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Brian Williams, Algebraic Quantum Field Theory
(Northeastern)

Theory of Observables in Quantum Field Theory

(1) Quantum Observables

0-dimensional QFT = Calculus

$Obs^{cl} = (\mathcal{O}(T^*[-1]X), \{S,-\})$ (-1)-shifted symplectic mfld

[Could only see calculus after quantization]
This is a $\hbar$-deformation of Obs^{cl}

$\downarrow$ $\qquad$ $PV^{-1}(X)$ algebra of poly-vector fields

$$Obs^{q} = (\mathcal{O}(T^*[-1]X)[[\hbar]]\ \{S,-\} + \hbar\Delta)$$

corresponds de Rham in finite dim

Δ = divergence (wrt volume elt)

Here

$\Delta|_{Sym^2} = \omega^{-1}$ $\qquad$ this symbol Δ is a distributional object

De Rham operator is a derivation
But Δ is shifted (-1), not a derivation

So the deformation is not an ordinary algebra (just a vector space ~~not a bv-alg~~)
It is a BV-algebra

Obs^{cl} is (-1)-shifted Poisson algebra
and it is deformed to a BV-algebra.

Must take into account "locality".
"Locality" prescribes some kind of
additional algebraic structure.

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In field theory.

Obs^{cl} is a ~~cosheaf~~ of (-1)-shifted Poisson structures on spacetime.

(Preserves colimits)

Lot of subtleties when we deform.
Have to work hard to define $\{S, -\}$
(renormalization)

$\rightsquigarrow$ Obs^q is no longer a commutative algebra (even in 0-dim case)

$\rightsquigarrow$ Obs^q is no longer a cosheaf [for gluing] (even of sets)
(forgetful functor does not preserve colimits)
[It is a pre-cosheaf for sets]

$\boxed{\text{Q}}$ What is the residual structure that remains in quantum observables?

The resulting structure is a factorization algebra.

② Factorization Algebras

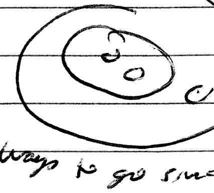
$\mathcal{C}^{\otimes} = \text{Vect}, \text{Ch}$

Defn. A factorization algebra $\mathcal{C}$ is   [(colored) Algebra over an operad]

① Given an open set $U \subset M \longmapsto \mathcal{F}(U) \in \mathcal{C}$.

② $\coprod U_i \subset V \longmapsto$ (structure map $\bigotimes_i \mathcal{F}(U_i) \to \mathcal{F}(V)$)
$\hat{M}$

Axioms.
(i) Operadic associativity axiom



agreement of ways to go smaller open sets to larger open sets

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(ii) Gluing axiom

$\mathcal{F}$ is a cosheaf with respect to some other topology

"Weiss topology"
which a massive topology
(many more open sets to have a cover,
much finer.)

(following Costello + Williams)

[related to the Ran space = {all finite sets of points on spacex}]

We will study these things in Riemannian manifolds

Subtleties of definition on Lorentzian manifolds
remains to be worked on. (Gwilliam-Kasia Rejzner)
make connection to other approaches to algebraic QFT.

It is hard to produce examples of factorization algebras in non-Riemannian manifolds. NO examples satisfying axioms are known for Lorentzian signature.

Example. 3D Abelian Chern-Simons CS theory

$A \in \Omega^1(\mathbb{R}^3)$ CS-Action $\int A\, dA$

Abelian gauge transformation (+ boundary condition)
$A \longmapsto A + dc$

$\rightsquigarrow$ BV-formulation

"fields" are the entire de Rham complex
$\Omega^{\cdot}(\mathbb{R}^3)[1]$.

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It has ~~a~~ formally shifted symplectic structure
but it is not shifted sym

Odd symplectic ~~pairing~~ $\int_{\mathbb{R}} \alpha \wedge \alpha'$

(with suitable boundary conditions, here compactly supported fns)

Key thing is to have it work for Vector bundle.

