On the origins of Langlands’ conjectures

by Bill Casselman

This talk can be found at

http://www.math.ubc.ca/~cass/harvard/harvard-2020.pdf

Langlands has made many contributions to number theory, but the principal one is probably his discovery in 1966–67, followed by work in subsequent years, of the role of the dual group in the theories of automorphic forms and L-functions. I shall try to explain what this amounted to by tracing the origins of this development through work of Ramanujan, Hecke, Siegel, Maass, Selberg, and other mathematicians of the twentieth century.

Here's a sobering thought:
more time separates us from Langlands' conjectures
than Langlands' from Ramanujan's!

Contents

1. Sums of squares

I begin with something elementary. *Have patience.*

Let

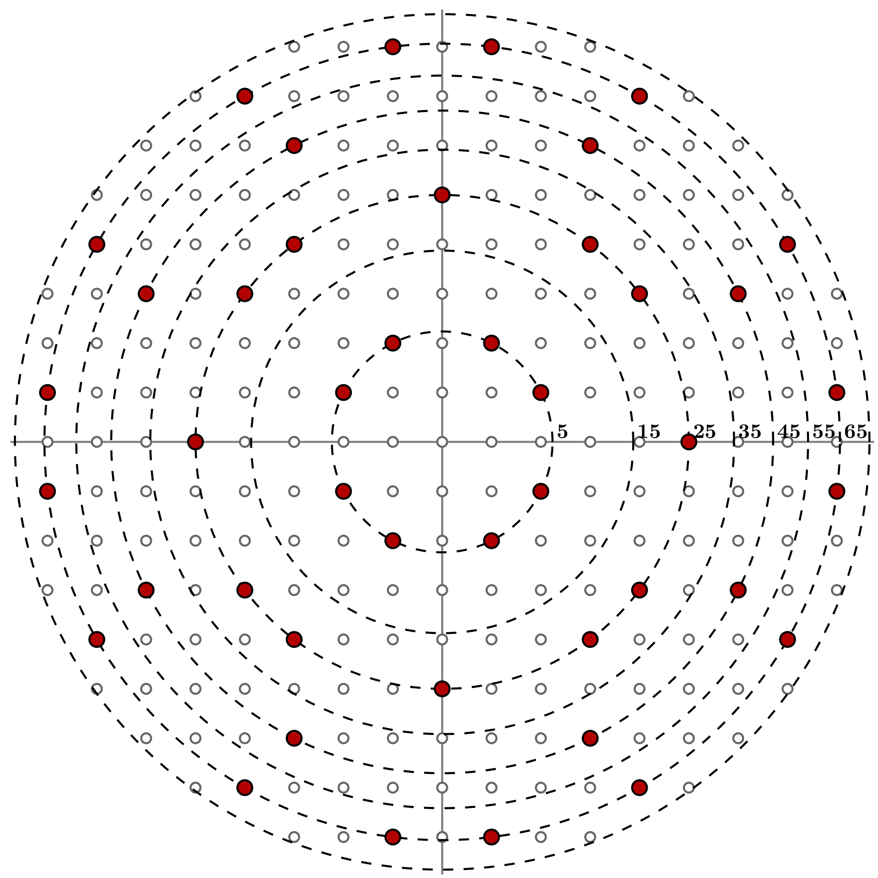
$r_\ell(n) =$ the number of ways n can be expressed as a sum of ℓ integral squares.

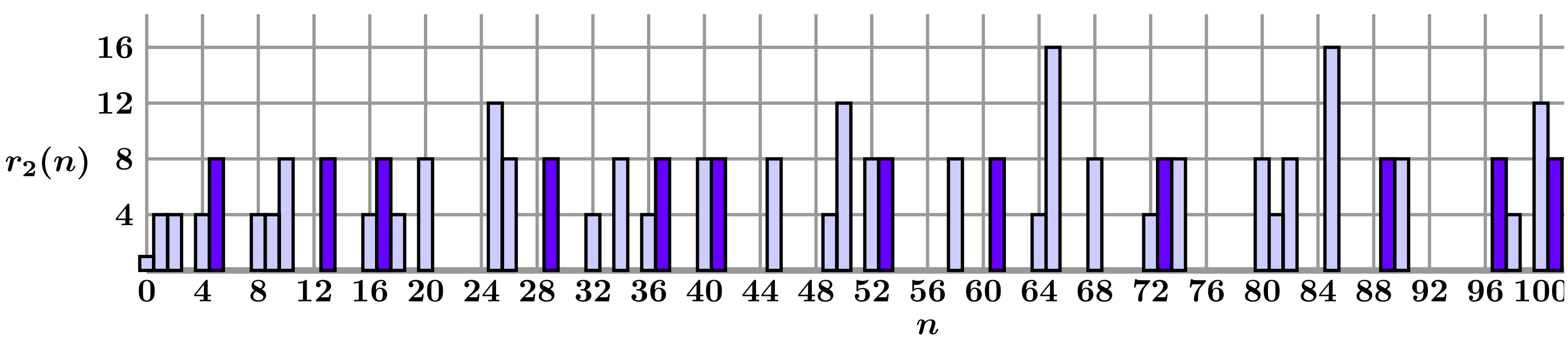
Thus $r_2(4) = 4$ because $4 = (\pm 2)^2 + (\pm 2)^2$.

