The ADHM construction of Yang-Mills instantons

Simon Donaldson

Simons Center for Geometry and Physics
Stony Brook

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I. The problem

Electromagnetism: field F_{ij} on space-time, given by a potential A_i:

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}.$$

$F = \sum F_{ij} dx_i dx_j$ is a 2-form.

Functional

$$\mathcal{E} = \int |F|^2$$

The Euler-Lagrange equations $\delta\mathcal{E} = 0$ are the source-free Maxwell equations.

Yang and Mills (1954): let A_i take values in the Lie algebra of a Lie Group G (we will take $U(r)$, and sometimes $GL(r, \mathbf{C})$) and add a term

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j]. \tag{1}$$

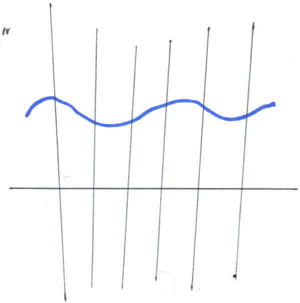
Define $\mathcal{E}$ as before. The Euler-Lagrange equations are the *Yang-Mills equations*.

Later it was realised that (1) is the formula for the curvature of a connection on a G-bundle $\tilde{E}$ over space time. In terms of a covariant derivative $\nabla_i = \frac{\partial}{\partial x_i} + A_i$ one has

$$F_{ij} = [\nabla_i, \nabla_j].$$

WAVE "FUNCTION"

→ section of $\tilde{E}$



$$\frac{\partial}{\partial x_i} \longmapsto \frac{\partial}{\partial x_i} + A_i$$

Work on Euclidean $\mathbf{R}^4$. The 2-forms split $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, the eigenspaces of $* : \Lambda^2 \to \Lambda^2$.

$$e.g. \quad *dx_1 dx_2 = dx_3 dx_4 \Rightarrow dx_1 dx_2 + dx_3 dx_4 \in \Lambda^2_+.$$

So $F = F_+ + F_-$. The functional

$$\kappa = \int |F_-|^2 - |F_+|^2,$$

is unchanged by compactly-supported variations. It follows that a connection with $F_+ = 0$ is a solution of the Yang-Mills equations. These connections are called *anti-self-dual (ASD) Yang-Mills instantons*.

In a “space and time” description (Euclidean version), F has electric and magnetic components $\underline{E}, \underline{B}$ and the ASD condition is

$$\underline{E} = -\underline{B}.$$

The equations are conformally invariant, so extend to the conformal compactification $S^4 = \mathbf{R}^4 \cup \{\infty\}$. According to Uhlenbeck (1979), solutions on S^4 are the same as finite-energy (i.e. $\mathcal{E} < \infty$) solutions on $\mathbf{R}^4$. The functional κ gives a characteristic class of the bundle over S^4. For $U(r)$:

$$\kappa = 8\pi^2 \langle c_2(\tilde{E}), S^4 \rangle.$$

The problem is to describe all ASD Yang-Mills instantons on S^4, for the compact classical groups

Solved by Atiyah, Hitchin, Drinfeld, Manin (1977).

From **Atiyah, Commentary on Vol. 5 of collected works (Oxford UP 1986)**

. . . with the help of Nigel Hitchin, I finally saw how Horrocks' method gave a very satisfactory and explicit solution to the problem. I remember our final discussion one morning when we had just seen how to fit together the last pieces of the puzzle. We broke off for lunch feeling very pleased with ourselves. On our return, I found a letter from Manin (whom I had earlier corresponded with on this subject) outlining essentially the same solution to the problem and saying "no doubt you have already realised this"! We replied at once and proposed that we should submit a short note from the four of us.

II. The ADHM construction

Standard co-ordinates x_i on $\mathbf{R}^4$.
Take a family of linear maps $\lambda_x : \mathbf{C}^{2k+r} \to \mathbf{C}^{2k}$ of the form

$$\lambda_x = \sum L_i x_i + M.$$

(So L_i, M are $2k \times (2k+r)$ matrices.)

Suppose that λ_x is surjective for all x and that the matrix data L_i, M satisfies certain conditions to be specified below.

