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Brian Williams, Arithmetic Quantum Field Theory

(Northwestern)

Disk Algebras ; Boundary Conditions

Talked about factorization algebras last time.

Simpler to talk about ~~disk~~ disk algebra A

1. Disk Algebras

These are

$$A : \mathrm{Disk}^{U} \longrightarrow C^{\odot} \qquad \text{(open sets } U)$$

The Category of disks $\underset{\text{disjoint union embeddings}}{\coprod} D^n$ has a monoidal structure $(\oplus, \otimes)$

The last time talked by $\mathbb{E}_n$-algebras, which are algebra over $\mathbb{E}_n$.

Look at framed embedding.

$\mathbb{E}_n$-algebras are a kind of disk algebra. (rectilinear with an trivial $SO(n)$-action

$\subseteq$ (disk algebra)

Important invariant of disk algebra is $\boxed{\text{factorization homology.}}$

Given M = any n-manifold

$$\int_M A \overset{\text{def}}{:=} \operatorname{colim} \otimes A(D^n$$

(Think of A are coefficients, M varies

$$\underbrace{\coprod D}_{\text{embed ~~finitely~~ many disjoint disks}} \subseteq M$$

Theorem (Ayala - Francis)

$$\left\{ \begin{array}{c} \text{framed} \\ \mathbb{E}_n\text{-algebra} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{homology theory} \\ \text{for oriented } n\text{-manifolds} \end{array} \right\}$$

HOMOLOGY AXIOMS satisfy excision

There are long exact sequences

The map is "factorization homology"

Remark. Locally constant factorization algebras in $\mathbb{R}^n$ $\xrightarrow{\sim}$ $\mathbb{E}_n$-algebras

$SO(n) \circlearrowleft$

The $SO(n)$ action with translation to define over a general manifold (arbitrary n-manifold)

disk algebra.

$$\mathcal{F}(M) = \int_M A_{\mathcal{F}}$$

Remark This construction behaves like a bar or co-bar construction.

A = associative algebra. Look at an a circle S^1
$\mathbb{E}_1$-algebra

$$\int_{S^1} A \simeq Hoch_\bullet(A)$$

Hochschild homology

'Circles' correspond to traces in the factorization algebra

(S^1 is automatically framed.)

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Now $A \in \mathbb{E}_{n-1}$-algebra $= \mathbb{E}_1$-algebra ($\mathbb{E}_n$-algebra).

$$\int_{S^1} A = \mathrm{Hoch}_\bullet(A) \quad \text{is an } \mathbb{E}_n\text{-algebra.}$$

(have only gotten rid of one direction.)

$\boxed{\text{"Dimension Reduction"}}$

Example Let $\mathfrak{g}$ = Lie algebra.

$\rightsquigarrow$ 2-dim'l locally constant factorization algebra

$$C_\bullet(\mathfrak{g} \otimes \Omega_c^\bullet(\mathbb{R}^2))$$

"locally" on a chart $\mathbb{R}^2$

Knudson $\rightarrow$ $\wr\wr$

Envelope algebra of $\mathfrak{g}$

$$\mathcal{U}_{\mathbb{E}_2}(\mathfrak{g}).$$

Here

$$\int \mathcal{U}_{\mathbb{E}_2}(\mathfrak{g}) = H_\bullet(\mathfrak{g} \otimes H(\Sigma))$$

Hochschild homology

(Automatically framed, can put it on any manifold)

"Compactification"

$$\begin{array}{c} X \\ f \big\downarrow \\ Y \end{array}$$

should be a proper map (don't need it for what I say)

$\mathcal{F} \in \mathrm{Fact}(X)$

~~Pushforwards exist:~~

$$f_* \mathcal{F} \in \mathrm{Fac}(Y)$$

$$f_* \mathcal{F}(U \subset Y) = \mathcal{F}(f^{-1}(U))$$

(Really a derived construction Done on a cochain complex)

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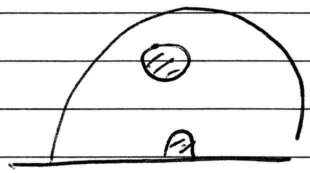
2. Boundary Conditions



Here have a factorization algebra

$$\mathcal{F}(D) \times \mathcal{F}(H) \to \mathcal{F}(H_{bdy})$$

Not going to define precisely

Have a residual multiplication on the boundary

(In the locally constant case this does return an $\mathbb{E}_n$-module.)

