Lie algebraic aspects of quantum control in interacting spin-$1/2$ (qubit) chains

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Outline of the talk

- **Introduction to quantum (coherent) control**

- **Local quantum control of Heisenberg spin-$1/2$ chains**

  R. Heule, C. Bruder, D. Burgarth, and VMS, PRA **82**, 052333 (2010);
  Eur. Phys. J. D **63**, 41 (2011).

  VMS, A. Fedorov, A. Wallraff, and C. Bruder, PRB **85**, 054504 (2012).

- **Pulse-sequence generated dynamics of stabilizer operators: application in measurement-based QC, i.e., one-way QC**

  T. Tanamoto, D. Becker, VMS, and C. Bruder, PRA **86**, 032327 (2012);
  T. Tanamoto, VMS, C. Bruder, and D. Becker, PRA **87**, 052305 (2013).

- **Summary**

Quantum control: generalities

- **State-to-state (state-selective) control**: How to steer a quantum system from a given initial- to a desired final state?
- **Operator (state-independent) control**: How to realize a pre-determined unitary transformation (target quantum gate)?

$$H(t) = H_0 + \sum_{j=1}^{p} f_j(t) H_j$$

$f_j(t)$ – control fields

The system is **completely controllable** if $H(t)$ can give rise to an arbitrary unitary transformation on its Hilbert space $\mathcal{H}$

i.e., the reachable set $\mathcal{R}$ is equal to $U(n)$ or $SU(n)$ ($n = \dim \mathcal{H}$)

General controllability theorems

$$\dot{U}(t) = -i[H_0 + \sum_{j=1}^{p} f_j(t) H_j] U(t) \quad , \quad U(0) = \mathbb{1} \qquad (\#)$$

Lie algebra rank condition

Theorem

The reachable set $\mathcal{R}$ of a quantum control system described by Eq. (#) is the connected Lie group associated with the Lie algebra $\mathcal{L}_0$ generated by $-iH_0, -iH_1, \ldots, -iH_p$, i.e., $\mathcal{R} = e^{\mathcal{L}_0}$.

$\Rightarrow$ complete (operator) controllability

Theorem

A system described by Eq. (#) is completely (operator) controllable iff $\mathcal{L}_0 = u(n)$ [or $\mathcal{L}_0 = su(n)$], where $\mathcal{L}_0$ is the Lie algebra generated by $-iH_0, -iH_1, \ldots, -iH_p$.

Interacting qubit arrays

Coupling between qubits:

$$H_{\text{int}} = \sum_{i<j} \sum_{\alpha,\beta} J_{ij}^{\alpha\beta} \sigma_i^{\alpha} \sigma_j^{\beta} \quad (\alpha, \beta = x, y, z)$$

qubit-qubit interaction	qubit system
Ising	charge
XY	flux, charge-flux, phase, cavity
Heisenberg	spin, donor atom

couplings beyond nearest neighbors can be induced using a "quantum bus" (e.g., cavity) [J. Majer *et al.*, Nature (2007)]

Local control in interacting systems: general aspects

composite system $S = C \cup \bar{C}$ with controls acting only on C

total Hamiltonian: $H = H_S + \sum_{j=1}^{p} f_j^C(t) H_j^C$

S is completely controllable iff $-iH_S$ and $-iH_j^C$ $(j = 1, \ldots, p)$ generate the Lie algebra $\mathcal{L}(S)$ of all skew-Hermitian operators on S

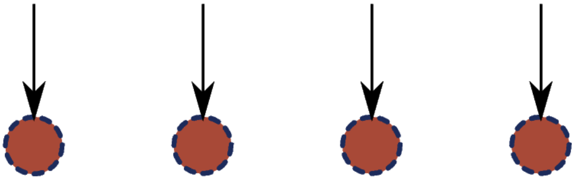
$$\langle iH_S, \mathcal{L}(C) \rangle = \mathcal{L}(S)$$

