Hodge structures and the topology of algebraic varieties

Claire Voisin

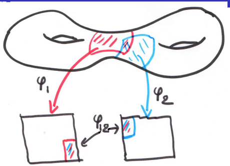
CNRS, Institut de mathématiques de Jussieu

CMSA
Harvard, 30 September 2020

- Analysis and differential geometry $\rightsquigarrow$ Hodge and Lefschetz decompositions.
- $+$ Algebra $\rightsquigarrow$ Consequences on topology.
- The importance of polarizations (signs and Hodge-Riemann relations).
- **Missing**. Variations of Hodge structures.

Kähler and projective complex manifolds

Complex manifold= manifold equipped with an atlas $U_i \cong V_i \subset \mathbb{C}^n$, with holomorphic change of coordinates maps.



- The tangent space at each point is endowed with a structure de $\mathbb{C}$-vector space, hence an operator I, $I^2 = -Id$, of **almost complex structure** acting on $T_{X,\mathbb{R}}$. Newlander-Nirenberg integrability condition.

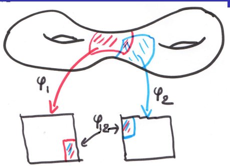
- Notion of **Hermitian metric** on X. In local holomorphic coordinates, $h = \sum_{ij} h_{ij} dz_i \otimes d\overline{z}_j$, with imaginary part $\omega = \frac{1}{\imath} \sum_{ij} \omega_{ij} dz_i \wedge d\overline{z}_j$, $\omega_{ij} = \operatorname{Im} h_{ij}$. This is a 2-form "of type $(1,1)$".

Definition. *The Hermitian metric is Kähler if $d\omega = 0$. $[\omega]$= Kähler class.*

- On $\mathbb{CP}^N$: Fubini-Study Kähler metric. *The Kähler class equals $c_1(\mathcal{H}^*)$, where $\mathcal{H}$ is the Hopf line bundle, hence is integral.* Idem for $X \subset \mathbb{CP}^N$ complex submanifold.

- **Kodaira embedding theorem.** *A compact Kähler manifold is projective iff it admits a Kähler form with integral cohomology class.*

Kähler and projective complex manifolds



Complex manifold= manifold equipped with an atlas $U_i \cong V_i \subset \mathbb{C}^n$, with holomorphic change of coordinates maps.

- The tangent space at each point is endowed with a structure de $\mathbb{C}$-vector space, hence an operator I, $I^2 = -Id$, of **almost complex structure** acting on $T_{X,\mathbb{R}}$. Newlander-Nirenberg integrability condition.

- Notion of **Hermitian metric** on X. In local holomorphic coordinates, $h = \sum_{ij} h_{ij} dz_i \otimes d\overline{z}_j$, with imaginary part $\omega = \frac{1}{\imath} \sum_{ij} \omega_{ij} dz_i \wedge d\overline{z}_j$, $\omega_{ij} = \operatorname{Im} h_{ij}$. This is a 2-form "of type $(1,1)$".

Definition. *The Hermitian metric is Kähler if $d\omega = 0$. $[\omega]$= Kähler class.*

- On $\mathbb{CP}^N$: Fubini-Study Kähler metric. *The Kähler class equals $c_1(\mathcal{H}^*)$, where $\mathcal{H}$ is the Hopf line bundle, hence is integral*. Idem for $X \subset \mathbb{CP}^N$ complex submanifold.

- **Kodaira embedding theorem.** *A compact Kähler manifold is projective iff it admits a Kähler form with integral cohomology class.*

• X complex manifold, $z_1, \dots z_n =$ local holomorphic coordinates. Holomorphic vector bundle Ω_X generated over $\mathcal{O}_X$ by dz_i. Transition matrices given by holomorphic Jacobian matrices.

• Holomorphic de Rham complex $\Omega_X^k := \bigwedge^k \Omega_X$, with exterior differential d.

Thm. (Holomorphic Poincaré lemma). *The complex*

$$\mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^n \to 0$$

is exact in degrees > 0. This is a resolution of the constant sheaf $\mathbb{C}$.

Corollary. $H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet)$.

• Filtration "bête" $F^p\Omega_X^\bullet := \Omega_X^{\bullet \geq p} \rightsquigarrow$ Frölicher spectral sequence. $E_1^{p,q} \Rightarrow H^{p+q}(X, \mathbb{C})$.

• $E_1^{p,q} = H^q(X, \Omega_X^p)$, $d_1 = d$.

• On the abutment : **"Hodge" filtration** $F^pH^k(X, \mathbb{C}) := \mathrm{Im}\,(\mathbb{H}^k(X, \Omega_X^{\bullet \geq p}) \to \mathbb{H}^k(X, \Omega_X^\bullet))$, $E_\infty^{p,q} = Gr_F^p H^k(X, \mathbb{C})$.

• X complex manifold, $z_1, \dots z_n =$ local holomorphic coordinates. Holomorphic vector bundle Ω_X generated over $\mathcal{O}_X$ by dz_i. Transition matrices given by holomorphic Jacobian matrices.
• Holomorphic de Rham complex $\Omega_X^k := \bigwedge^k \Omega_X$, with exterior differential d.
Thm. (Holomorphic Poincaré lemma). *The complex*

$$\mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^n \to 0$$

is exact in degrees > 0. This is a resolution of the constant sheaf $\mathbb{C}$.

Corollary. $H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet)$.

• Filtration "bête" $F^p\Omega_X^\bullet := \Omega_X^{\bullet \geq p} \rightsquigarrow$ Frölicher spectral sequence. $E_1^{p,q} \Rightarrow H^{p+q}(X, \mathbb{C})$.

• $E_1^{p,q} = H^q(X, \Omega_X^p)$, $d_1 = d$.

• On the abutment : **"Hodge" filtration** $F^pH^k(X, \mathbb{C}) := \mathrm{Im}\,(\mathbb{H}^k(X, \Omega_X^{\bullet \geq p}) \to \mathbb{H}^k(X, \Omega_X^\bullet))$, $E_\infty^{p,q} = Gr_F^p H^k(X, \mathbb{C})$.