Then $\tilde{E} = \ker \lambda$ is a rank r bundle, the induced connection is an ASD instanton with $c_2 = k$ and all $U(r)$ ASD instantons with $c_2 = k$ arise in this way.

“The induced connection”: this is the same construction as in classical differential geometry for the tangent bundle of a submanifold. We have:

$$\iota : \tilde{E} \to \underline{\mathbf{C}}^{2k+r} \quad \pi : \underline{\mathbf{C}}^{2k+r} \to \tilde{E}$$

and we set

$$\nabla s = \pi \circ \nabla_{\rm flat} \circ \iota,$$

where $\nabla_{\rm flat}$ is the usual derivative on the trivial bundle $\underline{\mathbf{C}}^{2k+r}$.

Significance of ADHM construction: one of the first applications of sophisticated modern geometry to a problem of interest in physics.

AND

. . . a beautiful piece of mathematics which can be approached from many directions.

- M.F. Atiyah, V. G. Drinfeld, N. J. Hitchin and Yu.I. Manin *Construction of instantons* Physics Letters **65 A** (1978) 185-7;
- M. F. Atiyah *Geometry of Yang-Mills fields* Lezioni Fermiane, Acadamia Nazionale dei Lincei & Scuola Normale Superiore, Pisa 1979, 96pp.

III. The Ward correspondence (Ward, 1977)

The approach of ADHM used the Penrose *twistor theory*. Recall that the holomorphic line bundles over the Riemann sphere $\mathbf{CP}^1$ are powers $\mathcal{O}(d)$ of the Hopf line bundle $H = \mathcal{O}(1)$.

Let Z be a 3-dimensional complex manifold and $L \subset Z$ a "line"—i.e. an embedded holomorphic sphere. Suppose that the normal bundle N of L in Z is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

From deformation theory, L moves in a family $\mathcal{M}$ of lines with tangent space at L isomorphic to $H^0(N)$, which is 4-dimensional; so $\mathcal{M}$ is a 4-dimensional complex manifold.

Notation $H^0(N)$ denotes the space of holomorphic sections.

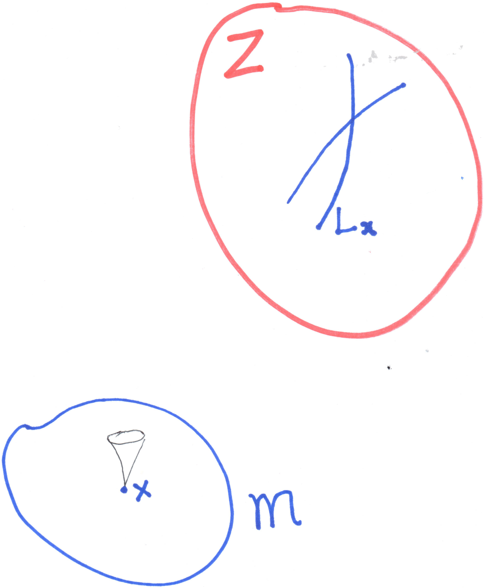
There is a *holomorphic conformal structure* on $\mathcal{M}$. If we write $N = S_- \otimes \mathcal{O}(1)$ and S_+ for the space of sections of $\mathcal{O}(1)$ over L then

$$T\mathcal{M} = S_+ \otimes S_-,$$

and $S_\pm$ are 2-dimensional.
We have skew symmetric maps $S_\pm \otimes S_\pm \to \Lambda^2 S_\pm$ whose tensor product gives a symmetric map

$$T\mathcal{M} \otimes T\mathcal{M} \to \Lambda^2 S_+ \otimes \Lambda^2 S_-,$$

which is our conformal structure.


Z
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x
M

Let E be a holomorphic vector bundle on Z which is trivial on all the lines in the family $\mathcal{M}$. Then we get a bundle $\tilde{E}$ over $\mathcal{M}$ with

$$\tilde{E}_L = H^0(L, E|_L).$$

There is a way to define a connection on $\tilde{E}$. (Any section of E over L has a unique extension to the first formal neighbourhood of L.) This is a holomorphic ASD connection over $\mathcal{M}$ and conversely any such connection comes from a holomorphic bundle $E \to Z$.

Note: triviality on a line is a open condition.

