Strategy selection for arbitrary initial condition

Yu-Ting Chen[a,b,*], Martin A. Nowak[b,c,**]

[a] *Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA, 02138*
[b] *Program for Evolutionary Dynamics, Harvard University, Cambridge, MA, 02138*
[c] *Department of Mathematics, Department of Organismic and Evolutionary Biology, Harvard University, Cambridge, MA, 02138*

Abstract

We study evolutionary games on graphs. The individuals of a population occupy the vertices of the graph and interact with their neighbors to receive payoff. We consider finite population size, regular graphs, probabilistic death-birth updating and weak selection. There are two types of strategies, A and B, and a payoff matrix $[(a,b),(c,d)]$. The initial condition is given by an arbitrary configuration where each vertex is occupied by either A or B. The conjugate initial condition is obtained by swapping A and B. We ask: when is the fixation probability of A for the original configuration greater than the fixation probability of B for the conjugate configuration? The answer is a linear condition of the form $\sigma a + b > c + \sigma d$. We calculate σ for any initial condition. For large population size we obtain the well known result $\sigma = (k+1)/(k-1)$, but now this result extends to any mixed initial condition. As a specific example we study evolution of cooperation. We calculate the critical benefit-to-cost ratio for natural selection to favor the fixation of cooperators for any initial condition. We obtain results that specify which initial conditions reduce and which initial conditions increase the critical benefit-to-cost ratio. Adding more cooperators to the initial condition does not necessarily favor cooperation. But strategic placing of cooperators in a network can enhance the takeover of cooperation.

Keywords: Evolution, games, evolutionary graph theory, spatial selection, social networks, voter model on graphs, coalescing random walk on graphs.

*Email address: yuting_chen@cmsa.fas.harvard.edu
**Email address: martin_nowak@harvard.edu

1. Introduction and main results

Population structure can affect the outcome of evolutionary dynamics (Nowak and May 1992; Hutson and Vickers, 2002; Hauert and Doebeli, 2004; Jansen and Van Baalen, 2006; Nowak *et al.*, 2010; Fu *et al.*, 2008). For example, strategies that lose in a well-mixed setting could win in a structured population by forming clusters or vice versa. Traditionally evolutionary games have been studied in well mixed populations (Cressman, 2003; Hofbauer and Sigmund, 1988; Hofbauer and Sigmund, 1998; Maynard Smith, 1982; Nowak and Sigmund, 2004; Nowak *et al.*, 2004; Skyrms, 1996; Weibull, 1995) or on regular grids (Nowak and May, 1993; Lindgren and Nordahl, 1994; Nakamaru *et al.*, 1997; Szabó *et al.*, 1998; Szabó *et al.*, 2000; Simon, 2008; Helbing *et al.*, 2010; Fu *et al.*, 2010). A generalization of this approach is evolutionary graph theory (Lieberman *et al.*, 2005; Ohtsuki and Nowak, 2006; Ohtsuki *et al.*, 2006; Santos *et al.*, 2008; Szabó, 2007; Szolnoki *et al.*, 2009; Broom *et al.*, 2011; Broom and Rychtář, 2012; Van Veelen *et al.*, 2012; Chen, 2013; Maciejewski *et al.*, 2014; Débarre *et al.*, 2014; Wardil and Hauert, 2014). The individuals occupy the vertices of the graph, and the edges determine who interacts with whom. There are two types of interactions: (i) those that generate payoff and (ii) those that specify evolutionary updating. Consequently there can be two different graphs, which are called interaction and replacement graph (Ohtsuki *et al.*, 2007). But in many papers and also here it is assumed that these two graphs are the same. Other extensions of evolutionary graph theory include situations where the population structure changes during evolutionary updating (Antal *et al.*, 2009; Tarnita *et al.*, 2009; Wu *et al.*, 2010; Wardil and Hauert, 2014), but here we study evolutionary dynamics on a constant graph. Moreover, we focus on regular graphs of degree k, which means that each individual has the same number, k, of neighbors. Under these assumptions we study how the initial configuration of individuals on the graph affects evolutionary outcomes.

Throughout this paper, we consider death-birth updating. At any time step a random individual is chosen for death. The neighbors compete for the empty site with probability proportional to payoff. In a social setting, death-birth updating means that a random individual decides to update its strategy: it adopts one of its neighbors strategies proportional to payoff. We consider the limit of weak selection. The effective payoff or fecundity is

$$1 + w \cdot \text{payoff}$$

(Nowak *et al.*, 2004). Here w is a parameter that scales the intensity of selection. The limit of weak selection is $w \to 0$ (Nowak *et al.*, 2004; Fu *et al.*, 2009; Wu *et al.*, 2010; Wu *et al.*, 2013). In a social setting, weak selection means that players are confused about payoffs or that the game under consideration is only a small contribution to overall success (Rand and Nowak, 2013).

At first we present our results for the game of cooperators versus defectors. Cooperators pay a cost, c, for the other individual to receive a benefit, b. Defectors pay no cost and distribute no benefit. We have $b > c > 0$. We will first focus on this game and then discuss the case of general payoffs which can be handled by a linear extension argument as in Tarnita *et al.* (2009).

The main objective of this paper is to compare the fixation probabilities of the two competing strategies for any initial condition. Fixation means that one of the two strategies takes over the entire population. Fixation of cooperators means that all individuals in the population eventually become cooperators. Fixation of defectors means that all individuals in the population eventually become defectors. These are the only two absorbing states. All mixed states are transient. For any initial condition, we can ask: what is the probability that starting from this configuration we reach the state of all-cooperators. This quantity is the fixation probability of cooperators for this initial condition. For the same initial condition we can also calculate the fixation probability of defectors. The two quantities add up to one, because the system must eventually reach one of the two absorbing states.

We assume that the underlying population structure is a finite, connected, regular graph. Regular means that each individual has exactly k neighbors. The parameter k denotes the degree of the graph.

In the simple case that only one cooperator is present in the initial condition, we say natural selection favors the fixation of cooperators, if the fixation probability of cooperators is greater than $1/N$, where N is population size. A neutral variant has fixation probability $1/N$. Whether or not natural selection favors cooperators depends on the benefit-to-cost ratio, the population size N and the degree of the graph, k. The critical benefit-to-cost ratio is given by

$$\left(\frac{b}{c}\right)^{\star} = \frac{k(N-2)}{N-2k} \tag{1.1}$$

provided that

$$N - 2k > 0. \tag{1.2}$$

If the benefit-to-cost ratio exceeds this critical value then natural selection favors the fixation of cooperators. The critical benefit-to-cost ratio (1.1) is independent of the location of the single cooperator. For large population size with fixed degree, the above formula leads to the limiting value

$$\lim_{N\to\infty} \left(\frac{b}{c}\right)^{\star} = k. \tag{1.3}$$

The critical benefit-to-cost ratio is given by the degree of the graph, k (Ohtsuki *et al.*, 2006; Chen, 2013). Note that (1.2) is a necessary and sufficient condition for achieving a finite critical benefit-to-cost ratio beyond which cooperation can be favored. If (1.2) does not hold, then the critical benefit-to-cost ratio is infinite.

More generally, we can consider initial conditions of n cooperators that are placed in certain positions on the graph. The analogous question is then: for which benefit-to-cost ratio is the fixation probability of cooperators greater than n/N? The value n/N is the fixation probability of n neutral variants regardless of their locations. In this case, the a-priori condition that cooperation can be favored for some choices of benefits and costs is given by a generalization of (1.2):

$$N\overline{f_1} \cdot \overline{f_0} - k\overline{f_{10}} - k\overline{f_1 f_0} > 0. \tag{1.4}$$

We use the following notation. The probability of finding a cooperator within neighbors of a random position on the graph is given by f_1. Averaging this probability over all positions we obtain $\overline{f_1} = n/N$, which is the frequency of cooperators in the initial condition. Likewise, the probability of finding a defector within one step of a random position on the graph is given by f_0. Averaging this probability over all positions we obtain $\overline{f_0} = (N-n)/N$, which is the frequency of defectors in the initial condition. Clearly $\overline{f_1} + \overline{f_0} = 1$. The quantity $\overline{f_{10}}$ is defined as follows: choose a random position on the graph, perform a random walk starting in that position and calculate the probability of finding a cooperator at step zero and a defector at step one; average over all positions. The quantity $\overline{f_1 f_0}$ is defined as follows: choose a random position on the graph, calculate the probability that one of its neighbors is a cooperator and one of its neighbors is a defector. Note that $\overline{f_1 f_0}$ is the same as $\overline{f_{1*0}}$, which is calculated as follows: choose a random position on the graph, perform a random walk starting in that position and calculate the probability of finding a cooperator at step zero and a defector at step two; average over all starting positions. In particular, calculating $\overline{f_{10}}$ and $\overline{f_1 f_0}$ amounts to counting the numbers of cooperator-defector paths and cooperator-anything-defector paths in the initial configuration. Computing $\overline{f_{10}}$ and $\overline{f_1 f_0}$ requires only a few elementary matrix operations of the adjacency matrix of the underlying graph.