Need Serre duality; [to get]

$$(\Omega^{\bullet}[1])^{\vee} = \Omega_c\,[2].$$

~~For~~ For each knot $K \subset \mathbb{R}^3$ have an observable $\mathcal{O}_K$

defined on a tubular neighborhood T of knot K.
so $K \subset T$

$\mathcal{O}_K \in Obs(T)$
is a linear observable

$\mathcal{O}_K(A) = \int_K A. \in \mathbb{C}$ a complex number

There is a structure map, embed T in $\mathbb{R}^3$

$$\begin{array}{ccc} \mathcal{O}_K & & Obs(T) \\ \downarrow & & \downarrow \\ \mathcal{O} & \in & Obs(\mathbb{R}^3) \end{array}$$

linear observable: power series fields or polynomial

in Cohomology goes to 0 (Poincaré lemma)

Think observable as dual to ~~fields~~ fields

$\mathcal{O}_K \in \Omega^2_c(T)$

On $\mathbb{R}^3$, $\mathcal{O}_K = d\rho$ for some other linear observable
ρ = degree(-1) "a homotopy"

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At least viewing knot observable inside the full manifolds $\mathbb{R}^3$, they are trivializable

Given (two knots) $K, K' \subset \mathbb{R}^3$

can compute

$$\underbrace{(d + \hbar\Delta)}_{\text{quantum differential}} (\rho\, O_K^i)$$

ρ compactly supported (distributional) 1-form

Factorization algebra gives you a (constant) product $O_K O_K$

$$(d + \hbar\Delta)(\rho O_{K'}) = O_K O_K' + \int_{\mathbb{R}^3} \rho O_K$$

product of distributions

Owen book computes this is exactly linking number

$$\Longrightarrow \quad [\underbrace{O_K O_{K'}}_{\text{quadratic observable}}] = \underbrace{\text{linking \#}}_{\text{constant observable}}$$

This obtains an interesting relation in the quantum algebra of observables.

3 $\mathbb{E}_n$-algebra

Local observables have an extra structure of an $\mathbb{E}_n$-algebra.

operations $A^{\otimes k} \to A$

for every collection of k disjoint polydisks in $\mathbb{R}^n$.

$\boxed{\begin{array}{l}\textbf{Theorem} \quad \text{(Lurie)} \\ \text{Locally constant factorization algebras in } \mathbb{R}^n \\ \text{are the same as } \mathbb{E}_n\text{-algebras.}\end{array}}$

This theorem is a statement on the pre-cosheaf of vector spaces.

'Locally constant' has a meaning.

Thm. allows to think of $\mathbb{E}_n$ algebras as depend on points (arbitrarily small nbhds of points)

LC factorization alg are algebras over operad

$\boxed{n=1}$

$\mathbb{E}_1$-algebras $\simeq$ associative algebras (A_∞-algebras)

Example TQM [Li, Li, etc.]

$\phi : \mathbb{R} \longrightarrow$ Symplectic Manifold (X, ω)

$\rightsquigarrow$

Observables ~~alg~~ $\cong \mathcal{O}_{pl}(X)$ Kontsevich deformation quantization of X

for any $\omega \in \hbar\, H^2(X)[\![\hbar]\!]$

(get usual Moyal product for $\emptyset$)

$\boxed{\begin{array}{l}\underline{\text{Fact}} \\ k\text{-ary operation on } \mathbb{E}_n\text{-algs} \\ \qquad \simeq \underbrace{\mathrm{Conf}_k(\mathbb{R}^n)}_{\text{configuration space}}\end{array}}$

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2-ary products $\simeq \text{Conf}_2(\mathbb{R}^n)$ in cohomology

$\simeq \mathbb{R}^2 \setminus \{pt\}$

$\simeq S^{n-1}$ parameterize

$\rightsquigarrow$ In cohomology, have two operations $\circ_1, \circ_2$ (at least when $n>1$.)

