

AN OPTIMAL REGULARITY RESULT ON THE QUASI-INVARIANT GAUSSIAN MEASURES FOR THE CUBIC FOURTH ORDER NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We study the transport properties of the Gaussian measures on Sobolev spaces under the dynamics of the cubic fourth order nonlinear Schrödinger equation on the circle. In particular, we establish an optimal regularity result for quasi-invariance of the mean-zero Gaussian measures on Sobolev spaces. The main new ingredient is an improved energy estimate established by performing an infinite iteration of normal form reductions on the energy functional. Furthermore, we show that the dispersion is essential for such a quasi-invariance result by proving non quasi-invariance of the Gaussian measures under the dynamics of the dispersionless model.

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1. INTRODUCTION

In this paper, we complete the study of the transport properties of Gaussian measures on Sobolev spaces for the cubic nonlinear Schrödinger equation (NLS) with quartic dispersion, initiated by the first and third authors in [26].

The question addressed in this work is motivated by a number of perspectives. In probability theory, absolute continuity properties for the pushforward of Gaussian measures under linear and nonlinear transformations have been studied extensively, starting with the classical work of Cameron-Martin; see [7, 22, 31]. More generally, questions of absolute continuity of the distribution of solutions to differential and stochastic differential equations with respect to a given initial distribution or some chosen reference measure are also central to stochastic analysis. For example, close to the topic of the current paper, see the work of Cruzeiro [11, 12].

On the other hand, in the analysis of partial differential equations (PDEs), Hamiltonian PDE dynamics with initial data distributed according to measures of Gibbs type have been studied intensively over the last two decades, starting with the work of Bourgain [4, 5]. See [26] for the references therein. These Gibbs-type measures are constructed as weighted Gaussian measures and are usually supported on Sobolev spaces of low regularity with the exception of completely integrable Hamiltonian PDEs such as the cubic NLS on the circle. In the approach initiated by Bourgain and successfully applied to many equations since then, invariance of such Gibbs-type measures under the flow of the equation has been established by combining the Hamiltonian structure of suitable finite dimensional approximations, in particular invariance of the finite dimensional Gibbs-type measures, with PDE approximation arguments. Invariance of such weighted Gaussian measures implies absolute continuity of the pushforward of the base Gaussian measures. If we substitute the underlying measure with a different Gaussian measure, however, the question of absolute continuity becomes non-trivial. See also [6] for a related question by Gel'fand on building a direct method to prove absolute continuity properties without relying on invariant measures.

In [34], the third author initiated the study of transport properties of Gaussian measures under the flow of a Hamiltonian PDE, combining probabilistic and PDE techniques. The result proved there for a specific Hamiltonian equation (the generalized BBM equation) went beyond general results on the pushforwards of Gaussian measures by nonlinear transformations such as Ramer's [31]. It was shown in [34] that a key step to showing absolute continuity is to establish a smoothing effect on the nonlinear part. In [26], the first and third authors studied the transport of Gaussian measures for the cubic NLS with quartic dispersion. An additional difficulty compared to [34] is the absence of explicit smoothing coming from the nonlinearity, thus requiring the use of *dispersion* in an explicit manner. In [26], such dispersion was manifested through the normal form method. In this paper, we improve the result in [26] to the optimal range of Sobolev exponents by pushing the normal form method to the limit. Furthermore, we present a result showing that, in the absence of dispersion, the distribution of the solution of the resulting dispersionless equation is *not*

absolutely continuous with respect to the Gaussian initial data for any non-zero time. This in particular establishes the necessity of dispersion for an absolute continuity property. Since the linear equation is easily seen to leave the distribution of the Gaussian initial data invariant, this highlights that the question of transport properties for a Hamiltonian PDE is a probabilistic manifestation of the competition between the dispersion and the nonlinear part, familiar for the study of nonlinear dispersive equations.

1.1. The equation. We consider the cubic fourth order nonlinear Schrödinger equation (4NLS) on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} i\partial_t u = \partial_x^4 u \pm |u|^2 u \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}, \quad (1.1)$$

where u is a complex-valued function on $\mathbb{T} \times \mathbb{R}$. The equation (1.1) is also called the biharmonic NLS and it was studied in [18, 33] in the context of stability of solitons in magnetic materials. See also [20, 21, 3, 13] for a more general class of fourth order NLS:

$$i\partial_t u = \lambda \partial_x^2 u + \mu \partial_x^4 u \pm |u|^2 u. \quad (1.2)$$

The equation (1.1) is a Hamiltonian PDE with the conserved Hamiltonian:

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} |\partial_x^2 u|^2 dx \pm \frac{1}{4} \int_{\mathbb{T}} |u|^4 dx.$$

In addition to the Hamiltonian, the flow of the equation (1.1) preserves the L^2 -norm, or the so-called “mass”:

$$M(u) = \int_{\mathbb{T}} |u|^2 dx.$$

This mass conservation law was used in [26] to prove the following sharp global well-posedness result.

Proposition 1.1. *The cubic 4NLS (1.1) is globally well-posed in $H^s(\mathbb{T})$ for $s \geq 0$.*

This global well-posedness result in $L^2(\mathbb{T})$ is sharp in the sense that the cubic 4NLS (1.1) is ill-posed in negative Sobolev spaces in the sense of non-existence of solutions. See [16, 26, 29].

The defocusing/focusing nature of the equation (1.1) does not play any role in the following. Hence, we assume that it is defocusing, i.e. with the + sign in (1.1).

1.2. Quasi-invariance of μ_s . Given $s > \frac{1}{2}$, we consider the mean-zero Gaussian measures μ_s on $L^2(\mathbb{T})$ with covariance operator $2(\text{Id} - \partial_x^2)^{-s}$, formally written as

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = \prod_{n \in \mathbb{Z}} Z_{s,n}^{-1} e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}_n|^2} d\hat{u}_n. \quad (1.3)$$

While the expression $d\mu_s = Z_s^{-1} \exp(-\frac{1}{2}\|u\|_{H^s}^2) du$ may suggest that μ_s is a Gaussian measure on $H^s(\mathbb{T})$, we need to enlarge the space in order to make sense of μ_s .

The covariance operator is diagonalized by the Fourier basis on \mathbb{T} and the Gaussian measure μ_s defined above is in fact the induced probability measure under the map¹

$$\omega \in \Omega \longmapsto u^\omega(x) = u(x; \omega) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx}, \quad (1.4)$$

¹Henceforth, we drop the harmless factor of 2π , if it does not play an important role.

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n \in \mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space (Ω, \mathcal{F}, P) , i.e. $\text{Var}(g_n) = 2$. From this random Fourier series representation, it is easy to see that u^ω in (1.4) lies in $H^\sigma(\mathbb{T})$ almost surely if and only if $\sigma < s - \frac{1}{2}$. Lastly, note that, for the same range of σ , the triplet (H^s, H^σ, μ_s) forms an abstract Wiener space. See [14, 23].

In the following, we continue to study the transport property of the Gaussian measure μ_s under the dynamics of the cubic 4NLS (1.1). Before proceeding further, recall the following definition of quasi-invariant measures; given a measure space (X, μ) , we say that the measure is *quasi-invariant* under a measurable transformation $T : X \rightarrow X$ if μ and the pushforward of μ under T , defined by $T_*\mu = \mu \circ T^{-1}$, are equivalent, i.e. mutually absolutely continuous with respect to each other.

Our first result improves the quasi-invariance result in [26] to the optimal range of Sobolev exponents.

Theorem 1.2. *Let $s > \frac{1}{2}$. Then, the Gaussian measure μ_s is quasi-invariant under the flow of the cubic 4NLS (1.1).*

Theorem 1.2 improves the main result in [26], where the first and the third authors proved quasi-invariance of μ_s under (1.1) for $s > \frac{3}{4}$. Moreover, the regularity $s > \frac{1}{2}$ is optimal since when $s = \frac{1}{2}$, the Gaussian measure μ_s is supported in $H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T})$, while the cubic 4NLS (1.1) is ill-posed in negative Sobolev spaces in the sense of non-existence of solutions.

As shown in [34], to prove quasi-invariance of μ_s , it is essential to exhibit a smoothing of the nonlinear part of the equation. This can be understood at an intuitive level by an analogy to the Cameron-Martin theorem: the Gaussian measures μ_s are quasi-invariant under translations by fixed vectors in their respective Cameron Martin spaces $H^s(\mathbb{T})$. Since a typical element under μ_s lies in $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$, one needs to show that the nonlinear part represents a perturbation which is smoother in the Sobolev regularity. The Cameron-Martin theorem applies only to translation by fixed vectors, but Ramer's quasi-invariance result [31] applies to a more general nonlinear transformation on an abstract Wiener space, although it requires the translations to be more regular. This was applied in [34] and [26], where it was noted that a direct application of Ramer's result yields a suboptimal range on s . In [26], we applied the normal form reduction to the equation and exhibited $(1 + \varepsilon)$ -smoothing on the nonlinearity when $s > 1$. We then proved quasi-invariance of μ_s by invoking Ramer's result. When $\frac{3}{4} < s \leq 1$, we followed the general approach introduced by the third author in the context of the (generalized) BBM equation [34]. This strategy combines an energy estimate with the analysis of the evolution of truncated measures. As in [26], showing a smoothing of the nonlinear part for (1.1) requires normal form reductions. The main improvements over [26] here comes from a more refined implementation of the normal form reductions, inspired by [15].

In the following, we first describe a rough idea behind this method introduced in [34]. Let $\Phi(t)$ denote the solution map of (1.1) sending initial data u_0 to the solution $u(t)$ at time $t \in \mathbb{R}$. Suppose that we have a measurable set $A \subset L^2(\mathbb{T})$ with $\mu_s(A) = 0$. Fix non-zero

$t \in \mathbb{R}$. In order to prove quasi-invariance of μ_s , we would like to prove $\mu_s(\Phi(t)(A)) = 0$.² The main idea is to establish the following two properties:

(i) Energy estimate (with smoothing):

$$\frac{d}{dt} \|\Phi(t)(u)\|_{H^s}^2 \text{ " \leq " } C(\|u\|_{L^2}) \|u\|_{H^{s-\frac{1}{2}-\varepsilon}}^{2-\theta} \quad (1.5)$$

for some $\theta > 0$,³

(ii) A change-of-variable formula:

$$\mu_s(\Phi(t)(A)) = Z_s^{-1} \int_{\Phi(t)(A)} e^{-\frac{1}{2}\|u\|_{H^s}^2} du \text{ " = " } Z_s^{-1} \int_A e^{-\frac{1}{2}\|\Phi(t)(u)\|_{H^s}^2} du.$$

Step (i) is an example of local analysis, studying a trajectory of a single solution, while Step (ii) is an example of global analysis on the phase space. Combining (i) and (ii), we can study the evolution of $\mu_s(\Phi(t)A)$ by estimating $\frac{d}{dt}\mu_s(\Phi(t)(A))$. In particular, by applying Yudovich's argument [37], we obtain

$$\mu_s(\Phi(t)(A)) \leq C(t, \delta)(\mu_s(A))^{1-\delta} \quad (1.6)$$

for any $\delta > 0$. In particular, if $\mu_s(A) = 0$, then we would have $\mu_s(\Phi(t)(A)) = 0$.

As the quotation marks indicate, both (i) and (ii) are not quite true as they are stated above. In [26], we first performed two transformations to (1.1) and transformed the equation into the following renormalized equation:

$$\partial_t \widehat{v}_n = -i \sum_{\substack{n=n_1-n_2+n_3 \\ n \neq n_1, n_3}} e^{-i\phi(\bar{n})t} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} + i|\widehat{v}_n|^2 \widehat{v}_n, \quad (1.7)$$

where the phase function $\phi(\bar{n})$ is given by

$$\phi(\bar{n}) = \phi(n_1, n_2, n_3, n) = n_1^4 - n_2^4 + n_3^4 - n^4. \quad (1.8)$$

Note that this reduction of (1.1) to (1.7) via two transformations on the phase space is another instance of global analysis. See Subsection 3.1. This reformulation exhibits resonant and non-resonant structure of the nonlinearity in an explicit manner and moreover it removes certain resonant interactions, which was crucial in establishing an effective energy estimate in Step (i). By applying a normal form reduction, we introduced a *modified* energy $E_t = \|u(t)\|_{H^s}^2 + R_t$ for some appropriate correction term R_t . See (1.9) - (1.11) below. We then established an energy estimate on the modified energy E_t , provided $s > \frac{3}{4}$. In Step (ii), in order to justify such a change-of-variable formula, we considered a truncated dynamics. Moreover, we needed to introduce and consider a change-of-variable formula for a *modified* measure associated with the modified energy E_t introduced in Step (i).