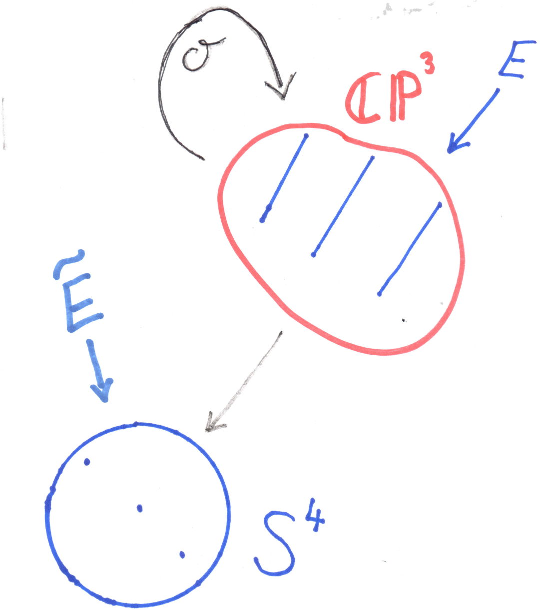
To get to real 4-manifolds and compact structure groups we introduce a “real” structure on Z; an anti-holomophic map $\sigma : Z \to Z$ with no fixed points. Then there is a real form $\mathcal{M}_{\mathbf{R}} \subset \mathcal{M}$ of σ-invariant lines (“real lines”) with an induced positive definite conformal structure. The Ward correspondence identifies ASD instantons on $\mathcal{M}_{\mathbf{R}}$ with “real” holomorphic bundles $E \to Z$ *i.e.* bundles such that

$$\sigma^*(E) \cong \overline{E}^*.$$

For the case at hand, take $Z = \mathbf{CP}^3$ regarded as $\mathbf{H}^2 \setminus \{0\}/\mathbf{C}^*$ where $\mathbf{H}$ is the quaternions. Then $\sigma : \mathbf{CP}^3 \to \mathbf{CP}^3$ is induced by multiplication by J. The real lines are the fibres of the fibration

$$\mathbf{CP}^3 \to \mathbf{HP}^1 = S^4.$$

So $\mathcal{M}_{\mathbf{R}} = S^4$ and the ASD instantons we seek correspond to certain holomorphic vector bundles over $\mathbf{CP}^3$.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IV. Construction of bundles over $\mathbf{CP}^3$.

Notation: For a bundle E over $\mathbf{CP}^n$ write $E \otimes \mathcal{O}(p) = E(p)$.

Let U, V, W be complex vector spaces and suppose that we have bundle maps

$$\underline{U}(-1) \xrightarrow{a} \underline{V} \xrightarrow{b} \underline{W}(1)$$

with a injective, b surjective and $b \circ a = 0$: a "*monad*" (Horrocks 1964).

Then we get a bundle $E \to \mathbf{CP}^3$ as the cohomology $\mathrm{ker} b / \mathrm{Im} a$.

Explicitly, with homogeneous co-ordinates Z_i, we write $a = \sum A_i Z_i, b = \sum B_i Z_i$ and a monad is given by a solution of the matrix equations

$$B_j A_i + B_i A_j = 0 \quad (i, j = 1 \dots 4).$$

Theorem (Barth 1977, Barth and Hulek 1978) *If* $E \to \mathbf{CP}^3$ *is a rank* r *bundle with* $c_2 = k$ *and*

$$H^0(E) = 0,\ H^0(E^*) = 0,\ H^1(E(-2)) = 0,\ H^1(E^*(-2)) = 0$$

then E *arises from a monad with* $\mathrm{dim} U = \mathrm{dim} W = k$ *and* $\mathrm{dim} V = 2k + r$.

The monad construction for a bundle E over $\mathbf{CP}^3$ goes over to the "induced connection" construction over S^4. If p_1, p_2 are distinct points on a line L over which E is trivial one finds that $E|_L$ can be described either as a subspace or as a quotient of the vector space V:

$$E|_L = \mathrm{ker} b_1 \cap \mathrm{ker} b_2 \quad E|_L = V/(\mathrm{Im} a_1 + \mathrm{Im} a_2),$$

where $a_i = a(p_i), b_i = b(p_i)$.