$\circ_1$ Product

$$H(A) \times H(A) \to H(A)$$

constants on S^{n-1}

degree 0 part

$\circ_2$ Bracket Product

$$\{-,-\} : H(A) \times H(A) \to H(A)[1-n]$$

Carries a degree which is 1 less than the dimension of

The cohomology has some algebraic structure

Fact 1. $\circ_1$ Product is commutative.

Fact 2. $\circ_2$ Product behaves like a Poisson Bracket
graded ~~symmetry~~ skew-symmetry

Call vector space V with two products $\circ_1, \circ_2$ a $\boxed{\mathbb{P}_n\text{-algebra}}$

$\boxed{n=2}$ is a Gerstenhaber algebra.
BV and $\mathbb{E}_2$ are close
Bracket differs by 2 in cohomological degree.

[2-7-202~~4~~]

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~~2.8~~

Example. $\boxed{\text{B-model}}$

= "twist" of $N=2$ SUSY σ-model.

[topological, no metric]

Σ = $\boxed{\text{oriented 2-manifold}}$

These are (fields in ghost degree 0.

[Space of fields]

$\phi : \Sigma \longrightarrow X$

there are odd functions (fermions) to.

2-manifold

(almost) complex manifold

(enough for definitions classical above!)

$\eta \in \Gamma(\Sigma, \phi^* T_X^* \otimes T_\Sigma^*)$ sections

with action

$$S = \int \eta \, d\phi$$

On a 2-disk. D^2, ~~quantum~~ classical observables are

$$Obs(D^2) = PV^0(X)$$

poly vector fields on X

η is gauge-trivial on a disk D^2

$$\eta \longmapsto \eta + dc$$

where $c \in \Gamma(\Sigma, \phi^* T_\Sigma^*)$

odd section

[odd directions all of this kind]

An operator carries cohomological degree 1.

The $\mathbb{E}_2$/rey $\mathbb{P}^c$ is $\{-,-\} : PV^i \times PV^j \to PV^{i+j-1}$

is Schouten bracket

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We get

$$Obs(D^2) \simeq HH^*(\mathcal{O}_X)$$

Hochschild cohomology
has Gerstenhaber-structure

Deformed quantum structure is

$$Obs^q(D^2) = PV^\circ(X)((\hbar)) \simeq HH^*(\mathcal{O})(\hbar)$$

<u>Example</u>. perturbation Chern-Simons on $\mathbb{R}^3$ [Kevin Costello, Gwilliam, France]

with Lie algebra g, + invariant symmetric form on g (the "level")

U an open set

$$Obs^{cl}(U) = (\mathcal{O}(\Omega^\circ(U) \otimes g[1]), \{S,-\})$$

$$\simeq C^\circ(\Omega^\circ(U) \otimes g))$$

commutative de Rham

~~Chevalley-algebra~~ Eilenberg complex
of dg-Lie algebra

For $U = D^3$ disk

$$Obs^{cl}(U) \simeq C(g)$$

all cohomology in large positive degree ( )
commutative algebra

Now, quantize, get an $\mathbb{E}^3$-algebra (claim not explain)

$C^\circ_\hbar(g)$

Want to explain interesting algebraic structure on this.

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Can view $\mathbb{E}_3$-alg

Poisson-Lie bracket

$\rotatebox{90}{$\simeq$}$

$(\mathbb{E}_2\text{-algebra } \cancel{\text{of}}\ \mathbb{E}_1\text{-algebra})$

$\Big\downarrow$ Left-Modules

An $\mathbb{E}_1$-algebra is a Category

$\mathbb{E}_2$-algebra (Category)

Work of Costello Willm

$$C^q_h(g) \longmapsto \mathrm{Mod}_{U_h(g)}$$

modules over a quantum group

$\Rightarrow$ Makes a connection to quantum groups.