Many well known mathematicians of the nineteenth century, among them Gauss, Jacobi, Eisenstein, Liouville, Smith, and Minkowski found formulas for this function, when k is small. Gauss found one for r_3, and Smith and Minkowski shared a prize for finding r_5.

Odd ℓ behave very, very differently from even ℓ, and I'll assume ℓ even from now on.

To give you an idea of the flavour of the problem, consider r_2.





The reason for the apparently erratic behaviour is that $r_2(n)$ depends on the factorization of n. For example, as Fermat knew, an odd prime p may be expressed the sum of squares if and only if $p \equiv_4 1$.

Jacobi proved that

$$r_2(n) = 4\Big(\sum_{d|n, d\equiv_4 1} 1 - \sum_{d|n, d\equiv_4 3} 1 \Big)$$

$$r_4(n) = 8\sigma_1(n) = 8 \sum_{d|n} d$$

He later went on to find formulas for r_6 and r_8, which we'll see in a moment.

One of the basic tools in Jacobi's work, and in all subsequent investigations, is the series

$$\vartheta = \sum_{\mathbb{Z}} q^{n^2} = 1 + 2\sum_{1} q^{n^2} \,.$$

This is because

$$\vartheta^\ell = \sum_{n=0} r_\ell(n) q^n \,.$$

For example

$$\vartheta^2 = \sum_{m,n} q^{m^2+n^2} \,.$$

So the problem can be reformulated: *How to find a formula for ϑ^ℓ?*

One hint as to what will be involved is the fact that ϑ^ℓ can be defined as a holomorphic function on the entire upper half-plane by setting $q = e^{2\pi i z}$. If $z = x + iy$ with $y > 0$ then

$$\vartheta^\ell(z) = \sum_{n=0} r_\ell(n)\, e^{2\pi i n x} e^{-2\pi n y} \,.$$

But $r_\ell(n)$ is certainly bounded by some power of n . . .

By 1907, answers of some kind were known for all even $\ell \leq 18$. The situation was summarized by J. W. L. Glaisher, a Fellow of Trinity College, Cambridge, who was particularly interested in numerical computation. I reproduce his notation exactly:

$$
\begin{aligned}
r_2(n) &= 4\,E_0(n) \\
r_4(n) &= (-1)^{n-1} 8\,\xi_1(n) \\
r_6(n) &= 4\big(E_2'(n) - E_2(n)\big) \\
r_8(n) &= (-1)^{n-1} 16\,\zeta_3(n) \\
r_{10}(n) &= (4/5)\big(E_4(n) + 16\,E_4'(n) + 8\,\chi_4(n)\big) \\
r_{12}(n) &= \begin{cases} -8\xi_5(n) & \text{if } n \text{ even} \\ 8\big(\Delta_5(n) + 2\Omega(n)\big) & \text{otherwise.} \end{cases} \\
r_{14}(n) &= (4/61)\big(64E_6'(n) - E_6(n) + 364\,W(n)\big) \\
r_{16}(n) &= (-1)^{n-1}(32/17)\big(\zeta_7(n) + 16\,\Theta(n)\big) \\
r_{18}(n) &= (4/18005)\big(13E_8(n) + 3328E_8'(n) \\
&\qquad + 97504\,\chi_8(n) - 61200\,G(n) - 6120\,G(2n)\big)
\end{aligned}
$$

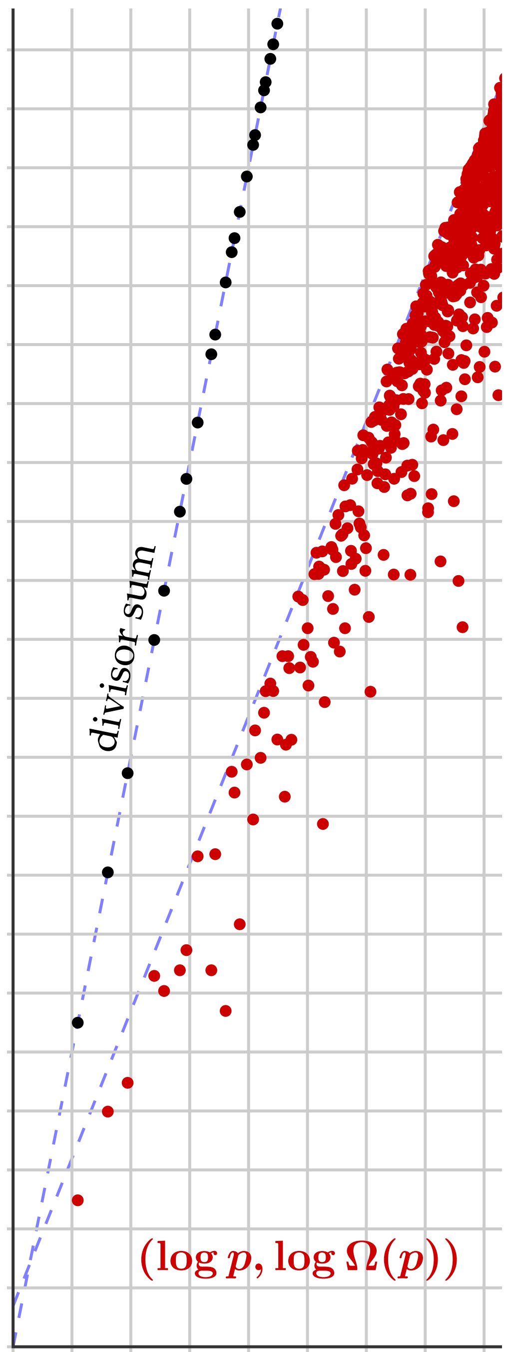
Here E_n, E_n', ξ_n, Δ_n, ζ_n are certain divisor sums. The functions in red all have vanishing constant term and are normalized so that the first term is equal to 1. The terms in red are also asymptotically much smaller than the others. There are simple formulas for $\chi_m(n)$, not the others.