The remaining problem is to show that the a bundle E corresponding to an instanton satisfies the vanishing criteria in Barth's Theorem. The essential case being $H^1(E(-2)) = 0$.

The argument for this goes back to the origins of Penrose's twistor theory.

Let $U \subset \mathbf{R}^4$ be an open set and $\mathcal{U} \subset \mathbf{CP}^3$ be the union of the corresponding lines. Let χ be a sheaf cohomology class in $H^1(\mathcal{U}; \mathcal{O}(-2))$. For a line L we have $H^1(L; \mathcal{O}(-2)) = \mathbf{C}$ so the restriction of χ to lines gives a function F_χ on U and Penrose showed that this satisfies the Laplace equation $\Delta F_\chi = 0$.

(If we take a Čech representation of χ by a holomorphic function g of three complex variables on an overlap, the function F_χ is given by a contour integral formula of a classical nature:

$$F_\chi(x) = \int_{\gamma_x} g,$$

where γ_x is a suitable contour in the line corresponding to x.)

Similarly, for a bundle E on $\mathbf{CP}^3$ corresponding to an instanton connection on $\tilde{E} \to S^4$ the cohomology $H^1(E \otimes \mathcal{O}(-2))$ is isomorphic to the space of sections s of $\tilde{E}$ satisfying the equation

$$\left(\nabla^*\nabla + \frac{R}{6}\right) s = 0$$

where R is the scalar curvature of S^4. (This is the "conformally invariant" coupled Laplace operator.)
Since $R > 0$ the only solution is $s = 0$, hence $H^1(E(-2)) = 0$, as desired.

V. The Beilinson spectral sequence

The quickest proof of Barth's Theorem comes from the application of a result of Beilinson (1978, post ADHM).

Let x be a point in projective space $\mathbf{CP}^n$. There is a section v of $T\mathbf{P}^n(-1)$ vanishing at x (unique up to a factor).

(If x is the origin in affine co-ordinates $\mathbf{C}^n$ this is just the radial vector field $v = \sum z_i \frac{\partial}{\partial z_i}$.)

If e is a non-zero vector in a vector space K there is an exact sequence

$$\ldots \Lambda^r K^* \to \Lambda^{r-1} K^* \cdots \to K^* \to \mathbf{C}.$$

The section v gives a *Koszul complex*, an exact sequence of sheaves:

$$\ldots \Omega^r(r) \to \Omega^{r-1}(r-1) \cdots \to \Omega^1(1) \to \mathcal{O} \to \mathcal{O}_x,$$

where $\mathcal{O}_x$ is the skyscraper sheaf at x.

(**Notation**: $\Omega^r = \Lambda^r T^* \mathbf{P}^n$.)

Let E be a holomorphic vector bundle over $\mathbf{CP}^n$ and take the tensor product with E to get an exact sequence of sheaves, say:

$$\dots \mathcal{F}_r \to \mathcal{F}_{r-1} \to \cdots \to \mathcal{F}_0 \to E_x.$$

In this general situation there is a "hypercohomology spectral sequence" with E_1 page

$$E_1^{p,q} = H^q(\mathcal{F}_{-p})$$

abutting to the vector space E_x (the fibre of E at x) in degree 0.

In our case

$$E_1^{p,q} = H^q(E \otimes \Omega^{-p}(p)),$$

and we have $n = 3$. Replace E by $E(-2)$. The E_1 page has rows

$$H^q(E(-3)) \to H^q(E \otimes \Omega^2) \to H^q(E \otimes \Omega^1(-1)) \to H^q(E(-2)).$$

Using the cohomology vanishing assumptions in Barth's Theorem one finds that the only terms are

$$H^2(E(-3)) \xrightarrow{a} H^2(E \otimes \Omega^2) \xrightarrow{b} H^2(E \otimes \Omega^1(-1)).$$

In other words $E(-2)_x = \mathrm{ker} b / \mathrm{Im} a$.

Now let x vary on $\mathbf{CP}^3$; keeping track of the scalar ambiguity in the section v we get a description of E as the cohomology of a monad:

$$\underline{U}(-1) \to \underline{V} \to \underline{W}(1),$$

with $U = H^2(E(-3)), V = H^2(E \otimes \Omega^2), W = H^2(E \otimes \Omega^1(-1))$.

