R,\ n, m \geq 0 \right\}.$$

We introduce now into this picture *a motivic measure*, as discussed in Section 7.8, namely a ring homomorphism from either the Grothendieck ring of varieties or that of varieties with exponentials, $\mu : K_0(\mathcal{V}_k) \to R$ or $\mu : KExp_k \to R$. We also consider, as in Section 7.8, the associated zeta function ζ_μ defined by applying the motivic measure μ to the Kapranov motivic zeta function

$$\zeta_\mu(X,t) = \sum_n \mu(Sym^n(X))t^n$$

in the first case, and

$$\zeta_\mu(X,f,t) = \sum_n \mu(Sym^n(X), f^{(n)})t^n$$

in the case with exponentials.

We can regard $\zeta_\mu(X,t)$ as defining an element in the Witt ring $W(R)$.

The addition in $K_0(\mathcal{V})$ is mapped by the zeta function to the addition in $W(R)$, which is the usual product of the power series,

$$\zeta_\mu(X \sqcup Y, t) = \zeta_\mu(X,t) \cdot \zeta_\mu(Y,t) = \zeta_\mu(X,t) +_{W(R)} \zeta_\mu(Y,t),$$

and similarly in the case with exponentials.

A motivic measure $\mu : K_0(\mathcal{V}) \to R$ is called *exponentiable* (see [Ram15], [RamTab15]), if the zeta function $\zeta_\mu(X,t)$ determines a *ring homomorphism*

$$\zeta_\mu : K_0(\mathcal{V}) \to W(R),$$

that is, if it satisfies $\zeta_\mu(X \sqcup Y, t) = \zeta_\mu(X,t) +_{W(R)} \zeta_\mu(Y,t)$ as above and it also satisfies

$$\zeta_\mu(X \times Y, t) = \zeta_\mu(X,t) \star_{W(R)} \zeta_\mu(Y,t).$$

with the product $\star_{W(R)}$ of the Witt ring.

Similarly, a motivic measure $\mu : KExp_k \to R$ is exponentiable if the associated zeta function ζ_μ is a ring homomorphism $\zeta_\mu : KExp_k \to W(R)$. (Note that the term "exponential" here has two different meanings: as exponential sums in $KExp_k$ and as exponentiability of measures in the properties of ζ_μ. The context should help the reader to avoid confusion.)

A motivic measure $\mu : K_0(\mathcal{V}) \to R$ or $\mu : KExp_k \to R$ is *rational* if the zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W(R)$ takes values in the subring $W_0(R)$ of the Witt ring $W(R)$.

Moreover, a motivic measure $\mu : K_0(\mathcal{V}) \to R$ or $\mu : KExp_k \to R$ is called *factorizable* if it is rational and it admit a factorization into linear factors

$$\zeta_\mu(X,t) = \frac{\prod_i (1-\alpha_i t)}{\prod_j (1-\beta_j t)}.$$

The latter expression given by a ratio of polynomials can also be written as a difference in the Witt ring

$$\zeta_{\mu,+}(X,t) \ -_W \ \zeta_{\mu,-}(X,t),$$

where $\zeta_{\mu,+}(X,t) = \prod_j (1-\beta_j t)^{-1}$ and $\zeta_{\mu,-}(X,t) = \prod_i (1-\alpha_i t)^{-1}$.

The motivic measure on $K_0(\mathcal{V}_K)$ given by the counting function is an exponentiable motivic measure, in the sense of [Ram15], [RamTab15].

One can consider the same question for the motivic measure $\mu : KExp_K \to \mathbf{C}$ discussed in Section 7.8, given by the exponential sum.

In general, a useful feature in the theory of zeta functions that makes it possible to write them in the form of ratios of polynomials, is based on a simple identity relating the generating series of traces of powers of an endomorphism σ and its characteristic polynomial,

$$\exp\left(\sum_{m=1}^{\infty} (tr\,\sigma^m) \frac{t^m}{m}\right) = \frac{1}{\det(1-\sigma t)}.$$

Exponential sums are usually very difficult to evaluate explicitly. Cases when this is possible typically reduce to expressing the exponential sums as traces of Frobenius on certain ℓ–adic sheaves.