Whenever (1.4) is valid, we find that the critical benefit-to-cost ratio for natural selection to favor the fixation of cooperators for any initial condition that contains some cooperators and some defectors is given by the simple equality

$$\left(\frac{b}{c}\right)^{\star} = \frac{k\big(N\overline{f_1} \cdot \overline{f_0} - \overline{f_{10}}\,\big)}{N\overline{f_1} \cdot \overline{f_0} - k\overline{f_{10}} - k\overline{f_1 f_0}}. \tag{1.5}$$

See Figure 6–8 for some examples of the critical benefit-to-cost ratios on regular graphs. In particular, (1.5) generalizes the earlier results of Chen (2013) for benefit-to-cost ratios where population configurations are uniformly chosen with a fixed number n of cooperators, for any $n \in \{1, 2, \cdots, N-1\}$ (see Proposition 2.2). In Section 2, we will discuss some specific implications of (1.5).

We point out two interesting features of (1.4) and (1.5). First, both expressions are invariant under reversal of players. Consider a particular initial condition, where some vertices are occupied by cooperators and the remaining ones are by defectors. Now consider the conjugate initial condition, where the types of all players are exactly reversed. For both initial conditions, we have the same critical benefit-to-cost ratio for favoring the fixation of cooperators. Second, both conditions, (1.4) and (1.5), depend only on local properties of the initial conditions. We only need to calculate features of the initial condition that depend on random walks of length at most two. Higher correlations do not matter.

Let us turn to the results for death-birth updating with general game payoffs. Now we assume that players interact according to the following payoff matrix

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}. \tag{1.6}$$

This payoff matrix has the following meaning. If an A-player meets another A player, both get payoff a. If an A player meets a B player, then the A player gets payoff b while the B player gets payoff c. If two B-players meet, both get payoff d.

We consider the following selection criterion: when is the fixation probability of A for a particular initial condition greater than the fixation probability of B for the conjugate initial condition? The conjugate configuration is generated by reversing all player types in the original initial condition (swapping A for B and vice versa). In particular, if there are n many A-players under a configuration, then the role-reversed configuration has precisely n many B-players.

Calculating the condition for the fixation probability of A to exceed n/N requires the exact evaluation of some expected meeting times of multiple random walks on regular graphs, which seem difficult to obtain in general. See Remark 3.5 for this issue.

For any mixed population configuration ξ and the conjugate initial condition $\widehat{\xi}$, we find that the fixation probability of A starting from ξ exceeds the fixation probability of B starting from $\widehat{\xi}$ if

$$\sigma a + b > c + \sigma d, \tag{1.7}$$

where

$$\sigma = \frac{N\overline{f_1}\cdot\overline{f_0}\left(1+\frac{1}{k}\right) - 2\overline{f_{10}} - \overline{f_1 f_0}}{N\overline{f_1}\cdot\overline{f_0}\left(1-\frac{1}{k}\right) + \overline{f_1 f_0}} \tag{1.8}$$

(see also Corollary 11 in Allen and Nowak, 2014). Here the average frequencies $\overline{f_1}, \overline{f_0}, \overline{f_{10}}, \overline{f_1 f_0}$ are defined as before (1.5). In particular, this result generalizes the so-called "sigma" theorem of Tarnita *et al.* (2009) where only single cooperators in initial conditions are considered. Notice that the σ values defined by (1.8) apply without any a-priori condition; in contrast recall that critical benefit-to-cost ratios are meaningful only if (1.4) holds. See Figure 7 (a) for an example.

This paper is organized as follows. In Section 2 we discuss some implications of the critical benefit-to-cost ratio (1.5). From Section 3 on, we give the proofs of the results stated in Section 1 and 2. In Section 3, we first recall a general perturbation result to study fixation probabilities. The proofs of (1.4) and (1.5) are particular consequences of Theorem 3.3. The proof of the alternative selection criterion for a general payoff matrix is given in Theorem 3.4. In Section 4, we consider applications of Theorem 3.3 and give the proofs of Proposition 2.1–2.6. Proposition 2.1 is restated as the stronger version Proposition 4.1.

2. The critical benefit-to-cost ratio

In this section, we discuss some implications of the critical benefit-to-cost ratio (1.5). The proofs are provided in Section 4.

The first implication concerns the limit of large population size, which is the emphasis of earlier studies (Ohtsuki *et al.*, 2006; Chen, 2013; Allen and Nowak, 2014).

Proposition 2.1. *Fix $k \geq 2$. In the limit of large population size, $N \to \infty$, the benefit-to-cost ratio for all mixed initial conditions on k-regular graphs with size N converges uniformly to k.*

Proposition 4.1 gives the rate of convergence for the critical benefit-to-cost ratio. Proposition 2.1 can be compared to an invariance of the critical benefit-to-cost ratio when the number of cooperators is increased (Chen, 2013):

Proposition 2.2. *For all initial conditions where we place an arbitrary number of cooperators uniformly at random we obtain the critical benefit-to-cost ratio given by (1.1).*

For a k-regular graph with N vertices, we can call the associated value in (1.1) the critical benefit-to-cost ratio for **random placing**, without referring to a number $n \in \{1, 2, \cdots, N-1\}$ of cooperators. See Section 4 for a proof of Proposition 2.2 which uses (1.5). On the other hand, if we consider a particular initial condition for the game, Proposition 2.2 does not apply and the use of (1.5) is necessary in order to obtain the critical benefit-to-cost ratio.

We now study how the critical benefit-to-cost ratio varies when increasing the number of cooperators in the initial condition.

Our first two results (Proposition 2.3 and Proposition 2.4) consider the question whether on small graphs, having more cooperators in the initial condition favors cooperators (by reducing the critical benefit-to-cost ratio). Proposition 2.3 below gives a rough confirmation of this question. Here and in what follows, on any graph we define N_0 to be the maximum number of vertices which can be chosen such that any of these vertices cannot find another within distance 2. For example, on a cycle with 10 vertices, we have $N_0 = 3$ (see Figure 4 (c)). See Remark 4.2 for a connection between the a-priori condition (1.2) and N_0.

Proposition 2.3. (1) *There exists a k-regular graph such that for some initial condition with one cooperator cooperation can never be favored, but for some initial condition with two cooperators cooperation can be favored.*

(2) *Consider a finite regular graph with $N - 2k > 0$. Then cooperation can be favored for a mixed initial condition with n cooperators for any*

$$n \in \{1, \cdots, N_0 + 1, N - N_0 - 1, \cdots, N - 1\}.$$

In particular, since $N_0 \geq 1$, cooperation can be favored under any initial condition with two cooperators or with two defectors.

Here in Proposition 2.3, “cooperation can be favored” means that the a-priori condition (1.4) holds for the population configuration under consideration. In this case there is a finite critical benefit-to-cost ratio. “Cooperation can never be favored” means otherwise. In addition, recall that $N - 2k > 0$ is the a-priori condition (1.4) for a single cooperator in the initial condition.

Proposition 2.3 (2) only assures cooperation under initial conditions with “small” numbers $1, 2, \cdots, N_0 + 1$ of cooperators or with “large” numbers $N - N_0 - 1, N - N_0, \cdots, N - 1$ of cooperators. It says nothing about the “intermediate” range

$$N_0 + 2, N_0 + 3, \cdots, N - N_0 - 2 \tag{2.1}$$

of numbers of cooperators in initial conditions such that cooperation can be favored. The following result shows that such an omission is necessary in general, and moreover, the lower bound and upper bound for the range (2.1) are sharp.

Proposition 2.4. *There exists a finite regular graph such that cooperation can be favored for initial conditions with one cooperator, but for some mixed initial conditions with N_0+2 cooperators and with $N-N_0-2$ cooperators, cooperation can be never favored.*

Our next two results (Proposition 2.5–2.6) describe the variation of the critical benefit-to-cost ratio when increasing the number of cooperators in the initial condition. We focus on regular graphs satisfying (1.2) and on having either large numbers of cooperators or small numbers of cooperators in the initial condition.

We introduce the initial placing of large numbers of cooperators which makes defectors **isolated** in the sense that any defector cannot find another defector within distance 2. For any configuration with only one defector, that defector is obviously always isolated.

Proposition 2.5. *Consider a finite k-regular graph with $N-2k>0$. Let ξ' and ξ be any mixed initial conditions with $n-1$ and n cooperators, respectively, such that the defectors under both configurations are isolated and condition (1.4) holds. Then we have:*

$$\left(\frac{b}{c}\right)^{\star}_{\xi'} > \left(\frac{b}{c}\right)^{\star}_{\xi}. \tag{2.2}$$

For the initial placing of a small number of cooperators, we compare the critical benefit-to-cost ratio with the critical benefit-to-cost ratio for random placing. The results are given in Proposition 2.6 below. Notice that on any regular graph, the integer N_0 defined above is equal to the maximal number of defectors which a configuration with isolated defectors can carry.