$\mathcal{L}(C) = \{-iH_1^C, \ldots, -iH_p^C\}_{\mathcal{L}}$

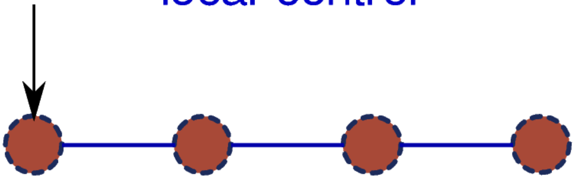
$\langle A, B \rangle$ – algebraic closure of the operator sets $\mathcal{A}$ and $\mathcal{B}$

Local control in qubit arrays with "always-on" interactions

conventional control



local control



Complete controllability of XXZ Heisenberg chains

$$H_0 = J \sum_{i=1}^{N-1} \left(S_{i,x} S_{i+1,x} + S_{i,y} S_{i+1,y} + \Delta S_{i,z} S_{i+1,z} \right)$$

$$H_c(t) = \underbrace{h_x(t)}_{f_1(t)} \underbrace{S_{1x}}_{H_1} + \underbrace{h_y(t)}_{f_2(t)} \underbrace{S_{1y}}_{H_2}$$

$$H_{\text{total}}(t) = H_0 + H_c(t)$$

Acting on the x- and y-components of an end spin in an XXZ Heisenberg spin chain renders the chain completely controllable!

sufficient to show that the dimension of the dynamical Lie algebra $\mathcal{L}_{xy}$ generated by $\{-iH_0, -iS_{1x}, -iS_{1y}\}$ is $d^2 - 1 \ (d \equiv 2^N)$

$\Rightarrow \mathcal{L}_{xy} \cong su(d) \ \Rightarrow e^{\mathcal{L}_{xy}} \cong SU(d)$ **(complete controllability)**

Generalization to graphs

Local controllability on a graph $G = (S, E)$ by acting on $C \subseteq S$

$$H = H_S + \sum_{j=1}^{p} f_j^C(t) H_j^C \qquad \boxed{H_S = \sum_{n,m \in E} H_{nm}}$$

graph criterion of controllability (sufficient condition):
algebraic property of H_{nm} + topological property of G

H_S is **algebraically propagating** if for all $n \in S$ and $(n, m) \in E$

$$\boxed{\langle [iH_{nm}, \mathcal{L}(n)], \mathcal{L}(n) \rangle = \mathcal{L}(n, m)}$$

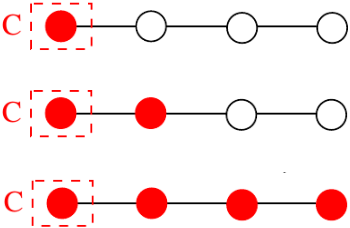
Heisenberg and Affleck-Kennedy-Lieb-Tasaki (AKLT) couplings are A.P. !

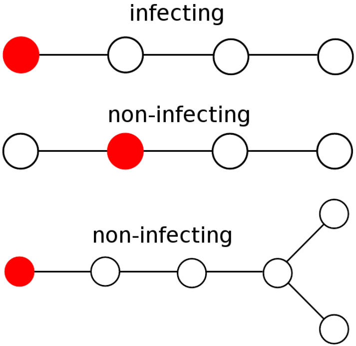
S is controllable by acting on C if H_S is A.P. and C is infecting

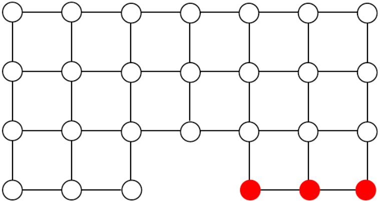
D. Burgarth *et. al.*, PRA **79**, 060305(R) (2009)

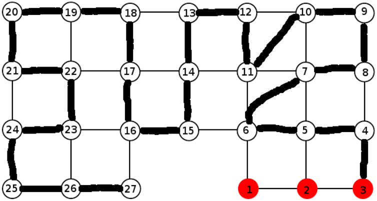
Infecting and non-infecting subgraphs

rules of infection:









Control objectives (target gates)

controlled-NOT on the last two qubits of the chain:

$$\mathbf{CNOT}_N \equiv \underbrace{\mathbf{I} \otimes \ldots \otimes \mathbf{I}}_{N-2} \otimes \Big(\underbrace{|0\rangle\langle 0| \otimes \mathbf{I} + |1\rangle\langle 1| \otimes \mathbf{X}}_{\mathbf{CNOT}} \Big)$$

($X \equiv \sigma_x$)

flip (**NOT**) of the last qubit $\mathbf{X}_N \equiv \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \mathbf{X}$
requires only an x control!