7.16.2. Categorification of Witt vectors and lifting of zeta functions. There are different ways to obtain a categorification and spectrification of Witt vectors. A spectrification of the ring $W(R)$ of Witt vectors was introduced in [Hess97]. We will describe here a different categorification and spectrification of $W_0(R)$ obtained in Section 6.2 of [LMM19], based on its description in terms of the K_0 of the endomorphism category End_R and of R, and the formalism of Segal Gamma–spaces.

Let $\mathcal{P}_R$ denote the category of finite projective modules over a commutative ring R with unit, and let End_R be the endomorphism category as above. By the Segal construction, we obtain associated Γ–spaces $F_{\mathcal{P}_R}$ and F_{End_R} and spectra that we write as $F_{\mathcal{P}_R}(\mathbf{S}) = K(R)$ (the K–theory spectrum of the ring R), and $F_{End_R}(\mathbf{S})$ (the spectrum of the endomorphism category) respectively.

The spectrum $\mathbf{W}(R)$ is then defined as the cofiber $\mathbf{W}(R) := F_{End_R}(\mathbf{S})/F_{\mathcal{P}_R}(\mathbf{S})$ obtained from these Γ–spaces. It is induced by the inclusion of the category $\mathcal{P}_R$ of finite projective modules as the subcategory of the endomorphism category. The spectrum $\mathbf{W}(R)$ has $\pi_0\mathbf{W}(R) = W_0(R)$. (We refer the reader to Section 6 of [LMM19] for a more detailed discussion.)

It is useful to consider also a variant of the above construction with $\mathcal{P}_R^{\pm}$ and $End_R^{\pm}$ the categories of $\mathbf{Z}/2\mathbf{Z}$–graded finite projective R–modules and $\mathbf{Z}/2\mathbf{Z}$–graded endomorphism category with objects given by pairs $\{(E_+, f_+), (E_-, f_-)\}$. Writing objects as $(E_\pm, f_\pm)$, the morphisms are given by morphisms $\phi : E_\pm \to E'_\pm$ of $\mathbf{Z}/2\mathbf{Z}$–graded finite projective modules that commute with $f_\pm$.

The map $\delta : K_0(End_R^{\pm}) \to K_0(End_R)$ given by $[E_\pm, f_\pm] \mapsto [E_+, f_+] - [E_-, f_-]$ induces a ring homomorphism

$$K_0(End_R^{\pm})/K_0(\mathcal{P}_R^{\pm}) \to K_0(End_R)/K_0(R) \simeq W_0(R).$$

As above, the categories $\mathcal{P}_R^{\pm}$ and $End_R^{\pm}$ have associated Γ-spaces $F_{\mathcal{P}_R^{\pm}}$ and $F_{End_R^{\pm}}$ and spectra. We write $\mathbf{W}^{\pm}(R) = F_{End_R^{\pm}}(\mathbf{S})/F_{\mathcal{P}_R^{\pm}}(\mathbf{S})$ for the cofiber of $F_{\mathcal{P}_R^{\pm}}(\mathbf{S}) \to F_{End_R^{\pm}}(\mathbf{S})$.

A factorizable motivic measure $\mu : K_0(\mathcal{V}) \to R$ determines a functor $\Phi_\mu : \mathcal{C}_{\mathcal{V}} \to End_R^{\pm}$ where $\mathcal{C}_{\mathcal{V}}$ is the assembler category connected to the Grothendieck ring $K_0(\mathcal{V})$ of varieties (or the assembler category underlying the Grothendieck ring with exponentials in the case of a motivic measure $\mu : KExp_k \to R$) and $End_R^{\pm}$ is the $\mathbf{Z}/2\mathbf{Z}$-graded endomorphism category described above.

As shown in Sections 6.2 and 6.3 of [LMM19], this is obtained, starting with a factorization

$$\zeta_\mu(X,t) = \frac{\prod_{i=1}^n (1-\alpha_i t)}{\prod_{j=1}^m (1-\beta_j t)},$$

by considering $E_+^{X,\mu} = R^{\oplus m}$ and $E_-^{X,\mu} = R^{\oplus n}$ with endomorphisms $f_\pm^{X,\mu}$ respectively given in matrix form by $M(f_+^{X,\mu}) = diag(\beta_j)_{j=1}^m$ and $M(f_-^{X,\mu}) = diag(\alpha_i)_{i=1}^n$. The pair $(E_\pm^{X,\mu}, f_\pm^{X,\mu})$ is an object of the endomorphism category $End_R^\pm$. Embeddings $Y \hookrightarrow X$ correspond to multiplicative decompositions of the zeta function, and the factorization of each term then determines the associated morphism in the endomorphism category.

