K-theory and characteristic classes in topology and complex geometry
(a tribute to Atiyah and Hirzebruch)

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Plan of the talk

I. **The early days of Riemann-Roch**

- Characteristic classes of complex vector bundles
- Hirzebruch-Riemann-Roch.

Ref. F. Hirzebruch. *Topological Methods in Algebraic Geometry* (German, 1956, English 1966)

II. **K-theory and cycle class**

- The Atiyah-Hirzebruch spectral sequence and cycle class with integral coefficients.
- Resolutions and Chern classes of coherent sheaves

Ref. M. Atiyah, F. Hirzebruch. *Analytic cycles on complex manifolds* (1962)

III. **Later developments on the cycle class**

- Complex cobordism ring. Kernel and cokernel of the cycle class map.
- Algebraic K-theory and the Bloch-Ogus spectral sequence

• X= compact Riemann surface (= smooth projective complex curve).
$E \to X$ a holomorphic vector bundle on X.

• $\mathcal{E}$ the sheaf of holomorphic sections of E. Sheaf cohomology $H^0(X, \mathcal{E})$= global sections, $H^1(X, \mathcal{E})$ (eg. computed as Čech cohomology).

Def. (*holomorphic Euler-Poincaré characteristic*)
$\chi(X, E) := h^0(X, \mathcal{E}) - h^1(X, \mathcal{E})$.

• E has a rank r and a degree $\deg E = \deg(\det E) := e(\det E)$.

• X has a genus related to the topological Euler-Poincaré characteristic:
$2 - 2g = \chi_{\text{top}}(X)$.

• **Hopf formula:** $2g - 2 = \deg K_X$, where K_X is the canonical bundle (dual of the tangent bundle).

Thm. (*Riemann-Roch formula*) $\chi(X, E) = \deg E + r(1 - g)$

Sketch of proof. (a) **Reduction to line bundles**: any E has a filtration by subbundles E_i such that E_i/E_{i+1} is a line bundle. The 3 quantities r, χ and $\deg$ are additive under short exact sequences.

(b) **Reduction to** $\mathcal{O}_X$: $L=$ holomorphic line bundle on X, $x \in X$. Line bundle $L(-x)$ whose sheaf of sections is $\mathcal{L} \otimes \mathcal{I}_x$, with short exact sequence $0 \to \mathcal{L} \otimes \mathcal{I}_x \to \mathcal{L} \to \mathbb{C}_x \to 0$. One has
$\deg L(-x) = \deg (L) - 1$, $\chi(X, L(-x)) = \chi(X, L) - 1$.
$\Rightarrow$ (*) $\chi(X, L) = \chi(X, \mathcal{O}_X) + \deg L$.

(c) **Serre duality** $\Rightarrow \chi(X, K_X) = -\chi(X, \mathcal{O}_X)$. Formula (*) for K_X then gives $2\chi(X, \mathcal{O}_X) = -\deg K_X$ hence $\chi(X, \mathcal{O}_X) = 1 - g$. **qed**

• **Surfaces.** For a holomorphic line bundle L on a projective surface X, one "easily" gets using the Riemann-Roch formula on curves,
(**) $\quad \chi(X, L) = \chi(X, \mathcal{O}_X) + \frac{L^2 - K_X \cdot L}{2}$.

• Serre duality gives $\chi(X, \mathcal{O}_X) = \chi(X, K_X)$; already contained in (**).

• Hirzebruch uses the *Hodge index theorem* + topological formulae for the signature $\Rightarrow$ Noether formula $\chi(X, \mathcal{O}_X) = \frac{c_1(X)^2 + c_2(X)}{12}$.

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Chern classes of complex vector bundles (Chern, Borel, Borel-Hirzebruch)

• **Chern.** $E=$ complex differentiable vector bundle on a manifold X. $\mathbb{C}$-linear Hermitian connection ∇ on $E \rightsquigarrow$ curvature $R_\nabla = \frac{1}{2i\pi}\nabla \circ \nabla$ and real closed forms $\mathrm{Tr}\, R_\nabla^k$ of degree $2k \rightsquigarrow$ real cohomology classes.