$\sqrt{\mathrm{SWAP}}$ on the last two qubits: $\sqrt{\mathrm{SWAP}}_{N-1,N}$

reminder: $\sqrt{\mathrm{SWAP}} \equiv e^{i\frac{\pi}{8}}\, e^{-i\frac{\pi}{8}(X\otimes X + Y\otimes Y + Z\otimes Z)}$

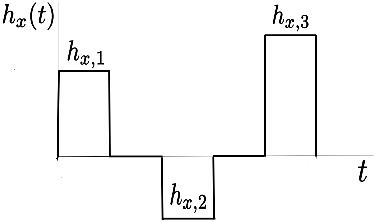
$N = 3$, $\Delta = 1$ case: $\dim \mathcal{L}_x = 18$, basis $\{-iH_0, \ldots, -iH_{17}\}$

Is there $A \in \mathcal{L}_x$ such that $\sqrt{\mathrm{SWAP}}_{2,3} = e^A$?

$$\mathbb{1}XX + \mathbb{1}YY + \mathbb{1}ZZ = \tfrac{1}{2}(H_0 - H_3 + H_6 - H_{16} + H_{17})$$

Control pulses and fidelity maximization

alternate x and y (or x only !) piecewise-constant controls:



full time evolution (total time $t_f \equiv N_t T$):

$$U(t_f) = U_{y,N_t/2} U_{x,N_t/2} \dots U_{y,1} U_{x,1}$$

$$\left[U_{j,n} \equiv e^{-iH_{j,n}T} \quad (j = x, y) \right]$$

gate fidelity:

$$F(t_f) = \frac{1}{d} \left| \mathrm{tr}\left[U^\dagger(t_f) U_{\mathrm{target}} \right] \right|$$

$$\left[0 \leq F(t_f) \leq 1 \right]$$

maximize $F = F(\{h_{x,n}; h_{y,n}\})$ numerically

frequency-filtered control fields:

$$\widetilde{h}_j(t) = \mathcal{F}^{-1}\left[f(\omega) \mathcal{F}[h_j(t)] \right]$$

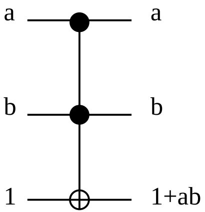
ideal low-pass filter:

$$f(\omega) = \theta(\omega + \omega_0) - \theta(\omega - \omega_0)$$

Toffoli-gate realization with superconducting qubits

state of the art: two-qubit gates with $F > 90\%$ [DiCarlo *et al.*, (2009)]

TOFFOLI ≡ controlled-controlled-NOT

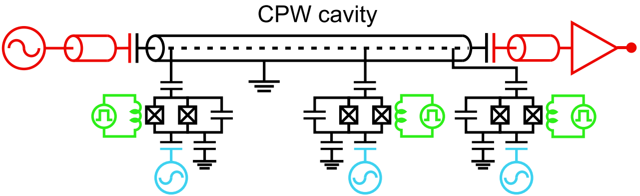


- trapped-ion [$F \approx 71\%$], photonic [$F \approx 81\%$] realizations in 2009!
- conventional **6 CNOTs + 10 single-qubit operations** approach not feasible due to long gate times!
- Way out: **use third level**
  A. Fedorov *et al.*, Nature 2012 : $F = 64.5 \pm 0.5\ \%$
  M. D. Reed *et al.*, Science 2012 : $F = 78 \pm 1\ \%$

Can quantum control be of help?