In Section 6.3 of [LMM19] it is then shown, that the functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \to End_R^\pm$ induces a map of Γ–spaces and of associated spectra $\Phi_\mu : K(\mathcal{V}) \to F_{End_R^\pm}(\mathbf{S})$. The induced maps on the homotopy groups have the property that the composition $\delta \circ \Phi_\mu : K(\mathcal{V}) \to K_0(End_R)$ followed by the quotient map $K_0(End_R) \to K_0(End_R)/K_0(R) = W_0(R)$, is given by the zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W_0(R)$. Thus, the functor Φ_μ and the induced map of spectra can be regarded as the appropriate categorification and spectrification of the zeta function.

7.17. The lifting problem for height zeta functions. In the same vein as the previous discussion of categorification and spectrification of certain classes of zeta functions, one can ask whether a form of categorification and spectrification may be possible for height zeta function as well. The case of the motivic height zeta function reviewed in the previous subsection is especially interesting, because it lies outside of the class of zeta functions discussed above, hence a different approach would be needed.

7.17.1. Question. *Is there a categorification and spectrification of the motivic height zeta function?*

While we do not provide an answer to this question in the present paper, through the rest of this section we discuss some general aspects of this question and we highlight what we consider to be the most relevant and useful properties of the motivic height zeta functions for approaching this problem.

Already by considering the case of the ordinary height zeta function

$$\zeta_{U,h}(s) = \sum_{x \in U(F)} h(x)^{-s}$$

rather than their motivic counterparts, one can identify two main problems with respect to the possible construction of a categorification and spectrification along the lines discussed in the previous subsections.

The first problem is the fact that these zeta functions in general do not exhibit the same nice behaviour of other arithmetic cases such as the Hasse–Weil zeta function, hence do not have the properties required for the categorification of [LMM19] reviewed in the previous subsection. The second main problem is the fact that one would like to be able to describe them in terms of a motivic measure on the $\pi_0 K(\mathcal{C})$ of the spectrum $K(\mathcal{C})$ of an assembler category $\mathcal{C}$.

For the first problem, the Poisson summation approach initiated in [BaTsch98], and applied in [ChamTsch02], [ChamTsch12] to equivariant compactifications of vector groups, suggests that the object of interest for a possible lifting of the zeta function should be the Fourier transform $\mathcal{F}(h^{-s})$ and its local factors, rather than directly working with the height zeta function itself, since the local Igusa zeta functions that express these local factors are better behaved functions. The reformulation in motivic terms can help highlighting some useful properties with respect to the second problem mentioned above.

7.17.2. Grothendieck classes and symmetric products. We first recall a convenient formalism for symmetric products introduced in [Bilu18] to the purpose of studying motivic Euler product decompositions. Consider again the Kapranov motivic zeta function

$$Z_X(t) = \sum_{n\geq 0} [Sym^n X]\, t^n,$$

as recalled in Section 7.8.

Given a positive integer n, let $\Pi(n)$ denote the set of partitions of n. For $\pi \in \Pi(n)$ a partition, one denotes by $Sym^\pi X$ the locally closed subset of $Sym^n X$ consisting of those effective zero–cycles of degree n that realise the partition π. Equivalently, for a partition π of the form $n = \sum_i n_i$, one obtains $Sym^\pi X$ as the quotient of the complement of the diagonals in $X^n = X^{\sum_i n_i} = \prod_i X^{n_i}$ by the action of the product of symmetric groups $\prod_i \Sigma_{n_i}$.