- $j : U \hookrightarrow X$, $U = X \setminus Y$, with $Y \subset X$ closed analytic.

- (Hironaka) By successive blow-ups of X along smooth centers supported over Y, one can assume that Y is a normal crossing divisor: i.e. Y is locally defined by a single holomorphic equation of the form $f = z_1 \ldots z_k$ in adequate holomorphic coordinates.

- Define $\Omega_X(\log Y)$ as the holomorphic vector bundle generated over $\mathcal{O}_X$ by $\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}$, $dz_{k+1}, \ldots, dz_n$.

- $\Omega_X^k(\log Y) = \bigwedge^k \Omega_X(\log Y)$, $d : \Omega_X^k(\log Y) \to \Omega_X^{k+1}(\log Y)$.

- Their sections (= forms with logarithmic growth) are the forms with pole order 1 along Y, whose differential also has pole order 1 along Y.

Thm. *The inclusion of the subcomplex $\Omega_X^\bullet(\log Y) \subset j_* \Omega_U^\bullet$ is a quasiisomorphism.*

Corollary. $H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log Y))$ *and Frölicher spectral sequence.*

- also $H^k(U, \mathbb{C}) = \mathbb{H}^k(U, \Omega_U^\bullet)$ hence two Hodge filtrations on $H^k(U, \mathbb{C})$.

- $j : U \hookrightarrow X$, $U = X \setminus Y$, with $Y \subset X$ closed analytic.

- (Hironaka) By successive blow-ups of X along smooth centers supported over Y, one can assume that Y is a normal crossing divisor: i.e. Y is locally defined by a single holomorphic equation of the form $f = z_1 \dots z_k$ in adequate holomorphic coordinates.

- Define $\Omega_X(\log Y)$ as the holomorphic vector bundle generated over $\mathcal{O}_X$ by $\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}$, $dz_{k+1}, \dots, dz_n$.

- $\Omega_X^k(\log Y) = \bigwedge^k \Omega_X(\log Y)$, $d : \Omega_X^k(\log Y) \to \Omega_X^{k+1}(\log Y)$.

- Their sections (= forms with logarithmic growth) are the forms with pole order 1 along Y, whose differential also has pole order 1 along Y.

Thm. *The inclusion of the subcomplex $\Omega_X^\bullet(\log Y) \subset j_*\Omega_U^\bullet$ is a quasiisomorphism.*

Corollary. $H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log Y))$ *and Frölicher spectral sequence.*

- also $H^k(U, \mathbb{C}) = \mathbb{H}^k(U, \Omega_U^\bullet)$ hence two Hodge filtrations on $H^k(U, \mathbb{C})$.

The Hodge decomposition theorem

- $X=$ compact oriented Riemannian manifold. $\rightsquigarrow L^2$-metric on forms. $(\alpha,\beta)_{L^2}=\int_X \alpha\wedge *\beta$.
- Formal adjoint $d^* = \pm * d *$. Laplacian $\Delta_d = d\circ d^* + d^*\circ d$.

Harmonic forms. $\Delta_d\alpha = 0$. X compact and α harmonic $\Rightarrow$ α is closed.

Thm. (Hodge) *Each de Rham cohomology class contains a unique harmonic representative.*

- Forms of type (p,q) on X =cplx mfld: $\alpha=\sum_{|I|=p,|J|=q}\alpha_{IJ}dz_I\wedge d\bar{z}_J$. Any k-form writes uniquely as a sum $\sum_{p+q=k}\alpha^{p,q}$.

Thm. (Hodge) *X Kähler $\Rightarrow \Delta_d\alpha^{p,q}$ is of type (p,q).*

Corollary. *α harmonic, $\alpha=\sum_{p,q}\alpha^{p,q}\Rightarrow$ each $\alpha^{p,q}$ is harmonic.*

Thm. (Hodge) *Let $H^{p,q}(X) := \{$classes of closed forms of type $(p,q)\}$. Then $H^{p,q}(X)\cong H^q(X,\Omega^p_X)$ and $H^k(X,\mathbb{C}) = \oplus_{p+q=k}H^{p,q}(X)$.*

- **Hodge symmetry.** $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Cor. *The Frölicher spectral sequence of X degenerates at E_1 ($E_1 = E_\infty$).*

- Consequences in deformation theory. For example BTT theorem.

- $X=$ compact oriented Riemannian manifold. $\rightsquigarrow$ L^2-metric on forms. $(\alpha,\beta)_{L^2}=\int_X \alpha\wedge *\beta$.
- Formal adjoint $d^*=\pm * d *$. Laplacian $\Delta_d = d\circ d^* + d^*\circ d$.

Harmonic forms. $\Delta_d\alpha=0$. X compact and α harmonic $\Rightarrow$ α is closed.

Thm. (Hodge) *Each de Rham cohomology class contains a unique harmonic representative.*

- Forms of type (p,q) on X =cplx mfld: $\alpha=\sum_{|I|=p,|J|=q}\alpha_{IJ}dz_I\wedge d\overline{z}_J$. Any k-form writes uniquely as a sum $\sum_{p+q=k}\alpha^{p,q}$.

Thm. (Hodge) *X Kähler $\Rightarrow$ $\Delta_d\alpha^{p,q}$ is of type (p,q).*

Corollary. *α harmonic, $\alpha=\sum_{p,q}\alpha^{p,q}\Rightarrow$ each $\alpha^{p,q}$ is harmonic.*

Thm. (Hodge) *Let $H^{p,q}(X):=\{$classes of closed forms of type $(p,q)\}$. Then $H^{p,q}(X)\cong H^q(X,\Omega^p_X)$ and $H^k(X,\mathbb{C})=\oplus_{p+q=k}H^{p,q}(X)$.*

- **Hodge symmetry.** $\overline{H^{p,q}(X)}=H^{q,p}(X)$.

Cor. *The Frölicher spectral sequence of X degenerates at E_1 ($E_1=E_\infty$)*.

- Consequences in deformation theory. For example BTT theorem.