Another way of saying this goes through a resolution of the diagonal in $\mathbf{CP}^3 \times \mathbf{CP}^3$.

VI. Some further developments

From a completely different direction, Mukai (1981) studied holomorphic bundles over a complex 2-torus T. The dual torus $\hat{T}$ parametrises flat line bundles L_ξ over T. Given $E \to T$ then in favourable circumstances we get a bundle $\hat{E} \to \hat{T}$ with fibres:

$$\hat{E}_\xi = H^1(T; E \otimes L_\xi).$$

Fourier-Mukai transform: Interchanging the roles of $T, \hat{T}$ the construction gives back E (up to a sign $x \mapsto -x$).

Riemannian version

If T has a compatible flat metric we can describe $\hat{E}$ using the Dirac operator:

$$\hat{E}_\xi = \ker\left(D : \Gamma(S_- \otimes E \otimes L_\xi) \to \Gamma(S_+ \otimes E \otimes L_\xi)\right).$$

This description is independent of the complex structure. Starting with an instanton connection on E we get an induced instanton connection on $\hat{E}$.

The instanton equation on $\mathbf{R}^4$ can be written in terms of the components ∇_i of the covariant derivative as

$$[\nabla_0, \nabla_i] + [\nabla_j, \nabla_k] = 0,$$

where (ijk) runs over cyclic permutations of (123).
Set

$$D_1 = \nabla_0 + \sqrt{-1}\nabla_1, D_2 = \nabla_2 + \sqrt{-1}\nabla_3.$$

The equations can be written as

$$[D_1, D_2] = 0 \quad , \quad [D_1, D_1^*] + [D_2, D_2^*] = 0.$$

Remark The connection to holomorphic structures can be seen through the first, commutator, equation. There are also connections with integrable systems such as KdV (Mason and Sparling, 1989).

The matrix equations arising from the ADHM construction can be put in a similar shape.

Data:

$$\alpha_1, \alpha_2 : \mathbf{C}^k \to \mathbf{C}^k;\ P : \mathbf{C}^r \to \mathbf{C}^k;\ Q : \mathbf{C}^k \to \mathbf{C}^r,$$

Equations:

$$[\alpha_1, \alpha_2] = PQ \qquad [\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] = PP^* - Q^*Q.$$

Then $\lambda_x = \sum L_i x_i + M$ with

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & P \\ -\alpha_2^* & \alpha_1^* & Q^* \end{pmatrix}$$

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}$$

Working in $\mathbf{R}^4$, the vector space U appears as the space of solutions of the coupled Dirac equation. One can write down a formula for the ADHM data and confirm the ADHM result by direct calculation (Corrigan and Goddard, 1984).

The *Nahm transform* (1982) takes **R**-invariant solutions of the instanton equations with suitable asymptotic conditions ("*monopoles*") to $\mathbf{R}^3$-invariant solutions (with suitable boundary conditions). The latter are solutions of *Nahm's ODE*

$$\frac{dT_i}{ds} = [T_j, T_k],$$

(ijk) cyclic permutations of (123).

One review of this “ADHM-Fourier-Mukai-Nahm transform” in various contexts:

M. Jardim: A survey on Nahm transform J. Geom. Phys. 2004

We have *moduli spaces* $M_{k,r}$ of $U(r)$-instantons on S^4 with $c_2 = k$.

Examples of application of the ADHM theory;

- Boyer, Hurtubise, Milgram, Mann (1993), Kirwan (1994): calculation of homology groups $H_i(M_{k,r})$ in a “stable range” $k >> i$. (Confirmation of the *Atiyah-Jones conjecture*.)
- Nekrasov (2003)—evaluation of the *Seiberg-Witten prepotential* by integration over $M_{k,r}$ and localisation.

Some current and future directions:

- Work of Taubes (2013) and others on instanton equations(and variants) for non-compact groups such as $SL(2,\mathbf{C})$.
- Holomorphic bundles on $\mathbf{CP}^n$ for $n = 2, 3, 4$ are important in the study of higher dimensional versions of the instanton equation, related to exceptional holonomy.