$$F \xrightarrow{\text{decoherence}} F * \exp(-t_g/T_2)$$

Three-qubit (transmon) circuit QED setup



effective XY-type model:

$$H_0 = \sum_{i<j} J_{ij}(\sigma_{ix}\sigma_{jx} + \sigma_{iy}\sigma_{jy})$$

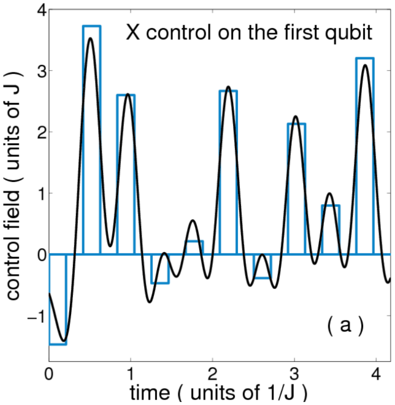
$J_{12} = J_{23} = J \approx 30$ MHz, $J_{13} \approx 5$ MHz

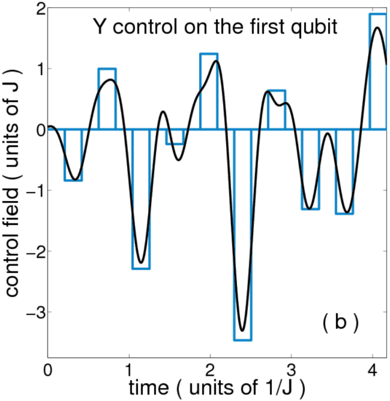
$$H_c(t) = \sum_{i=1}^{3} \left[\Omega_x^{(i)}(t)\,\sigma_{ix} + \Omega_y^{(i)}(t)\,\sigma_{iy}\right]$$

$$\sqrt{\Omega_x^2 + \Omega_y^2} \lesssim 130 \text{ MHz}$$

VMS, A. Fedorov, A. Wallraff, and C. Bruder, PRB **85**, 054504 (2012)

Toffoli gate in circuit QED: results





cutoff frequency:

$\omega_0 = 500\,\mathrm{MHz}$

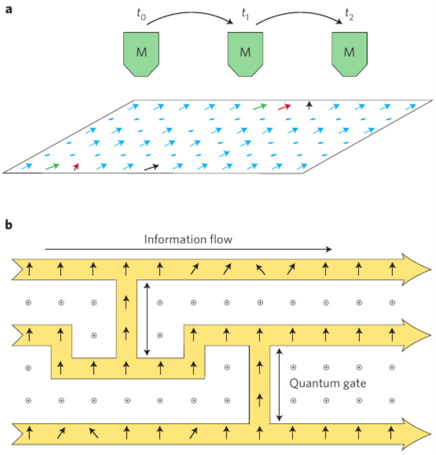
$\boldsymbol{\omega_0 \approx 17J\ !}$

$t_g \approx 140\,\mathrm{ns}$ $\qquad F \approx 99\%$

$t_g = 75\,\mathrm{ns}$ $\qquad F \approx 92\%$

Measurement-based quantum computation (MBQC)

R. Raussendorf and H. J. Briegel, PRL **86**, 5188 (2001)



H. J. Briegel *et al.*, Nature Phys. **5**, 19 (2009)

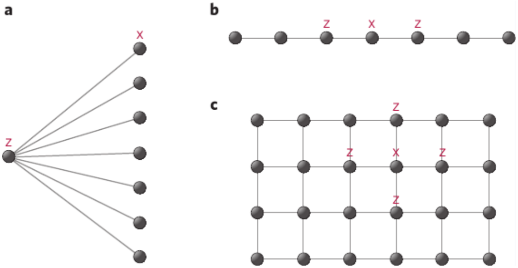
with local (single-qubit) measurements:
MBQC $\longrightarrow$ one-way QC

2D cluster state is a universal resource for MBQC!

Other candidates, e.g., the AKLT state, are difficult to produce in solid-state systems!

one-way quantum computing

H. J. Briegel and R. Raussendorf, PRL **86**, 910 (2001)



initial preparation:

$$|G\rangle = \prod_{\{i,j\}} U_{\mathrm{PG}}^{(ij)} |+\rangle^{\otimes N}$$

$$|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$$

$$U_{\mathrm{PG}} = \mathrm{diag}(1,1,1,-1)$$

correlation operators $K_i \equiv \sigma_i^x \bigotimes_{j \in \mathrm{nghd}(i)} \sigma_j^z$ satisfy $K_i|\psi_c\rangle = \pm|\psi_c\rangle$