Definition. (Hodge structure) *A Hodge structure of weight k= lattice L + decomposition $L_{\mathbb{C}} = \oplus_{p+q=k} L^{p,q}$, with $\overline{L^{p,q}} = L^{q,p}$.*

• Hodge decomposition $\rightsquigarrow$ Hodge filtration: $F^p L_{\mathbb{C}} := \oplus_{r \geq p} L^{r,k-r}$.
Conversely $L^{p,q} = F^p L_{\mathbb{C}} \cap \overline{F^q L_{\mathbb{C}}}$, $p + q = k$.
Condition on $F^\bullet$: $L_{\mathbb{C}} = F^p L_{\mathbb{C}} \oplus \overline{F^{k-p+1} L_{\mathbb{C}}}$.

• **Variants.** (a) Rational coefficients.
(b) **Effective** Hodge structure : $L^{p,q} = 0$ if $p < 0$ or $q < 0$.

Definition. $(L, F^p L_{\mathbb{C}})$, $(L', F^p L'_{\mathbb{C}})$ *Hodge structures of weights k, $k + 2r$. A morphism of Hodge structures between them is $\phi : L \to L'$, s.t. $\phi_{\mathbb{C}}(L^{p,q}) \subset L'^{p+r,q+r}$.*

Example $T = \mathbb{C}^n/\Gamma \rightsquigarrow \Gamma_{\mathbb{C}} \twoheadrightarrow \mathbb{C}^n$ with kernel $\Gamma^{1,0} \subset \Gamma_{\mathbb{C}}$ is an equivalence of categories (Complex tori)$\leftrightsquigarrow$ (**effective** weight 1 Hodge structures).

• Complex tori up to isogeny $\leftrightsquigarrow$ Weight 1 rational Hodge structures.

Fact. *The category of* **rational** *Hodge structures is not semi-simple. There are morphisms of complex tori $T \twoheadrightarrow T'$ which do not split up to isogeny.*

Definition. (Hodge structure) *A Hodge structure of weight k= lattice L + decomposition $L_{\mathbb{C}} = \oplus_{p+q=k} L^{p,q}$, with $\overline{L^{p,q}} = L^{q,p}$.*

• Hodge decomposition $\rightsquigarrow$ Hodge filtration: $F^p L_{\mathbb{C}} := \oplus_{r \geq p} L^{r,k-r}$.
Conversely $L^{p,q} = F^p L_{\mathbb{C}} \cap \overline{F^q L_{\mathbb{C}}}$, $p + q = k$.
Condition on $F^\bullet$: $L_{\mathbb{C}} = F^p L_{\mathbb{C}} \oplus \overline{F^{k-p+1} L_{\mathbb{C}}}$.

• **Variants.** (a) Rational coefficients.
(b) **Effective** Hodge structure : $L^{p,q} = 0$ if $p < 0$ or $q < 0$.

Definition. *$(L, F^p L_{\mathbb{C}})$, $(L', F^p L'_{\mathbb{C}})$ Hodge structures of weights k, $k + 2r$. A morphism of Hodge structures between them is $\phi : L \to L'$, s.t. $\phi_{\mathbb{C}}(L^{p,q}) \subset L'^{p+r,q+r}$.*

Example $T = \mathbb{C}^n/\Gamma \rightsquigarrow \Gamma_{\mathbb{C}} \twoheadrightarrow \mathbb{C}^n$ with kernel $\Gamma^{1,0} \subset \Gamma_{\mathbb{C}}$ is an equivalence of categories (Complex tori)$\leftrightsquigarrow$ (**effective** weight 1 Hodge structures).

• Complex tori up to isogeny $\leftrightsquigarrow$ Weight 1 rational Hodge structures.

Fact. *The category of* **rational** *Hodge structures is not semi-simple. There are morphisms of complex tori $T \twoheadrightarrow T'$ which do not split up to isogeny.*

Thm. *X compact Kähler. The cohomology $H^k(X,\mathbb{Z})/\mathrm{Tors}$ carries an effective Hodge structure of weight k.*

• $\phi : X \to Y$ holomorphic map, with X, Y compact Kähler.
$\phi^* : H^k(Y,\mathbb{Z})_{\mathrm{tf}} \to H^k(X,\mathbb{Z})_{\mathrm{tf}}$ is a morphism of Hodge structures.

Prop. *The Gysin morphism $\phi_* : H^k(X,\mathbb{Z})_{\mathrm{tf}} \to H^{k-2d}(Y,\mathbb{Z})_{\mathrm{tf}}$, $d = \dim X - \dim Y$, is a morphism of Hodge structures.*

• **Explanation.** (a) Via Poincaré duality, ϕ_* is the transpose of ϕ^*.

(b) weight k Hodge structure on $L \leftrightsquigarrow$ weight $-k$ Hodge structure on L^*: $L^{*-p,-q}$ is defined as the orthogonal of $\oplus_{(r,s)\neq(p,q)} L^{r,s}$.

(c) the Hodge structure on $H^{2n-k}(X,\mathbb{Z})_{\mathrm{tf}}$ is dual to the Hodge structure on $H^k(X,\mathbb{Z})_{\mathrm{tf}}$ up to a shift of bidegree (for "type reasons": $\int_X \alpha^{p,q} \wedge \beta^{p',q'} = 0$ for $(p',q') \neq (n-p,n-q)$). **qed**

Construction. Hodge structure on L, resp. M of weights k, resp. $k' \rightsquigarrow$ Weight $k+k'$ Hodge structure on $L \otimes M$:

$$(L_{\mathbb{C}} \otimes M_{\mathbb{C}})^{p,q} = \oplus_{r+r'=p,s+s'=q} L^{r,s} \otimes L'^{r',s'}.$$

Thm. *X compact Kähler. The cohomology $H^k(X,\mathbb{Z})/\mathrm{Tors}$ carries an effective Hodge structure of weight k.*

• $\phi : X \to Y$ holomorphic map, with X, Y compact Kähler.
$\phi^* : H^k(Y,\mathbb{Z})_{\mathrm{tf}} \to H^k(X,\mathbb{Z})_{\mathrm{tf}}$ is a morphism of Hodge structures.