Consider then the case of a k–variety X and a family $\mathcal{X}$ of X–varieties $\mathcal{X} = (X_i)_{i\in\mathbf{N}}$ with structure morphisms $u_i : X_i \to X$. Given a partition $\pi = \{n_i\}$ with $\sum_i n_i = n$, consider the space $\prod_i X_i^{n_i}$ and the complement of the diagonals $(\prod_i X^{n_i}) \smallsetminus \Delta$. As is customary in the case of configuration spaces, here Δ stands for the union of all the diagonals (that is, the locus where two or more of the coordinates coincide), rather than just the deepest diagonal where all coordinates coincide. We can construct the fibered product

$$\prod_i X_i^{n_i} \times_{\prod_i X^{n_i}} ((\prod_i X^{n_i}) \smallsetminus \Delta)$$

with the maps $u = (u_i)$ and the inclusion. One defines $Sym^\pi \mathcal{X}$ as the quotient of this fibered product by the group $\prod_i \Sigma_{n_i}$. There is a natural map $Sym^\pi \mathcal{X} \to Sym^\pi X$. One can define a multivariable Kapranov zeta function

$$Z_{\mathcal{X}}(\underline{t}) := \sum_\pi [Sym^\pi \mathcal{X}] \underline{t}^\pi,$$

with $\underline{t}^\pi = \prod_i t_i^{n_i}$.

Given a closed embedding $Y \subset X$ with open complement $U = X \smallsetminus Y$, one has the relation between Grothendieck classes of symmetric products

$$[Sym^n X] = \sum_{r=0}^n [Sym^r Y] \cdot [Sym^{n-r} U],$$

which extends to the case of partitions as

$$[Sym^\pi X] = \sum_{\pi' \leq \pi} [Sym^{\pi'} Y] \cdot [Sym^{\pi - \pi'} U].$$

Similarly, for a family $\mathcal{X}$ of X–varieties with $\mathcal{Y} \subset \mathcal{X}$, $\mathcal{Y} = (Y_i)_{i \in \mathbf{N}}$, and their complements $\mathcal{U} = (U_i = X_i \smallsetminus Y_i)_{i \in \mathbf{N}}$, one has

$$[Sym^\pi (\mathcal{X})] = \sum_{\pi' \subset \pi} [Sym^{\pi'} (\mathcal{Y})] \cdot [Sym^{\pi - \pi'} \mathcal{U}],$$

which implies the factorization

$$Z_{\mathcal{X}}(\underline{t}) = Z_{\mathcal{Y}}(\underline{t}) \cdot Z_{\mathcal{U}}(\underline{t}).$$

The reason for introducing these multivariable Kapranov zeta functions (see [Bilu18]) is summarized in the next subsection.

7.17.3. Summary of the geometric setting. In order to better identify the underlying geometry, relevant to the possible construction of a related assembler category, we recall the following general setting from [Bilu18]. As above we have a smooth projective curve C over k with function field $F = k(C)$. We also consider a smooth equivariant compactification X of $G = \mathbf{G}_a^n$ as a smooth projective F–scheme, with $X \smallsetminus G = D_G$ a strict normal crossings divisor of components $(D_\alpha)_{\alpha\in\mathcal{A}}$, as well as a partial compactification given by a G–invariant quasi–projective scheme $U \subset X$, with strict normal crossings divisor $X \smallsetminus U = D \subset D_G$, where $-K_X(D)$ is the log–anticanonical class of U.

This geometry is also described in terms of a good model given by a k–scheme $\mathcal{X}$ with $u : \mathcal{X} \to C$ with an open $\mathcal{U} \subset \mathcal{X}$, with $\mathcal{X} \smallsetminus \mathcal{U}$ a strict normal crossings divisor, together with a line bundle $\mathcal{L}$ over $\mathcal{X}$ that restricts to $-K_X(D)$ on the generic fiber X. For $v \in C(k)$, if $\mathcal{B}_v$ denotes the set of irreducible components of $u^{-1}(v)$ and $\mathcal{B} = \sqcup_{v\in C(k)}\mathcal{B}_v$, the line bundle $\mathcal{L}$ can be written as $\sum_\alpha \lambda_\alpha \mathcal{L}_\alpha$, for integers λ_α and for line bundles $\mathcal{L}_\alpha = \mathcal{D}_\alpha + \sum_{\beta\in\mathcal{B}} c_{\alpha,\beta} E_\beta$ with E_β the component corresponding to $\beta \in \mathcal{B}$.