For example, the case $\ell = 12$:

$$\begin{aligned}\Omega = q &- 12\,q^{3} + 54\,q^{5} - 88\,q^{7} - 99\,q^{9} + 540\,q^{11} \\ &- 418\,q^{13} - 648\,q^{15} + 594\,q^{17} + 836\,q^{19} + 1056\,q^{21} \\ &- 4104\,q^{23} - 209\,q^{25} + 4104\,q^{27} - 594\,q^{29} + 4256\,q^{31} \\ &- 6480\,q^{33} - 4752\,q^{35} - 298\,q^{37} + 5016\,q^{39} + 17226\,q^{41} \\ &- 12100\,q^{43} - 5346\,q^{45} - 1296\,q^{47} - 9063\,q^{49} - 7128\,q^{51} \\ &+ 19494\,q^{53} + 29160\,q^{55} - 10032\,q^{57} - 7668\,q^{59} - 34738\,q^{61} \\ &+ 8712\,q^{63} - 22572\,q^{65} + 21812\,q^{67} + 49248\,q^{69} - 46872\,q^{71} \\ &+ 67562\,q^{73} + 2508\,q^{75} - 47520\,q^{77} - 76912\,q^{79} - 25191\,q^{81} \\ &+ 67716\,q^{83} + 32076\,q^{85} + 7128\,q^{87} + 29754\,q^{89} + 36784\,q^{91} \\ &- 51072\,q^{93} + 45144\,q^{95} - 122398\,q^{97} - 53460\,q^{99} + 11286\,q^{101} \pm \cdots\end{aligned}$$

Glaisher noticed empirically that $\Omega(mn) = \Omega(m)\Omega(n)$ when $(m,n) = 1$. This is best interpreted in terms of a Dirichlet series with Euler product:

$$\sum_{1} \frac{\Omega(n)}{n^s} = \prod_p \left(1 + \Omega(p)p^{-s} + \Omega(p^2)p^{-2s} + \Omega(p^3)p^{-3s} + \cdots\right).$$



This is a log-log plot for $\ell = 12$. The divisor sums in all cases are of order $p^{\ell/2-1}$, here p^5.

Because of the multiplicative property, the terms $\Omega(p)$ are of greatest interest.

And as I have already remarked, the 'supplementary term' is asymptotically much smaller than this divisor sum.

$$
\begin{aligned}
r_2(n) &= 4\,E_0(n)\\
r_4(n) &= (-1)^{n-1} 8\,\xi_1(n)\\
r_6(n) &= 4\big(E_2'(n) - E_2(n)\big)\\
r_8(n) &= (-1)^{n-1} 16\,\zeta_3(n)\\
r_{10}(n) &= (4/5)\big(E_4(n) + 16\,E_4'(n) + 8\,\chi_4(n)\big)\\
r_{12}(n) &= \begin{cases} -8\xi_5(n) & \textbf{if } n \textbf{ even}\\ 8\big(\Delta_5(n) + 2\Omega(n)\big) & \textbf{otherwise.}\end{cases}\\
r_{14}(n) &= (4/61)\big(64E_6'(n) - E_6(n) + 364\,W(n)\big)\\
r_{16}(n) &= (-1)^{n-1}(32/17)\big(\zeta_7(n) + 16\,\Theta(n)\big)\\
r_{18}(n) &= (4/18005)\big(13E_8(n) + 3328E_8'(n)\\
&\qquad + 97504\,\chi_8(n) - 61200\,G(n) - 6120\,G(2n)\big)
\end{aligned}
$$

- **What about larger ℓ?**
- **Can we find a single formula for all divisor sums?**
- **Why sometimes none, sometimes many supplemental terms?**
- **Can we estimate them?**
- **To what extent does the ‘multiplicative property’ hold?**
- **Is there anything to be said about the p-series like $\sum_n \Omega(p^n)p^{-ns}$?**
- **What about other quadratic forms?**

Can we bring some organization into this apparently chaotic business?

The situation doesn't seem very satisfactory to us, but it is definitely intriguing. Glaisher wasn't very happy about it, either, but for reasons different from ours.

He commented: "The only complete and effective method of research in such investigations is afforded by the processes of the theory of numbers, and any method dependent upon elliptic functions . . . is necessarily partial and inadequate."

This contrasts completely with what we now believe: *In the course of the twentieth century, it became clear that automorphic forms (as such functions are called very generally) make up an an inevitable component of number theory.*

Unlike Glaisher, we have probably become used to simple problems with insanely complicated solutions.

In 1916, about ten years later, Ramanujan read Glaisher's note (at Hardy's suggestion, I imagine), and found formulas for r_{20}, r_{22}, r_{24}. The case of r_{24} is especially striking, because in this case the supplementary series involved a well known entity.

$$r_{24}(n) = \left(\frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}(n/2)\right) + \left(\frac{33152}{691}(-1)^{n-1}\tau(n) - \frac{66536}{691}\tau(n/2)\right)$$

where $\sum_1 \tau(n)q^n = q\prod_k (1-q^\ell)^{24} =$ **(say)** Δ.