Prop. *The Gysin morphism $\phi_* : H^k(X,\mathbb{Z})_{\mathrm{tf}} \to H^{k-2d}(Y,\mathbb{Z})_{\mathrm{tf}}$, $d = \dim X - \dim Y$, is a morphism of Hodge structures.*

• **Explanation.** (a) Via Poincaré duality, ϕ_* is the transpose of ϕ^*.

(b) weight k Hodge structure on $L \leftrightsquigarrow$ weight $-k$ Hodge structure on L^*:
$L^{*-p,-q}$ is defined as the orthogonal of $\oplus_{(r,s)\neq(p,q)} L^{r,s}$.

(c) the Hodge structure on $H^{2n-k}(X,\mathbb{Z})_{\mathrm{tf}}$ is dual to the Hodge structure on $H^k(X,\mathbb{Z})_{\mathrm{tf}}$ up to a shift of bidegree (for "type reasons":
$\int_X \alpha^{p,q} \wedge \beta^{p',q'} = 0$ for $(p',q') \neq (n-p,n-q)$). **qed**

Construction. Hodge structure on L, resp. M of weights k, resp. $k' \rightsquigarrow$
Weight $k+k'$ Hodge structure on $L \otimes M$:

$$(L_{\mathbb{C}} \otimes M_{\mathbb{C}})^{p,q} = \oplus_{r+r'=p,s+s'=q} L^{r,s} \otimes L'^{r',s'}.$$

Thm. *X compact Kähler. The cohomology $H^k(X,\mathbb{Z})/\mathrm{Tors}$ carries an effective Hodge structure of weight k.*

• $\phi : X \to Y$ holomorphic map, with X, Y compact Kähler.
$\phi^* : H^k(Y,\mathbb{Z})_{\mathrm{tf}} \to H^k(X,\mathbb{Z})_{\mathrm{tf}}$ is a morphism of Hodge structures.

Prop. *The Gysin morphism $\phi_* : H^k(X,\mathbb{Z})_{\mathrm{tf}} \to H^{k-2d}(Y,\mathbb{Z})_{\mathrm{tf}}$, $d = \dim X - \dim Y$, is a morphism of Hodge structures.*

• **Explanation.** (a) Via Poincaré duality, ϕ_* is the transpose of ϕ^*.

(b) weight k Hodge structure on $L \leftrightsquigarrow$ weight $-k$ Hodge structure on L^*: $L^{*-p,-q}$ is defined as the orthogonal of $\oplus_{(r,s)\neq(p,q)} L^{r,s}$.

(c) the Hodge structure on $H^{2n-k}(X,\mathbb{Z})_{\mathrm{tf}}$ is dual to the Hodge structure on $H^k(X,\mathbb{Z})_{\mathrm{tf}}$ up to a shift of bidegree (for "type reasons": $\int_X \alpha^{p,q} \wedge \beta^{p',q'} = 0$ for $(p',q') \neq (n-p, n-q)$). **qed**

Construction. Hodge structure on L, resp. M of weights k, resp. $k' \rightsquigarrow$ Weight $k+k'$ Hodge structure on $L \otimes M$:

$$(L_{\mathbb{C}} \otimes M_{\mathbb{C}})^{p,q} = \oplus_{r+r'=p,s+s'=q} L^{r,s} \otimes L'^{r',s'}.$$

Definition. (Hodge classes) *A Hodge class on a weight $2k$ Hodge structure L is an element of $L \cap L^{k,k}$.*

Example. Hodge classes on $L^* \otimes M$, L of weight k, M of weight $k+2r$, are the morphisms of Hodge structures $L \to M$.

Corollary. *Hodge classes on a product $X \times Y$ of compact Kähler manifolds identify with the morphisms of Hodge structures $H^*(X, \mathbb{Z})_{\mathrm{tf}} \to H^{*+2r}(Y, \mathbb{Z})_{\mathrm{tf}}$.*

Example. $Z \subset X$ closed analytic subset of codimension k has a class $[Z] \in H^{2k}(X, \mathbb{Z})$. If X is compact Kähler, this is a Hodge class.

Conjecture. (Hodge conjecture) *X smooth complex projective.* **Rational** *Hodge classes on X are* **algebraic**, *i.e. generated by cycles classes.*

Example. Künneth components of the diagonal. $\delta_k \rightsquigarrow Id_{H^k(X,\mathbb{Z})}$.

• Known in degree 2 (Lefschetz $(1,1)$-thm) and $2n-2$ by hard Lefschetz.

• Wrong in the compact Kähler setting, even in a weaker form replacing cycle classes by Chern classes of coherent sheaves (Voisin).

Definition. (Hodge classes) *A Hodge class on a weight $2k$ Hodge structure L is an element of $L \cap L^{k,k}$.*

Example. Hodge classes on $L^* \otimes M$, L of weight k, M of weight $k+2r$, are the morphisms of Hodge structures $L \to M$.

Corollary. *Hodge classes on a product $X \times Y$ of compact Kähler manifolds identify with the morphisms of Hodge structures $H^*(X,\mathbb{Z})_{\mathrm{tf}} \to H^{*+2r}(Y,\mathbb{Z})_{\mathrm{tf}}$.*

Example. $Z \subset X$ closed analytic subset of codimension k has a class $[Z] \in H^{2k}(X,\mathbb{Z})$. If X is compact Kähler, this is a Hodge class.

Conjecture. (Hodge conjecture) *X smooth complex projective.* **Rational** *Hodge classes on X are* **algebraic***, i.e. generated by cycles classes.*

Example. Künneth components of the diagonal. $\delta_k \rightsquigarrow Id_{H^k(X,\mathbb{Z})}$.

• Known in degree 2 (Lefschetz $(1,1)$-thm) and $2n-2$ by hard Lefschetz.

• Wrong in the compact Kähler setting, even in a weaker form replacing cycle classes by Chern classes of coherent sheaves (Voisin).