The regularity restriction $s > \frac{3}{4}$ in the previous paper [26] comes from the energy estimate in Step (i), where we applied the normal form reduction (namely integration by parts in time) once to the equation: $\partial_t \|u\|_{H^s}^2 = \dots$ satisfied by the H^s -energy functional $\|u\|_{H^s}^2$. In the following, we prove Theorem 1.2 by performing normal form reductions *infinitely many*

²By time reversibility, this would also yield $\Phi(t)_* \mu_s(A) = \mu_s(\Phi(-t)(A)) = 0$.

³In [27], the first and third authors recently proved quasi-invariance of $\mu_s \otimes \mu_{s-1}$ on $(u, \partial_t u)$ under the dynamics of the two-dimensional cubic nonlinear wave equation (NLW), where they showed that even when $\theta = 0$, we can still apply Yudovich's argument in the limiting case and establish a desired estimate of the form (1.6). This was crucial in proving quasi-invariance of $\mu_s \otimes \mu_{s-1}$ under the cubic NLW on \mathbb{T}^2 .

times. Our normal form approach is analogous to the approach employed in [1, 24, 15]. In particular, in [15], the first author (with Guo and Kwon) implemented an infinite iteration scheme of normal form reductions to prove unconditional well-posedness of the cubic NLS on \mathbb{T} in low regularity. In [15], we performed integration by parts in a successive manner, introducing nonlinear terms of higher and higher degrees. While the nonlinear terms thus introduced are of higher degrees, they satisfy better estimates. In order to keep track of all possible ways to perform integration by parts, we introduced the notion of ordered trees. See also [8] for another example of an infinite iteration of normal form reductions to prove unconditional well-posedness.

In establishing an improved energy estimate (Proposition 3.4), we perform an infinite iteration of normal form reductions. It is worthwhile to note that, unlike [15], we do not work at the level of the equation (1.1). Instead, we work at the level of the evolution equation $\partial_t \|u\|_{H^s}^2 = \dots$ satisfied by the H^s -energy functional. Let us first go over the computation performed in [26] to show a flavor of this method. Using (1.7), we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|u(t)\|_{H^s}^2 \right) &= \frac{d}{dt} \left(\frac{1}{2} \|v(t)\|_{H^s}^2 \right) \\ &= -\operatorname{Re} i \sum_{n \in \mathbb{Z}} \sum_{\substack{n=n_1-n_2+n_3 \\ n \neq n_1, n_3}} e^{-i\phi(\bar{n})t} \langle n \rangle^{2s} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} \overline{\widehat{v}_n}, \end{aligned} \quad (1.9)$$

where v is the renormalized variable as in (1.7). Then, differentiating by parts, i.e. integrating by parts without an integral symbol,⁴ we obtain

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|v(t)\|_{H^s}^2 \right) &= \operatorname{Re} \frac{d}{dt} \left[\sum_{n \in \mathbb{Z}} \sum_{\substack{n=n_1-n_2+n_3 \\ n \neq n_1, n_3}} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} \overline{\widehat{v}_n} \right] \\ &\quad - \operatorname{Re} \sum_{n \in \mathbb{Z}} \sum_{\substack{n=n_1-n_2+n_3 \\ n \neq n_1, n_3}} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} \partial_t (\widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} \overline{\widehat{v}_n}). \end{aligned} \quad (1.10)$$

This motivates us to define the first modified energy $E_t^{(1)}(v)$ with the correction term $R_t^{(1)}(v)$ by

$$\begin{aligned} E_t^{(1)}(v) &= \frac{1}{2} \|v\|_{H^s}^2 + R_t^{(1)}(v) \\ &:= \frac{1}{2} \|v\|_{H^s}^2 - \operatorname{Re} \sum_{n \in \mathbb{Z}} \sum_{\substack{n=n_1-n_2+n_3 \\ n \neq n_1, n_3}} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} \overline{\widehat{v}_n}. \end{aligned} \quad (1.11)$$

This is the modified energy used in the previous work [26] (up to a constant factor). Note that the time derivative of $E_t^{(1)}(v)$ is given by the second term on the right-hand side of (1.10).

In the second step, we divide the the second term on the right-hand side of (1.10) into nearly resonant and non-resonant parts and apply differentiation by parts only to the non-resonant part. When we apply differentiation by parts as in (1.10) in an iterative

⁴This is indeed a Poincaré-Dulac normal form reduction applied to the evolution equation (1.9) for $\frac{1}{2} \|v(t)\|_{H^s}^2$. See Section 1 in [15] for a discussion on the relation between differentiation by parts and normal form reductions.

manner, the time derivative may fall on any of the factors \widehat{v}_{n_j} and \widehat{v}_n , generating higher order nonlinear terms. In general, the structure of such terms can be very complicated, depending on where the time derivative falls. In [15], ordered (ternary) trees played an important role for indexing such terms. In our case, we work on the evolution equation satisfied by the H^s -energy functional and we need to consider tree-like structures that grow in two directions. In Section 4, we introduce the notion of bi-trees and ordered bi-trees for this purpose.

After J steps of the normal form reductions, we arrive at

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|v(t)\|_{H^s}^2 \right) &= \frac{d}{dt} \left(\sum_{j=2}^{J+1} \mathcal{N}_0^{(j)}(v)(t) \right) + \sum_{j=2}^{J+1} \mathcal{N}_1^{(j)}(v)(t) \\ &\quad + \sum_{j=2}^{J+1} \mathcal{R}^{(j)}(v)(t) + \mathcal{N}_2^{(J+1)}(v)(t). \end{aligned} \quad (1.12)$$

Here, $\mathcal{N}_0^{(j)}(v)$ consists of certain $2j$ -linear terms, while $\mathcal{N}_1^{(j)}(v)$ and $\mathcal{R}^{(j)}(v)$ consist of $(2j+2)$ -linear terms. In practice, we obtain (1.12) for smooth functions with a truncation parameter $N \in \mathbb{N}$. Here, we can only show that the remainder term $\mathcal{N}_2^{(J+1)}$ satisfies the bound of the form:

$$|\mathcal{N}_2^{(J+1)}| \leq F(N, J),$$

with the upper bound $F(N, J)$ satisfying

$$\lim_{N \rightarrow \infty} F(N, J) = \infty$$

for each fixed $J \in \mathbb{N}$. This, however, does not cause an issue since we also show that

$$\lim_{J \rightarrow \infty} F(N, J) = 0$$

for each *fixed* $N \in \mathbb{N}$. Therefore, by first taking the limit $J \rightarrow \infty$ and then $N \rightarrow \infty$, we conclude that the error term $\mathcal{N}_2^{(J+1)}$ vanishes in the limit. See Subsection 4.4. While it is simple, this observation is crucial in an infinite iteration of the normal form reductions.

At the end of an infinite iteration of the normal form reductions, we can rewrite (1.9) as

$$\frac{d}{dt} \left(\frac{1}{2} \|v(t)\|_{H^s}^2 \right) = \frac{d}{dt} \left(\sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(v)(t) \right) + \sum_{j=2}^{\infty} \mathcal{N}_1^{(j)}(v)(t) + \sum_{j=2}^{\infty} \mathcal{R}^{(j)}(v)(t), \quad (1.13)$$

involving infinite series. The main point of this normal form approach is that, while the degrees of the nonlinear terms appearing in (1.13) can be arbitrarily large, we can show that they are all bounded in $L^2(\mathbb{T})$ (in a summable manner over j). In particular, by defining the modified energy $\mathcal{E}_t(v)$ by

$$\mathcal{E}_t(v) := \frac{1}{2} \|v(t)\|_{H^s}^2 - \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(v)(t), \quad (1.14)$$

we see that its time derivative is bounded:

$$\left| \frac{d}{dt} \mathcal{E}_t(v) \right| \leq C_s(\|u\|_{L^2}),$$

satisfying the energy estimate (1.5) in Step (i) with $\theta = 2$. See Proposition 3.4 below. This is the main new ingredient for proving Theorem 1.2. See also the recent work [29]

by the first author (with Y. Wang) on an infinite iteration of normal form reductions for establishing a crucial energy estimate on the difference of two solutions in proving enhanced uniqueness for the renormalized cubic 4NLS (see (1.16) below) in negative Sobolev spaces.

Remark 1.3. (i) Heuristically speaking, this infinite iteration of normal form reductions allows us to exchange analytical difficulty with algebraic/combinatorial difficulty.

(ii) The “correction term” $R_t^{(1)}$ in (1.11) is nothing but the correction term in the spirit of the I -method [9, 10]. In fact, at each step of normal form reductions, we obtain a correction term $\mathcal{N}_0^{(j)}(v)$. Hence, our improved energy estimate (Proposition 3.4) via an infinite iteration of normal form reductions can be basically viewed as an implementation of the I -method with an infinite sequence $\{\mathcal{N}_0^{(j)}(v)\}_{j=2}^\infty$ of correction terms. Namely, the modified energy $\mathcal{E}_t(v)$ defined in (1.14) is an modified energy of an infinite order in the I -method terminology.⁵

(iii) We point out that a finite iteration of normal form reductions is not sufficient to go below $s > \frac{3}{4}$. See (6.14) in [26], showing the restriction $s - \frac{1}{2} > \frac{1}{4}$.

Remark 1.4. Let us briefly discuss the situation for the more general cubic fourth order NLS (1.2). For this equation, the following phase function

$$\phi_{\lambda,\mu}(\bar{n}) = -\lambda(n_1^2 - n_2^2 + n_3^2 - n^2) + \mu(n_1^4 - n_2^4 + n_3^4 - n^4)$$

plays an important role in the analysis. In view of Lemma 3.1 below, we have

$$\phi_{\lambda,\mu}(\bar{n}) = (n_1 - n_2)(n_1 - n) \left\{ -2\lambda + \mu(n_1^2 + n_2^2 + n_3^2 + n^2 + 2(n_1 + n_3)^2) \right\}. \quad (1.15)$$

If the last factor in (1.15) does not vanish for any $n_1, n_2, n_3, n \in \mathbb{Z}$, then we can establish quasi-invariance of μ_s under (1.2) for $s > \frac{1}{2}$ with the same proof as in [26] and this paper. It suffices to note that, while we make use of the divisor counting argument in the proof, we only apply it to $\mu(\bar{n}) = (n_1 - n_2)(n_1 - n)$ and thus the integer/non-integer character of the last factor in (1.15) is irrelevant.

For example, when $\lambda\mu < 0$, the last factor in (1.15) does not vanish and thus Theorem 1.2 applies to this case. When $\lambda\mu > 0$, the non-resonant condition $2\lambda \notin \mu\mathbb{N}$ also guarantees the non-vanishing of the last factor in (1.15). It seems of interest to investigate the transport property of the Gaussian measure μ_s in the resonant case $2\lambda \in \mu\mathbb{N}$. In this case, there are more resonant terms and thus further analysis is required.

Remark 1.5. On the one hand, the cubic 4NLS (1.1) is ill-posed in negative Sobolev spaces and hence the quasi-invariance result stated in Theorem 1.2 is sharp. On the other hand, the first author and Y. Wang [29] considered the following renormalized cubic 4NLS on \mathbb{T} :

$$i\partial_t u = \partial_x^4 u + (|u|^2 - 2 \int_{\mathbb{T}} |u|^2 dx) u, \quad (1.16)$$

where $\int_{\mathbb{T}} f(x) dx := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) dx$. In particular, they proved global well-posedness of (1.16) in $H^s(\mathbb{T})$ for $s > -\frac{1}{3}$. In a very recent work [28], the first and third authors with Y. Wang went further and constructed global-in-time dynamics for (1.16) almost surely with respect to the white noise, i.e. the Gaussian measure μ_s with $s = 0$ supported on $H^\sigma(\mathbb{T})$, $\sigma < -\frac{1}{2}$. As a result, they proved invariance of the white noise μ_0 under the renormalized cubic

⁵The highest order of modified energies used in the literature is three in the application of the I -method to the KdV equation [10], corresponding to two iterations of normal form reductions.