One-way QC: an illustration

two-qubit cluster state: $K_1 = \sigma_1^x \sigma_2^z$, $K_2 = \sigma_1^z \sigma_2^x$

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}(|+\rangle|0\rangle + |-\rangle|0\rangle)$$

measure qubit 1 in a basis rotated by a $\mathrm{SU}(2)$ matrix:

$$U(\theta, \phi) \equiv \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

measurement results: $m = 0 \longrightarrow$ qubit 2 in $|\psi_0\rangle = P(\phi)R(\theta)|0\rangle$

$$m = 1 \longrightarrow |\psi_1\rangle = \sigma_1^z P(-\phi)\sigma_1^x R(\theta)|1\rangle$$

$$P(\phi) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{i\phi} \\ 1 & -e^{i\phi} \end{pmatrix} \quad R(\theta) \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

arbitrary single-qubit rotation on the second qubit can be performed by measuring the first one in an appropriate basis!

Stabilizer Hamiltonian

cluster states are eigenstates of the stabilizer Hamiltonian

$$H_{\rm stab} = -\sum_i K_i$$

How to "generate" a stabilizer Hamiltonian starting from a "natural" two-body spin-1/2 (qubit) Hamiltonian?

$$H = H_0 + H_{\rm int}$$

$$H_0 = \sum_i (\Omega_x \sigma_i^x + \Omega_y \sigma_i^y + \varepsilon_i \sigma_i^z)$$

Ising: $H_{\rm int} = J \sum_i \sigma_i^z \sigma_{i+1}^z$

XY: $H_{\rm int} = J \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y)$

Heisenberg: $H_{\rm int} = J \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z)$

Stabilizer Hamiltonian from Ising-type interactions

What $e^{-i\theta\sum_i \sigma_i^z\sigma_{i+1}^z} H_\mathrm{s}\, e^{i\theta\sum_i \sigma_i^z\sigma_{i+1}^z}$ amounts to?

basic relations:

$$e^{-i\theta\sigma_1^z\sigma_2^z}\sigma_1^{x,y}e^{i\theta\sigma_1^z\sigma_2^z} = \cos(2\theta)\sigma_1^{x,y} \pm \sin(2\theta)\sigma_1^{y,x}\sigma_2^z$$

special case $\theta = \pi/4$ – increasing the order of the Pauli-matrix terms:

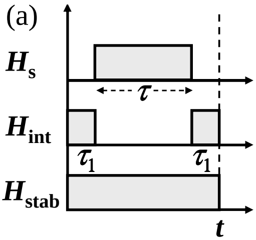
$$e^{-i\frac{\pi}{4}\sigma_1^z\sigma_2^z}\sigma_1^x e^{i\frac{\pi}{4}\sigma_1^z\sigma_2^z} = \sigma_1^y\sigma_2^z \quad ; \quad e^{-i\frac{\pi}{4}\sigma_1^z\sigma_2^z}\sigma_1^y e^{i\frac{\pi}{4}\sigma_1^z\sigma_2^z} = -\sigma_1^x\sigma_2^z$$

$\Rightarrow$ 1D stabilizer Hamiltonian; 1D $\rightarrow$ 2D straightforward!

Q: But how can we physically "generate" ($\equiv$ induce effective dynamics of)

$$H_\mathrm{stab} = e^{-i(\pi/4)\sum_i \sigma_i^z\sigma_{i+1}^z} H_\mathrm{s}\, e^{i(\pi/4)\sum_i \sigma_i^z\sigma_{i+1}^z}\ ?$$

Stabilizer Hamiltonian as the effective Hamiltonian



Ising-interaction pulses with $\tau_1 \equiv \pi/(4J)$:

$$H_{\rm stab} = e^{-i\frac{\pi}{4}\sum_i \sigma_i^z\sigma_{i+1}^z} H_{\rm s}\, e^{i\frac{\pi}{4}\sum_i \sigma_i^z\sigma_{i+1}^z}$$

state evolution:

$$\rho(0) \xrightarrow{\tau_1 H_{\rm int}} \xrightarrow{\tau H_{\rm s}} \xrightarrow{-\tau_1 H_{\rm int}} \rho(t=\tau+2\tau_1)$$

pulse-induced effective evolution:

$$e^{-i\tau H_{\rm stab}} = \exp\left(-i\frac{\pi}{4}\sum_i \sigma_i^z\sigma_{i+1}^z\right) e^{-i\tau H_{\rm s}} \exp\left(i\frac{\pi}{4}\sum_i \sigma_i^z\sigma_{i+1}^z\right)$$

unitary tsf. with a generator S, analytic operator function $f(A)$:

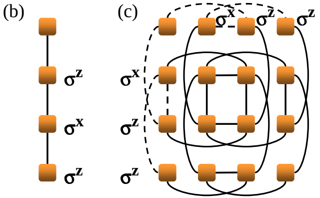
$$e^S\, f(A)\, e^{-S} = f(e^S\, A\, e^{-S})$$

Stabilizer Hamiltonian from XY-type interactions

main difference from the Ising case: $e^{-i\theta \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y)}$ does not factorize as

$[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y, \sigma_{i+1}^x \sigma_{i+2}^x + \sigma_{i+1}^y \sigma_{i+2}^y] \neq 0 \Rightarrow H_{\text{stab}}^{2D}$ step by step:

1. generate H_{stab} for adjacent qubit pairs
2. connect the pairs to obtain H_{stab}^{1D}
3. generate multiple H_{stab}^{1D} and connect them pairwise into ladders
4. connect the ladders to obtain H_{stab}^{2D}



the obtained H_{stab} is **twisted**!

the corresponding cluster state is twisted too!

Case with always-on interactions

Q: What if H_0 and $H_{\rm int}$ cannot be switched on/off at will?
A: The method can be applied in a stroboscopic fashion!

based on the Baker-Campbell-Hausdorff (BCH) formula: $e^A e^B =$

$$= \exp\left(A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}[A,[A,B]] + \tfrac{1}{12}[B,[B,A]] + \ldots\right)$$

realizing interaction pulse ("extracting" $H_{\rm int}$):

$$A = i(H_0 + H_{\rm int})\tau \ , \ B = i(-H_0 + H_{\rm int})\tau \quad \Rightarrow \quad e^A e^B =$$

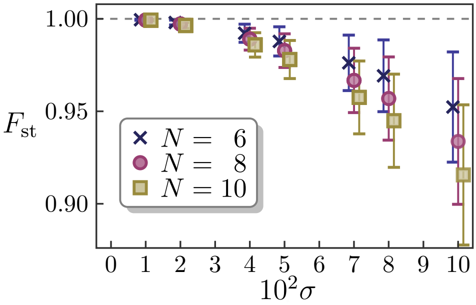
$$\exp\left(2iH_{\rm int}\tau + (i\tau)^2[H_0, H_{\rm int}] + \frac{(i\tau)^3}{3}[H_0,[H_0,H_{\rm int}]] + \ldots\right)$$

repeating n-times with $n\Omega\tau = \pi/4$: in $(e^A e^B)^n$

k-th term $\sim \left(\frac{\pi}{4n}\right)^k \quad \Rightarrow \quad$ the effects of H_0 are eliminated!

Cluster-state fidelity: numerical results in the XY case

$$F_{\mathrm{st}}(\tau) \equiv |\langle \Psi_c | U_\tau(\delta) | \Psi_c \rangle|^2 \qquad U_\tau(\delta) \equiv e^{-i\tau H_{\mathrm{stab}}(\pi/4+\delta)}$$



analytical (perturbative) result:

$$1 - F_{\mathrm{st}} \propto \delta^2 \quad (\, \delta \ll \pi/4 \,)$$

$$\delta \;\rightarrow\; \delta_i \quad (i = 1, \ldots, N-1)$$

averaged over 10000 random realizations of the δ_i
taken from a Gaussian distribution of width σ

T. Tanamoto, D. Becker, VMS, and C. Bruder, PRA **86**, 032327 (2012)

Summary

- Local-control approach allows for an efficient realization of quantum gates in qubit arrays with XXZ Heisenberg interaction!
- 2D cluster states, a universal resource for MBQC, can be preserved with high fidelity in XY- and Ising-coupled qubit arrays!
- The two approaches to quantum control in interacting qubit arrays are complementary!