Proposition 2.6. *Consider a finite k-regular graph with $N-2k>0$.*

(1) *If $N_0 \geq 2$, then for any $n \in \{2,3,\cdots,N_0\}$ there exists a configuration with n cooperators such that its critical benefit-to-cost ratio is smaller than the critical benefit-to-cost ratio for random placing.*
(2) *For any configuration with two cooperators where the cooperators are neighbors to each other, cooperation can be favored and the corresponding critical benefit-to-cost ratio is smaller than the critical benefit-to-cost ratio for random placing.*

Proposition 2.5 and 2.6 can be read in a different way if we take into account the invariance of critical benefit-to-cost ratio under reversal of players and recall that the critical benefit-to-cost ratios for random placing and for any configuration with one cooperator are the same (Proposition 2.1). Proposition 2.5 is equivalent to a statement for the initial placing of a small number of cooperators: an increase in the number of isolated cooperators gives a larger critical benefit-to-cost ratio. Similarly, Proposition 2.6 can be stated alternatively in terms of initial conditions with large numbers of cooperators which have critical benefit-to-cost ratios smaller than that for any configuration with $N-1$ cooperators. By the invariance of critical benefit-to-cost ratio under reversal of players, we can translate methods which enhance cooperation by placing more cooperators to arrangements such that cooperation becomes discouraged after more cooperators are placed, and conversely.

3. Methods

In the rest of this paper, we give proofs of the results stated in Section 1 and Section 2. As announced in Section 1, the underlying graph is assumed to be a finite, connected, simple k-regular graph $G=(\mathsf{V},\mathsf{E})$ with $\#\mathsf{V}=N$, for $k\geq 2$, throughout this paper. We denote $\{1,0\}$-valued configurations on G by ξ,η with "1" standing for cooperator and "0" for defector, and vertices in G by x,y. In addition, $\widehat{\xi}$ stands for the configuration obtained from ξ by changing 1's to 0's and 0's to 1's.

Death-birth evolutionary game under weak selection depends on effective payoffs of players which are defined as follows. If individuals play games according to the payoff matrix (1.6) and sufficiently small intensity of selection $w\in[0,1]$, then, under population configuration ξ, the effective payoff $e_i^w(x,\xi)$ of an i-player at x is given by

$$\begin{aligned} e_1^w(x,\xi)&=1+w[an_1(x,\xi)+bn_0(x,\xi)],\\ e_0^w(x,\xi)&=1+w[cn_1(x,\xi)+dn_0(x,\xi)], \end{aligned}\tag{3.1}$$

where $n_i(x,\xi)$ denotes the number of i-neighbors of x under configuration ξ.

We work with death-birth game under weak selection $w\to 0+$. It defines a rate-N pure-jump Markov chain such that at each updating time, a random individual x is chosen to die and its neighbors compete for the vacant site with probability proportional to effective payoff. Hence, when a random site

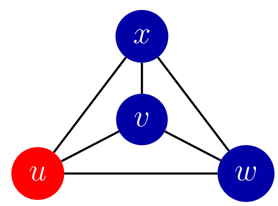

Figure 1: Consider the above configuration, say ξ, where the red vertex is occupied by a defector and the other blue vertices are occupied by cooperators. Let vertex x be a focal vertex. To find $f_{10}(x,\xi)$ and $f_1(x,\xi)\cdot f_0(x,\xi)$, consider the following. Note that each random-walk path of length 2 has probability $\frac{1}{9}$ and there are two x-cooperator-defector paths: $x\to v\to u$ and $x\to w\to u$. This gives $f_{10}(x,\xi)=\frac{2}{9}$. Also, there are two ordered pairs of (x-cooperator, x-defector) paths of length 1: $(x\to w, x\to u)$ and $(x\to v, x\to u)$. Each pair has probability $\frac{1}{9}$. Hence, $f_1(x,\xi)\cdot f_0(x,\xi)=\frac{2}{9}$

x is chosen to update, a new i-player is born at x with probability

$$\pi_i^w(x,\xi)=\frac{\sum_{y\sim x} e_i^w(y)\mathbb{1}_{\{\xi(y)=i\}}}{\sum_{y\sim x}[e_1^w(y)\xi(y)+e_0^w(y)\widehat{\xi}(y)]} \tag{3.2}$$

for $i\in\{1,0\}$, where $y\sim x$ means that y and x are neighbors to each other. In particular, for $w=0$, the evolutionary game reduces to the voter model (or equivalently the neutral model) defined by random walk on G: at each updating time, a random individual chooses to adopt a random neighbor's strategy (Liggett, 1985).

We write $\mathbb{P}_\xi^w$ and $\mathbb{E}_\xi^w$ for the probability and expectation, respectively, of the game under which selection strength is w and initial population configuration is ξ.

Let us define some local frequencies of configurations in order to study the fixation probabilities of the game. Recall that the number of neighbors of each vertex on G is k. We set

$$\begin{aligned} f_i(x,\xi)=&\frac{1}{k}\#\{y; x\sim y,\ \xi(y)=i\},\\ f_{ij}(x,\xi)=&\frac{1}{k^2}\#\{(y,z); x\sim y\sim z,\ \xi(y)=i,\xi(z)=j\}, \end{aligned} \tag{3.3}$$

for $i,j\in\{1,0\}$, and for any function $f(x,\xi)$,

$$\overline{f}(\xi)=\frac{1}{N}\sum_{x\in\mathsf{V}} f(x,\xi) \tag{3.4}$$

as its arithmetic average with respect to site x. See Figure 1 for an example of $f_{10}(x,\xi)$ and $f_1f_0(x,\xi) = f_1(x,\xi)\cdot f_0(x,\xi)$.

The particular average frequencies $\overline{f_1}$, $\overline{f_0}$, $\overline{f_{10}}$ and $\overline{f_1f_0}$ play a crucial role in the subsequent arguments. With respect to a given population configuration, they have the following interpretations. If we run a random walk on the graph with a uniformly chosen starting point, then $\overline{f_i}$ gives the probability of finding an i-player at step 1, and $\overline{f_{10}}$ gives the probability of finding a cooperator at step 1 and a defector at step 2. If we run two independent copies of random walks on the graph with the same starting point and the shared starting point is randomly chosen, then $\overline{f_1f_0}$ is equal to the probability of finding one cooperator at step 1 along the first random walk and one defector at step 1 along the second random walk. These interpretations are equivalent to those introduced in Section 1, thanks to the reversibility of the walks and the fact that stationary distributions of random walks on regular graphs are uniform.

Lemma 3.1. *Consider death-birth game, and assume payoff matrix (1.6) for players. For any configuration ξ, we have the following first-order expansion of fixation probabilities as $w \to 0+$:*

$$
\begin{aligned}
&\mathbb{P}^w_\xi(\textit{cooperators fixate}) = \mathbb{P}^0_\xi(\textit{cooperators fixate})\\
&+ w\Bigg(ak\int_0^\infty \mathbb{E}^0_\xi\left[\overline{f_0f_{11}}(\xi_t)\right]dt + bk\int_0^\infty \mathbb{E}^0_\xi\left[\overline{f_0f_{10}}(\xi_t)\right]dt\\
&\quad - ck\int_0^\infty \mathbb{E}^0_\xi\left[\overline{f_1f_{01}}(\xi_t)\right]dt - dk\int_0^\infty \mathbb{E}^0_\xi\left[\overline{f_1f_{00}}(\xi_t)\right]dt\Bigg) + \mathcal{O}(w^2).
\end{aligned}
\tag{3.5}
$$

Proof. We work with the following expansion:

$$
\begin{aligned}
\mathbb{P}^w_\xi(\text{cooperators fixate}) =&\,\mathbb{P}^0_\xi(\text{cooperators fixate})\\
&+ w\int_0^\infty \mathbb{E}^0_\xi\left[\overline{D}(\xi_t)\right]dt + \mathcal{O}(w^2)
\end{aligned}
\tag{3.6}
$$

as $w \to 0+$, where the function $\overline{D}(\xi)$ is defined by

$$
\overline{D}(\xi) = \frac{1}{N}\sum_{x\in\mathsf{V}}\Big(\widehat{\xi}(x)h_1(x,\xi) - \xi(x)h_0(x,\xi)\Big), \tag{3.7}
$$

$$
h_i(x,\xi) = \frac{d}{dw}\pi^w_i(x,\xi)\big|_{w=0} \tag{3.8}
$$

(cf. Theorem 3.8 in Chen (2013) for the expansion (3.6)). Then by (3.2) and (3.8), we have

$$\begin{aligned}
&h_1(x,\xi) \equiv akf_0f_{11}(x,\xi) + bkf_0f_{10}(x,\xi) - ckf_1f_{01}(x,\xi) - dkf_1f_{00}(x,\xi),\\
&h_0(x,\xi) \equiv -h_1(x,\xi).
\end{aligned}$$

Apply the foregoing explicit forms of h_i to (3.7), and then we obtain the required expansion (3.5) from (3.6). □

We remark that the approach to study fixation probabilities by expansion as in (3.6) also appears in Rousset (2003), Lessard and Ladret (2007), and Ladret and Lessard (2008). The proof for the expansion (3.6) of fixation probabilities in Chen (2013) was obtained independently, and considers an expansion of transition probabilities of the game dynamics which leads to a series-like expansion for fixation probabilities.