(Swiss-cheese operad in the locally constant case.)

$\boxed{Q}$ How does QFT produce such objects?

It prescribes a boundary condition for fields on the boundary.

Example (Top. Quantum Mechanics)

$$\phi : \mathbb{R}_+ = [0, \infty) \longrightarrow (X, \omega)$$

symplectic manifold

Equation of motion (EOM): ϕ is locally constant.

Phase Space. $$EOM\Big|_{t=0} \simeq X$$

Boundary Condition: Choosing a Lagrangian $L \subset X$ inside phase space.

(To define ODE)

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Boundary Condition: At time $t=0$ the field lies in that Lagrangian.

Module $\mathcal{O}(\mathcal{L}) \circlearrowright \mathcal{Q}_{\substack{\text{deformation} \\ \text{quantization}}}(X)$

"Fock module"

More generally: (can prove in special cases)

$$M = \partial M \times [0,\infty)_t$$

then

$$EOM\big|_{t=0} \text{ is } \underline{0\text{-symplectic}}$$

The topological direction "killing off" (I am) changes a (-1)-symplectic structure to a 0-symplectic structure.

"Topologically normal to the boundary."

$$\mathcal{E}\big|_M \boxtimes \Omega^\bullet([0,\infty))$$

[Theory has one topological direction but may not be topological]

The pairing

$$\underset{(-1)\text{-symplectic}}{\omega_{BV}} = \omega_{\mathcal{E}|_M} \boxtimes \int_{[0,\infty)}$$

You should think about boundary conditions as choosing a Lagrangian on phase space.

To get factorization algebra the Lagrangian must satisfy extra "locality conditions" of same kind that the fields satisfy.

Example CS on $\Sigma \times \mathbb{R}^+$

$$EoM|_{t=0} \cong Flat_G(\Sigma)$$

$$\underset{\substack{\uparrow \\ \text{Atiyah-Bott}}}{\omega_{AB}} = \int_\Sigma \delta A \times \delta A'$$

The WZW boundary condition is $\boxed{\text{not}}$ topological, it is not locally constant.

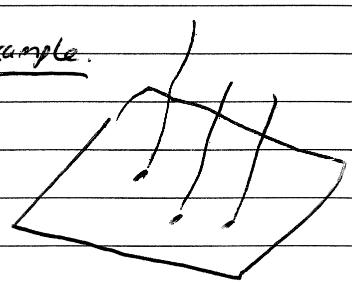
[We produce theories that are still 'local'! This is a (strong) condition on a 3-dimension theory.

Remark. $Flat_G(\Sigma) = \underset{\text{twist by level}}{T^*} Bun_G(\Sigma)$.

G is complex here.

Thm. [Rabinowitz, Bertram-Yeo]
For nice BV theories on manifold with boundary one can produce Factorization algebra.

Example.



The chiral disk boundary algebra.

~~On $\mathbb{C}$ the chiral WZW boundary condition is~~ Not locally constant, but holomorphic.

They give vertex algebras
Kac-Moody vertex algebras of some level

- point defects on boundary are line operators over the bulk.

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Line operators on bulk topological structure

Extra multiplication operators allowing to multiply lines that collide.

It has an $\mathbb{E}_2$-structure.
Related to whole Kazhdan-Lusztig theory

Example (Topological BF theory on Σ)

This is sort of field theory so it has fields

$$\underline{A} \in \Omega^\circ(\Sigma) \otimes \mathfrak{g}[1]$$

(Will think perturbatively)

Cochain complex, not an algebra.