When q is set to $e^{2\pi i z}$ this becomes the discriminant of elliptic function theory. Following Glaisher, Ramanujan then conjectured that

$$\tau(m)\tau(n) = \tau(mn) \qquad \textbf{if } (m,n) = 1\,.$$

Ramanujan also looked at the p-series and found empirically that it satisfied a difference equation of order two. Combining these observations, he arrived at the conjecture that

$$\sum_1 \frac{\tau(n)}{n^s} = \prod \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}\,.$$

It is difficult to exaggerate how consequential this observation was.

He also found an simple expression for the divisor sum when $\ell \equiv_8 4$. partially answering one of our questions. I am not sure why he couldn't deal with the arbitrary case $\ell \equiv_4 0$.

Even more interesting, he wrote

$$x^2 - \tau(p)x + p^{11} = (x - \alpha_p p^{11/2})(x - \beta_p p^{11/2}),$$

and then asserted that it was "highly probable" that the roots were complex conjugates. In this case, we can set

$$\alpha_p = e^{2\pi i\theta_p}$$

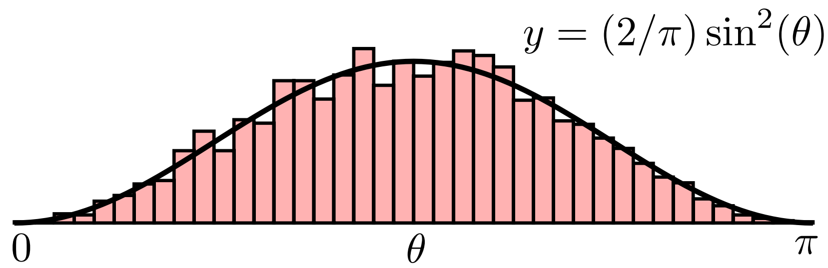
for some θ_p in $[0, \pi]$. Equivalently, $\cos\theta_p = \tau(p)/2p^{11/2}$. This conjecture is certainly strongly suggested by the graph I showed earlier.

It is not clear to me that there was much excitement at the time attached to this guess.

Ramanujan's paper attracted more attention than Glaisher's. One reason is that the function Δ is a well known modular form. Hardy remarked in his book on Ramanujan (1939), "We may seem to be straying into one of the back waters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect."

Still, my impression is that even as late as 1939, as well as in 1917, these matters were not thought to be of much significance. (Well $\ldots$ not by any English mathematician. The Germans were doing much better.)

One thing that Ramanujan might have done, but apparently did not, was examine the statistical distribution of the angles θ_p involved in the supplementary series. Here is a bar graph of the distribution for about 2500 values of $\Omega(p)$, the supplementary term for r_{12}.



Superimposed is a conjectured asymptotic limit.

This contrasts with the case of r_{10}. Here the supplementary term is

$$\chi_4(p) = \sum_{\substack{z \in \mathbb{Z}[i] \\ |z|^2 = p}} z^4$$

It vanishes if $p \equiv_4 3$, and is uniformly distributed for $p \equiv_4 1$.

I see no mention of questions like this in the classical literature, but since around 1965 it has become a major issue in the subject.

2. Hecke operators

Ramanujan's first two conjectures were proved by L. J. Mordell (British, but American-born) a year later.

If we set $q = e^{2\pi iz}$, the series for Δ becomes a modular form on the complex upper half-plane $\mathcal{H}$, with respect to the group $\mathrm{SL}_2(\mathbb{Z})$. (I'll explain what this means in a moment. This had been known for a very long time.) Mordell defined operators $T(n)$ on such forms. It is easy to see that they satisfy the multiplicative property, and Ramanujan's conjecture reduces to the equation $T(n)\Delta = \tau(n) \cdot \Delta$. Mordell proved this by showing that the ratio $T(n)\Delta/\Delta$, which is a holomorphic function on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$, is bounded, hence constant.

Nowadays, once the operators have been defined, the final step is trivial, because we know the space of cusp forms (with constant term 0) to have dimension one. But this reasoning is anachronistic, and Mordell's proof was rather special to the problem at hand. One notable feature is that he did not use any relationship with sums of squares, but just well known properties of Δ.

It is important that you understand how the operator $T(n)$ is defined. I have to tell you first what a modular form is.

A lattice in $\mathbb{C}$ is a copy of $\mathbb{Z}^2$ in it. It may be specified by a pair $\omega = (\omega_1, \omega_2)$ with $z = \omega_1/\omega_2$ of positive imaginary part. Two such pairs give rise to the same lattice when one is the transform of the other by a matrix in $\mathrm{SL}_2(\mathbb{Z})$.

A modular form of weight $k \geq 1$ with respect to any group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function F of ω satisfying some mild growth condition and the two conditions

$$F(\lambda\omega) = \lambda^{-k} F(\omega) \quad (\lambda \in \mathbb{C}^\times)$$
$$F(\gamma\omega) = F(\omega) \quad (\gamma \in \Gamma)$$

Effectively, it is a function on the set of Γ-equivalence classes of lattices.