4NLS (1.16). Invariance is of course a stronger property than quasi-invariance and hence the white noise is in particular quasi-invariant under (1.16). The question of quasi-invariance of μ_s for $s \in (0, \frac{1}{2}]$ under the dynamics of the renormalized cubic 4NLS (1.16) is therefore a natural sequel of the analysis of this paper.

1.3. Non quasi-invariance under the dispersionless model. To motivate our second result, note that, by invariance of the complex-valued Gaussian random variable g_n in (1.4) under rotations, it is clear that the Gaussian measure μ_s is invariant under the linear dynamics:

$$\begin{cases} i\partial_t u = \partial_x^4 u \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \quad (1.17)$$

See Lemma 3.2 (i) below. In particular, μ_s is quasi-invariant under the linear dynamics (1.17).

In the proof of the quasi-invariance of μ_s under the cubic 4NLS (1.1) (Theorem 1.2 above), the dispersion plays an essential role. The strong dispersion allows us to show that the nonlinear part in (1.1) is a perturbation to the linear equation (1.17). Our next result shows that the dispersion is indeed *essential* for Theorem 1.2 to hold.

Consider the following dispersionless model:

$$\begin{cases} i\partial_t u = |u|^2 u \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \quad (1.18)$$

Recall that there is an explicit solution formula for (1.18) given by:

$$u(x, t) = e^{-it|u_0(x)|^2} u_0(x) \quad (1.19)$$

at least for continuous initial data such that the pointwise product makes sense.

Let $s > \frac{1}{2}$. Then, it is easy to see that the random function u^ω in (1.4) is continuous almost surely. Indeed, by the equivalence of Gaussian moments and the mean value theorem, we have

$$\begin{aligned} \mathbb{E}[|u^\omega(x) - u^\omega(y)|^p] &\leq C_p \left(\mathbb{E}[|u^\omega(x) - u^\omega(y)|^2] \right)^{\frac{p}{2}} \sim \left(\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^{2s-2\varepsilon}} \right)^{\frac{p}{2}} |x - y|^{\varepsilon p} \\ &\lesssim |x - y|^{\varepsilon p}, \end{aligned}$$

provided that $\varepsilon > 0$ is sufficiently small such that $2s - 2\varepsilon > 1$. Now, by choosing $p \gg 1$ such that $\varepsilon p > 1$, we can apply Kolmogorov's continuity criterion and conclude that u^ω in (1.4) is almost surely continuous when $s > \frac{1}{2}$. This in particular implies that the solution formula (1.19) is well defined for initial data distributed according to μ_s , $s > \frac{1}{2}$, and the corresponding solutions exist globally in time. We denote by $\tilde{\Phi}(t)$ the solution map for the dispersionless model (1.18).

We now state our second result.

Theorem 1.6. *Let $s > \frac{1}{2}$. Then, given $t \neq 0$, the pushforward measure $\tilde{\Phi}(t)_* \mu_s$ under the dynamics of the dispersionless model (1.18) is not absolutely continuous with respect to the Gaussian measure μ_s . Namely, the Gaussian measure μ_s is not quasi-invariant under the dispersionless dynamics (1.18).*

This is a sharp contrast with the quasi-invariance result for the cubic 4NLS in Theorem 1.2 and for the cubic NLS for $s \in \mathbb{N}$ (see Remark 1.4 in [26]). In particular, Theorem 1.6 shows that dispersion is essential for establishing quasi-invariance of μ_s .

We prove this negative result in Theorem 1.6 by establishing that typical elements under μ_s , $s > \frac{1}{2}$, possess an almost surely constant modulus of continuity at each point. This is the analogue of the classical law of the iterated logarithm for the Brownian motion. We show that this modulus of continuity is destroyed with a positive probability by the nonlinear transformation (1.19) for any non-zero time $t \in \mathbb{R} \setminus \{0\}$.

Our proof is based on three basic tools: the Fourier series representation of the (fractional) Brownian loops, the law of the iterated logarithm, and the solution formula (1.19) to the dispersionless model (1.18). We will use three different versions of the law of the iterated logarithm, depending on (i) $s = 1$ corresponding to the Brownian/Ornstein-Uhlenbeck loop, (ii) $\frac{1}{2} < s < \frac{3}{2}$, corresponding to the fractional Brownian loop, and (iii) the critical case $s = \frac{3}{2}$. In Cases (i) and (ii), we make use of the mutual absolute continuity of the function u given in the random Fourier series (1.4) and the (fractional) Brownian motion on $[0, 2\pi)$ to deduce the law of the iterated logarithm for the random function u . In Case (iii), we directly establish the relevant law of the iterated logarithm for u in (1.4).⁶ See Proposition 5.7.

On the one hand, the law of the iterated logarithm yields almost sure constancy of the modulus of continuity at time $t = 0$. On the other hand, we combine this almost sure constancy of the modulus of continuity at time $t = 0$ and the solution formula (1.19) to show that the modulus of continuity at non-zero time $t \neq 0$ does not satisfy the conclusion of the law of the iterated logarithm with a positive probability. Lastly, for $s > \frac{3}{2}$, we reduce the proof to one of Cases (i), (ii), or (iii) by differentiating the random function.

Remark 1.7. The existence of a quasi-invariant measure shows a delicate persistence property of the dynamics. In particular, this persistence property due to the quasi-invariance is stronger than the persistence of regularity⁷ obtained by the usual well-posedness theory. While the dispersionless model (1.18) enjoys the persistence of regularity in $H^\sigma(\mathbb{T})$, $\sigma > \frac{1}{2}$, Theorem 1.6 shows that the Gaussian measure μ_s is not quasi-invariant under the dynamics of (1.18).

1.4. Organization of the paper. In Section 2, we introduce some notations. In Section 3, we prove Theorem 1.2, assuming the improved energy estimate (Proposition 3.4). We then present the proof of Proposition 3.4 in Section 4 by implementing an infinite iteration of normal form reductions. Lastly, by studying the random Fourier series (1.4) and the relevant law of the iterated logarithm, we prove non quasi-invariance of the Gaussian measure μ_s under the dispersionless model (Theorem 1.6) in Section 5.

⁶We could apply this argument to directly establish the relevant law of the iterated logarithm in Cases (i) and (ii) as well.

⁷In the scaling sub-critical case, by persistence of regularity, we mean the following; if one proves local well-posedness in H^{s_0} for some $s_0 \in \mathbb{R}$ and if u_0 lies in a smoother space H^s for some $s > s_0$, then the corresponding solution remains smoother and lies in $C([-T, T]; H^s)$, where the local existence time $T > 0$ depends only on the H^{s_0} -norm of the initial condition u_0 .

2. NOTATIONS

Given $N \in \mathbb{N}$, we use $\mathbf{P}_{\leq N}$ to denote the Dirichlet projection onto the frequencies $\{|n| \leq N\}$ and set $\mathbf{P}_{>N} := \text{Id} - \mathbf{P}_{\leq N}$. When $N = \infty$, it is understood that $\mathbf{P}_{\leq N} = \text{Id}$. Define E_N and E_N^\perp by

$$\begin{aligned} E_N &= \mathbf{P}_{\leq N} L^2(\mathbb{T}) = \text{span}\{e^{inx} : |n| \leq N\}, \\ E_N^\perp &= \mathbf{P}_{>N} L^2(\mathbb{T}) = \text{span}\{e^{inx} : |n| > N\}. \end{aligned}$$

Given $s > \frac{1}{2}$, let μ_s be the Gaussian measure on $L^2(\mathbb{T})$ defined in (1.3). Then, we can write μ_s as

$$\mu_s = \mu_{s,N} \otimes \mu_{s,N}^\perp,$$

where $\mu_{s,N}$ and $\mu_{s,N}^\perp$ are the marginal distributions of μ_s restricted onto E_N and E_N^\perp , respectively. In other words, $\mu_{s,N}$ and $\mu_{s,N}^\perp$ are the induced probability measures under the following maps:

$$\begin{aligned} u_N : \omega \in \Omega &\longmapsto u_N(x; \omega) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx}, \\ u_N^\perp : \omega \in \Omega &\longmapsto u_N^\perp(x; \omega) = \sum_{|n| > N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx}, \end{aligned}$$

respectively. Formally, we can write $\mu_{s,N}$ and $\mu_{s,N}^\perp$ as

$$d\mu_{s,N} = Z_{s,N}^{-1} e^{-\frac{1}{2} \|\mathbf{P}_{\leq N} u_N\|_{H^s}^2} du_N \quad \text{and} \quad d\mu_{s,N}^\perp = \widehat{Z}_{s,N}^{-1} e^{-\frac{1}{2} \|\mathbf{P}_{>N} u_N^\perp\|_{H^s}^2} du_N^\perp. \quad (2.1)$$

Given $r > 0$, we also define a probability measure $\mu_{s,r}$ with an L^2 -cutoff by

$$d\mu_{s,r} = Z_{s,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} d\mu_s.$$

Given a function $v \in L^2(\mathbb{T})$, we simply use v_n to denote the Fourier coefficient \widehat{v}_n of v , when there is no confusion. This shorthand notation is especially useful in Section 4.

We use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon \ll 1$, where an implicit constant is allowed to depend on $\varepsilon > 0$ (and it usually diverges as $\varepsilon \rightarrow 0$). Given $x \in \mathbb{R}$, we use $\lfloor x \rfloor$ to denote the integer part of x .

In view of the time reversibility of the equations (1.1) and (1.18), we only consider positive times in the following.

 3. PROOF OF THEOREM 1.2: QUASI-INVARIANCE OF μ_s UNDER THE CUBIC 4NLS

In this section, we present the proof of Theorem 1.2. The main new ingredient is the improved energy estimate (Proposition 3.4) whose proof is postponed to Section 4. The remaining part of the proof follows closely the presentation in [26] and thus we keep our discussion concise.

3.1. Basic reduction of the problem. We first go over the basic reduction of the problem from [26]. Given $t \in \mathbb{R}$, we define a gauge transformation \mathcal{G}_t on $L^2(\mathbb{T})$ by setting

$$\mathcal{G}_t[f] := e^{2itf} |f|^2 f.$$

Given a function $u \in C(\mathbb{R}; L^2(\mathbb{T}))$, we define \mathcal{G} by setting

$$\mathcal{G}[u](t) := \mathcal{G}_t[u(t)].$$

Note that \mathcal{G} is invertible and its inverse is given by $\mathcal{G}^{-1}[u](t) = \mathcal{G}_{-t}[u(t)]$.

Given a solution $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ to (1.1), let $\tilde{u} = \mathcal{G}[u]$. Then, it follows from the mass conservation that \tilde{u} is a solution to the following renormalized fourth order NLS:

$$i\partial_t \tilde{u} = \partial_x^4 \tilde{u} + (|\tilde{u}|^2 - 2 \int_{\mathbb{T}} |\tilde{u}|^2 dx) \tilde{u}. \quad (3.1)$$

This is precisely the renormalized cubic 4NLS in (1.16).

Let $S(t) = e^{-it\partial_x^4}$ be the solution operator for the linear fourth order Schrödinger equation (1.17). Denoting by $v = S(-t)\tilde{u}$ the interaction representation of \tilde{u} , we see that v satisfies the following equation for $\{v_n\}_{n \in \mathbb{Z}}$:

$$\begin{aligned} \partial_t v_n &= -i \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n \\ &=: \mathcal{N}(v)_n + \mathcal{R}(v)_n, \end{aligned} \quad (3.2)$$

where the phase function $\phi(\bar{n})$ is as in (1.8) and the plane $\Gamma(n)$ is given by

$$\Gamma(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\}.$$

Recall that the phase function $\phi(\bar{n})$ admits the following factorization. See [26] for the proof.

Lemma 3.1. *Let $n = n_1 - n_2 + n_3$. Then, we have*

$$\phi(\bar{n}) = (n_1 - n_2)(n_1 - n)(n_1^2 + n_2^2 + n_3^2 + n^2 + 2(n_1 + n_3)^2).$$

It follows from Lemma 3.1 that $\phi(\bar{n}) \neq 0$ on $\Gamma(\bar{n})$. Namely, $\mathcal{N}(v)$ and $\mathcal{R}(v)$ on the right-hand side of (3.2) correspond to the non-resonant and resonant parts, respectively. It follows from Lemma 3.1 that there is a strong smoothing property on the non-resonant term $\mathcal{N}(v)$ due to the fast oscillation caused by $\phi(\bar{n})$.

