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# Algebraic Quantum Field Theory

## 0. Opening Remarks

There are no von Neumann algebras in this talk.
So not on Algebraic Quantum Field Theory

Derived Algebraic Quantum Field Theory more accurate title

This talk is traditional QFT, nothing arithmetic here.

Explain a Langlands duality from this perspective.

## 1. Dualities + Langlands Duality

Schematically, duality is equivalence between two different

G = complex reductive group ( Langlands dual group )

$$\mathcal{A}_G \simeq \mathcal{B}_G$$

Not a 2-dim duality like mirror symmetry

It is a 4-dim TFT.

Axiomatic Functional Approach to QFT

4-mfld $\rightsquigarrow$ number (partition fn.
$Z(M^4) \in \mathbb{C}$

3-mfld $\rightsquigarrow$ $Z(M^3) \in \mathrm{Vect}$
(vector space)
Differential Graded vector spaces
Cochain complexes.

If $M^4$ bdd 4-mld $\Rightarrow$ $Z(M^4) \in Z(M^3)$

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2-mfld ⟶ Category
(∞-category or dg-category)
$Z(M^3) \in Z(M^2) \in \mathrm{Cat}$.

1-mfld ⟶ [won't go there]

Duality implies (for 2-mflds)

$$\mathcal{A}_G(\underset{\substack{\uparrow \\ \text{2-dim'l} \\ \text{surface}}}{Z}) \cong \mathcal{B}_G(\underset{\substack{\uparrow \\ \text{2-dim'l} \\ \text{surface}}}{Z})$$

as an Equivalence of Categories

Best Hope
Conjecture

$\mathcal{A}_G(Z)$ → Category of Principal G-bundles
$\mathrm{Bun}_G(Z)$

$\mathcal{B}_G(Z)$ → Quasicoherent sheaves on flat $\check{G}$-bundles
$\mathrm{QCoh}(\mathrm{Flat}_G(Z))$
Flat G-bundle

A-side Geometric
B-side Spectral or Galois

Very hard to do; extra corrections needed to the conjecture
Not even clear how to state conjecture formally.

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**Question**

What does the theory do on "defects".
[Submanifolds of the 4-manifold]

**Defects** [Names for these defects)

dim 2 ⇝ surface operators
(ramifications in geometric Langlands)

dim 1 ⇝ line operator
(= Hecke operators in geometric Langlands)

dim 0 ⇝ local operators
(less understood in Langlands side
"singular support conditions on categories"

**GOAL**

Explain a mathematical package to access these operators

Do classical in this lecture; quantum at the end

## 2. Classical Field Theory

Some very nice ~~theory~~ system of partial diffl eqn.

Like Variational principle formulation

$$S[\phi] = \int_M \mathcal{L}(\phi)$$

✓ totally local
depends on jets of field $\phi$.

$M$ manifold

(local) action functional

output is a density.

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$\boxed{\text{Example 1}}$ $\overset{\text{Sigma}}{\sigma}$-models ( since space denoted $\underset{\text{Sigma}}{\Sigma}$ )

$$\phi : \Sigma \to X$$

$$S(\phi) = \int_\Sigma \phi \, \Delta\phi.$$

Need to fix both a metric on $X$
and a metric on $\Sigma$.

We find harmonic functions, that's the PDE we
find here

$\boxed{\text{Example 2}}$ Chern-Simons gauge theory

$$S(A) = \int_{\underset{\text{3-mfld}}{M^3}} CS(A) \quad A = \text{is a one form,}$$

a connection on some principle
G-bundle on

$$\begin{tikzcd} P \arrow[d] \\ M^3 \end{tikzcd}$$

The $\overset{\text{(PDE says)}}{\text{connection}}$ must be flat.

$$F_A = 0$$

( $\overset{\text{This is a}}{\text{Topological}}$ Field Theory )

Supersymmetry gives more examples.