Restriction to points $(z, 1)$ identifies F with a function f on the upper half-plane transforming in a certain way under integral fractional linear transformations. If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ then invariance under matrices $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ implies that $f(z+n) = f(z)$ for all n in $\mathbb{Z}$. This implies that

$$f(z) = \sum\nolimits_{\mathbb{Z}} c_n e^{2\pi i n z} \,.$$

The growth condition means that $c_n = 0$ for $n < 0$.

The function ϑ^{2k} is a modular form of weight k for some Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$.

I'll now follow Mordell, with a slight modification, in defining certain operators on modular forms, which are now called Hecke operators. If $M \subseteq L$ are two lattices, the principal divisor theorem tells us that $M = n_1\omega_1 + n_2\omega_2$ for some basis ω_1, ω_2 of L and $n_2 | n_1$ in $\mathbb{N}$. In these circumstances, I'll write $[L : M] = (n_1, n_2)$.

If F is any function on the set of lattices invariant under $\mathrm{SL}_2(\mathbb{Z})$, define

$$[T(n_1, n_2)F](L) = \sum_{[L:M]=(n_1,n_2)} F(M)\,.$$

For example, if $(n_1, n_2) = (p, 1)$ and $L = \mathbb{Z}^2$ then M will have one of the bases

$$(1, p), \quad (p, x) \text{ (with } 0 \le x < p)\,.$$

Then define the simple and normalized Hecke operators

$$T(n) = \sum_{n_1 n_2 = n} T(n_1, n_2)$$
$$T_k(n) = n^{k-1}\, T(n)$$

The Chinese remainder theorem implies that $T(mn) = T(m)T(n)$ if m, n are relatively prime.

The factor n^{k-1} is chosen so that if F is a cusp form of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ such that $T_k(n)F = t_n F$ then

$$F(z,1) = \sum_1^\infty t_n e^{2\pi i n z} \,.$$

if F is scaled so $c_1 = 1$.

What about the p-series? You can see on geometric grounds that

$$T(p,1)T(p,1) = T(p^2,1) + (p+1)T(p,p)$$
$$T(p,1)T(p^n,1) = T(p^{n+1},1) + p\,T(p^n,p) \quad (n \geq 2)\,.$$

which implies

$$T_k(p)T_k(p^n) = T_k(p^{n+1}) + pT_k(p^{n-1}) \quad (n \geq 1)\,.$$

This leads to Ramanujan's difference equation, and hence to his Euler product.

Mordell's proof leaves unanswered a lot of interesting questions. Here is a simple one:

Why aren't the $T(n)$ called Mordell operators?

At the end of his paper Mordell remarks that Ramanujan made similar conjectures for functions related to $r_{10}(n)$ and $r_{16}(n)$. (He doesn't seem to have read either Glaisher or Ramanujan carefully.) He then went on

> **"These results can be proved by the aid of the principles used in** *[his proof of the conjecture about τ]*. **We should however have to consider new invariants of a sub-group of the modular group, and it seems hardly worth while to go into details."**

There are indeed some small technical problems involved in defining the right 'Hecke operators' for congruence groups, but hardly insuperable. (Although I think these problems weren't completely understood until the nineteen-sixties, and primarily through work of Langlands.) Mordell, however, just didn't understand that he was standing on top of a gold mine. He seems to have never again looked at problems involving modular forms, and in some late reminiscences he does not refer to this paper.

Let $\mathcal{M}_k(1)$ be the space of modular forms of weight k for $\Gamma(1)$, $\mathcal{S}_k(1)$ the subspace of cusp forms, with vanishing constant term. Both have finite-dimension.

The difference in dimensions is at most 1, since $\Gamma(1)$ has exactly one cusp. The complement of $\mathcal{S}_k$ is spanned by a function called an Eisenstein series, which was in fact first defined by Eisenstein:

$$G_k(L) = \sum_{\omega \neq 0 \in L} \frac{1}{\omega^{2k}}$$

with series expansion

$$G_k(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_1 \sigma_{2k-1}(n) e^{2\pi niz}$$

The series expansion is attributed to Eisenstein by Weil, but I have not been able to locate it in his Collected Works. It is included in Hurwitz' Habilitationschrift (1881), referred to by Mordell.

The point here is that analogues of these series are responsible for the 'divisor sums' occurring in the asymptotic approximations for all $r_{2k}(n)$. The complexities in Glaisher's table are largely due to the complexities of Bernoulli numbers.

3. More about GL(2)

Erich Hecke took up the subject of quadratic forms and modular forms around 1926, and changed completely the map of the country. Among his contributions was the extension to arbitrary congruence groups $\Gamma(N)$ of the theory we have seen for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. This was a major and thoroughgoing accomplishment.

- **He defined Eisenstein series, therefore a basis of the complement of $\mathcal{S}_k(N)$ in $\mathcal{M}_k(N)$.**
- **He defined operators $T(n)$ for $(n, N) = 1$ satisfying the same relations as those for $\Gamma(1)$. The operator $T(p, p)$ is now $R_p F(p\omega)$ where R_p is the action of**

$$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \quad (pq \equiv_N 1)$$

  on functions fixed by $\Gamma(N)$.
- **His student Hans Petersson defined a Euclidean norm on $\mathcal{S}_k(N)$, according to which the $T(n)$ are self-adjoint. Thus $\mathcal{S}_k$ is the direct sum of eigenspaces.**

Hecke, cont'd

- **He presumably knew a formula for the dimension of $\mathcal{S}_k$. although I have seen only specific cases among his papers. (It depends on the theorem of Riemann-Roch, and I see no reason why Hurwitz couldn't have discovered it.)**
- **He defined L-functions as Euler products**

$$\prod_{(p,N)=1} \frac{1}{1 - c_p p^{-s} + \varepsilon(p) p^{k-1} p^{-2s}} \quad (\varepsilon\colon (\mathbb{Z}/N)^\times \to \mathbb{C}^\times)\,,$$

  found a good form of functional equation, and showed they were entire.