Given $t, \tau \in \mathbb{R}$, let $\Phi(t) : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ be the solution map for (1.1) and $\Psi(t, \tau) : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ be the solution map for (3.2),⁸ sending initial data at time τ to solutions at time t . When $\tau = 0$, we may denote $\Psi(t, 0)$ by $\Psi(t)$ for simplicity. Then, from $v = S(-t) \circ \mathcal{G}[u]$, we have

$$\Phi(t) = \mathcal{G}^{-1} \circ S(t) \circ \Psi(t). \quad (3.3)$$

Recall the following lemma from [26].

Lemma 3.2. (i) *Let $s > \frac{1}{2}$ and $t \in \mathbb{R}$. Then, the Gaussian measure μ_s defined in (1.3) is invariant under the linear map $S(t)$ and the gauge transformation \mathcal{G}_t .*

(ii) *Let (X, μ) be a measure space. Suppose that T_1 and T_2 are measurable maps on X into itself such that μ is quasi-invariant under T_j for each $j = 1, 2$. Then, μ is quasi-invariant under $T = T_1 \circ T_2$.*

See Section 4 in [26] for the proof of Lemma 3.2. In view of this lemma, Theorem 1.2 follows once we prove quasi-invariance of μ_s under $\Psi(t)$. Therefore, we focus our attention to (3.2) in the following.

⁸Note that (3.2) is non-autonomous. We point out that this non-autonomy does not play an essential role in the remaining part of the paper, since all the estimates hold uniformly in $t \in \mathbb{R}$.

3.2. Truncated dynamics. Let us first introduce the following truncated approximation to (3.2):

$$\partial_t v_n = \mathbf{1}_{|n| \leq N} \left\{ -i \sum_{\Gamma_N(n)} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n \right\}, \quad (3.4)$$

where $\Gamma_N(n)$ is defined by

$$\begin{aligned} \Gamma_N(n) &= \Gamma(n) \cap \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \leq N\} \\ &= \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3, n_1, n_3 \neq n, \text{ and } |n_j| \leq N\}. \end{aligned}$$

Note that (3.4) is an infinite dimensional system of ODEs for the Fourier coefficients $\{v_n\}_{n \in \mathbb{Z}}$, where the flow is constant on the high frequencies $\{|n| > N\}$. We also consider the following finite dimensional system of ODEs:

$$\partial_t v_n = -i \sum_{\Gamma_N(n)} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n, \quad |n| \leq N. \quad (3.5)$$

with $v|_{t=0} = \mathbf{P}_{\leq N} v|_{t=0}$, i.e. $v_n|_{t=0} = 0$ for $|n| > N$.

Given $t, \tau \in \mathbb{R}$, denote by $\Psi_N(t, \tau)$ and $\tilde{\Psi}_N(t, \tau)$ the solution maps of (3.4) and (3.5), sending initial data at time τ to solutions at time t , respectively. For simplicity, we set

$$\Psi_N(t) = \Psi_N(t, 0) \quad \text{and} \quad \tilde{\Psi}_N(t) = \tilde{\Psi}_N(t, 0) \quad (3.6)$$

when $\tau = 0$. Then, we have the following relations:

$$\Psi_N(t, \tau) = \tilde{\Psi}_N(t, \tau) \mathbf{P}_{\leq N} + \mathbf{P}_{> N} \quad \text{and} \quad \mathbf{P}_{\leq N} \Psi_N(t, \tau) = \tilde{\Psi}_N(t, \tau) \mathbf{P}_{\leq N}.$$

We now recall the following approximation property of the truncated dynamics (3.4).

Lemma 3.3 (Proposition 6.21/B.3 in [26]). *Given $R > 0$, let $A \subset B_R$ be a compact set in $L^2(\mathbb{T})$. Fix $t \in \mathbb{R}$. Then, for any $\varepsilon > 0$, there exists $N_0 = N_0(t, R, \varepsilon) \in \mathbb{N}$ such that we have*

$$\Psi(t)(A) \subset \Psi_N(t)(A + B_\varepsilon)$$

for all $N \geq N_0$. Here, B_r denotes the ball in $L^2(\mathbb{T})$ of radius r centered at the origin.

3.3. Energy estimate. In this subsection, we state a crucial energy estimate. The main goal is to establish an energy estimate of the form (1.5) by introducing a suitable modified energy functional. We achieve this goal by performing normal form reductions infinitely many times and thus constructing an infinite sequence of correction terms.

Let $N \in \mathbb{N} \cup \{\infty\}$. In the following, we simply say that v is a solution to (3.5) if v is a solution to (3.5) when $N \in \mathbb{N}$ and to (3.2) when $N = \infty$.

Proposition 3.4. *Let $\frac{1}{2} < s < 1$. Then, given $N \in \mathbb{N} \cup \{\infty\}$, there exist multilinear forms $\{\mathcal{N}_{0,N}^{(j)}\}_{j=2}^\infty$, $\{\mathcal{N}_{1,N}^{(j)}\}_{j=2}^\infty$, and $\{\mathcal{R}_N^{(j)}\}_{j=2}^\infty$ such that*

$$\frac{d}{dt} \left(\frac{1}{2} \|v(t)\|_{H^s}^2 \right) = \frac{d}{dt} \left(\sum_{j=2}^\infty \mathcal{N}_{0,N}^{(j)}(v)(t) \right) + \sum_{j=2}^\infty \mathcal{N}_{1,N}^{(j)}(v)(t) + \sum_{j=2}^\infty \mathcal{R}_N^{(j)}(v)(t),$$

for any solution $v \in C(\mathbb{R}; H^s(\mathbb{T}))$ to (3.5). Here, $\mathcal{N}_{0,N}^{(j)}$ are $2j$ -linear forms, while $\mathcal{N}_{1,N}^{(j)}$ and $\mathcal{R}_N^{(j)}$ are $(2j+2)$ -linear forms, satisfying the following bounds on $L^2(\mathbb{T})$; there exist positive

constants $C_0(j)$, $C_1(j)$, and $C_2(j)$ decaying faster than any exponential rate⁹ as $j \rightarrow \infty$ such that

$$\begin{aligned} |\mathcal{N}_{0,N}^{(j)}(v)(t)| &\lesssim C_0(j) \|v\|_{L^2}^{2j}, \\ |\mathcal{N}_{1,N}^{(j)}(v)(t)| &\lesssim C_1(j) \|v\|_{L^2}^{2j+2}, \\ |\mathcal{R}_N^{(j)}(v)(t)| &\lesssim C_2(j) \|v\|_{L^2}^{2j+2}, \end{aligned}$$

for $j = 2, 3, \dots$. Note that these constants $C_0(j)$, $C_1(j)$, and $C_2(j)$ are independent of the cutoff size $N \in \mathbb{N} \cup \{\infty\}$ and $t \in \mathbb{R}$.

Define the modified energy $\mathcal{E}_{N,t}(v)$ by

$$\mathcal{E}_{N,t}(v) := \frac{1}{2} \|v(t)\|_{H^s}^2 - \sum_{j=2}^{\infty} \mathcal{N}_{0,N}^{(j)}(v)(t). \quad (3.7)$$

Then, the following energy estimate holds:

$$\left| \frac{d}{dt} \mathcal{E}_{N,t}(v) \right| \leq C_s (\|v\|_{L^2}) \quad (3.8)$$

for any solution $v \in C(\mathbb{R}; H^s(\mathbb{T}))$ to (3.5), uniformly in $N \in \mathbb{N} \cup \{\infty\}$ and $t \in \mathbb{R}$.

In the remaining part of this section, we continue with the proof of Theorem 1.2, assuming Proposition 3.4. We present the proof of Proposition 3.4 in Section 4. See Lemmas 4.9 and 4.10. In the following, we simply denote $\mathcal{E}_{\infty,t}$ by \mathcal{E}_t and drop the subscript $N = \infty$ from the multilinear forms, when $N = \infty$. For example, we write $\mathcal{N}_0^{(j)}$ for $\mathcal{N}_{0,\infty}^{(j)}$.

3.4. Weighted Gaussian measures. Let $s > \frac{1}{2}$. As in [26], we would like to define the weighted Gaussian measures associated with the modified energies $\mathcal{E}_{N,t}(v)$ and $\mathcal{E}_t(v)$:¹⁰

$$\begin{aligned} d\rho_{s,N,r,t} &= "Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\mathcal{E}_{N,t}(\mathbf{P}_{\leq N} v)} dv" \\ &= Z_{s,N,r}^{-1} F_{N,r,t} d\mu_s \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} d\rho_{s,r,t} &= "Z_{s,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\mathcal{E}_t(v)} dv" \\ &= Z_{s,r}^{-1} F_{r,t} d\mu_s, \end{aligned} \quad (3.10)$$

where $\mathcal{E}_{N,t}$ is the modified energy defined in (3.7) and $F_{N,r,t}$ and $F_{r,t}$ are given by

$$\begin{aligned} F_{N,r,t}(v) &:= \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} \exp \left(\sum_{j=2}^{\infty} \mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\leq N} v)(t) \right), \\ F_{r,t}(v) &= F_{r,\infty,t}(v) := \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} \exp \left(\sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(v)(t) \right). \end{aligned} \quad (3.11)$$

⁹In fact, by slightly modifying the proof, we can make $C_0(j)$, $C_1(j)$, and $C_2(j)$ decay as fast as we want as $j \rightarrow \infty$.

¹⁰Noting that we have $\mathbf{P}_{\leq N} v = v$ for all solutions to (3.5), we have $\mathcal{E}_{N,t}(\mathbf{P}_{\leq N} v) = \mathcal{E}_{N,t}(v)$. In the following, we explicitly insert $\mathbf{P}_{\leq N}$ for clarity. A similar comment applies to $\mathcal{N}_{0,N}^{(j)}$.

It follows from Proposition 3.4 that

$$\mathbf{1}_{\{\|v\|_{L^2} \leq r\}} \exp \left(\sum_{j=2}^{\infty} |\mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\leq N} v)| \right) \leq \exp \left(\sum_{j=2}^{\infty} C_0(j) r^{2j} \right) \leq C(s, r),$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$ and $t \in \mathbb{R}$. Hence, we have

$$Z_{s,N,r} = \int_{H^{s-\frac{1}{2}-}} F_{N,r,t}(v) d\mu_s \leq C(s, r),$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$ and $t \in \mathbb{R}$. See also Remark 3.6 below. This shows that $\rho_{s,N,r,t}$ and $\rho_{s,r,t}$ in (3.9) and (3.10) are well defined probability measures on $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, $\varepsilon > 0$. Moreover, the following lemma immediately follows from the computation above as in [26, Proposition 6.2 and Corollary 6.3]. See Subsection 4.6.

Lemma 3.5. *Let $s > \frac{1}{2}$ and $r > 0$.*

(i) *Given any finite $p \geq 1$, $F_{N,r,t}(v)$ converges to $F_{r,t}(v)$ in $L^p(\mu_s)$, uniformly in $t \in \mathbb{R}$, as $N \rightarrow \infty$.*

(ii) *For any $\gamma > 0$, there exists $N_0 \in \mathbb{N}$ such that*

$$|\rho_{s,N,r,t}(A) - \rho_{s,r,t}(A)| < \gamma$$

for any $N \geq N_0$ and any measurable set $A \subset L^2(\mathbb{T})$, uniformly in $t \in \mathbb{R}$.

Remark 3.6. The normalizing constants $Z_{s,N,r}$ and $Z_{s,r}$ a priori depend on $t \in \mathbb{R}$. It is, however, easy to see that they are indeed independent of $t \in \mathbb{R}$ by (i) noticing that the correction terms $\{\mathcal{N}_{0,N}^{(j)}\}_{j=2}^{\infty}$ defined in Proposition 3.4 is in fact autonomous in terms of $\tilde{u}(t) = S(t)v(t)$ and (ii) the invariance of μ_s under $S(t)$ (Lemma 3.2). See also Remark 4.8 below. The same comment applies to the normalizing constant $\widehat{Z}_{s,N,r}$ defined in (3.12).