• **Chern classes** $c_k(E)$:="k-th symmetric functions of the eigenvalues of R_∇". Related to the classes above by the Newton formulas.

• L=complex line bundle on $X \rightsquigarrow$ **first Chern class** $c_1(L) \in H^2(X, \mathbb{Z})$. Defined using the map $H^1(X, \underline{(\mathcal{C}^0)^*}) \to H^2(X, \mathbb{Z})$ induced by the exponential exact sequence $0 \to \mathbb{Z} \to \underline{\mathcal{C}^0} \to \underline{(\mathcal{C}^0)^*} \to 1$.

Thm. (*Axiomatic construction/characterization of Chern classes*) *There exist unique Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for any E, X, with total Chern class $c(E) = \sum_i c_i(E)$ satisfying the following axioms.*

(i) Contravariant functoriality.

(ii) (Whitney formula) $c(E \oplus F) = c(E) \cdot c(F)$.

(iii) $c(L) = 1 + c_1(L)$, where $c_1(L)$ is as defined above.

The proof uses the **splitting principle** : *Given $E \to X$, there exists a $f : Y \to X$ such that $f^* : H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ is injective and f^*E is a direct sum of line bundles.*

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• **Virtual roots and symmetric functions.** For any symmetric polynomial f in k variables $\lambda_1, \ldots, \lambda_k$, one has a polynomial P_f in the symmetric functions σ_i of $\lambda_1, \ldots, \lambda_k$, such that $P_f(\sigma.) = f(\lambda.)$.

• Works as well with formal series. If f has coefficients in A, so does P_f.

• E a vector bundle of rank k on X with Chern classes $c_i(E) \in H^*(X, \mathbb{Q})$. For any f as above $\rightsquigarrow P_f(c.(E)) \in H^*(X, \mathbb{Q})$. The λ_i implicitly used in the function f are called the *virtual roots of the Chern polynomial*. When the vector bundle is a direct sum of line bundles, one can take $\lambda_i = c_1(L_i)$.

• In general, the λ_i can be realized as cohomology classes only on a splitting manifold $Y \to X$ for E.

• **Chern character**: $\mathrm{ch}\, E = \sum_i \exp \lambda_i$. Obviously $\mathrm{ch}\,(E \oplus F) = \mathrm{ch}\, E + \mathrm{ch}\, F$, $\mathrm{ch}\,(E \otimes F) = \mathrm{ch}\, E \cdot \mathrm{ch}\, F$.

• **Todd genus**. $\mathrm{td}\, E = \prod_i \frac{\lambda_i}{1-\exp(-\lambda_i)}$. Obviously $\mathrm{td}\,(E \oplus F) = \mathrm{td}\, E \cdot \mathrm{td}\, F$.

• E=complex vector bundle on X= complex manifold. T_X has a complex structure $\rightsquigarrow$ Chern classes $c_i(E)$, $c_j(T_X)$.

• Holomorphic structure on $E \rightsquigarrow$ sheaf $\mathcal{E}$ of holomorphic sections, cohomology groups $H^i(X, \mathcal{E})$ and holomorphic Euler-Poincaré characteristic $\chi(X, E) := \chi(X, \mathcal{E}) = \sum_i (-1)^i h^i(X, \mathcal{E})$ (X compact).

Thm. (*Hirzebruch-Riemann-Roch formula*) *One has* $\chi(X, E) = \int_X \mathrm{ch}\, E \cdot \mathrm{td}\, X =: T_0(X, E)$.

• **The χ_y-genus**. T_X is a holomorphic vector bundle, hence also $\Omega_X = T_X^*$. Define $\chi_y(E) := \sum_p y^p \chi(X, E \otimes \Omega_X^p)$.

• **Obvious.** $\chi(X, E) = \chi_0(E)$.