Next, let us recall coalescing random walks on graphs in order to compute voter-model integrals as those on the right-hand side of (3.5), and introduce some auxiliary random walks to simplify notation. First, let $\{B^x; x \in \mathsf{V}\}$ be a system of rate-1 coalescing random walks on G, where B^x starts at x. These interacting random walks move independently of each other until they meet another and move together afterwards. The duality between the voter model and the coalescing random walks is through the following equation:

$$\mathbb{E}^0_\xi\left[\prod_{x\in S}\xi_t(x)\right] = \mathbb{E}\left[\prod_{x\in S}\xi(B^x_t)\right] \tag{3.9}$$

for $S \subseteq \mathsf{V}$, $t \in \mathbb{R}_+$ and configurations $\xi \in \{1,0\}^{\mathsf{V}}$. See Section III.4 in Liggett (1985) for the identity (3.9), and also Section III.6 in the same reference for its interpretation by graphical representations.

Second, we introduce two independent discrete-time random walks

$$(X_n; n \geq 0) \quad \text{and} \quad (Y_n; n \geq 0)$$

starting at the same vertex, both independent of $\{B^x; x \in \mathsf{V}\}$. We will write $\mathbb{E}_x$ for the expectation under which the common starting point for (X_n) and (Y_n) is x, and $\mathbb{E}_\pi$ for the expectation under which the common starting point is randomized according to the stationary distribution π, namely uniform distribution, of random walk. The random walk probabilities $\mathbb{P}_x$ and $\mathbb{P}_\pi$ are understood in the same way. These random walks (X_n) and (Y_n) will be used to save notation when we compute local frequencies of configurations. For example, we can write $\sum_x \frac{1}{N}\xi(x)\sum_{y\sim x}\frac{1}{k}\widehat{\xi}(y)$ as $\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_1)]$.

In Lemma 3.2 we evaluate some voter-model integrals which will be used in the proof of Theorem 3.3 below. See also Remark 3.5 (1) for discussions of this lemma.

Lemma 3.2. *We have the following exact formulas:*

$$\int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{f_{10}}(\xi_t)\right]dt = \frac{N\overline{f_1}(\xi)\overline{f_0}(\xi)}{2}, \tag{3.10}$$

$$\int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{f_1f_0}(\xi_t)\right]dt = \frac{N\overline{f_1}(\xi)\overline{f_0}(\xi)}{2} - \frac{\overline{f_{10}}(\xi)}{2}, \tag{3.11}$$

$$\begin{aligned}\int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{f_1f_{*0}}(\xi_t)\right]dt = {}& \frac{N\overline{f_1}(\xi)\overline{f_0}(\xi)}{2} - \frac{\overline{f_{10}}(\xi)}{2}\\ &- \frac{\overline{f_1f_0}(\xi)}{2} + \frac{N\overline{f_1}(\xi)\overline{f_0}(\xi)}{2k},\end{aligned} \tag{3.12}$$

for any initial configuration ξ, where $f_{*0} = f_{10} + f_{00}$.

Proof. Let us state some preliminaries. First, we recall the following equation under the voter model: for any initial condition ξ,

$$\mathbb{E}^0_\xi\left[\,\overline{f_1}(\xi_t)\overline{f_0}(\xi_t)\right] = \overline{f_1}(\xi)\overline{f_0}(\xi) - \frac{2}{N}\int_0^t \mathbb{E}^0_\xi\left[\,\overline{f_{10}}(\xi_s)\right]ds \quad \forall\, t\in\mathbb{R}_+ \tag{3.13}$$

(cf. Theorem 3.1 in Chen, Choi and Cox (2014)). Second, for any vertices $x \neq y$, we have the integral equation

$$\begin{aligned}&\mathbb{E}[\xi(B^x_t)\widehat{\xi}(B^y_t)] = e^{-2t}\xi(x)\widehat{\xi}(y)\\ &+\int_0^t e^{-2(t-s)}\left(\sum_{z\sim x}\frac{1}{k}\mathbb{E}[\xi(B^z_s)\widehat{\xi}(B^y_s)] + \sum_{z\sim y}\frac{1}{k}\mathbb{E}[\xi(B^x_s)\widehat{\xi}(B^z_s)]\right)ds,\end{aligned} \tag{3.14}$$

which follows by considering whether the first epoch time of the bivariate Markov chain (B^x, B^y) occurs before time t or not. Notice that the above equality is false if $x = y$ since the left-hand side is zero but the integral term on the right-hand side is not in general. This fact needs be taken into account when (3.14) is applied. Third, we can use duality (3.9) and rewrite the voter-model integrals in question as

$$\int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{f_{10}}(\xi_t)\right]dt = \int_0^\infty \mathbb{E}_\pi[\xi(B^{X_0}_t)\widehat{\xi}(B^{X_1}_t)]dt, \tag{3.15}$$

$$\int_0^\infty \mathbb{E}^0_\xi\left[\overline{f_1 f_0}(\xi_t)\right] dt = \int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_2})]dt, \tag{3.16}$$

$$\int_0^\infty \mathbb{E}^0_\xi\left[\overline{f_1 f_{*0}}(\xi_t)\right] dt = \int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_3})]dt. \tag{3.17}$$

We are ready to prove (3.10)–(3.12). The first equality (3.10) follows upon passing $t \to \infty$ for both sides of (3.13), since $\overline{f_1} \cdot \overline{f_0}$ is zero for a homogeneous configuration. For (3.11), we use the assumption that the graph has no self-loops to see $X_0 \neq X_1$ a.s. and then obtain from (3.14) that

$$\begin{aligned}\mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_1})] =& e^{-2t}\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_1)] \\ &+ \int_0^t e^{-2(t-s)}\left(\mathbb{E}_\pi[\xi(B_s^{Y_1})\widehat{\xi}(B_s^{X_1})] + \mathbb{E}_\pi[\xi(B_s^{X_0})\widehat{\xi}(B_s^{X_2})]\right) ds \\ =& e^{-2t}\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_1)] + \int_0^t 2e^{-2(t-s)}\mathbb{E}_\pi[\xi(B_s^{X_0})\widehat{\xi}(B_s^{X_2})]ds\end{aligned}$$

by the reversibility of the chain (X_n) under $\mathbb{P}_\pi$. Integrating both sides of the last equality with respect to t over $(0,\infty)$ implies that

$$\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_2})]dt = \int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_1})]dt - \frac{\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_1)]}{2},$$

which gives (3.11) by (3.10), (3.15) and (3.16).

The proof of the last equality (3.12) is similar except that we have to take into account the fact $\mathbb{P}_\pi(X_0 = X_2) > 0$ in applying (3.14):

$$\begin{aligned}&\mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_2})] = e^{-2t}\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)] \\ &+ \int_0^t e^{-2(t-s)}\left(\mathbb{E}_\pi[\xi(B_s^{Y_1})\widehat{\xi}(B_s^{X_2})\mathbb{1}_{\{X_0\neq X_2\}}] + \mathbb{E}_\pi[\xi(B_s^{X_0})\widehat{\xi}(B_s^{X_3})\mathbb{1}_{\{X_0\neq X_2\}}]\right) ds \\ &= e^{-2t}\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)] \\ &+ \int_0^t e^{-2(t-s)}\left(\mathbb{E}_\pi[\xi(B_s^{X_3})\widehat{\xi}(B_s^{X_0})\mathbb{1}_{\{X_2\neq X_0\}}] + \mathbb{E}_\pi[\xi(B_s^{X_0})\widehat{\xi}(B_s^{X_3})\mathbb{1}_{\{X_0\neq X_2\}}]\right) ds,\end{aligned}$$

where the last equality follows since by reversibility (Y_1, X_0, X_1, X_2) under $\mathbb{P}_\pi$ has the same distribution as (X_3, X_2, X_1, X_0) under $\mathbb{P}_\pi$. Integrating both sides of the foregoing equality with respect to t over $(0,\infty)$, we get

$$\begin{aligned}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_2})]dt =&\frac{\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)]}{2}\\&+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_3})\widehat{\xi}(B_t^{X_0})\mathbb{1}_{\{X_0\neq X_2\}}]dt\\&+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_3})\mathbb{1}_{\{X_0\neq X_2\}}]dt.\end{aligned}\tag{3.18}$$