3.5. A change-of-variable formula. Next, we go over an important global aspect of the proof of Theorem 1.2. Given $N \in \mathbb{N}$, let $dL_N = \prod_{|n| \leq N} d\widehat{u}_n$ denote the Lebesgue measure on $E_N \cong \mathbb{C}^{2N+1}$. Then, from (3.9) and (3.11) with (2.1), we have

$$\begin{aligned} d\rho_{s,N,r,t} &= Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{\sum_{j=2}^{\infty} \mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\leq N} v)} d\mu_s \\ &= \widehat{Z}_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\mathcal{E}_{N,t}(\mathbf{P}_{\leq N} v)} dL_N \otimes d\mu_{s,N}^{\perp}, \end{aligned}$$

where $\widehat{Z}_{s,N,r}$ is a normalizing constant defined by

$$\widehat{Z}_{s,N,r} = \int_{L^2} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\mathcal{E}_{N,t}(\mathbf{P}_{\leq N} v)} dL_N \otimes d\mu_{s,N}^{\perp}. \quad (3.12)$$

Then, proceeding as in [26] and exploiting invariance of L_N under the map $\widetilde{\Psi}_N(t, \tau)$ for (3.5), we have the following change-of-variable formula.

Lemma 3.7. *Let $s > \frac{1}{2}$, $N \in \mathbb{N}$, and $r > 0$. Then, we have*

$$\begin{aligned} \rho_{s,N,r,t}(\Psi_N(t, \tau)(A)) &= Z_{s,N,r}^{-1} \int_{\Psi_N(t, \tau)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{\sum_{j=2}^{\infty} \mathcal{N}_{0,N}^{(j)}(\mathbf{P}_{\leq N} v)} d\mu_s(v) \\ &= \widehat{Z}_{s,N,r}^{-1} \int_A \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\mathcal{E}_{N,t}(\mathbf{P}_{\leq N} \Psi_N(t, \tau)(v))} dL_N \otimes d\mu_{s,N}^{\perp} \end{aligned}$$

for any $t, \tau \in \mathbb{R}$ and any measurable set $A \subset L^2(\mathbb{T})$. Here, $\Psi_N(t, \tau)$ is the solution map to (3.4) defined in (3.6).

3.6. On the measure evolution property and the proof of Theorem 1.2. In this subsection, we use the energy estimate (Proposition 3.4) and the change-of-variable formula (Lemma 3.7) to establish a growth estimate on the truncated weighted Gaussian measure $\rho_{s,N,r,t}$ under $\Psi_N(t) = \Psi_N(t, 0)$ for (3.4). Thanks to the improved energy estimate, the following estimates are simpler than those presented in [26].

Lemma 3.8. *Let $s > \frac{1}{2}$. Then, given $r > 0$, there exists $C = C(r) > 0$ such that*

$$\frac{d}{dt} \rho_{s,N,r,t}(\Psi_N(t)(A)) \leq C \rho_{s,N,r,t}(\Psi_N(t)(A)) \quad (3.13)$$

for any $N \in \mathbb{N}$, any $t \in \mathbb{R}$, and any measurable set $A \subset L^2(\mathbb{T})$. As a consequence, we have the following estimate; given $t \in \mathbb{R}$ and $r > 0$, there exists $C = C(t, r) > 0$ such that

$$\rho_{s,N,r,t}(\Psi_N(t)(A)) \leq C \rho_{s,N,r,t}(A) \quad (3.14)$$

for any $N \in \mathbb{N}$ and any measurable set $A \subset L^2(\mathbb{T})$.

Proof. As in [35, 36, 34, 26], the main idea of the proof of Lemma 3.8 is to reduce the analysis to that at $t = 0$ in the spirit of the classical Liouville theorem on Hamiltonian dynamics. Let $t_0 \in \mathbb{R}$. By the definition of $\Psi(t, \tau)$, Lemma 3.7, and Proposition 3.4, we have

$$\begin{aligned} & \left. \frac{d}{dt} \rho_{s,N,r,t}(\Psi_N(t)(A)) \right|_{t=t_0} \\ &= \left. \frac{d}{dt} \rho_{s,N,r,t_0+t}(\Psi_N(t_0+t, t_0)(\Psi_N(t_0)(A))) \right|_{t=0} \\ &= \widehat{Z}_{s,N,r}^{-1} \left. \frac{d}{dt} \int_{\Psi_N(t_0)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\mathcal{E}_{N,t_0+t}(\mathbf{P}_{\leq N} \Psi_N(t_0+t, t_0)(v))} dL_N \otimes d\mu_{s,N}^\perp \right|_{t=0} \\ &= - \left. \int_{\Psi_N(t_0)(A)} \frac{d}{dt} \mathcal{E}_{N,t_0+t}(\mathbf{P}_{\leq N} \Psi_N(t_0+t, t_0)(v)) \right|_{t=0} d\rho_{s,N,r,t_0} \\ &\leq C_r \rho_{s,N,r,t_0}(\Psi_N(t_0)(A)). \end{aligned}$$

This proves (3.13). The second estimate (3.14) follows from a direct integration of (3.13). \square

As in [26], we can upgrade Lemma 3.8 to the untruncated measure $\rho_{s,r,t}$.

Lemma 3.9. *Let $s > \frac{1}{2}$. Then, given $t \in \mathbb{R}$ and $r > 0$, there exists $C = C(t, r) > 0$ such that*

$$\rho_{s,r,t}(\Psi(t)(A)) \leq C \rho_{s,r,t}(A)$$

for any measurable set $A \subset L^2(\mathbb{T})$.

This lemma follows from the approximation properties of $\Psi_N(t)$ to $\Psi(t)$ (Lemma 3.3) and $\rho_{s,N,r,t}$ to $\rho_{s,r,t}$ (Lemma 3.5 (ii)) along with some limiting argument. See [26, Lemma 6.10] for the details of the proof.

Once we have Lemma 3.9, the proof of Theorem 1.2 follows just as in [26]. We present its proof for the convenience of readers. Recall that in view of (3.3) and Lemmas 3.2, that it suffices to prove that μ_s is quasi-invariant under $\Psi(t)$, i.e. under the dynamics of (3.2).

Fix $t \in \mathbb{R}$. Let $A \subset L^2(\mathbb{T})$ be a measurable set such that $\mu_s(A) = 0$. Then, for any $r > 0$, we have

$$\mu_{s,r}(A) = 0.$$

By the mutual absolute continuity of $\mu_{s,r}$ and $\rho_{s,r,t}$, we obtain

$$\rho_{s,r,t}(A) = 0$$

for any $r > 0$. Then, by Lemma 3.9, we have

$$\rho_{s,r,t}(\Psi(t)(A)) = 0.$$

By invoking the mutual absolute continuity of $\mu_{s,r}$ and $\rho_{s,r,t}$ once again, we have

$$\mu_{s,r}(\Psi(t)(A)) = 0.$$

Then, the dominated convergence theorem yields

$$\mu_s(\Psi(t)(A)) = \lim_{r \rightarrow \infty} \mu_{s,r}(\Psi(t)(A)) = 0.$$

This proves Theorem 1.2, assuming Proposition 3.4. In the next section, we implement an infinite iteration of normal form reductions and prove the improved energy estimate (Proposition 3.4).

4. PROOF OF PROPOSITION 3.4: NORMAL FORM REDUCTIONS

In this section, we present the proof of Proposition 3.4 by implementing an infinite iteration scheme of normal form reductions. This procedure allows us to construct an infinite sequences of correction terms and thus build the desired modified energies $\mathcal{E}_{N,t}(v)$ and $\mathcal{E}_t(v)$ in (3.7).

4.1. Notations: index by ordered bi-trees. In [15], the first author with Guo and Kwon implemented an infinite iteration of normal form reductions to study the cubic NLS on \mathbb{T} , where differentiation by parts was applied to the evolution equation satisfied by the interaction representation. In [15], (ternary) trees and ordered trees played an important role for indexing various multilinear terms and frequencies arising in the general steps of the Poincaré-Dulac normal form reductions.

In the following, we instead implement an infinite iteration scheme of normal form reduction applied to the H^s -energy functional¹¹ $\|v(t)\|_{H^s}^2$. In particular, we need tree-like structures that grow in two directions. For this purpose, we introduce the notion of bi-trees and ordered bi-trees in the following. Once we replace trees and ordered trees by bi-trees and ordered bi-trees, other related notions can be defined in a similar manner as in [15] with certain differences to be noted.

Definition 4.1. Given a partially ordered set \mathcal{T} with partial order \leq , we say that $b \in \mathcal{T}$ with $b \leq a$ and $b \neq a$ is a child of $a \in \mathcal{T}$, if $b \leq c \leq a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that a is the parent of b .

¹¹More precisely, to the evolution equation satisfied by the H^s -energy functional.

As in [15], our trees in this paper refer to a particular subclass of usual trees with the following properties.

Definition 4.2. (i) A tree \mathcal{T} is a finite partially ordered set satisfying the following properties:

- (a) Let $a_1, a_2, a_3, a_4 \in \mathcal{T}$. If $a_4 \leq a_2 \leq a_1$ and $a_4 \leq a_3 \leq a_1$, then we have $a_2 \leq a_3$ or $a_3 \leq a_2$,
- (b) A node $a \in \mathcal{T}$ is called terminal, if it has no child. A non-terminal node $a \in \mathcal{T}$ is a node with exactly three ordered¹² children denoted by a_1, a_2 , and a_3 ,
- (c) There exists a maximal element $r \in \mathcal{T}$ (called the root node) such that $a \leq r$ for all $a \in \mathcal{T}$,
- (d) \mathcal{T} consists of the disjoint union of \mathcal{T}^0 and \mathcal{T}^∞ , where \mathcal{T}^0 and \mathcal{T}^∞ denote the collections of non-terminal nodes and terminal nodes, respectively.

(ii) A *bi-tree* $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is a disjoint union of two trees \mathcal{T}_1 and \mathcal{T}_2 , where the root nodes r_j of \mathcal{T}_j , $j = 1, 2$, are joined by an edge. A bi-tree \mathcal{T} consists of the disjoint union of \mathcal{T}^0 and \mathcal{T}^∞ , where \mathcal{T}^0 and \mathcal{T}^∞ denote the collections of non-terminal nodes and terminal nodes, respectively. By convention, we assume that the root node r_1 of the tree \mathcal{T}_1 is non-terminal, while the root node r_2 of the tree \mathcal{T}_2 may be terminal.

(iii) Given a bi-tree $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, we define a projection Π_j , $j = 1, 2$, onto a tree by setting

$$\Pi_j(\mathcal{T}) = \mathcal{T}_j. \quad (4.1)$$

In Figure 1, $\Pi_1(\mathcal{T})$ corresponds to the tree on the left under the root node r_1 , while $\Pi_2(\mathcal{T})$ corresponds to the tree on the right under the root node r_2 .

Note that the number $|\mathcal{T}|$ of nodes in a bi-tree \mathcal{T} is $3j + 2$ for some $j \in \mathbb{N}$, where $|\mathcal{T}^0| = j$ and $|\mathcal{T}^\infty| = 2j + 2$. Let us denote the collection of trees in the j th generation (namely, with j parental nodes) by $BT(j)$, i.e.

$$BT(j) := \{\mathcal{T} : \mathcal{T} \text{ is a bi-tree with } |\mathcal{T}| = 3j + 2\}.$$

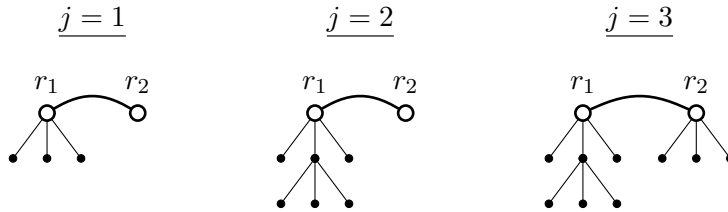


FIGURE 1. Examples of bi-trees of the j th generation, $j = 1, 2, 3$.

Next, we introduce the notion of ordered bi-trees, for which we keep track of how a bi-tree “grew” into a given shape.