• **Less obvious, due to Serre.** *For the trivial bundle $\mathcal{O}_X$, one has* $\chi_{-1}(X, \mathcal{O}_X) = \chi_{\mathrm{top}}(X)$.

Proof. Holomorphic de Rham complex $0 \to \mathcal{O}_X \to \Omega_X \to \ldots \to \Omega_X^n \to 0$. This is a resolution of the constant sheaf $\mathbb{C}$. **qed**

• T_y-**genus** $T_y(X, E)$: plug-in y in the formal expression for $\mathrm{ch}\, E \cdot \mathrm{td}\, X$, eg $\mathrm{ch}_y(E) = \sum_i \exp(1 + y)\lambda_i$.

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Strategy of the proof in the projective case

• **Reduction to the line bundle case.** Work on $\mathbb{P}(E)$ and the Hopf line bundle H on $\mathbb{P}(E)$. Leray spectral sequence $\Rightarrow$ $\chi(\mathbb{P}(E), H) = \chi(X, E)$.

• **Reduction to the absolute case (trivial line bundle)**. If $D \subset X$ is a smooth hypersurface, and $\mathcal{L} = \mathcal{O}_X(-D) = \mathcal{I}_D$, one has $0 \to \mathcal{L} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ so $\chi(X, L) = \chi(X, \mathcal{O}_X) - \chi(D, \mathcal{O}_D)$. Use also $0 \to \mathcal{L}_{|D} \to \Omega_{X|D} \to \Omega_D \to 0$.

• **Absolute case.** Index $\tau(X)$ for X real oriented of dimension $2n$: $\tau(X) = 0$ if n is odd, otherwise $\tau(X) :=$ signature of intersection pairing on $H^n(X, \mathbb{R})$. Thom cobordism $\Rightarrow \tau(X)=$ polynomial in the Pontryagin classes of X. If X is almost complex: get Chern number of X. Hirzebruch: *this is* $T_1(X)$.

Thm. (*Hodge index thm*) *If X is a complex projective manifold, one has* $\tau(X) = \sum_p \chi(X, \Omega_X^p) =: \chi_1(X, \mathcal{O}_X)$. (**True for X complex compact**).

• $\Rightarrow$ equality $\chi_1(X, \mathcal{O}_X) = T_1(X)$.

• Functional equation for χ_y-genus and T_y-genus + equality for $y = 1 \Rightarrow$ $\chi_0(X, \mathcal{O}_X) = T_0(X)$ for X a split manifold, and finally for any X. **qed**

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• For a topological space X, $K^0(X)$ is the abelian group with generators the isomorphism classes $[E]$ of complex vector bundles E on X, and relations $[E \oplus F] = [E] + [F]$. For pointed space (X, x), $\overline{K}^0 =$ rank 0 at x.

• **Holomorphic variant**. $X =$ complex manifold. $K^0_{an}(X)$ is the abelian group with generators the isomorphism classes $[E]$ of holomorphic vector bundles E on X, and relations $[G] = [E] + [F]$ whenever there exists an exact sequence $0 \to E \to G \to F \to 0$ of holomorphic vector bundles.

• Due to the Whitney axiom, Chern classes factor through K^0. The Chern character gives a ring homomorphism to **rational** cohomology.

• Atiyah-Hirzebruch introduce K^*:
$K^1(X) := \mathrm{Ker}\,(K^0(X \times \mathbb{S}^1) \to K^0(X))$ + Bott periodicity. For a pair (X, Y) (say of CW-complexes), let $K^0(X, Y) := \overline{K}^0(X/Y)$. Long exact sequence (*)

$$K^{-1}(Y) \to K^0(X, Y) \to K^0(X) \to K^0(Y) \to K^1(X, Y) \to \dots$$

• X a CW-complex. $X^i \subset X$ is the i-skeleton of X, union of cells of dimension $\leq i$.