Then we obtain

$$\begin{aligned}&\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_3})]dt\\=&\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_3})\widehat{\xi}(B_t^{X_0})]dt+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_3})]dt\\=&\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_3})\widehat{\xi}(B_t^{X_0})\mathbb{1}_{\{X_0\neq X_2\}}]dt+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_3})\widehat{\xi}(B_t^{X_0})\mathbb{1}_{\{X_0=X_2\}}]dt\\&+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_3})\mathbb{1}_{\{X_0\neq X_2\}}]dt+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_3})\mathbb{1}_{\{X_0=X_2\}}]dt\\=&\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_2})]dt-\frac{\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)]}{2}\\&+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_3})\widehat{\xi}(B_t^{X_2})\mathbb{1}_{\{X_0=X_2\}}]dt+\frac{1}{2}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_2})\widehat{\xi}(B_t^{X_3})\mathbb{1}_{\{X_0=X_2\}}]dt\\=&\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_2})]dt-\frac{\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)]}{2}\\&+\frac{1}{2k}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_1})\widehat{\xi}(B_t^{X_0})]dt+\frac{1}{2k}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_1})]dt\\=&\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_2})]dt-\frac{\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)]}{2}+\frac{1}{k}\int_0^\infty \mathbb{E}_\pi[\xi(B_t^{X_0})\widehat{\xi}(B_t^{X_1})]dt,\end{aligned}$$

where the first and last equalities follow from reversibility, the third equality from (3.18), and the fourth equality from the Markov property of (X_n) at $n=2,0$ and the implication of the graph spatial structure: $\mathbb{P}_x(X_0=X_2)=1/k$ for any x. Hence (3.12) follows upon recalling (3.17) and applying (3.10) and (3.11) to the right-hand side of the last equality. The proof is complete. □

Now we prove the main results of this paper. They are stated as the following two theorems.

Theorem 3.3. *Consider death-birth game between cooperators ("C"s) and defectors ("D"s) subject to payoff matrix*

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix}, \end{array} \tag{3.19}$$

and assume that the underlying social network is a simple, connected, k-regular graph $G = (\mathsf{V}, \mathsf{E})$ on N vertices. Then under weak selection ($w \to 0+$), the fixation probability of cooperators for any initial configuration ξ satisfies the first-order expansion:

$$\begin{aligned} &\mathbb{P}^w_\xi(\text{cooperators fixate}) = \mathbb{P}^0_\xi(\text{cooperators fixate}) \\ &+ \frac{w}{2}\Big\{ b\Big[N\overline{f_1}(\xi)\overline{f_0}(\xi) - k\overline{f_{10}}(\xi) - k\overline{f_1 f_0}(\xi) \Big] \\ &\qquad\qquad - c\Big[kN\overline{f_1}(\xi)\overline{f_0}(\xi) - k\overline{f_{10}}(\xi) \Big] \Big\} + \mathcal{O}(w^2). \end{aligned} \tag{3.20}$$

As immediate applications of (3.20), we obtain the a-priori condition (1.4) for cooperation to be favored, and the equation (1.5) for the critical benefit-to-cost ratio whenever (1.4) holds.

Proof of Theorem 3.3. By Lemma 3.1, it is enough to obtain the required explicit form for the first-order coefficient of w on the right-hand side of (3.5). Since the payoff matrix under present consideration is given by (3.19), a simple computation shows that

$$\begin{aligned} \int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{D}(\xi_t)\right] dt = &-ck\mathbb{E}^0_\xi \int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{f_1 f_0}(\xi_t)\right] dt \\ &+ bk\int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{f_1 f_{*0}}(\xi_t)\right] dt - bk\int_0^\infty \mathbb{E}^0_\xi\left[\,\overline{f_{10}}(\xi_t)\right] dt. \end{aligned} \tag{3.21}$$

Applying the exact formulas in Lemma 3.2 to the right-hand side of (3.21) proves the theorem. □

Theorem 3.4. *Consider death-birth game on a simple connected k-regular graph over N vertices with general payoff matrix (1.6). Then*

$$\mathbb{P}^w_\xi(\text{cooperators fixate}) > \mathbb{P}^w_{\widehat{\xi}}(\text{defectors fixate})$$

for all small $w > 0$ if and only if

$$\begin{aligned}(a-d)&\left[N\overline{f_1}(\xi)\cdot\overline{f_0}(\xi)\left(1+\frac{1}{k}\right)-2\overline{f_{10}}(\xi)-\overline{f_1f_0}(\xi)\right]\\+(b-c)&\left[N\overline{f_1}(\xi)\cdot\overline{f_0}(\xi)\left(1-\frac{1}{k}\right)+\overline{f_1f_0}(\xi)\right]>0.\end{aligned}\tag{3.22}$$

An analogous equivalence holds if we reverse both of the inequalities in the foregoing two displays.

Proof. The idea of this proof is due to Tarnita *et al.* (2009). We begin with an application of (3.6): for all small $w > 0$,

$$\mathbb{P}^w_\xi(\text{cooperators fixate}) > \mathbb{P}^w_{\widehat{\xi}}(\text{defectors fixate}) = 1 - \mathbb{P}^w_{\widehat{\xi}}(\text{cooperators fixate})$$

if and only if

$$\begin{aligned}&\mathbb{P}^0_\xi(\text{cooperators fixate}) + w\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{D}(\xi_t)\big]\,dt + \mathcal{O}(w^2)\\&\qquad > \mathbb{P}^0_{\widehat{\xi}}(\text{defectors fixate}) - w\int_0^\infty \mathbb{E}^0_{\widehat{\xi}}\big[\,\overline{D}(\xi_t)\big]\,dt + \mathcal{O}(w^2).\end{aligned}\tag{3.23}$$

An analogous equivalence holds if we replace "<" in both of the last two displays by ">". On the other hand, the neutrality of players under the voter model implies that

$$\mathbb{P}^0_{\widehat{\xi}}(\text{defectors fixate}) = \mathbb{P}^0_\xi(\text{cooperators fixate}).$$

Hence by (3.23), to prove the theorem, we need to find

$$\begin{aligned}&\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{D}(\xi_t)\big]\,dt + \int_0^\infty \mathbb{E}^0_{\widehat{\xi}}\big[\,\overline{D}(\xi_t)\big]\,dt\\=&(a-d)k\left(\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_0f_{11}}(\xi_t)\big]\,dt + \int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_1f_{00}}(\xi_t)\big]\,dt\right)\\&+(b-c)k\left(\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_0f_{10}}(\xi_t)\big]\,dt + \int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_1f_{01}}(\xi_t)\big]\,dt\right).\end{aligned}\tag{3.24}$$

Here, the foregoing equality follows from (3.5), (3.6) and the neutrality of players under the voter model.

To determine the coefficients of $a-d$ and $b-c$ on the right-hand side of (3.24), we consider the special payoff matrix (3.19). Under (3.19), the foregoing equation (3.24) reads

$$\begin{aligned}
&\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{D}(\xi_t)\big]\,dt+\int_0^\infty \mathbb{E}^0_{\widehat{\xi}}\big[\,\overline{D}(\xi_t)\big]\,dt\\
=&(b-c)k\left(\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_0f_{11}}(\xi_t)\big]\,dt+\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_1f_{00}}(\xi_t)\big]\,dt\right)\\
&-(b+c)k\left(\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_0f_{10}}(\xi_t)\big]\,dt+\int_0^\infty \mathbb{E}^0_\xi\big[\,\overline{f_1f_{01}}(\xi_t)\big]\,dt\right).
\end{aligned}\tag{3.25}$$

On the other hand, we can use (3.20) (see also (3.6)) to write the sum of voter-model integrals on the left-hand side of (3.25) as a linear combination of b and c with coefficients in local frequencies of ξ. Comparing this linear combination with (3.25) and letting b and c vary, we can solve for the coefficients of $(b-c)$ and $(b+c)$ in (3.25) explicitly in terms of local frequencies of ξ. Theorem 3.4 follows if we insert the solutions into (3.24). The proof is complete. □

Let us close this section with some remarks on the methods in this section.

Remark 3.5. (1). Earlier results of critical benefit-to-cost ratios are for death-birth game with suitably randomized initial configurations (Chen, 2013; Allen and Nowak, 2014), and the methods there have strong dependence on links to coalescence times for coalescing random walks. In contrast, the readers may observe that the main objects in our arguments here are some integrated spatial locations visited by coalescing walks with randomized initial conditions as in (3.15)–(3.17). Their use is essential to obtain explicit first-order expansions of fixation probabilities under arbitrary initial conditions.

(2). The result for general payoff matrices in Theorem 3.4 could be reinforced to explicit first-order expansions as in Theorem 3.3 if the voter-model integrals on the right-hand side of (3.5) were evaluated explicitly. For this case, the use of a system of *three* coalescing random walks is required if we apply the duality equation (3.9).