Given such a setting, one can consider the moduli spaces $M_{\underline{n},U}$, for $\underline{n} = (n_\alpha)_{\alpha\in\mathcal{A}}$ of sections $\sigma : C \to \mathcal{X}$ with $\sigma(\eta_C) \subset G$, satisfying the integrality condition $\sigma(C_0) \subset \mathcal{U}$ for $C_0 \subset C$ a subset of places, and satisfying $\deg(\sigma^*\mathcal{L}_\alpha) = n_\alpha$. The moduli spaces $M_{n,U}$ recalled earlier in this section are decomposed as a disjoint union of the $M_{\underline{n},U}$ for $n = \sum_\alpha \lambda_\alpha n_\alpha$. As shown in Section 6 of [Bilu18], one can further decompose the $M_{\underline{n},U}$ into level sets of intersection indices of the sections with the components of the boundary divisor. Namely, given an element $g \in G(F)$ and the corresponding section $\sigma_g : C \to \mathcal{X}$ that extends it, one has

$$\deg(\sigma_g^*(\mathcal{D}_\alpha)) = \sum_{v\in C(k)} (g, \mathcal{D}_\alpha)_v,$$

where $(g, \mathcal{D}_\alpha)_v$ denote the intersections indices. Similarly, one has intersection indices $(g, E_\beta)_v$, which have value 1 for a single E_β and zero otherwise ([Bilu18], Sec.6.2.2 and [ChamLoe15], Sec.3.3). One can define the level sets of these intersection indices as

$$G(\underline{m}_v, \beta)_v = \{g \in G(F_v) \,|\, (g, E_\beta)_v = 1 \;\text{ and }\; (g, \mathcal{D}_\alpha)_v = m_{v,\alpha}\},$$

where $\underline{m}_v = (m_{v,\alpha})_{\alpha\in\mathcal{A}}$ and $\sum_{v\in C(k)} \underline{m}_v = \underline{n} = (n_\alpha)_{\alpha\in\mathcal{A}}$. When taking also into account the property $\sigma(C_0) \subset \mathcal{U}$, these points $\underline{m}_v = (m_{v,\alpha})_{\alpha\in\mathcal{A}}$ are parametrised by

$$Sym^{\underline{n}'}(C \smallsetminus C_0) \times Sym^{\underline{n}''}(C).$$

Here $\underline{n}'$ consists of those n_α, for which D_α is a component of $X \smallsetminus U$, and $\underline{n}''$ consists of the n_α of the remaining components of $X \smallsetminus G$. The sets $H(\underline{m}, \beta)_v$ are defined as $G(\underline{m}_v, \beta)_v$ when $(\underline{m}, \beta)$ satisfy the integrality condition and $\emptyset$ otherwise. It is shown in Section 6.2.6 of [Bilu18] that the Grothendieck classes of the subsets $M_{\underline{n},\beta}$ of the moduli spaces of sections $M_{n,U}$ with assigned intersection indices decompose as

$$[M_{\underline{n},\beta}] = \sum_{\underline{m} \in Sym^{\underline{n}^\beta}(C)} [H(\underline{m}, \beta) \cap G(F)],$$

where $\underline{n}^\beta = (n^\beta_\alpha)_{\alpha \in \mathcal{A}}$, where $\deg(\sigma^* \mathcal{D}_\alpha) = n^\beta_\alpha := n_\alpha - \sum_v c_{\alpha,\beta_v}$. Thus, for the multivariable version of the motivic height zeta function one obtains

$$\begin{aligned} Z_{U,L}(\underline{T}) &= \sum_{\underline{n} \in \mathbf{N}^{\mathcal{A}}} [M_{\underline{n}.U}] \underline{T}^{\underline{n}} \\ &= \sum_{\underline{n},\beta} \sum_{\underline{m} \in Sym^{\underline{n}^\beta}(C)} [H(\underline{m}, \beta) \cap G(F)] \underline{T}^{\underline{n}}. \end{aligned}$$