Definition. *A rational mixed Hodge structure = a $\mathbb{Q}$-vector space L with an increasing (weight) filtration $W_i L$ and a decreasing (Hodge) filtration $F^p L_{\mathbb{C}}$, such that: the induced filtration on $\mathrm{Gr}_i^W L$ defines a Hodge structure of weight i.*

Thm. (Deligne) *The cohomology of quasiprojective complex varieties, or analytic-Zariski open in compact Kähler manifolds, or relative (co)homology of such pairs, carries functorial mixed Hodge structures.*

- **Smooth case**: $U = X \setminus Y \overset{j}{\hookrightarrow} X$, Y=normal crossing divisor. Use $H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log Y))$. Filtration F on $\Omega_X^\bullet(\log Y)$ is the usual one ("bête"). Filtration W on $\Omega_X^\bullet(\log Y)$: up to a shift, this is given by $W_i \Omega_X^\bullet(\log Y) = \Omega_X^i(\log Y) \wedge \Omega_X^{\bullet - i}$.
- A posteriori, the induced W- filtration is defined on rational cohomology and related to the Leray filtration of j.
- *The s.s. for F degenerates at E_1, the s.s. for W degenerates at E_2.*
- In this case, the smallest weight part of $H^k(U, \mathbb{Q})$ is $\mathrm{Im}\,(j^* : H^k(X, \mathbb{Q}) \to H^k(U, \mathbb{Q}))$ (weight k).

Definition. *A rational mixed Hodge structure = a $\mathbb{Q}$-vector space L with an increasing (weight) filtration $W_i L$ and a decreasing (Hodge) filtration $F^p L_{\mathbb{C}}$, such that: the induced filtration on $\mathrm{Gr}_i^W L$ defines a Hodge structure of weight i.*

Thm. (Deligne) *The cohomology of quasiprojective complex varieties, or analytic-Zariski open in compact Kähler manifolds, or relative (co)homology of such pairs, carries functorial mixed Hodge structures.*

- **Smooth case**: $U = X \setminus Y \overset{j}{\hookrightarrow} X$, Y=normal crossing divisor. Use $H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log Y))$. Filtration F on $\Omega_X^\bullet(\log Y)$ is the usual one ("bête"). Filtration W on $\Omega_X^\bullet(\log Y)$: up to a shift, this is given by $W_i \Omega_X^\bullet(\log Y) = \Omega_X^i(\log Y) \wedge \Omega_X^{\bullet - i}$.
- A posteriori, the induced W- filtration is defined on rational cohomology and related to the Leray filtration of j.
- *The s.s. for F degenerates at E_1, the s.s. for W degenerates at E_2.*
- In this case, the smallest weight part of $H^k(U, \mathbb{Q})$ is $\mathrm{Im}\,(j^* : H^k(X, \mathbb{Q}) \to H^k(U, \mathbb{Q}))$ (weight k).

- Morphisms of MHS: $\phi : L \to L'$, $\phi(W_iL) \subset W_iL'$, $\phi_{\mathbb{C}}(F^pL_{\mathbb{C}}) \subset F^pL'_{\mathbb{C}}$.

Thm. (Deligne) *Morphisms of mixed Hodge structures are strict for both filtrations* (i.e.: $F^pL'_{\mathbb{C}} \cap \operatorname{Im}\phi_{\mathbb{C}} = \phi_{\mathbb{C}}(F^pL_{\mathbb{C}})$, $W_iL' \cap \operatorname{Im}\phi = \phi(W_iL)$).

Sketch of proof. Follows from an algebra lemma: *There exists a functorial decomposition $L_{\mathbb{C}} = \oplus_{p,q}L^{p,q}$ for mixed Hodge structures (L, W, F), with $F^pL_{\mathbb{C}} = \oplus_{r\geq p,q}L^{r,q}$, $W_iL_{\mathbb{C}} = \oplus_{p+q\leq i}L^{p,q}$.*
Let $\alpha \in W_iL' \cap \operatorname{Im}\phi$. Write $\alpha = \phi(\beta)$, $\beta = \sum_{p,q}\beta^{p,q}$. Then $\phi(\beta^{p,q}) = 0$ for $p + q > i$ so $\alpha = \phi(\beta')$ with $\beta' = \sum_{p+q\leq i}\beta^{p,q} \in W_iL_{\mathbb{C}}$. **qed**

Definition. *A class $\alpha \in H^k(X, \mathbb{Q})$ is of coniveau $\geq c$ if $\alpha_{|X\setminus Y} = 0$ with Y closed analytic of codim $\geq c$.*
If X is smooth compact, $j : Y \hookrightarrow X$, equivalent condition: $\alpha = j_*\beta$ in $H_{2n-k}(X, \mathbb{Q})$ for some $\beta \in H_{2n-k}(Y, \mathbb{Q})$.

Strictness $\Rightarrow$ *If X is smooth projective, $j : Y \hookrightarrow X$ with desingularization $\tilde{j} : \widetilde{Y} \to X$, then $\operatorname{Im} j_* = \operatorname{Im}\tilde{j}_* \subset H_{2n-k}(X, \mathbb{Q})$.*

Corollary. (Deligne) *The set of cohomology classes of coniveau $\geq c$ is a Hodge substructure of $H^k(X, \mathbb{Q})$, of Hodge coniveau $\geq c$.*

- Morphisms of MHS: $\phi : L \to L'$, $\phi(W_iL) \subset W_iL'$, $\phi_{\mathbb{C}}(F^pL_{\mathbb{C}}) \subset F^pL'_{\mathbb{C}}$.

Thm. (Deligne) *Morphisms of mixed Hodge structures are strict for both filtrations* (i.e.: $F^pL'_{\mathbb{C}} \cap \operatorname{Im}\phi_{\mathbb{C}} = \phi_{\mathbb{C}}(F^pL_{\mathbb{C}})$, $W_iL' \cap \operatorname{Im}\phi = \phi(W_iL)$).