In this direction, it can be shown that the evaluation problem for the voter-model integrals in (3.5) can be simplified if we consider the context of single random cooperators in initial conditions of the game (see Section

3 in Chen (2013)), since, in this case, each of them can be written as an explicit sum of expected coalescence times of three coalescing random walks, or equivalently, expected first meeting times of three *independent* random walks on the same graph. Nonetheless, it seems that so far, exact formulas for expected first meeting times by multiple random walks are possible only in very special cases (cf. Karlin and McGregor (1959)). □

4. Proofs for Section 2

In this section, we prove Proposition 2.1–2.6. We begin with a quantitative version of Proposition 2.1.

Proposition 4.1. *For fixed $k \geq 2$ and $N \in \mathbb{N}$ such that there exists at least one k-regular graph on N vertices and $(N-1)^{1/2} - k > 0$, we have*

$$\max_{G} \max_{\xi} \left| \left(\frac{b}{c} \right)^{\star}_{\xi} - k \right| \leq \frac{k + 2k^2}{(N-1)^{1/2} - k}, \tag{4.1}$$

where G range over all k-regular graphs on N vertices, and for such a k-regular graph G, ξ range over all configurations different from the all-1 configuration and the all-0 configuration.

Proof. Observe that the functions $\overline{f_{10}}$ and $\overline{f_1 f_0}$ are bounded by $\left(\overline{f_1} \cdot \overline{f_0}\right)^{1/2}$ by the Cauchy-Schwarz inequality and the reversibility of random walk. Hence on any k-regular graph with N vertices and for any population configuration ξ different from the all-i configuration for $i = 1, 0$, it follows from (1.5) that

$$\left| \left(\frac{b}{c} \right)^{\star}_{\xi} - k \right| \leq \max_{1 \leq n \leq N} \frac{(k + 2k^2) \left(\frac{n(N-n)}{N^2} \right)^{1/2}}{N \frac{n(N-n)}{N^2} - 2k \left(\frac{n(N-n)}{N^2} \right)^{1/2}} = \frac{k + 2k^2}{(N-1)^{1/2} - 2k},$$

as required in (4.1), since the bound on the right most side is independent of G and ξ. □

Proof of Proposition 2.2 using (1.5). Fix a k-regular graph on N vertices. Let $\mathbf{u}_n$ denote the uniform distribution on the set of configurations ξ

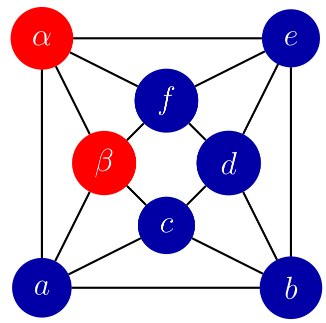

Figure 2: This figure is for the proof of Proposition 2.3. Cooperation can be favored under this initial condition. Blue vertices are occupied by cooperators and red vertices by defectors.

with exactly n many 1's, for $n \in \{1, 2, \cdots, N-1\}$. Since graph geometry is irrelevant under $\mathbf{u}_n$, it follows that

$$\mathbf{u}_n[\xi(x)\widehat{\xi}(y)] = \frac{n(N-n)}{N(N-1)}, \quad \forall\, x \neq y.$$

Hence by the definitions of f_i and f_{ij} in (3.3), we get

$$\mathbf{u}_n\big[\,\overline{f_{10}}(\xi)\big] = \frac{n(N-n)}{N(N-1)} \quad \text{and} \quad \mathbf{u}_n\big[\,\overline{f_1 f_0}(\xi)\big] = \frac{k-1}{k} \cdot \frac{n(N-n)}{N(N-1)}. \tag{4.2}$$

Then by (1.5), the critical benefit-to-cost ratio for initial condition $\mathbf{u}_n$ is given by

$$\left(\frac{b}{c}\right)^{\star}_{\mathbf{u}_n} = \frac{\mathbf{u}_n\big[k\big(N\overline{f_1} \cdot \overline{f_0} - \overline{f_{10}}\,\big)\big]}{\mathbf{u}_n\big[N\overline{f_1} \cdot \overline{f_0} - k\overline{f_{10}} - k\overline{f_1 f_0}\,\big]} = \frac{k(N-2)}{N-2k}.$$

These critical benefit-to-cost ratios are independent of $n \in \{1, 2, \cdots, N-1\}$ and coincide with the value in (1.1).

On the other hand, one can use the equalities (4.2) to see that the coefficient of w on the right-hand side of (3.20) under initial condition $\mathbf{u}_n$ is equal to

$$\frac{n(N-n)}{2N(N-1)}[b(N-2k) - ck(N-2)], \tag{4.3}$$

which is consistent with Theorem 1 (1) in Chen (2013). □

Recall the paragraph before Proposition 2.5 for the placing which makes defectors isolated. The placing which makes cooperators isolated is defined analogously.

Proof of Proposition 2.3. (1) We consider the 4-regular graph on $N = 8$ vertices in Figure 2, where we have two defectors in the graph located at vertex α and β. In this case, $\overline{f_{10}} = \frac{6}{32}$ since we have 6 cooperator-defector paths as

$$(a, \alpha),\ (a, \beta),\ (c, \beta),\ (e, \alpha),\ (f, \alpha),\ (f, \beta),$$

and $\overline{f_1 f_0} = \frac{20}{128}$ since we have 20 cooperator-anything-defector paths as

$$\begin{array}{lllll} (a, c, \beta), & (a, \alpha, \beta), & (a, \beta, \alpha), & (b, a, \alpha), & (b, a, \beta), \\ (b, c, \beta), & (b, e, \alpha), & (c, a, \alpha), & (c, a, \beta), & (c, \beta, \alpha), \\ (d, c, \beta), & (d, e, \alpha), & (d, f, \alpha), & (d, f, \beta), & (e, f, \alpha), \\ (e, f, \beta), & (e, \alpha, \beta), & (f, e, \alpha), & (f, \alpha, \beta), & (f, \beta, \alpha). \end{array}$$

The a-priori condition (1.4) for cooperation to be favored under the configuration in Figure 2 holds since the left-hand side of (1.4) is equal to $\frac{1}{8}$. On the other hand, in this case, $N - 2k = 0$ and so the a-priori condition (1.2) for cooperation to be favored under a configuration with one cooperator fails.

(2) We begin with the observation that for any configuration with n cooperators,

$$\overline{f_{10}} \leq \frac{n}{N} \quad \text{and} \quad \overline{f_1 f_0} \leq \frac{n(k-1)}{Nk}. \tag{4.4}$$

Notice that for (4.4), equalities are attained by a configuration with n isolated cooperators. Indeed $\overline{f_{10}}$ and $\overline{f_1 f_0}$ depend on the numbers of cooperator-defector paths and cooperator-anything-defector paths, respectively, in the underlying configuration, and such a path is defined by an edge or two incident edges. On the other hand, at least one of the equalities (4.4) is a strict inequality, for any configuration which is not a configuration with isolated cooperators. For a configuration where there are two cooperators adjacent to each other, the first inequality in (4.4) is strict; for a configuration where there is a pair of cooperators facing the same defecting neighbor, the second inequality in (4.4) is strict.

Now the proof of (2) follows if we can show that the a-priori condition (1.4) holds for any configuration with n cooperators for $n \in \{2, 3, \cdots, N_0+1\}$, since the left-hand side of (1.4) is invariant under role reversal. Moreover, to verify (1.4) for configurations with these numbers of cooperators, it suffices to show that

$$N - N_0 - 2k \geq 0 \tag{4.5}$$

if we consider the inequality

$$N\overline{f_1} \cdot \overline{f_0} - k\overline{f_{10}} - k\overline{f_1 f_0} \geq \frac{n(N-n-2k+1)}{N}, \tag{4.6}$$

which follows from (4.4). Indeed, for any configuration with $n = N_0 + 1$ cooperators, the foregoing inequality (4.6) is strict by the discussion after (4.4) since the cooperators are not isolated.

We now prove that (4.5) holds, using the assumption that the underlying graph satisfies (1.2). Suppose the converse of (4.5) holds for the graph: $N - 2k < N_0$. Then by the definition of N_0, we can find a configuration with $N - 2k + 1$ isolated cooperators. Counting these isolated cooperators and their neighboring defectors (they do no repeat) yields

$$(N - 2k + 1)(1 + k) \leq N \iff N - 2k \leq 0,$$

which contradicts (1.2). We have proved (2). □

Remark 4.2. The proof of Proposition 2.3 (2) shows that whenever $N - 2k > 0$, the stronger inequality $N - 2k \geq \max_G N_0$ holds, where N_0 is defined for G as before and G ranges over all k-regular graphs on N vertices. This lower bound for $N - 2k$ is sharp by the example in Figure 3, since the graph is a 4-regular graph on 9 vertices which satisfies $N_0 = 1$. □

Proof of Proposition 2.4. The required graph and configuration are given in Figure 3. Some features of the graph has been stated in Remark 4.2, and there are $3 = N_0 + 2$ cooperators in the configuration.