  (Hardy asserts in his book that the functional equation for Ramanujan's "must have been familiar to him, but I cannot find it anywhere in his papers." Refers to an obscure 1928 paper by Wilkin.)
- **He gave fair estimates on the magnitudes $|c_p|$.**
- **He extended earlier results on r_{2k} to other positive definite integral quadratic forms.**

After the war, Hecke's student Hans Maass extended the theory to include eigenfunctions of the non-Euclidean Laplacian on $\Gamma\backslash\mathcal{H}$. This turned out to be a very valuable idea, and led to a fruitful generalization of the notion of automorphic (as opposed to modular) form.

Doing this, Maass introduced spectral analysis into number theory—for example, in analyzing the spectrum of the Laplacian on cusp forms. This was particularly interesting, since $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ is not compact. Most noticeable was his theory of Eisenstein series. Here his analyis relied on difficult properties of $\zeta(s)$ and Whittaker functions.

Apparently unsatisfied with this, he got his student Roelcke to try to prove analytically a Plancherel theorem for arithmetic quotients. Roelcke succeeded only partially, and it was Selberg who finished this off. Even today, this does not look quite trivial.

Selberg used his results to arrive at his Trace Formula, which enables us to calculate eigenvalues of Hecke operators on classical modular forms. At this point, Glaisher's table became a graduate course exercise.

The set of all lattice bases in $\mathbb{C}$ is a principal homogeneous space over $G = \mathrm{GL}_2(\mathbb{R})$. Let ω_0 be the basis $(i, 1)$. If f is a function on this space, the function

$$F(g) = f(g(\omega_0))$$

is a function on G. Modular forms of weight k become functions on $\Gamma\backslash G$ satisfying the equation

$$F(g\lambda) = \lambda^{-k} F(g)$$

for

$$\lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

(a copy of $\mathbb{C}^\times$ in G), and in addition satisfying some differential equation $\mathfrak{C}F = CF$ equivalent to holomorphicity, as well as some growth condition.

More generally, I'll define an automorphic form of level N to be a function of moderate growth on $\Gamma(N)\backslash G$ such that

$$F(g\lambda) = F(g)\chi(\lambda)$$

for some multiplicative character

$$\chi\colon \mathbb{C}^\times \to \mathbb{C}^\times ,$$

and which is an eigenfunction for the same differential operator $\mathfrak{C}$. The Hecke operators act also on this space, and I'll assume as well that F is an eigenfunction. This means that for each prime p not dividing N there exists c_p and ε such that

$$T(p)F = c_p F, \quad T(p,p)F = \varepsilon(p)\chi(p)F \,.$$

The subspace of cusp forms is that of F rapidly decreasing at infinity.

This includes both Hecke's and Maass' cases. For a given N, χ, and eigenvalue C of $\mathfrak{C}$, the space of automorphic forms has finite dimension.

We can now define

$$L(s,F) = \prod_{p \nmid N} \frac{1}{1 - c_p p^{-s} + \varepsilon(p)\chi(p)p^{-2s}} \, .$$

and wonder about its analytic properties. In fact, as long as F is a cusp form and after throwing in some extra factors for $p|N$, it becomes entire, and satisfies a relatively simple functional equation (Hecke, Maass).

4. The L-group

Suppose F to be a cusp form of type N, χ, $\{c_p, \varepsilon(p) \,|\, (p,N) = 1\}$, and C. Following Ramnujan, factor

$$x^2 - c_p x + \varepsilon(p)\chi(p) = (x - \alpha_p)(x - \beta_p)\,.$$

The pair (α_p, β_p) determines a conjugacy class

$$g_p = \begin{bmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{bmatrix}$$

in $\mathrm{GL}_2(\mathbb{C})$.

In this way, an automorphic form is characterized by the infinite family of conjugacy classes $\{g_p\}$ for $(p,N) = 1$. Now we can always change F harmlessly to some $\det^{-\sigma}(g)F(g)$ so as to arrange $|\chi| = 1$. With this normalization, it is tempting to speculate that the conjugacy class g_p is always unitary. This is the generalization of Ramanujan's conjecture about $|\tau(p)|$ to arbitrary automorphic forms on GL_2.

One of Langlands' ideas is that the set of conjugacy classes $\{g_p\}$ in $\mathrm{GL}_2(\mathbb{C})$ is a very strong characteristic of the automorphic form. Furthermore, a similar construction works for arbitrary reductive groups. One can, for example, define the notion of an automorphic form on $\mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R})$, define a polynomial algebra of Hecke operators for each p, and specify for all but a finite number of p a conjugacy class g_p in $\mathrm{GL}_n(\mathbb{C})$ associated to the form. Something like this seems to have been first done by Tamagawa, although not in this terminology.