Sketch of proof. Follows from an algebra lemma: *There exists a functorial decomposition $L_{\mathbb{C}} = \oplus_{p,q}L^{p,q}$ for mixed Hodge structures (L, W, F), with $F^pL_{\mathbb{C}} = \oplus_{r\geq p,q}L^{r,q}$, $W_iL_{\mathbb{C}} = \oplus_{p+q\leq i}L^{p,q}$.*
Let $\alpha \in W_iL' \cap \operatorname{Im}\phi$. Write $\alpha = \phi(\beta)$, $\beta = \sum_{p,q}\beta^{p,q}$. Then $\phi(\beta^{p,q}) = 0$ for $p + q > i$ so $\alpha = \phi(\beta')$ with $\beta' = \sum_{p+q\leq i}\beta^{p,q} \in W_iL_{\mathbb{C}}$. **qed**

Definition. *A class $\alpha \in H^k(X, \mathbb{Q})$ is of coniveau $\geq c$ if $\alpha_{|X\setminus Y} = 0$ with Y closed analytic of codim $\geq c$.*
If X is smooth compact, $j : Y \hookrightarrow X$, equivalent condition: $\alpha = j_*\beta$ in $H_{2n-k}(X, \mathbb{Q})$ for some $\beta \in H_{2n-k}(Y, \mathbb{Q})$.

Strictness $\Rightarrow$ *If X is smooth projective, $j : Y \hookrightarrow X$ with desingularization $\tilde{j} : \widetilde{Y} \to X$, then $\operatorname{Im} j_* = \operatorname{Im}\tilde{j}_* \subset H_{2n-k}(X, \mathbb{Q})$.*

Corollary. (Deligne) *The set of cohomology classes of coniveau $\geq c$ is a Hodge substructure of $H^k(X, \mathbb{Q})$, of Hodge coniveau $\geq c$.*

Thm. (Hard Lefschetz, proved by Hodge) *Let X be compact Kähler of dimension n, ω a Kähler form on X. Then $\forall k \leq n$, $\cup[\omega]^{n-k} := L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$ is an isomorphism.*

• **Projective case** : One can take $[\omega]$ rational. Then the Lefschetz isomorphism is an isomorphism of Hodge structures.

Coro. (Lefschetz decomp.) $H^k(X, \mathbb{R}) = \oplus_{k-2r \geq 0} L^r H^{k-2r}(X, \mathbb{R})_{\text{prim}}$, *where* $H^{k-2r}(X, \mathbb{R})_{\text{prim}} := \operatorname{Ker} L^{n-k+2r+1} \subset H^{k-2r}(X, \mathbb{R})$.

• Lefschetz intersection pairing on H^k: $(\alpha, \beta)_{\text{Lef}} = \int_X L^{n-k} \alpha \cup \beta$. $h_{\text{Lef}}(\alpha, \beta) := i^k (\alpha, \overline{\beta})_{\text{Lef}}$.

• **easy:** *The Lefschetz decomposition is orthogonal for $(\, , \,)_{\text{Lef}}$, and the Hodge decomposition is orthogonal for h_{Lef}.* (HR1).

Thm. *2nd H-R bilinear relations: $(-1)^{p+r} h_{\text{Lef}|L^r H^{p-r,q-r}(X,\mathbb{R})_{\text{prim}}}$ is positive definite Hermitian (up to a global sign depending on k).* (HR2).

Corollary. *Let $[\omega]$ be rational. On $L^r H^{k-2r}(X, \mathbb{Q})_{\text{prim}}$, multiply $(\, , \,)_{\text{Lef}}$ by $(-1)^r$: one gets a* **polarized Hodge structure** *on $H^k(X, \mathbb{Q})$.*

Thm. (Hard Lefschetz, proved by Hodge) *Let X be compact Kähler of dimension n, ω a Kähler form on X. Then $\forall k \leq n$, $\cup[\omega]^{n-k} := L^{n-k} : H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R})$ is an isomorphism.*

• **Projective case** : One can take $[\omega]$ rational. Then the Lefschetz isomorphism is an isomorphism of Hodge structures.

Coro. (Lefschetz decomp.) *$H^k(X,\mathbb{R}) = \oplus_{k-2r\geq 0} L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}$, where $H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}} := \mathrm{Ker}\, L^{n-k+2r+1} \subset H^{k-2r}(X,\mathbb{R})$.*

• Lefschetz intersection pairing on H^k: $(\alpha,\beta)_{\mathrm{Lef}} = \int_X L^{n-k}\alpha \cup \beta$.
$h_{\mathrm{Lef}}(\alpha,\beta) := i^k(\alpha,\overline{\beta})_{\mathrm{Lef}}$.

• **easy:** *The Lefschetz decomposition is orthogonal for $(\,,\,)_{\mathrm{Lef}}$, and the Hodge decomposition is orthogonal for h_{Lef}.* (HR1).

Thm. *$2nd$ H-R bilinear relations: $(-1)^{p+r} h_{\mathrm{Lef}|L^r H^{p-r,q-r}(X,\mathbb{R})_{\mathrm{prim}}}$ is positive definite Hermitian (up to a global sign depending on k).* (HR2).

Corollary. *Let $[\omega]$ be rational. On $L^r H^{k-2r}(X,\mathbb{Q})_{\mathrm{prim}}$, multiply $(\,,\,)_{\mathrm{Lef}}$ by $(-1)^r$: one gets a* ***polarized Hodge structure*** *on $H^k(X,\mathbb{Q})$.*

Thm. *Let H=rational polarized Hodge structure, $H' \subset H$ a Hodge substructure, then $H = H' \oplus H''$ for some Hodge substructure $H'' \subset H$. (The category of polarized Hodge structures is semisimple).*

Proof. Choose a polarization $(\ ,\ )$ on H. First prove that $(\ ,\ )_{|H'}$ is nondegenerate using HR2, then define $H'' = H'^{\perp}$. H'' is a Hodge substructure by HR1. **qed**

• Polarizations on the cohomology of smooth projective varieties are almost motivic, but one needs the Lefschetz decomposition and the change of signs. To make them **motivic**, one needs:

Lefschetz standard conjecture. *X projective. There exists a codimension k closed algebraic subset $Z_{\mathrm{Lef}} \subset X \times X$ such that $[Z_{\mathrm{Lef}}]^* : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ is the inverse $(L^{n-k})^{-1}$ of the Lefschetz isomorphism.*

• ($[Z_{\mathrm{Lef}}] \in H^{2k}(X \times X, \mathbb{Q})$ = cohomology class of Z_{Lef}.)