¹²For example, we simply label the three children as a_1, a_2 , and a_3 by moving from left to right in the planar graphical representation of the tree \mathcal{T} . As we see below, we assign the Fourier coefficients of the interaction representation v at a_1 and a_3 , while we assign the complex conjugate of the Fourier coefficients of v at the second child a_2 .

Definition 4.3. (i) We say that a sequence $\{\mathcal{T}_j\}_{j=1}^J$ is a chronicle of J generations, if

- (a) $\mathcal{T}_j \in BT(j)$ for each $j = 1, \dots, J$,
- (b) \mathcal{T}_{j+1} is obtained by changing one of the terminal nodes in \mathcal{T}_j into a non-terminal node (with three children), $j = 1, \dots, J-1$.

Given a chronicle $\{\mathcal{T}_j\}_{j=1}^J$ of J generations, we refer to \mathcal{T}_J as an *ordered bi-tree* of the J th generation. We denote the collection of the ordered trees of the J th generation by $\mathfrak{BT}(J)$. Note that the cardinality of $\mathfrak{BT}(J)$ is given by $|\mathfrak{BT}(1)| = 1$ and

$$|\mathfrak{BT}(J)| = 4 \cdot 6 \cdot 8 \cdots 2J = 2^{J-1} \cdot J! =: c_J, \quad J \geq 2. \quad (4.2)$$

(ii) Given an ordered bi-tree $\mathcal{T}_J \in \mathfrak{BT}(J)$ as above, we define projections π_j , $j = 1, \dots, J-1$, onto the previous generations by setting

$$\pi_j(\mathcal{T}_J) = \mathcal{T}_j \in \mathfrak{BT}(j).$$

We stress that the notion of ordered bi-trees comes with associated chronicles. For example, given two ordered bi-trees \mathcal{T}_J and $\tilde{\mathcal{T}}_J$ of the J th generation, it may happen that $\mathcal{T}_J = \tilde{\mathcal{T}}_J$ as bi-trees (namely as planar graphs) according to Definition 4.2, while $\mathcal{T}_J \neq \tilde{\mathcal{T}}_J$ as ordered bi-trees according to Definition 4.3. In the following, when we refer to an ordered bi-tree \mathcal{T}_J of the J th generation, it is understood that there is an underlying chronicle $\{\mathcal{T}_j\}_{j=1}^J$.

Given a bi-tree \mathcal{T} , we associate each terminal node $a \in \mathcal{T}^\infty$ with the Fourier coefficient (or its complex conjugate) of the interaction representation v and sum over all possible frequency assignments. In order to do this, we introduce index functions, assigning integers to *all* the nodes in \mathcal{T} in a consistent manner.

Definition 4.4. (i) Given a bi-tree $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, we define an index function $\mathbf{n} : \mathcal{T} \rightarrow \mathbb{Z}$ such that

- (a) $n_{r_1} = n_{r_2}$, where r_j is the root node of the tree \mathcal{T}_j , $j = 1, 2$,
- (b) $n_a = n_{a_1} - n_{a_2} + n_{a_3}$ for $a \in \mathcal{T}^0$, where a_1, a_2 , and a_3 denote the children of a ,
- (c) $\{n_a, n_{a_2}\} \cap \{n_{a_1}, n_{a_3}\} = \emptyset$ for $a \in \mathcal{T}^0$,

where we identified $\mathbf{n} : \mathcal{T} \rightarrow \mathbb{Z}$ with $\{n_a\}_{a \in \mathcal{T}} \in \mathbb{Z}^{\mathcal{T}}$. We use $\mathfrak{N}(\mathcal{T}) \subset \mathbb{Z}^{\mathcal{T}}$ to denote the collection of such index functions \mathbf{n} on \mathcal{T} .

Given $N \in \mathbb{N}$, we define a subcollection $\mathfrak{N}_N(\mathcal{T}) \subset \mathfrak{N}(\mathcal{T})$ by imposing $|n_a| \leq N$ for any $a \in \mathcal{T}$. We also define $\mathfrak{N}_N^0(\mathcal{T}) \subset \mathfrak{N}(\mathcal{T})$ by imposing $|n_a| \leq N$ for any non-terminal nodes $a \in \mathcal{T}^0$.

(ii) Given a tree \mathcal{T} , we also define an index function $\mathbf{n} : \mathcal{T} \rightarrow \mathbb{Z}$ by omitting the condition (a) and denote by $\mathfrak{N}(\mathcal{T}) \subset \mathbb{Z}^{\mathcal{T}}$ the collection of index functions \mathbf{n} on \mathcal{T} .

Remark 4.5. (i) In view of the consistency condition (a), we can refer to $n_{r_1} = n_{r_2}$ as the frequency at the root node without ambiguity. We shall simply denote it by n_r in the following.

(ii) Just like index functions for (ordered) trees considered in [15], an index function $\mathbf{n} = \{n_a\}_{a \in \mathcal{T}}$ for a bi-tree \mathcal{T} is completely determined once we specify the values $n_a \in \mathbb{Z}$ for all the terminal nodes $a \in \mathcal{T}^\infty$. An index function \mathbf{n} for a bi-tree $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is basically a pair $(\mathbf{n}_1, \mathbf{n}_2)$ of index functions \mathbf{n}_j for the trees \mathcal{T}_j , $j = 1, 2$, (omitting the non-resonance condition in [15, Definition 3.5 (iii)]), satisfying the consistency condition (a): $n_{r_1} = n_{r_2}$.

(iii) Given a bi-tree $\mathcal{T} \in \mathfrak{BT}(J)$ and $n \in \mathbb{Z}$, consider the summation of all possible frequency assignments $\{\mathbf{n} \in \mathfrak{N}(\mathcal{T}) : n_r = n\}$. While $|\mathcal{T}^\infty| = 2J + 2$, there are $2J$ free variables in this summation. Namely, the condition $n_r = n$ reduces two summation variables. It is easy to see this by separately considering the cases $\Pi_2(\mathcal{T}) = \{r_2\}$ and $\Pi_2(\mathcal{T}) \neq \{r_2\}$.

Given an ordered bi-tree \mathcal{T}_J of the J th generation with a chronicle $\{\mathcal{T}_j\}_{j=1}^J$ and associated index functions $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)$, we would like to keep track of the “generations” of frequencies. In the following, we use superscripts to denote such generations of frequencies.

Fix $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)$. Consider \mathcal{T}_1 of the first generation. Its nodes consist of the two root nodes r_1, r_2 , and the children r_{11}, r_{12} , and r_{13} of the first root node r_1 . See Figure 1. We define the first generation of frequencies by

$$(n^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}) := (n_{r_1}, n_{r_{11}}, n_{r_{12}}, n_{r_{13}}).$$

From Definition 4.4, we have

$$n^{(1)} = n_{r_2}, \quad n_1^{(1)} = n_1^{(1)} - n_2^{(1)} + n_3^{(1)}, \quad n_2^{(1)} \neq n_1^{(1)}, n_3^{(1)}.$$

Next, we construct an ordered bi-tree \mathcal{T}_2 of the second generation from \mathcal{T}_1 by changing one of its terminal nodes $a \in \mathcal{T}_1^\infty = \{r_2, r_{11}, r_{12}, r_{13}\}$ into a non-terminal node. Then, we define the second generation of frequencies by setting

$$(n^{(2)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

Note that we have $n^{(2)} = n^{(1)}$ or $n_k^{(1)}$ for some $k \in \{1, 2, 3\}$,

$$n^{(2)} = n_1^{(2)} - n_2^{(2)} + n_3^{(2)}, \quad n_2^{(2)} \neq n_1^{(2)}, n_3^{(2)},$$

where the last identities follow from Definition 4.4. This extension of $\mathcal{T}_1 \in \mathfrak{BT}(1)$ to $\mathcal{T}_2 \in \mathfrak{BT}(2)$ corresponds to introducing a new set of frequencies after the first differentiation by parts, where the time derivative may fall on any of v_n and v_{n_j} , $j = 1, 2, 3$.¹³

In general, we construct an ordered bi-tree \mathcal{T}_j of the j th generation from \mathcal{T}_{j-1} by changing one of its terminal nodes $a \in \mathcal{T}_{j-1}^\infty$ into a non-terminal node. Then, we define the j th generation of frequencies by

$$(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

As before, it follows from Definition 4.4 that

$$n^{(j)} = n_1^{(j)} - n_2^{(j)} + n_3^{(j)}, \quad n_2^{(j)} \neq n_1^{(j)}, n_3^{(j)}.$$

Given an ordered bi-tree \mathcal{T} , we denote by $B_j = B_j(\mathcal{T})$ the set of all possible frequencies in the j th generation.

We denote by ϕ_j the phase function for the frequencies introduced at the j th generation:

$$\phi_j = \phi_j(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_1^{(j)})^4 - (n_2^{(j)})^4 + (n_3^{(j)})^4 - (n^{(j)})^4.$$

¹³The complex conjugate signs on v_n and v_{n_j} do not play any significant role. Hereafter, we drop the complex conjugate sign, when it does not play any important role.

Note that we have $|\phi_1| \geq 1$ in view of Definition 4.4 and Lemma 3.1. We also denote by μ_j the phase function corresponding to the usual cubic NLS (at the j th generation):

$$\begin{aligned} \mu_j &= \mu_j(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_1^{(j)})^2 - (n_2^{(j)})^2 + (n_3^{(j)})^2 - (n^{(j)})^2 \\ &= -2(n^{(j)} - n_1^{(j)})(n^{(j)} - n_3^{(j)}). \end{aligned}$$

Then, by Lemma 3.1, we have

$$|\phi_j| \sim (n_{\max}^{(j)})^2 \cdot |(n^{(j)} - n_1^{(j)})(n^{(j)} - n_3^{(j)})| \sim (n_{\max}^{(j)})^2 \cdot |\mu_j|, \quad (4.3)$$

where $n_{\max}^{(j)} := \max(|n^{(j)}|, |n_1^{(j)}|, |n_2^{(j)}|, |n_3^{(j)}|)$.

Given an ordered bi-tree $\mathcal{T} \in \mathfrak{BT}(J)$ for some $J \in \mathbb{N}$, define $C_j \subset \mathfrak{N}(\mathcal{T})$ by

$$C_j = \{|\tilde{\phi}_{j+1}| \leq (2j+4)^3\}, \quad (4.4)$$

where $\tilde{\phi}_j$ is defined by

$$\tilde{\phi}_j = \sum_{k=1}^j \phi_k. \quad (4.5)$$

In Subsection 4.3, we perform normal form reductions in an iterative manner. At each step, we divide multilinear forms into “nearly resonant” part (corresponding to the frequencies belonging to C_j) and highly non-resonant part (corresponding to the frequencies belonging to C_j^c) and apply a normal form reduction only to the highly non-resonant part.

4.2. Arithmetic lemma. As we see in the next subsection, normal form reductions generate multilinear forms of higher and higher degrees, where we need to sum over all possible ordered bi-trees in $\mathfrak{BT}(J)$. The main issue is then to control the rapidly growing cardinality $c_J = |\mathfrak{BT}(J)|$ defined in (4.2). On the one hand, we utilize the divisor counting estimate (see (4.8) below) as in [15]. On the other hand, we split the argument into two parts. The following lemma shows the heart of the matter in the multilinear estimates presented in the next subsection. This allows us to show that there is a sufficiently fast decay at each step of normal form reductions.

Lemma 4.6. *Let $s < 1$ and $J \in \mathbb{N}$. Then, the following estimates hold:*

$$(i) \quad \sup_{\mathcal{T}_J \in \mathfrak{BT}(J)} \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J) \\ n_r = n}} \mathbf{1}_{\cap_{j=1}^{J-1} C_j^c} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \prod_{j=2}^J \frac{1}{|\tilde{\phi}_j|^2} \lesssim \frac{1}{\prod_{j=2}^J (2j+2)^{3-}}, \quad (4.6)$$

$$(ii) \quad \sup_{\mathcal{T}_{J+1} \in \mathfrak{BT}(J+1)} \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1}) \\ n_r = n}} \mathbf{1}_{(\cap_{j=1}^{J-1} C_j^c) \cap C_J} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \prod_{j=2}^J \frac{1}{|\tilde{\phi}_j|^2} \lesssim \frac{J^{3+}}{\prod_{j=2}^J (2j+2)^{3-}}. \quad (4.7)$$

Before proceeding further, let us recall the following arithmetic fact [17]. Given $n \in \mathbb{N}$, the number $d(n)$ of the divisors of n satisfies

$$d(n) \leq C_\delta n^\delta \quad (4.8)$$

for any $\delta > 0$. This divisor counting estimate will be used iteratively in the following proof.