The fields form a sheaf on spacetime

The solns to eqns of motion also form a
sheaf on spacetime
( totally local action functional )

We assume

$$\text{fields} = \underset{\text{sections}}{\Gamma}\left(M, \text{possibly graded vector bundle}\right)$$

(Examples 1 and 2 do NOT satisfy this globally)

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**Defn.** The $\boxed{\text{classical observables}}$ $\mathrm{Obs}^{cl}$ of a theory are

$$\mathrm{Obs}^{cl} := \mathrm{Fun}(\underset{\substack{\text{equations of} \\ \text{motion}}}{\mathrm{EOM}})$$

Here EOM have algebraic definition:
(polynomials) or (power series)

**Remark**. What physicists really care about are solutions to equations of motion.

**Remark.** Answers are cohomological, there is a differential around

$\boxed{\text{Property of } \mathrm{Obs}^{cl}}$

- It is a cosheaf on spacetime of commutative algebras

[There is one more structure, save later)

This structure all comes from the equations of motion

**Remark** ① Really important to work in a derived sense. The EOM will be implemented cohomologically.

② $\boxed{\text{Critical Locus}}$

$$\mathrm{Crit}(S) \sim \mathrm{Crit}^{dg}(S)$$

This is $(-1)$-shifted symplectic

"Classical B-V formalism"

[Batalin-Volkovsky 1980's gauge theory]

**Example:** (Toy Model)

In "0-dim", fields = manifold $X$

$$S : X \longrightarrow \mathbb{R}$$

explicit Koszel ~~representation~~ resolution of singularities

$$\mathrm{Crit}(S) = \underset{\text{in } T^*X}{\mathrm{Graph}(dS) \cap \{0\}}$$

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Derived Critical Locus in example:

$$\mathrm{Crit}^{dg}(S) \cong T^\circ[-1]X = \mathrm{Obs}^{cl}$$

$$\circlearrowleft$$

$$\{S,-\}$$

($T^\circ X$ = cotangent complex)

"bracket with $S$"

For a dg-object, look at its algebra of functions.
(The functions are algebraic in fiber direction
algebra of polyvector function,

$$\varphi(\mathrm{Crit}^{dg}(S)) = (PV^{-\circ}(X)$$

↑ starts in deg $-\dim X$

$$PV^{\dim X}(X) \xrightarrow{dS} PV^{\dim X-1}(X) \to \cdots \to \overset{\deg 0}{\mathcal{O}(X)}$$

Resolution has information on how transverse the intersection is.

- This $\mathrm{Obs}^{cl}$ is commutative dg-algebra.

  (We are missing "locality" principle in this picture.)

  Here it is 0-dimensional so not relevant.

  In the quantum level we have to modify this.

② $\boxed{\text{QFT}}$

Classical: to understand correlation functions out observables.

Quantum: It is expectation value

$$\langle O \rangle = \int_{\substack{\text{space}\\ \text{fields}}} O e^{-S(O)/\hbar} [D_p]$$

(don't have defn in general)

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## Challenges

(1) Infinite-dim'l integration space

(2) how define measure $[Dp] = ?$

**Solutions:** B-V formulation [Costello]

Give a flavor of how this calculus looks.

0-dim'l: Gaussian integral

Kinetic term is action functional.

$$\int_{\mathbb{R}^n} \prod \frac{dx^i}{\sqrt{2\pi}}\, e^{-x^TAx} = \frac{1}{\sqrt{\mathrm{Det}(A)}}$$

space of fields

here measure is well-defined;

$A$ = symmetric real matrix

('Simplest 0-dim'l free field theory')

Modification:

$$S(x) = x^TAx + I(x)$$

cubic ~~or~~ higher order polynomial

Then have an expansion

$$\int \prod_i \frac{dx^i}{\sqrt{2\pi}}\, e^{-S(x)} = \frac{1}{\sqrt{\det A}}\left(\sum_{\substack{\text{connected graph}\\ \Gamma}} \frac{W_\Gamma}{|\mathrm{Aut}(\Gamma)|}\hbar^{\alpha(\Gamma)}\right)$$