Notice that (1.2) holds, but the a-priori condition (1.4) for cooperation under the configuration in Figure 3 fails since $N\overline{f_1} \cdot \overline{f_0} - k\overline{f_{10}} - k\overline{f_1 f_0} = 0$. Role-reversing this configuration produces a configuration with $6 = N - N_0 - 2$ cooperators such that the corresponding a-priori condition (1.4) fails. □

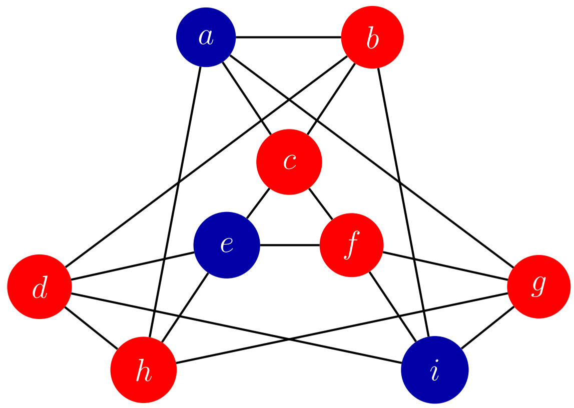

Figure 3: This figure is for the proof of Proposition 2.4. Cooperation cannot be favored under this initial condition. Blue vertices are occupied by cooperators and red vertices by defectors.

Proof of Proposition 2.5. For any configuration η with m isolated defectors, Proposition 2.3 (2) implies that the a-priori condition (1.4) holds for $n = N - m$. Also, equalities occur for both of the inequalities in (4.4) with n replaced by m by role reversal.

Now, if there are $m+1$ and m isolated defectors under ξ' and ξ, respectively, then by (1.5), we obtain

$$\left(\frac{b}{c}\right)^{\star}_{\xi'} = \frac{k(N-m-2)}{N-m-2k} \quad \text{and} \quad \left(\frac{b}{c}\right)^{\star}_{\xi} = \frac{k(N-m-1)}{N-m+1-2k}.$$

The foregoing equalities imply (2.2). □

Proof of Proposition 2.6. (1) Since $N - 2k > 0$, Proposition 2.3 (2) applies and the a-priori condition (1.4) holds for all of the initial conditions considered in (1) and (2).

First we claim that the critical benefit-to-cost ratio for any configuration ξ with n isolated cooperators exceeds the critical benefit-to-cost ratio (1.1) for any configuration with one cooperator. To see this, consider the sequence of configurations $\{\xi^{(n)}, \xi^{(n-1)}, \cdots, \xi^{(1)}\}$ obtained from ξ by converting the isolated cooperators of ξ to defectors one after the other in an arbitrary order, such that $\xi^{(j)}$ has j cooperators. Cooperators under these derived configurations remain isolated. Hence, by role reversal and Proposition 2.5, the sequence of the critical benefit-to-cost ratios associated with $\{\xi^{(j)}\}$ decreases as we decrease the number of cooperators. This is enough for our claim.

We prove (1) now. Recall that the critical benefit-to-cost ratio for any configuration with one cooperator is the same as the critical benefit-to-cost ratio for the random placing of any number $n \in \{1, 2, \cdots, N-1\}$ of cooperators. In addition, note that by (3.21) the critical benefit-to-cost ratio under configuration ξ takes the form $\left(\frac{b}{c}\right)^\star_\xi = \frac{\mathsf{N}_\xi}{\mathsf{D}_\xi}$, for some voter-model expectations N_ξ and D_ξ. By these two observations, the above claim and a simple averaging argument, we deduce the existence of the required configurations with n cooperators for all $n \in \{2, 3, \cdots, N_0\}$.

(2) Let ξ be a configuration with two cooperators placed at adjacent vertices x and y. Then

$$\overline{f_{10}}(\xi) = \frac{2k-2}{Nk} \quad \text{and} \quad \overline{f_1 f_0}(\xi) = \frac{2k(k-1) - 2T(x,y)}{Nk^2}$$

(cf. (4.4)). Here, $T(x, y)$ is the number of vertices which are adjacent to both x and y. Then applying the foregoing display to (1.5), we get

$$\left(\frac{b}{c}\right)^\star_\xi = \frac{k\left(N - 3 + \frac{1}{k}\right)}{N - 2k + \frac{T(x,y)}{k}}. \tag{4.7}$$

It is now readily checked that $\left(\frac{b}{c}\right)^\star_\xi$ is smaller than the critical benefit-to-cost ratio (1.1) for random placing by the assumption $N - 2k > 0$. □

5. Conclusions

In summary, we have calculated how the initial condition affects strategy selection for evolutionary games on graphs. Our calculations apply to regular graphs, where each individual has the same number of neighbors, k, which is also called the degree of the graph. We assume death-birth updating and weak selection. For evolution of cooperation, we have calculated how the critical benefit-to-cost ratio depends on the population size, N, the degree k and the initial condition. Previous work has always calculated the critical benefit-to-cost ratio for the initial condition of placing a single cooperator in a population of defectors. Here we have provided a method that applies to every initial condition. Thus, several cooperators can be placed in an arbitrary configuration on the graph. We show that placing additional cooperators can increase or decrease the critical benefit-to-cost ratio depending on the exact configuration. For general two-strategy games, we prove the existence of a σ value and calculate how that σ value depends on the population size, N, the degree k and the initial condition. Both the value and the critical benefit-to-cost ratio only depend on local properties of the initial configuration, which are evaluated by performing random walks of up to length two.

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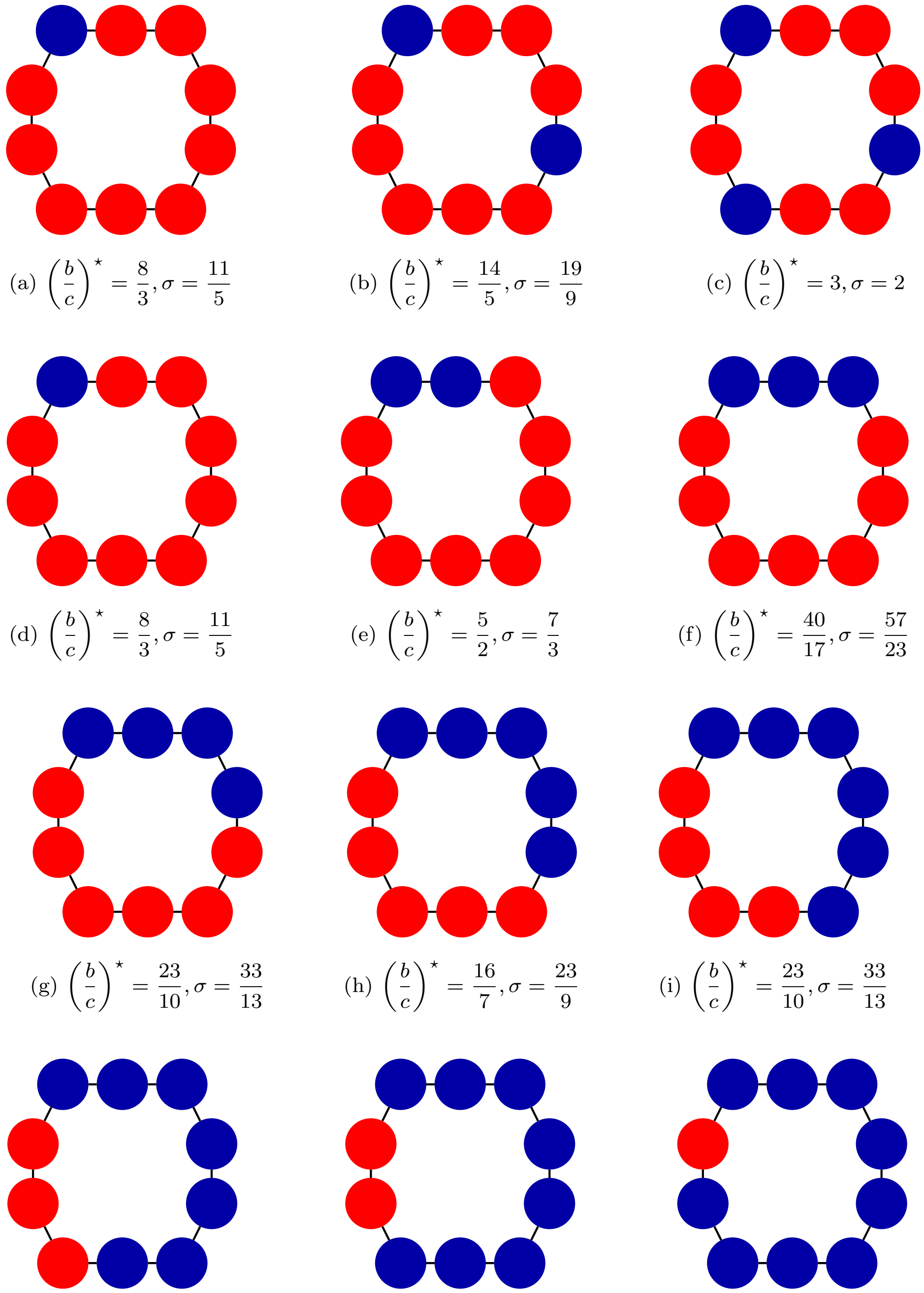