• Implied by the Hodge conjecture because $(L^{n-k})^{-1}$ is an iso of Hodge structures hence produces a Hodge class on $X \times X$.

Thm. *Let H=rational polarized Hodge structure, $H' \subset H$ a Hodge substructure, then $H = H' \oplus H''$ for some Hodge substructure $H'' \subset H$. (The category of polarized Hodge structures is semisimple).*

Proof. Choose a polarization $(\,,\,)$ on H. First prove that $(\,,\,)_{|H'}$ is nondegenerate using HR2, then define $H'' = H'^{\perp}$. H'' is a Hodge substructure by HR1. **qed**

• Polarizations on the cohomology of smooth projective varieties are almost motivic, but one needs the Lefschetz decomposition and the change of signs. To make them **motivic**, one needs:

Lefschetz standard conjecture. *X projective. There exists a codimension k closed algebraic subset $Z_{\mathrm{Lef}} \subset X \times X$ such that $[Z_{\mathrm{Lef}}]^* : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ is the inverse $(L^{n-k})^{-1}$ of the Lefschetz isomorphism.*

• ($[Z_{\mathrm{Lef}}] \in H^{2k}(X \times X, \mathbb{Q})$ = cohomology class of Z_{Lef}.)

• Implied by the Hodge conjecture because $(L^{n-k})^{-1}$ is an iso of Hodge structures hence produces a Hodge class on $X \times X$.

• A cohomology algebra = graded, graded commutative, algebra of finite dimension over $\mathbb{Q}$, with $A^{2n} = \mathbb{Q}$ and Poincaré duality.

Definition. *A Hodge structure on a cohomology algebra A^*, = Hodge structure of weight k on A^k, such that $A^k \otimes A^l \to A^{k+l}$ is a morphism of Hodge structures.*

Example. $H^*(X, \mathbb{Q})$ for X compact Kähler.

Thm. (Voisin) *There exist compact Kähler manifolds (dim ≥ 4) whose cohomology algebra is not isomorphic to $H^*(X, \mathbb{Q})$ for X complex projective.*

Idea of proof. (1) Construct an X such that the structure of its cohomology algebra $\Rightarrow$ the Hodge structure on $H^1(X, \mathbb{Q})$ (or $H^2(X, \mathbb{Q})$ for simply connected examples) has endomorphisms.
(2) Certain endomorphisms on weight 1 (or weight 2) HS prevent the existence of a polarization.

Case of $\dim 2$ (Kodaira), $\dim 3$ (Lin): *Any compact Kähler X has small deformations which are projective.*

• A cohomology algebra = graded, graded commutative, algebra of finite dimension over $\mathbb{Q}$, with $A^{2n} = \mathbb{Q}$ and Poincaré duality.

Definition. *A Hodge structure on a cohomology algebra A^*, = Hodge structure of weight k on A^k, such that $A^k \otimes A^l \to A^{k+l}$ is a morphism of Hodge structures.*

Example. $H^*(X, \mathbb{Q})$ for X compact Kähler.

Thm. (Voisin) *There exist compact Kähler manifolds (dim ≥ 4) whose cohomology algebra is not isomorphic to $H^*(X, \mathbb{Q})$ for X complex projective.*

Idea of proof. (1) Construct an X such that the structure of its cohomology algebra $\Rightarrow$ the Hodge structure on $H^1(X, \mathbb{Q})$ (or $H^2(X, \mathbb{Q})$ for simply connected examples) has endomorphisms.
(2) Certain endomorphisms on weight 1 (or weight 2) HS prevent the existence of a polarization.

Case of $\dim 2$ (Kodaira), $\dim 3$ (Lin): *Any compact Kähler X has small deformations which are projective.*

Thm. (Blanchard, Deligne) *If $f : X \to Y$ is smooth projective, the Leray spectral sequence of f with $\mathbb{Q}$-coefficients degenerates at E_2.*

Proof. Relative Lefschetz operator $L = c_1(\mathcal{L})\cup$ acts on the whole spectral sequence, and induces Lefschetz decomposition
$R^k f_*\mathbb{Q} = \oplus_r L^r(R^{k-2r} f_*\mathbb{Q})_{\mathrm{prim}}$. Suffices to prove $d_2\alpha = 0$ for
$\alpha \in H^p(Y, R^q f_*\mathbb{Q}_{\mathrm{prim}})$. But $L^{n-q+1}\alpha = 0 \Rightarrow L^{n-q+1} d_2\alpha = 0$. But
$d_2\alpha \in H^{p+2}(Y, R^{q-1} f_*\mathbb{Q})$ and $L^{n-q+1} : R^{q-1} f_*\mathbb{Q}) \cong R^{2n-q+1} f_*\mathbb{Q}$. **qed**

• **Monodromy.** Local system $R^k f_*\mathbb{Q} \rightsquigarrow$ monodromy representation
$\rho : \pi_1(Y, 0) \to \mathrm{Aut}\, H^k(X_0, \mathbb{Q})$. Thus $H^k(X_0, \mathbb{Q})^\rho = H^0(Y, R^k f_*\mathbb{Q})$
$= \mathrm{Im}\,(H^k(X, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}))$ by degeneracy at E_2.