Observables will anticommute.

$$\underline{B} \in \Omega^*(\Sigma) \otimes \mathfrak{g}^*$$ B-field

Now

$$S = \int_{\Sigma} BF_A$$

Now on $S \times \mathbb{R}_+ = \Sigma$

(1-manifold)

Phase space for BF-theory

$$T^* Flat_G(S).$$

This phase spaces have two nice choices of boundary conditions

$Flat_G(\Sigma)$ = base Neumann-boundary condition

$\cup$

Canonical BC is Dirichlet boundary condition

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Perturbatively ~~work~~ can work near the boundary of bundle

Neumann b.c. $\Omega^0(S)\otimes\mathfrak{g}[1]$ $\subset$

$T^*(\Omega^0(S)\otimes\mathfrak{g}[1])$

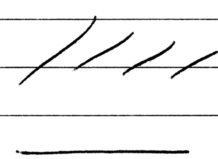
$\supset$

Dirichlet Lagrangian $\Omega^0(S)\otimes\mathfrak{g}$

[There will be one leftover field]

More generally
There is an important boundary condition.

$\boxed{\text{3. Hamiltonian G-space } (X, \omega) \text{ boundary condition}}$



Neumann
boundary condition (1)-shifted symplectic

"Totally well-defined theory"

Lagrangians in this derived context

Neumann ($1/2$-dimensional) Boundary conditions [Lagrangian in the weak sense]

Choose coupled 1-d topological quantum mechanics
Totally well-defined 1-dimensional theory.

TQM of target (X, ω).

Action:

$$S_{\text{TQM}} = \int_S \omega(\phi, d\phi) + \int_S \mu(A)$$

$\boxed{\text{1-dim}} \rightarrow S$

There are no propagating fields in this theory.

~~These are~~ boundary theories

They are degenerate versions of BV-theories

[Use symplectic pairing to contract vectors]

4 Quantum Hamiltonian Reduction

To a Hamiltonian G-space, there is an important construction called **Hamiltonian reduction**

It uses the moment map in an essential way

For this picture need to add another boundary

Dirichlet + $TQM(X, \omega)$

BF-theory in bulk $\quad S \times [0,1]$

Neumann + $TQM(X, \omega)$

This is a well-defined system.

Let (for compactification of factorization algebra)

$$\pi : S \times [0,1] \to S$$

Now

$\pi_* \mathrm{Obs}$ is a factorization algebra on $S_0 = \mathbb{R}$

Claim: $\pi_* \mathrm{Obs}$ is quantum Hamiltonian reduction

Can phrase this in terms of shifted symplectic.

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The pushforward is a derived reduction

$$\mathcal{L}_N \cap \mathcal{L}_D$$

~~Remark~~ This is a toy model for Langlands correspondence
It has a 4-D TFT has a Langlands program.

There is a higher version of quantum Hamiltonian reduction with a shift by 2.

5. Question Time

Q: Why does k-shifted quantum reduction give k-categories?

A: I don't have a good answer

Q: Why BV-quantization?

A: Point of BV-quantization is to take a QFT and produce a factorization algebra.

$$\begin{array}{ccc} \text{QFT} & \rightsquigarrow & \text{Factorization algebra.} \\ \cup & & \\ \text{TQFT} & \longrightarrow & \underset{\text{Factorization algebra}}{\text{locally constant}} \simeq \mathbb{E}_n\text{-algebra} \end{array}$$

Thm. (Nick Rozenblyum + Pavel Safronov)

Classical Field Theory $=$ $\mathbb{P}_0$-factorization algebra

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Classical Field Theory

$\cup$ $\cong$ $\mathbb{P}_0$-algebra ($\mathbb{E}_n$-algebra)

Topological associative alg(?) $\quad \mathbb{P}_n$-algebra.