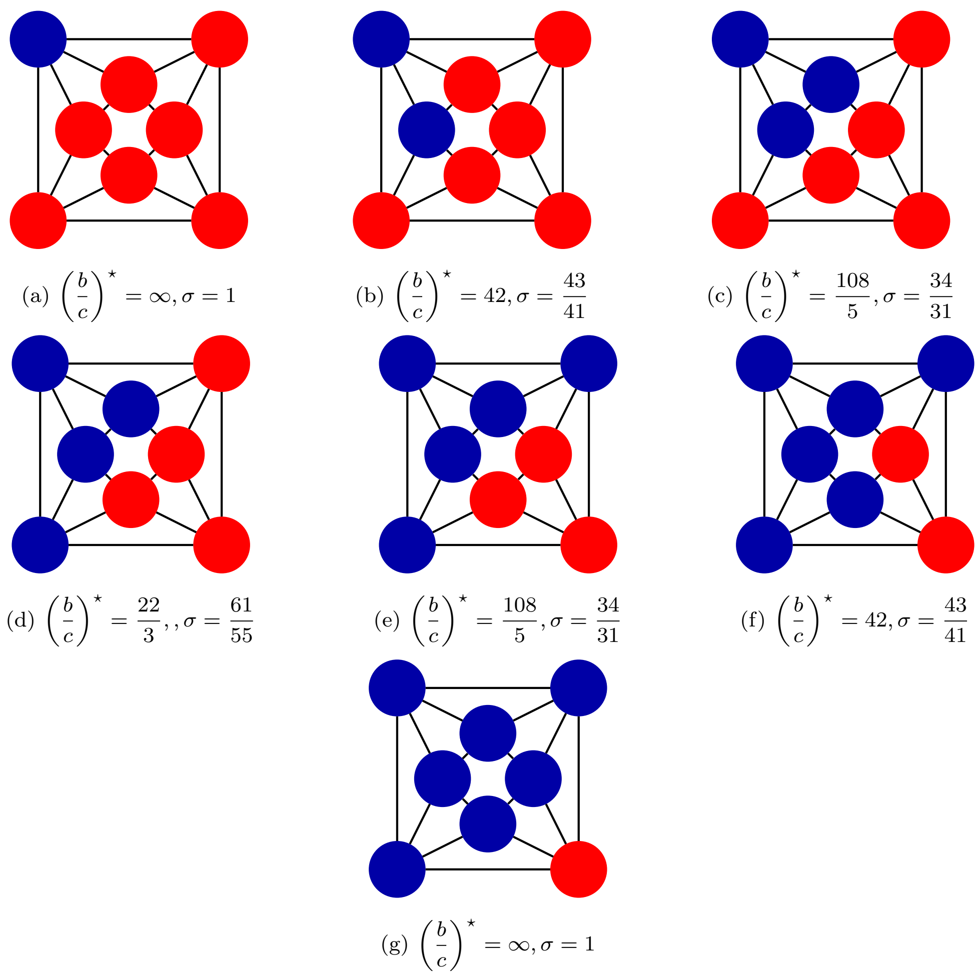
(a) $\left(\frac{b}{c}\right)^{\star} = \frac{8}{3}, \sigma = \frac{11}{5}$
(b) $\left(\frac{b}{c}\right)^{\star} = \frac{14}{5}, \sigma = \frac{19}{9}$
(c) $\left(\frac{b}{c}\right)^{\star} = 3, \sigma = 2$

(d) $\left(\frac{b}{c}\right)^{\star} = \frac{8}{3}, \sigma = \frac{11}{5}$
(e) $\left(\frac{b}{c}\right)^{\star} = \frac{5}{2}, \sigma = \frac{7}{3}$
(f) $\left(\frac{b}{c}\right)^{\star} = \frac{40}{17}, \sigma = \frac{57}{23}$

(g) $\left(\frac{b}{c}\right)^{\star} = \frac{23}{10}, \sigma = \frac{33}{13}$
(h) $\left(\frac{b}{c}\right)^{\star} = \frac{16}{7}, \sigma = \frac{23}{9}$
(i) $\left(\frac{b}{c}\right)^{\star} = \frac{23}{10}, \sigma = \frac{33}{13}$

(j) $\left(\frac{b}{c}\right)^{\star} = \frac{40}{17}, \sigma = \frac{57}{23}$
(k) $\left(\frac{b}{c}\right)^{\star} = \frac{5}{2}, \sigma = \frac{7}{3}$
(l) $\left(\frac{b}{c}\right)^{\star} = \frac{8}{3}, \sigma = \frac{11}{5}$

Figure 4: Finite cycle with $N = 10$. Blue vertices are occupied by cooperators and red vertices by defectors.



(a) $\left(\frac{b}{c}\right)^{\star} = \infty, \sigma = 1$

(b) $\left(\frac{b}{c}\right)^{\star} = 42, \sigma = \frac{43}{41}$

(c) $\left(\frac{b}{c}\right)^{\star} = \frac{108}{5}, \sigma = \frac{34}{31}$

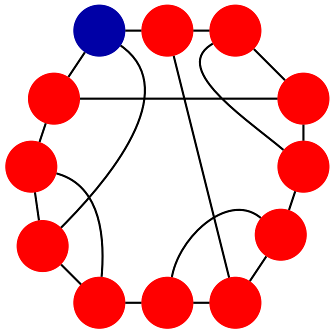
(d) $\left(\frac{b}{c}\right)^{\star} = \frac{22}{3},, \sigma = \frac{61}{55}$

(e) $\left(\frac{b}{c}\right)^{\star} = \frac{108}{5}, \sigma = \frac{34}{31}$

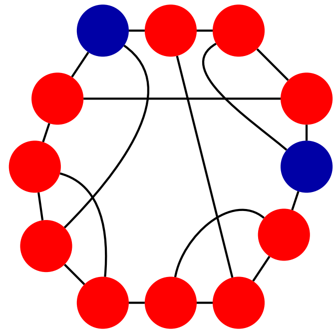
(f) $\left(\frac{b}{c}\right)^{\star} = 42, \sigma = \frac{43}{41}$

(g) $\left(\frac{b}{c}\right)^{\star} = \infty, \sigma = 1$

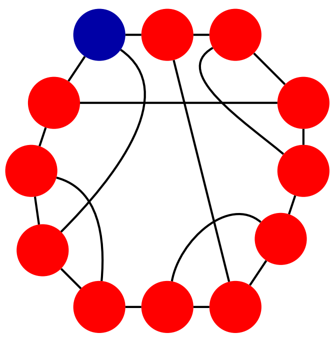
Figure 5: A 4-regular graph on 8 vertices with diameter 2. Blue vertices are occupied by cooperators and red vertices by defectors.



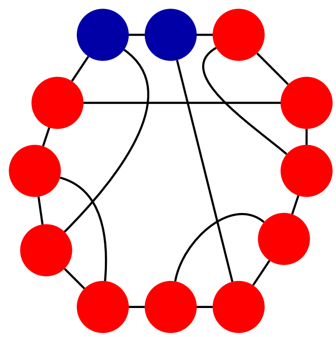
(a) $\left(\frac{b}{c}\right)^{\star} = 5, \sigma = \frac{3}{2}$



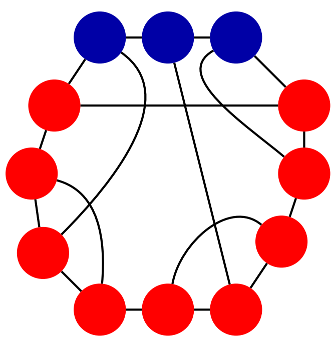
(b) $\left(\frac{b}{c}\right)^{\star} = \frac{27}{5}, \sigma = \frac{16}{11}$



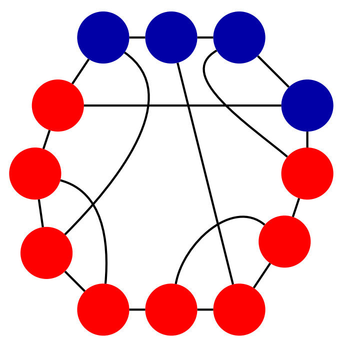
(c) $\left(\frac{b}{c}\right)^{\star} = 5, \sigma = \frac{3}{2}$



(d) $\left(\frac{b}{c}\right)^{\star} = \frac{14}{3}, \sigma = \frac{17}{11}$



(e) $\left(\frac{b}{c}\right)^{\star} = \frac{114}{25}, \sigma = \frac{139}{89}$



(f) $\left(\frac{b}{c}\right)^{\star} = \frac{135}{31}, \sigma = \frac{83}{52}$

Figure 6: A 3-regular graph on 12 vertices (see Theorem 2.3 in Frucht, 1949). Blue vertices are occupied by cooperators and red vertices by defectors.