"path integral"

The weights $W_\Gamma$ are combinatorial to evaluate in this setting (Feynman)

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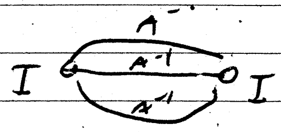
**Sub-example** $I \sim \lambda^3$. (this is allowed)

for cubics, only nonzero weights $w_\Gamma$ come from trivalent graphs $\Gamma$.



insert $A^{-1}$ on edges
~~make~~ $A^{-1}$ is plays role of the propagator.

Takes values in $\mathbb{C}$.

Doing $\mathbb{R}^n \otimes \mathbb{R}^n$ contracting with all the tensors from the interaction.

**Subexample** (Free propagator)

$$\phi : M \to \mathbb{R}.$$

$$S(\phi) = \int_M \phi \Delta \phi + \overbrace{\int \tfrac{1}{4} \phi^4}^{\text{deformed interaction}}$$

In finite-dims replace with each infinite quantity in these eqns

$\Delta^{-1}$ replaces $A^{-1}$ $\sim G(x,y)$

Inverse Laplacian

Green's Function

If $M = \mathbb{R}^d$ there is asymptotic expansion of Greens functs

$$G(x,y) \sim \frac{1}{|x-y|^{d-2}}$$

**Problem**. Hard to multiply distributions with overlapping support.

Get divergences called UV-divergences

(correct with counterterms)

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K. Costello and Si Li
have combined divergence cancellation
with cohomological formalism
to do renormalization

There is no <u>metric around</u> in statements I am making.

$\boxed{\text{Question Time}}$

Ⓐ Talk
Not just about topological theory.

Ⓑ <u>Homological BV</u> how it appears in the quantum settings

0-diml setting

$X =$ "space of fields"

$$Obs^{cl} = ( \mathcal{O}(T^*[-1]X, \{S,-\})$$

$$= ( PV^{-}(x) \quad , \text{contracting with 1-form } dS(V,-))$$

polyvector fields

To make sense of

$$\int_X e^{-S(x)/\hbar} vol_X .$$

need a volume form. Let $n = \dim(X)$.

$$\begin{tikzcd}
 & -n & & & -1 & 0 & \\
 & PV^n \arrow[r, "\Delta"] \arrow[d, phantom, "\cong"] & PV^{n-1} \arrow[r, "\Delta"] \arrow[d, phantom, "\cong"] & {} & PV^1 \arrow[r] \arrow[d, phantom, "\cong"] & PV^0 \arrow[d, phantom, "\cong"] & \\
\text{de Rham complex} & \Omega^0 \arrow[r, "d"'] & \Omega^1 \arrow[r, "d"'] & {} \arrow[r, "d"'] & \Omega^{n-1} \arrow[r] & \Omega^n \arrow[r, "\int"', "dvol"'] & \mathbb{R} \\
 & & & & & & \text{reals}
\end{tikzcd}$$

Contract against volume form

Don't know how to integrate against volume form in <ins>infinite dimensions</ins>

So use the BV-formalism, top row, using the <ins>BV-Laplacian $\Delta$</ins>

$$\rightsquigarrow \int_M = \int_{BV} = [-]_{[PV^{-1}, \Delta]}$$

Thus BV-formalism gives another way to interpret integration.

Now how to interpret degree 0 classical [illegible]

Integrals are non-local: this is <ins>quantum</ins>.

Put $\hbar$ in and work formally (very asymptotic)

$$\int_M e^{-S/\hbar} = [-]_{[PV^{-1}(\ ],\ \hbar\Delta + '\{S, -\}]}$$

deformed differential

Have to check <ins>quantum Master equation</ins> to check differential property $\partial\partial = 0$.

Have to do a renormalization to get an <ins>effective action</ins> (this is Wilson's approach).