If σ is the standard complex representation of $\mathrm{GL}_2(\mathbb{C})$, its symmetric powers $\sigma_m = S^m(\sigma)$ are also irreducible, and this gives us an embedding σ_m of $\mathrm{GL}_2(\mathbb{C})$ into $\mathrm{GL}_{m+1}(\mathbb{C})$. The set $\{g_p\}$ gives rise also to the set $\{\sigma_m(g_p)\}$. One example of Langlands' functoriality conjecture is that there should exist an automorphic form on GL_{m+1} corresponding to it. (This has apparently been verified for classical modular forms just within the past few weeks.)

One consequence in turn would be that

$$\prod_p \frac{1}{\det\left(I - \sigma(g_p)p^{-s}\right)}$$

is an entire function with functional equation, and a yet further consequence would be a verification of the conjectured statistical distribution of the g_p for most classical modular forms.

This argument is reminiscent of one by Serre, closely related to work by Tate and Mumford, concerning the statistical distribution of Frobenius automorphisms of ℓ-adic cohomology.

The exceptions are interesting.

Any quadratic field extension $K/\mathbb{Q}$ determines an integral quadratic form of dimension two. Theta functions determine a certain subspace of automorphic forms, defined by Hecke when K is imaginary, by Maass when it is real.

For example, the supplementary form χ_4 mentioned earlier comes from $\mathbb{Q}(\sqrt{-1})$. Explicitly

$$\chi_4(n) = \left(\frac{1}{4}\right) \sum_{z \in \mathbb{Z}[i], |z|^2 = n} z^4 .$$

By quadratic reciprocity, the associated L-function is

$$\prod_{\mathfrak{p}} \frac{1}{1 - \varpi^4 \cdot N\mathfrak{p}^{-s}} ,$$

in which the product is over prime ideals $\mathfrak{p} = (\varpi)$ of $\mathbb{Z}[i]$.

The conjugacy classes determined by this form lie in the group

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} .$$

This is Langlands' L-group for the algebraic torus determined by the algebraic torus $K^\times$.

There is a more famous example.

Any two-dimensional representation π of the rational Galois group gives the set of $\pi(\mathfrak{F}_p)$ in $\mathrm{GL}_2(\mathbb{C})$. Artin has conjectured that the associated L-function is entire and satisfies a good functional equation. Langlands has pointed out that this happens if and only if the set is that of some $\{g_p\}$. This was shown to be true in many cases, by Deligne and Serre, and then Joe Buhler. Langlands showed it was so for solvable Galois extensions.

This represents a kind of non-abelian reciprocity. Along similar lines, the L-functions associated to modular forms were shown by Eichler and Shimura to be the Hasse-Weil ζ-functions of modular varieties.

For reductive groups G other than GL_n, and in dealing with groups defined over arbitrary number fields, there are unavoidable technical difficulties. To every one of these Langlands has associated a complex group ${}^L G$, which in general will be an extension of a connected reductive group by something related to Galois groups. The connected group is that whose root system is the dual of the one defining G. For example, the dual of $\mathrm{Sp}(2n)$ is $\mathrm{SO}(2n+1)$.

Loosely speaking, every automorphic form on $\Gamma\backslash G(\mathbb{R})$ determines a family of conjugacy classes $\{g_p\}$, now in ${}^L G$. Proper L-functions are of the form

$$\prod_{p \notin S} \frac{1}{\det(I - \pi(g_p)p^{-s})} \,,$$

where π is a finite-dimensional representation of ${}^L G$. These are expected to have good properties. A homomorphism of L-groups should give rise to an embedding of automorphic forms.

The form χ_4 is an example. In this case, the algebraic group H is the multiplicative group of $\mathbb{Q}(\sqrt{-1})$, considered as a group defined over $\mathbb{Q}$. The group ${}^L H$ is the extension of $\mathbb{C}^\times$ by the Galois group. This is consistent with what I said about the distribution of the $\chi_4(p)$.

In general an automorphic form determines not only a set $\{g_p\}$, but also for every completion of $\mathbb{Q}$ arepresentation of $G(\mathbb{Q}_v)$. Langlands' proposal was that these also are characterized in terms of ${}^LG$, and to classify such representations, relating them to the local Galois group. This allowed him, for example, to make Hecke's functional equation more explicit.

In general, certain subtle phenomena have made this proposal a bit complicated. This involves Langlands' notion of endoscopy, concerning which work of Ngô Bảo Châuwon him a Fields Medal.

5. Unfinished business

What about Ramanujan's conjecture on the size of $\tau(p)$? Arthur has proposed a set of unitary representations of local groups among which the ones occurring in automorphic forms have to occur. There are a number of unsolved problems involved in this, and the global version, at least for those forms that are not motivic (related to algebraic geometry), seems completely out of sight.

It is not at all apparent how to verify functoriality in general, or the expected properties of arbitrary automorphic L-functions. About the year 2000 Langlands introduced a number of suggestions that tried to configure the Selberg trace formula suitably. So far, this has produced some striking results, and it seems likely to many of us that the trace formula must be involved in any attack on the problems. But there are few precise results.

Finally, I return to the opening of this talk. We have seen one case of how theta functions match with Langlands' conjectures, and I might have mentioned the case of r_4 to account for another. But is there a general explanation in Langlands's terms about how theta functions and quadratic forms give rise to automorphic forms? What is known is largely due to Stephen Rallis, who applied classical results of Martin Eichler and Carl Ludwig Siegel as well as more recent work of André Weil. But this work seems to me somewhat incomplete.

6. Appendix: Mordell's subsequent fame

Mordell became famous in the nineteen twenties when he proved his half of the Mordell-Weil theorem.