• One gets a decreasing filtration of the cochain complex by subcomplexes $C^*(X, X^p)$ and a spectral sequence with $E_1^{p,q} = 0$ for $q \neq 0$, $E_1^{p,0} = C^p(X^p/X^{p-1})$, $E_2^{p,0} = E_\infty^{p,0} = H^p(X, \mathbb{Z})$.

• Using (*), Atiyah and Hirzebruch construct a similar spectral sequence for K-theory.

Thm. *There exists a spectral sequence $E_2^{pq} \Rightarrow K^{p+q}(X)$ with $E_2^{pq} = 0$ if q is odd, $E_2^{pq} = H^p(X, \mathbb{Z})$ if q is even.*

• (**Formal**). *The differential d_r vanishes for even r.*
• With $\mathbb{Q}$-coefficients, the differentials must vanish (compare with cohomology).

Cor. One has $E_2^{pq} = E_\infty^{pq}$ if
(i) $H^{\text{odd}}(X, \mathbb{Z}) = 0$ or
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• Let $Z \subset X$ be a closed analytic subset in a complex manifold. Coherent sheaves $\mathcal{I}_Z \subset \mathcal{O}_X$, $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$.

• In the smooth projective case: any coherent sheaf admits a (finite) locally free resolution. Follows from (a) local statement, (b) any coherent sheaf $\mathcal{H}$ admits a surjective quotient map $\mathcal{F} \to \mathcal{H} \to 0$ with $\mathcal{F}$ locally free.

• Not true in the general compact complex case:
Thm. (Voisin 2002) *Take $X = T$ very general complex torus of dimension 3, $x \in T$ a point. Then $\mathcal{I}_x$ does not admit a locally free resolution.*

• X complex compact. Atiyah-Hirzebruch use locally free resolutions of coherent sheaves by **real analytic** complex vector bundles:
$0 \to \mathcal{F}_n \to \dots \mathcal{F}_i \dots \to \mathcal{F}_0 \to \mathcal{H}_\omega \to 0$. Thus any coherent sheaf $\mathcal{H}$ has a class in $K^0(X)$. One has $c(\mathcal{H}) = \prod_i c(\mathcal{F}_i)^{\epsilon_i}$, $\epsilon_i = (-1)^i$.

Thm. (Atiyah-Hirzebruch, Grothendieck-Riemann-Roch) *$Z \subset X$ closed analytic of codimension k. $\mathcal{O}_Z$ has a class in $K^0(X, X \setminus Z)$ and (*) $c_k(\mathcal{O}_Z) = (-1)^{k-1}(k-1)![Z]$ in $H^{2k}(X, \mathbb{Z})$.*

• Here $[Z] \in H^{2k}(X, \mathbb{Z})$ is the **cycle class** of Z.

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Conj. (*Hodge conjecture*) *Let X=projective complex manifold and $\alpha \in H^{2k}(X,\mathbb{Q})$ be of Hodge type (k,k). Then $\alpha = \sum_i \alpha_i [Z_i]$, with $\alpha_i \in \mathbb{Q}$, $Z_i \subset X$ closed of codim. k.*

Rem. Equivalent formulations, using resolutions and formula (*): *$\alpha \in \langle c_k(\mathcal{F})\rangle_{\mathbb{Q}}$, $\mathcal{F}$= coherent sheaf on X*, or *$\alpha \in \langle c_k(\mathcal{F})\rangle_{\mathbb{Q}}$, $\mathcal{F}$= locally free coherent sheaf on X.*

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- **$\mathbb{Z}$-coefficients**. **Wrong** (Atiyah-Hirzebruch).

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- **The example**. 2-torsion cohomology class α of degree 4 on a smooth projective manifold X which is not annihilated by $d_3 \Rightarrow$ not a cycle class.

- **Construction**: Serre's trick : finite group G acting on projective space $\mathbb{CP}^N \rightsquigarrow$ complete intersection $X \subset \mathbb{CP}^N$ on which G acts freely $\rightsquigarrow X/G$.