Proof of Lemma 4.6. (i) We first consider the case $J = 1$. In this case, from Lemma 3.1 with $s < 1$, we have

$$\text{LHS of (4.6)} \leq \sup_{n \in \mathbb{Z}} \sum_{\substack{n_1, n_3 \in \mathbb{Z} \\ n_1, n_3 \neq n \\ |\phi_1| \geq 1}} \frac{\langle n \rangle^{4s}}{|\phi_1|^{2-}} \lesssim \sup_{n \in \mathbb{Z}} \sum_{\substack{n_1, n_3 \in \mathbb{Z} \\ n_1, n_3 \neq n}} \frac{1}{|(n - n_1)(n - n_3)|^{2-}} \lesssim 1. \quad (4.9)$$

Next, we consider the case $J \geq 2$. Fix $\mathcal{T}_J \in \mathfrak{BT}(J)$. For simplicity of notations, we drop the supremum over $\mathcal{T}_J \in \mathfrak{BT}(J)$ in the following with the understanding that the implicit constants are independent of $\mathcal{T}_J \in \mathfrak{BT}(J)$. A similar comment applies to the proof of the estimate (4.7) presented in (ii) below.

The main idea is to apply the divisor counting argument in an iterative manner. It follows from the divisor counting estimate (4.8) with the factorization of ϕ_j (Lemma 3.1) that for fixed $n^{(j)}$ and ϕ_j , there are at most $O(|\phi_j|^{0+})$ many choices for $n_1^{(j)}$, $n_2^{(j)}$, and $n_3^{(j)}$ on B_j . Also, note that ϕ_j is determined by $\tilde{\phi}_1, \dots, \tilde{\phi}_j$ and

$$|\phi_j| \leq \max(|\tilde{\phi}_{j-1}|, |\tilde{\phi}_j|). \quad (4.10)$$

since $\phi_j = \tilde{\phi}_j - \tilde{\phi}_{j-1}$. In the following, we apply the divisor counting argument to sum over the frequencies in B_J, B_{J-1}, \dots, B_2 . From Definition 4.3 (ii) and (4.4), we have

$$\text{LHS of (4.6)} = \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\pi_1(\mathcal{T}_J)) \\ n_r = n \\ |\phi_1| \geq 1}} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \sum_{\substack{\psi_2 \in \mathbb{Z} \\ |\psi_2| > 6^3}} \sum_{B_2} \frac{1}{|\psi_2|^2} \cdots \sum_{\substack{\psi_J \in \mathbb{Z} \\ |\psi_J| > (2J+2)^3}} \sum_{B_J} \frac{1}{|\psi_J|^2}$$

By applying the divisor counting argument in B_J with (4.10), we have

$$\begin{aligned} &\lesssim \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\pi_1(\mathcal{T}_J)) \\ n_r = n \\ |\phi_1| \geq 1}} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \sum_{\substack{\psi_2 \in \mathbb{Z} \\ |\psi_2| > 6^3}} \sum_{B_2} \frac{1}{|\psi_2|^2} \cdots \\ &\quad \sum_{\substack{\psi_{J-1} \in \mathbb{Z} \\ |\psi_{J-1}| > (2J)^3}} \sum_{B_{J-1}} \frac{1}{|\psi_{J-1}|^2} \sum_{\substack{\psi_J \in \mathbb{Z} \\ |\psi_J| > (2J+2)^3}} \frac{1}{|\psi_J|^2} |\psi_{J-1}|^{0+} |\psi_J|^{0+} \end{aligned}$$

By iteratively applying the divisor counting argument in B_{J-1}, \dots, B_2 , we have

$$\begin{aligned} &\lesssim \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\pi_1(\mathcal{T}_J)) \\ n_r = n \\ |\phi_1| \geq 1}} \frac{\langle n \rangle^{4s}}{|\phi_1|^{2-}} \sum_{\substack{\psi_2, \dots, \psi_J \in \mathbb{Z} \\ |\psi_j| > (2j+2)^3 \\ j=2, \dots, J}} \prod_{j=2}^J \frac{1}{|\psi_j|^{2-}} \\ &\lesssim \frac{1}{\prod_{j=2}^J (2j+2)^{3-}}, \end{aligned}$$

where the last inequality follows from (4.9).

(ii) Fix $\mathcal{T}_{J+1} \in \mathfrak{BT}(J+1)$. We proceed with the divisor counting argument as in (i). From (4.4), we have $|\phi_{J+1}| \lesssim |\tilde{\phi}_J| + J^3$ on C_J and thus for fixed $n^{(J+1)}$ and ϕ_{J+1} , there are at most $O(J^{0+} |\tilde{\phi}_J|^{0+})$ many choices for $n_1^{(J+1)}$, $n_2^{(J+1)}$, and $n_3^{(J+1)}$ on B_{J+1} . Also, on C_J , there are at most $O(J^3)$ many choices for $\tilde{\phi}_{J+1}$. Hence, for fixed $\tilde{\phi}_J$, there are also at

most $O(J^3)$ many choices for $\phi_{J+1} = \tilde{\phi}_{J+1} - \tilde{\phi}_J$ on C_J . Then, the contribution to (4.7) in this case is estimate by

$$\begin{aligned} \text{LHS of (4.7)} &= \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\pi_1(\mathcal{T}_{J+1})) \\ n_r = n \\ |\phi_1| \geq 1}} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \sum_{\substack{\psi_2 \in \mathbb{Z} \\ |\psi_2| > 6^3}} \sum_{\substack{B_2 \\ \tilde{\phi}_2 = \psi_2}} \frac{1}{|\psi_2|^2} \cdots \\ &\quad \sum_{\substack{\psi_J \in \mathbb{Z} \\ |\psi_J| > (2J+2)^3}} \sum_{\substack{B_J \\ \tilde{\phi}_J = \psi_J}} \frac{1}{|\psi_J|^2} \sum_{\substack{B_{J+1} \\ |\phi_{J+1} + \psi_J| \leq (2J+4)^3}} 1 \end{aligned}$$

By applying the divisor counting argument in B_{J+1} , we have

$$\begin{aligned} &\lesssim J^{3+} \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\pi_1(\mathcal{T}_{J+1})) \\ n_r = n \\ |\phi_1| \geq 1}} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \sum_{\substack{\psi_2 \in \mathbb{Z} \\ |\psi_2| > 6^3}} \sum_{\substack{B_2 \\ \tilde{\phi}_2 = \psi_2}} \frac{1}{|\psi_2|^2} \cdots \\ &\quad \sum_{\substack{\psi_J \in \mathbb{Z} \\ |\psi_J| > (2J+2)^3}} \sum_{\substack{B_J \\ \tilde{\phi}_J = \psi_J}} \frac{1}{|\psi_J|^2} |\psi_J|^{0+} \end{aligned}$$

By iteratively applying the divisor counting argument in B_J, \dots, B_2 and then applying (4.9), we have

$$\begin{aligned} &\lesssim J^{3+} \sup_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\pi_1(\mathcal{T}_{J+1})) \\ n_r = n \\ |\phi_1| \geq 1}} \frac{\langle n \rangle^{4s}}{|\phi_1|^{2-}} \sum_{\substack{\psi_j \in \mathbb{Z} \\ |\psi_j| > (2j+2)^3 \\ j=2, \dots, J}} \prod_{j=2}^J \frac{1}{|\psi_j|^{2-}} \\ &\lesssim \frac{J^{3+}}{\prod_{j=2}^J (2j+2)^{3-}}. \end{aligned}$$

This proves (4.7). \square

Remark 4.7. In [15], we applied the divisor counting argument even to the frequencies of the first generation. On the other hand, we did not apply the divisor counting argument to the frequencies of the first generation in the proof of Lemma 4.6 above. Instead, we simply used (4.9) to control the first generation. By using only the factor $\mu_1 = -2(n^{(1)} - n_1^{(1)})(n^{(1)} - n_3^{(1)})$ (and not the entire ϕ_1) for the summation, (4.9) allows us to exhibit the required smoothing in Proposition 3.4.

4.3. Normal form reductions. Our main goal is to obtain an effective estimate on the growth of the H^s -norm of a solution v to the truncated equation (3.5), independent of $N \in \mathbb{N}$. We first implement a formal infinite iteration scheme of normal form reductions without justifying switching of limits and summations. For simplicity of presentation, we work on the equation (3.2) in the following, in particular, without the cutoff $\mathbf{1}_{|n| \leq N}$. We point out that our argument does not employ any symmetrization argument (unlike the one in [8]). Therefore, the same normal form reductions and estimates hold for the truncated equation (3.5), uniformly in $N \in \mathbb{N}$, with straightforward modifications: (i) set $\hat{v}_n = 0$ for all $|n| > N$ and (ii) the multilinear forms for (3.5) are obtained by inserting the frequency

cutoff $\mathbf{1}_{|n| \leq N}$ in appropriate places.¹⁴ In the following, we introduce multilinear forms such as $\mathcal{N}_0^{(j)}$, $\mathcal{N}_1^{(j)}$, and $\mathcal{R}^{(j)}$ for the untruncated equation (3.2). With a small modification, these multilinear forms give rise to $\mathcal{N}_{0,N}^{(j)}$, $\mathcal{N}_{1,N}^{(j)}$, and $\mathcal{R}_N^{(j)}$, $N \in \mathbb{N}$, for the truncated equation (3.5), appearing in Proposition 3.4. See Subsection 4.6.

Let $v \in C(\mathbb{R}; H^\infty(\mathbb{T}))$ be a global solution to (3.2).¹⁵ Using the notations introduced in Subsection 4.1, we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|v(t)\|_{H^s}^2 \right) &= -\operatorname{Re} i \sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \langle n \rangle^{2s} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n} \\ &= -\operatorname{Re} i \sum_{\mathcal{T}_1 \in \mathfrak{B}\mathfrak{T}(1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \langle n_r \rangle^{2s} e^{-i\phi_1 t} \prod_{a \in \mathcal{T}_1^\infty} v_{n_a} =: \mathcal{N}^{(1)}(v)(t). \end{aligned}$$

In view of Lemma 3.1 and Definition 4.4, we have $|\phi_1| \geq 1$ in the summation above. Then, by performing a normal form reduction, namely, differentiating by parts as in (1.10), we obtain

$$\begin{aligned} \mathcal{N}^{(1)}(v)(t) &= \operatorname{Re} \partial_t \left[\sum_{\mathcal{T}_1 \in \mathfrak{B}\mathfrak{T}(1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \frac{\langle n_r \rangle^{2s} e^{-i\phi_1 t}}{\phi_1} \prod_{a \in \mathcal{T}_1^\infty} v_{n_a} \right] \\ &\quad - \operatorname{Re} \sum_{\mathcal{T}_1 \in \mathfrak{B}\mathfrak{T}(1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \frac{\langle n_r \rangle^{2s} e^{-i\phi_1 t}}{\phi_1} \partial_t \left(\prod_{a \in \mathcal{T}_1^\infty} v_{n_a} \right) \\ &= \operatorname{Re} \partial_t \left[\sum_{\mathcal{T}_1 \in \mathfrak{B}\mathfrak{T}(1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \frac{\langle n_r \rangle^{2s} e^{-i\phi_1 t}}{\phi_1} \prod_{a \in \mathcal{T}_1^\infty} v_{n_a} \right] \\ &\quad - \operatorname{Re} \sum_{\mathcal{T}_1 \in \mathfrak{B}\mathfrak{T}(1)} \sum_{b \in \mathcal{T}_1^\infty} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)} \frac{\langle n_r \rangle^{2s} e^{-i\phi_1 t}}{\phi_1} \mathcal{R}(v)_{n_b} \prod_{a \in \mathcal{T}_1^\infty \setminus \{b\}} v_{n_a} \\ &\quad + \operatorname{Re} i \sum_{\mathcal{T}_2 \in \mathfrak{B}\mathfrak{T}(2)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2)} \frac{\langle n_r \rangle^{2s} e^{-i(\phi_1 + \phi_2)t}}{\phi_1} \prod_{a \in \mathcal{T}_2^\infty} v_{n_a} \\ &=: \partial_t \mathcal{N}_0^{(2)}(v)(t) + \mathcal{R}^{(2)}(v)(t) + \mathcal{N}^{(2)}(v)(t). \end{aligned} \tag{4.11}$$

In the second equality, we applied the product rule and used the equation (3.2) to replace $\partial_t v_{n_b}$ by the resonant part $\mathcal{R}(v)_{n_b}$ and the non-resonant part $\mathcal{N}(v)_{n_b}$. In substituting the non-resonant part $\mathcal{N}(v)_{n_b}$, we turned the terminal node $b \in \mathcal{T}_1^\infty$ into a non-terminal node with three children b_1, b_2 , and b_3 , which corresponds to extending the tree $\mathcal{T}_1 \in \mathfrak{B}\mathfrak{T}(1)$ (and $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1)$) to $\mathcal{T}_2 \in \mathfrak{B}\mathfrak{T}(2)$ (and to $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2)$, respectively).