The Poisson summation formula, in the more general form proved in [Bilu18], can then be applied to

$$\sum_{\underline{m} \in Sym^{\underline{n}^\beta}(C)} [H(\underline{m}, \beta) \cap G(F)] = \sum_{\underline{m} \in Sym^{\underline{n}^\beta}(C)} \sum_{x \in F^n} 1_{H(\underline{m},\beta)}(x),$$

and analyzed in terms of the properties of the motivic Fourier transforms $\mathcal{F}(1_{H(\underline{m},\beta)})$, where $1_{H(\underline{m},\beta)}$ denotes the family of motivic Bruhat–Schwartz functions parameterized by $Sym^{\underline{n}^\beta}(C)$. This family can also be described (Section 6.2.5 of [Bilu18]) in terms of symmetric products $Sym^{\underline{n}}(\mathcal{H}_\beta)$, where $\mathcal{H}_\beta = (H_{\underline{m},\beta})_{\underline{m} \in \mathbf{N}^{\mathcal{A}}}$ is the family over C with $H(\underline{m}, \beta)$ over $v \in C(k)$. Thus, we can consider the associated multivariable zeta functions as in Section 7.12.2 above,

$$Z_{\mathcal{H}_\beta}(\underline{T}) = \sum_{\underline{n} \in \mathbf{N}^{\mathcal{A}}} [Sym^{\underline{n}}(\mathcal{H}_\beta)] \underline{T}^{\underline{n}}.$$

7.17.4. Poisson summation and rationality. Another step towards achieving a suitable setting for a lifting of the motivic height zeta function is provided by the following result of [ChamLoe15] and [Bilu18]. Using motivic Poisson summation, the motivic height zeta function is rewritten in the form

$$\mathbf{L}^{(1-g)N} \sum_{\xi \in F^n} Z(T, \xi),$$

where the $Z(T, \xi) = \prod_v Z_v(T, \xi)$ are sums of the motivic Fourier transforms of the form $\mathcal{F}(1_{H(\underline{m},\beta)})$ mentioned above. It is then shown (see Proposition 5.3.4 of [ChamLoe15]) that for each ξ the multivariable $Z(\underline{T}, \xi)$ are rational functions with denominators given by products of terms of the form $1 - \mathbf{L}^n \underline{T}^{\underline{m}}$. with $n \in \mathbf{N}$ and $\underline{m} \in \mathbf{N}^{\mathcal{A}}$.

7.17.5. Scissor congruences and assembler. We finally discuss briefly the problem of finding a suitable scissor congruence relations group (or ring) $\pi_0 K(\mathcal{C})$ and an underlying assembler $\mathcal{C}$ and spectrum $K(\mathcal{C})$, that makes it possible to realize the zeta functions one wants to lift as a morphism of additive groups (or possibly of rings) from $\pi_0 K(\mathcal{C})$ to $Exp\mathcal{M}_k[[T]][T^{-1}]$ endowed with the product of Laurent series as addition (and a Witt–ring multiplication). The compatibility with addition means the requirement that the zeta function splits multiplicatively under scissor congruence relations in $\pi_0 K(\mathcal{C})$.

As mentioned in Sections 7.17.2 and 7.17.3 above, using the zeta functions $Z_{\mathcal{H}_\beta}(\underline{T})$ briefly described at the end of Section 7.17.3, has the advantage that these do satisfy a scissor congruence relation as recalled in Section 7.17.2.

One expects to obtain a suitable assembler $\mathcal{C} = \mathcal{C}_{X,G}$, for X over $F = k(C)$, an equivariant compactification of $G = G_F$ with $X \smallsetminus G = D_G$ a strict normal crossings divisor. This may be built with objects given by choices of data (C_0, D) with $C_0 \subset C$ and $D = X \smallsetminus U$ with U a partial compactification, with morphisms given by inclusions and with a Grothendieck topology generated by covering families of the form

$$(C_0', D') \hookrightarrow (C_0, D) \hookleftarrow (C_0 \smallsetminus C_0', D \smallsetminus D').$$

In fact, this idea for a construction of an assembler $\mathcal{C} = \mathcal{C}_{X,G}$ may be further refined by taking into consideration the structure of the Clemens complex considered in [ChamLoe15] and [Bilu18] that is reflected in the structure of the $Z(T, \xi)$ as rational functions. This is the simplicial complex with vertices the components of a normal crossings divisor D and a n–simplex for each irreducible component of a nonempty $(n + 1)$-fold intersection of components of D. The combinatorics of this complex influences the structure of the local factors, where for instance one has (Proposition 4.3.2 of [ChamLoe15])

$$Z_v(\underline{T},0) = \sum_A P_{v,A}(\underline{T}) \prod_{\alpha\in A} \frac{1}{1-\mathbf{L}^{\rho_\alpha - 1}T_\alpha}$$

with A ranging over the set of maximal faces of the Clemens complex.

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