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• **Milnor construction of** $MU_*(pt)$. 1) **Generators**: compact differentiable manifolds M of dim $*$ + a virtual complex structure on T_M.
2) **Complex cobordism relations**: N = differentiable manifold with boundary ∂N and virtual complex structure on T_N, hence virtual complex structure on $T_{\partial N}$.

• **Complex cobordism group** $MU_*(X)$, X**=manifold**: 1) **Generators**: compact diff. manifolds M of dim. $*$, + diff. map $f : M \to X$ + virtual complex structure on virtual normal bundle $N_f := f^*T_X - T_M$.
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• Map $o : MU_*(X) \otimes_{MU_*(\mathrm{pt})} \mathbb{Z} \to H_*(X, \mathbb{Z})$, $(M, f) \to f_*[M]$. Iso. $\otimes\mathbb{Q}$.

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• The Atiyah-Hirzebruch-Totaro obstruction for an integral cohomology class α on X =complex compact manifold to be algebraic is **topological**.
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• **Kollár's examples.** Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a smooth hypersurface of degree d. Lefschetz thm on hyperplane sections $\Rightarrow H^{2n-4}(X, \mathbb{Z}) = \mathbb{Z}\alpha$, with $\deg \alpha = 1$. If X contains a line Δ, $\alpha = [\Delta]$.

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• The proof is by specialization of X to the image of a generic map $\mathbb{P}^{n-1} \to \mathbb{P}^n$ given by polynomials of degree p.

• $n = 4$. X as above, S = surface with $0 \neq \beta \in H^2(S, \mathbb{Z})$ of p-torsion. $\rightsquigarrow$ p-torsion class $\gamma = \mathrm{pr}_1^*\alpha \smile \mathrm{pr}_2^*\beta \in H^6(Y, \mathbb{Z})$, $Y = X \times S$.

Thm. (Soulé-Voisin) *The p-torsion class γ is not algebraic on Y for X very general.* (The prime p is arbitrarily large, the dimension is fixed.)

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Nontopological obstructions to the algebraicity of integral Hodge classes

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- Let X be a complex algebraic manifold. $X(\mathbb{C})$ has two topologies, the Euclidean and Zariski topologies $\rightsquigarrow$ continuous map $f : X_{\mathrm{an}} \to X_{\mathrm{Zar}}$.
- The Bloch-Ogus spectral sequence is the Leray spectral sequence of f.
- A abelian group. $\mathcal{H}^i(A) := R^i f_* A$, sheaf associated to presheaf $U \mapsto H^i(U_{\mathrm{an}}, A)$ on X_{Zar}.
- $E_2^{p,q} = H^p(X_{\mathrm{Zar}}, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(X_{\mathrm{an}}, A)$.

Thm. (Bloch-Ogus) *(a) One has $E_2^{p,q} = 0$ for $p > q$.*
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- Group of cycles $\mathcal{Z}^k(X) = \{\sum_i n_i Z_i, \operatorname{codim} Z_i = k\}$.

Def. *X projective. Z, $Z' \subset X$ are algebraically equivalent if $\exists$ smooth projective curve C, a cycle $\mathcal{Z}$ in $C \times X$ (flat over C) and two points t, t' of C such that $\mathcal{Z}_t - \mathcal{Z}_{t'} = Z - Z'$ as cycles of X.*

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• Let $\mathrm{Griff}^k(X) := \mathrm{Ker}\,[\,] \subset \mathcal{Z}^k(X)/\mathrm{alg}$. Analyzing the Bloch-Ogus spectral sequence in degree 4, get:

Cor. (Bloch-Ogus) *($k = 2$) Exact sequence*
$H^3(X, \mathbb{Z}) \to H^0(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z})) \stackrel{d_2}{\to} \mathrm{Griff}^2(X) \to 0$.

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Cor. (Bloch-Ogus) *The group $H^0(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Q}))/H^3(X, \mathbb{Q})$ can be nonzero.*

• Define $H^k(\mathbb{C}(X), A) := \lim_{\to\, \emptyset \neq U \subset X,\ \mathrm{Zar.\,open}} H^k(U, A)$.

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