And then, unfortunately, again in the mid nineteen sixties. Serge Lang wrote a book Diophantine geometry about diophantine approximations, largely characterized by its use of algebraic geometry. Mordell wrote a review for the *Bulletin of the AMS*, which panned it. Many of his criticisms were quite legitimate, since in truth Lang (who wrote faster than normal people can read) had been rather sloppy. But he also railed a bit about modern tools in number theory, and thereby caused some controversy.

What really blew things up was that Carl Ludwig Siegel wrote a letter of support to Mordell which—in modern terminology—went viral. I bring this up because it was an hysterical amplification of Glaisher's lament:

> “The whole style of the author contradicts the sense for simplicity and honesty which we admire in the works of the masters in number theory."

> “I see a pig broken into a beautiful garden and rooting up all flowers and trees."

> “These people remind me of the impudent behaviour of the national socialists . . . "

7. References

19th century

- G. Eisenstein, ‘Genaue Untersuchung der unendlichen Doppelprodukte, aus welchen die elliptischen Funktionen als Quotienten zusammengesetzt sind, und der mit ihnen zusammenhängenden Doppelreihen’, *Crelle Journal* 35 (1847), 153–274.
- Adolf Hurwitz, ‘Grundlagen einer independenten Theorie der elliptischen Modulfunctionen und Theorie der Multiplicatorgleichungen erster Stufe’, *Mathematische Annalen* 18 (1881), 528–592.
- André Weil, Elliptic functions according to Eisenstein and Kronecker, Springer, 1976.

Early 20th century

- J. W. Glaisher, '**On the numbers of representations of a number as a sum of $2r$ squares, where $2r$ does not exceed 18.**', *Proceedings of the London Mathematical Society* 2 (5) (1907), 479-490.
- Srinivasa Ramanujan, '**On certain arithmetical functions**', *Transactions of the Cambridge Philosophical Society* XXII (1916).
- Louis Joel Mordell, '**On Mr. Ramanujan's empirical expansions of modular functions**'. *Proceedings of the Cambridge Philosophical Society* 19 (1917), 117–124.
- Geoffrey H. Hardy, Ramanujan, Cambridge, 1940

Mid 20th century

- **Carl Ludwig Siegel, Lectures on the analytic theory of quadratic forms,** *Mathematisches Institut der Universität Göttingen*, **1934/1995.**
- **Erich Hecke, 'Über die Modulfunktionen und die Dirichletschen Reihen mit Eulerschen Produktentwicklung II',** *Mathematische Annalen* **114 (1937), 316–351.**
- **Hans Maass, 'Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen',** *Mathematische Annalen* **121 (1949), 141–183.**
- **Louis J. Mordell, 'Reminiscences',**

`https://mathshistory.st-andrews.ac.uk/Extras/Mordell_reminiscences/`

- **Robert Rankin, 'On the representation of a number as the sum of any number of squares, and in particular of twenty',** *Acta Arithmetica* **VII (1962), 401–407.**

1960s

- Israel M. Gelfand, ‘Automorphic functions and theory of representations’, pp. 74–85 in the Proceedings of the ICM, Stockholm, 1962.
- ———–, M. L. Graev, and Ilya Piatetskii-Shapiro, ‘Representations of adele groups’, *Doklady Akademia Nauk SSSR* 156 (1964), 487–490.
- Roger Godement, ‘Les fonctions ζ des algèbres simple II’, *Seminaire Bourbaki*, exposé 176, 1959.
- Ichiro Satake, ‘Theory of spherical functions on reductive groups over $\mathfrak{p}$-adic fields’, *Publications Mathématiques de l’IHES* 18 (1963), 5–70.
- Jean-Pierre Serre, ‘Lettre à Armand Borel’, pp. 10–18 in Frobenius distributions: Lang-Trotter and Sato-Tate conjectures, in the series *Contemporary Mathematics* 663, American Mathematical Society, 2016.
- Tsuneo Tamagawa, ‘On Selberg’s trace formula’, *Journal of the Faculty of Science of the University of Tokyo* 8 (1963), 363–386.
- ———–, ‘On the ζ-function of a division algebra’, *Annals of Mathematics* 77 (1960), 387–405.
- John Tate, ‘Algebraic cycles and poles of zeta functions’, pp. 93-110 in Arithmetical algebraic geometry, edited by O. F. G. Schilling, Harper & Row, 1965

Langlands' work

- James G. Arthur, notes on Langlands' work, commissioned by the Abel Prize committee. To appear soon.
- Roger Godement, 'Formes automorphes et produits Euleriennes', *Seminaire Bourbaki*, exposé 349, 1968/69.
- Robert P. Langlands, 'Eisenstein series', pp. 235–252 in [Borel-Mostow:1966].
- ———-, 'Funktorialität in der Theorie der automorphen Formen: Ihre Entdeckung und ihre Ziele', available at

  https://publications.ias.edu/rpl/paper/451
- Stephen Rallis, 'Langlands functoriality and the Weil representation', *American Journal of Mathematics* 104 (1982), 469–515.

Very recent work

- **Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, 'A family of Calabi-Yau varieties and potential automorphy II',** *Publications of the Research Institute for Mathematical Sciences (Kyoto)* **47 (2011), 29–98.**

- **James Newton and Jack A. Thorne, 'Symmetric power functoriality for holomorphic modular forms, II',** `arXiv:2009.07180`**.**