Thm (Deligne) *$X \subset \overline{X}$ smooth projective, $f : X \to Y$ as above with Y quasi-projective. Then $H^k(X_0, \mathbb{Q})^\rho = \mathrm{Im}\,(H^k(\overline{X}, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}))$. This is a Hodge substructure of $H^k(X_0, \mathbb{Q})$.*

Proof. $H^k(X, \mathbb{Q}) \to H^k(X_0, \mathbb{Q})$ is a morphism of mixed Hodge structures. On the right, pure of weight k. On the left, the weight k part is $\mathrm{Im}\,(H^k(\overline{X}, \mathbb{Q}) \to H^k(X, \mathbb{Q}))$. Then apply strictness. **qed**

Thm. (Blanchard, Deligne) *If $f : X \to Y$ is smooth projective, the Leray spectral sequence of f with $\mathbb{Q}$-coefficients degenerates at E_2.*

Proof. Relative Lefschetz operator $L = c_1(\mathcal{L})\cup$ acts on the whole spectral sequence, and induces Lefschetz decomposition $R^k f_*\mathbb{Q} = \oplus_r L^r(R^{k-2r} f_*\mathbb{Q})_{\mathrm{prim}}$. Suffices to prove $d_2\alpha = 0$ for $\alpha \in H^p(Y, R^q f_*\mathbb{Q}_{\mathrm{prim}})$. But $L^{n-q+1}\alpha = 0 \Rightarrow L^{n-q+1} d_2\alpha = 0$. But $d_2\alpha \in H^{p+2}(Y, R^{q-1} f_*\mathbb{Q})$ and $L^{n-q+1} : R^{q-1} f_*\mathbb{Q}) \cong R^{2n-q+1} f_*\mathbb{Q}$. **qed**

• **Monodromy.** Local system $R^k f_*\mathbb{Q} \rightsquigarrow$ monodromy representation $\rho : \pi_1(Y, 0) \to \mathrm{Aut}\, H^k(X_0, \mathbb{Q})$. Thus $H^k(X_0, \mathbb{Q})^\rho = H^0(Y, R^k f_*\mathbb{Q})$ $= \mathrm{Im}\,(H^k(X, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}))$ by degeneracy at E_2.

Thm (Deligne) *$X \subset \overline{X}$ smooth projective, $f : X \to Y$ as above with Y quasi-projective. Then $H^k(X_0, \mathbb{Q})^\rho = \mathrm{Im}\,(H^k(\overline{X}, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}))$. This is a Hodge substructure of $H^k(X_0, \mathbb{Q})$.*

Proof. $H^k(X, \mathbb{Q}) \to H^k(X_0, \mathbb{Q})$ is a morphism of mixed Hodge structures. On the right, pure of weight k. On the left, the weight k part is $\mathrm{Im}\,(H^k(\overline{X}, \mathbb{Q}) \to H^k(X, \mathbb{Q}))$. Then apply strictness. **qed**

• Algebraic de Rham complex $\Omega^\bullet_{X/\mathbb{C}}$, relative version $\Omega^\bullet_{X/Y}$ for $f : X \to Y$ algebraic, smooth morphism.

Thm. (Serre-Grothendieck) *X smooth quasiprojective over $\mathbb{C}$. Then $\mathbb{H}^k(X, \Omega^\bullet_X/\mathbb{C}) \cong H^k_B(X, \mathbb{C})$.*

• So, for X projective, the Hodge filtration and Frölicher s.s. are algebraic.

• Relative version $\Rightarrow$ *If $f : X \to Y$ is algebraic, smooth projective, then the Hodge bundles $\mathcal{H}^k$, $F^p\mathcal{H}^k$, $\mathcal{H}^{p,q}$ are* **algebraic** *on Y.*

• Katz-Oda construction : relative holomorphic de Rham complex $\Omega^\bullet_{X/Y}$. $R^k f_* \Omega^\bullet_{X/Y} \cong \mathcal{H}^k := H^k \otimes \mathcal{O}_Y$. Hodge filtration $F^p\mathcal{H}^k = R^k f_* \Omega^{\bullet \geq p}_{X/Y}$ with fiber $F^p H^k(X_t)$.

• Let $L^2\Omega^\bullet_X := f^*\Omega^2_Y \wedge \Omega^{\bullet-2}_X$.

• **Exact sequence**. $0 \to \Omega^{\bullet-1}_{X/Y} \otimes f^*\Omega_Y \to \Omega^\bullet_X / L^2\Omega^\bullet_X \to \Omega^\bullet_{X/Y} \to 0$

Thm. (Katz-Oda) *The Gauss-Manin connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_Y$ is the connecting map.*

Corollary. The Gauss-Manin connection is algebraic.

• Algebraic de Rham complex $\Omega^\bullet_{X/\mathbb{C}}$, relative version $\Omega^\bullet_{X/Y}$ for $f : X \to Y$ algebraic, smooth morphism.

Thm. (Serre-Grothendieck) *X smooth quasiprojective over $\mathbb{C}$. Then $\mathbb{H}^k(X, \Omega^\bullet_X/\mathbb{C}) \cong H^k_B(X, \mathbb{C})$.*

• So, for X projective, the Hodge filtration and Frölicher s.s. are algebraic.

• Relative version $\Rightarrow$ *If $f : X \to Y$ is algebraic, smooth projective, then the Hodge bundles $\mathcal{H}^k$, $F^p\mathcal{H}^k$, $\mathcal{H}^{p,q}$ are **algebraic** on Y.*

• Katz-Oda construction : relative holomorphic de Rham complex $\Omega^\bullet_{X/Y}$. $R^k f_* \Omega^\bullet_{X/Y} \cong \mathcal{H}^k := H^k \otimes \mathcal{O}_Y$. Hodge filtration $F^p\mathcal{H}^k = R^k f_* \Omega^{\bullet \geq p}_{X/Y}$ with fiber $F^p H^k(X_t)$.

• Let $L^2\Omega^\bullet_X := f^*\Omega^2_Y \wedge \Omega^{\bullet-2}_X$.

• **Exact sequence**. $0 \to \Omega^{\bullet-1}_{X/Y} \otimes f^*\Omega_Y \to \Omega^\bullet_X / L^2\Omega^\bullet_X \to \Omega^\bullet_{X/Y} \to 0$

Thm. (Katz-Oda) *The Gauss-Manin connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_Y$ is the connecting map.*

Corollary. The Gauss-Manin connection is algebraic.