Remark 4.8. (i) Strictly speaking, the phase factor appearing in $\mathcal{N}^{(2)}(v)$ may be $\phi_1 - \phi_2$ when the time derivative falls on the terms with the complex conjugate. In the following, however, we simply write it as $\phi_1 + \phi_2$ since it does not make any difference in our analysis. Also, we often replace ± 1 and $\pm i$ by 1 for simplicity when they do not play an important

¹⁴It follows from (3.5) that we simply need to insert the frequency cutoff $\mathbf{1}_{|n^{(j)}| \leq N}$ on the parental frequency $n^{(j)}$ assigned to each non-terminal node $a \in \mathcal{T}^0$.

¹⁵While we work with (3.2) without a frequency cutoff in the following, it follows from the uniform boundedness of the frequency truncation operator $\mathbf{P}_{\leq N}$ that our argument and estimates hold for (3.5), uniformly in $N \in \mathbb{N}$. Noting that any solution to (3.5) (for some $N \in \mathbb{N}$) is smooth, the following computation can be easily justified for solutions to (3.5).

role. Lastly, for notational simplicity, we drop the real part symbol on multilinear forms with the understanding that all the multilinear forms appear with the real part symbol.

(ii) Due to the presence of $e^{-i\phi_1 t}$ in their definitions, the multilinear forms such as $\mathcal{N}_0^{(2)}(v)$ are non-autonomous in t . Therefore, strictly speaking, they should be denoted as $\mathcal{N}_0^{(2)}(t)(v(t))$. In the following, however, we establish nonlinear estimates on these multilinear forms, uniformly in $t \in \mathbb{R}$, by simply using $|e^{-i\phi_1 t}| = 1$. Hence, we simply suppress such t -dependence when there is no confusion. The same comment applies to other multilinear forms. Note that this convention was already used in Proposition 3.4.

It is worthwhile to note that the multilinear forms introduced in this section are non-autonomous when they are expressed in terms of the interaction representation v , solving (3.2). When they are expressed in terms of the original solution u to (1.1) (or \tilde{u} to (3.1)), however, it is easy to see that these multilinear terms are indeed autonomous.

As we see in Lemma 4.9 below, we can estimate the boundary term $\mathcal{N}_0^{(2)}$ and the contribution $\mathcal{R}^{(2)}$ from the resonant part in a straightforward manner thanks to Lemma 4.6. It turns out, however, that we can not handle the last term $\mathcal{N}^{(2)}$ as it is. Let C_1 be as in (4.4). Now, write

$$\mathcal{N}^{(2)} = \mathcal{N}_1^{(2)} + \mathcal{N}_2^{(2)},$$

where $\mathcal{N}_1^{(2)}$ is the restriction of $\mathcal{N}^{(2)}$ onto C_1 and $\mathcal{N}_2^{(2)} := \mathcal{N}^{(2)} - \mathcal{N}_1^{(2)}$. On the one hand, we can estimate the contribution $\mathcal{N}_1^{(2)}$ from C_1 in an effective manner (Lemma 4.10), thanks to the frequency restriction on C_1 . On the other hand, $\mathcal{N}_2^{(2)}$ can not be handled as it is and thus we apply the second step of normal form reductions to $\mathcal{N}_2^{(2)}$. After differentiation by parts, we obtain

$$\begin{aligned} \mathcal{N}_2^{(2)}(v) &= \partial_t \left[\sum_{\mathcal{T}_2 \in \mathfrak{B}\mathfrak{I}(2)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2)} \mathbf{1}_{C_1^c} \frac{\langle n_r \rangle^{2s} e^{-i(\phi_1 + \phi_2)t}}{\phi_1(\phi_1 + \phi_2)} \prod_{a \in \mathcal{T}_2^\infty} v_{n_a} \right] \\ &\quad - \sum_{\mathcal{T}_2 \in \mathfrak{B}\mathfrak{I}(2)} \sum_{b \in \mathcal{T}_2^\infty} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2)} \mathbf{1}_{C_1^c} \frac{\langle n_r \rangle^{2s} e^{-i(\phi_1 + \phi_2)t}}{\phi_1(\phi_1 + \phi_2)} \mathcal{R}(v)_{n_b} \prod_{a \in \mathcal{T}_2^\infty \setminus \{b\}} v_{n_a} \\ &\quad - \sum_{\mathcal{T}_3 \in \mathfrak{B}\mathfrak{I}(3)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_3)} \mathbf{1}_{C_1^c} \frac{\langle n_r \rangle^{2s} e^{-i(\phi_1 + \phi_2 + \phi_3)t}}{\phi_1(\phi_1 + \phi_2)} \prod_{a \in \mathcal{T}_3^\infty} v_{n_a} \\ &=: \partial_t \mathcal{N}_0^{(3)}(v) + \mathcal{R}^{(3)}(v) + \mathcal{N}^{(3)}(v). \end{aligned}$$

As in the previous step, we can estimate $\mathcal{N}_0^{(3)}$ and $\mathcal{R}^{(3)}$ in a straightforward manner (Lemma 4.9), while we split $\mathcal{N}^{(3)}$ as

$$\mathcal{N}^{(3)} = \mathcal{N}_1^{(3)} + \mathcal{N}_2^{(3)},$$

where $\mathcal{N}_1^{(3)}$ is the restriction of $\mathcal{N}^{(3)}$ onto C_2 defined in (4.4). In this way, we iterate normal form reductions in an indefinite manner.

After the J th step, we have

$$\begin{aligned}
\mathcal{N}_2^{(J)}(v) &= \partial_t \left[\sum_{\mathcal{T}_J \in \mathfrak{B}\mathfrak{T}(J)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)} \mathbf{1}_{\cap_{j=1}^{J-1} C_j^c} \frac{\langle n_r \rangle^{2s} e^{-i\tilde{\phi}_J t}}{\prod_{j=1}^J \tilde{\phi}_j} \prod_{a \in \mathcal{T}_J^\infty} v_{n_a} \right] \\
&\quad - \sum_{\mathcal{T}_J \in \mathfrak{B}\mathfrak{T}(J)} \sum_{b \in \mathcal{T}_J^\infty} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)} \mathbf{1}_{\cap_{j=1}^{J-1} C_j^c} \frac{\langle n_r \rangle^{2s} e^{-i\tilde{\phi}_J t}}{\prod_{j=1}^J \tilde{\phi}_j} \mathcal{R}(v)_{n_b} \prod_{a \in \mathcal{T}_J^\infty \setminus \{b\}} v_{n_a} \\
&\quad - \sum_{\mathcal{T}_{J+1} \in \mathfrak{B}\mathfrak{T}(J+1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1})} \mathbf{1}_{\cap_{j=1}^{J-1} C_j^c} \frac{\langle n_r \rangle^{2s} e^{-i\tilde{\phi}_{J+1} t}}{\prod_{j=1}^J \tilde{\phi}_j} \prod_{a \in \mathcal{T}_{J+1}^\infty} v_{n_a} \\
&=: \partial_t \mathcal{N}_0^{(J+1)}(v) + \mathcal{R}^{(J+1)}(v) + \mathcal{N}^{(J+1)}(v), \tag{4.12}
\end{aligned}$$

where $\tilde{\phi}_J$ is as in (4.5). In the following, we first estimate $\mathcal{N}_0^{(J+1)}$ and $\mathcal{R}^{(J+1)}$ by applying Cauchy-Schwarz inequality and then applying the divisor counting argument (Lemma 4.6).

Lemma 4.9. *Let $\mathcal{N}_0^{(J+1)}$ and $\mathcal{R}^{(J+1)}$ be as in (4.12). Then, for any $s < 1$, we have*

$$|\mathcal{N}_0^{(J+1)}(v)| \lesssim \frac{c_J}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \|v\|_{L^2}^{2J+2}, \tag{4.13}$$

$$|\mathcal{R}^{(J+1)}(v)| \lesssim \frac{(2J+2) \cdot c_J}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \|v\|_{L^2}^{2J+4}. \tag{4.14}$$

Proof. We split the proof into the following two cases:

$$(i) \Pi_2(\mathcal{T}_J) = \{r_2\} \quad \text{and} \quad (ii) \Pi_2(\mathcal{T}_J) \neq \{r_2\},$$

where Π_2 denotes the projection defined in (4.1).

• **Case (i):** We first consider the case $\Pi_2(\mathcal{T}_J) = \{r_2\}$. Recall that for general $J \in \mathbb{N}$, we need to control the rapidly growing cardinality $c_J = |\mathfrak{B}\mathfrak{T}(J)|$ defined in (4.2). By Cauchy-Schwarz inequality and Lemma 4.6, we have

$$\begin{aligned}
|\mathcal{N}_0^{(J+1)}(v)| &\lesssim \|v\|_{L^2} \sum_{\substack{\mathcal{T}_J \in \mathfrak{B}\mathfrak{T}(J) \\ \Pi_2(\mathcal{T}_J) = \{r_2\}}} \left\{ \sum_{n \in \mathbb{Z}} \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J) \\ n_r = n}} \mathbf{1}_{\cap_{j=1}^{J-1} C_j^c} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \prod_{j=2}^J \frac{1}{|\tilde{\phi}_j|^2} \right) \right. \\
&\quad \left. \times \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J) \\ n_r = n}} \prod_{a \in \mathcal{T}_J^\infty \setminus \{r_2\}} |v_{n_a}|^2 \right) \right\}^{\frac{1}{2}} \\
&\lesssim \frac{c_J}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \|v\|_{L^2}^{2J+2}.
\end{aligned}$$

• **Case (ii):** Next, we consider the case $\Pi_2(\mathcal{T}_J) \neq \{r_2\}$. In this case, we need to modify the argument above since the frequency $n_r = n$ does not correspond to a terminal node. Note that $\mathcal{T}_J^\infty = \Pi_1(\mathcal{T}_J)^\infty \cup \Pi_2(\mathcal{T}_J)^\infty$ and

$$\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J) \\ n_r = n}} \prod_{a \in \mathcal{T}_J^\infty} |v_{n_a}|^2 = \prod_{j=1}^2 \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\Pi_j(\mathcal{T}_J)) \\ n_{r_j} = n}} \prod_{a_j \in \Pi_j(\mathcal{T}_J)^\infty} |v_{n_{a_j}}|^2 \right). \tag{4.15}$$

Then, by Cauchy-Schwarz inequality and Lemma 4.6 with (4.15), we have

$$\begin{aligned}
 |\mathcal{N}_0^{(J+1)}(v)| &\lesssim \sum_{\substack{\mathcal{T}_J \in \mathfrak{B}\mathfrak{T}(J) \\ \Pi_2(\mathcal{T}_J) = \{r_2\}}} \sum_{n \in \mathbb{Z}} \left\{ \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J) \\ n_r = n}} \mathbf{1}_{\cap_{j=1}^{J-1} C_j^c} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \prod_{j=2}^J \frac{1}{|\tilde{\phi}_j|^2} \right) \right. \\
 &\quad \left. \times \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J) \\ n_r = n}} \prod_{a \in \mathcal{T}_J^\infty} |v_{n_a}|^2 \right) \right\}^{\frac{1}{2}} \\
 &\lesssim \frac{c_J}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \sup_{\substack{\mathcal{T}_J \in \mathfrak{B}\mathfrak{T}(2) \\ \Pi_2(\mathcal{T}_J) \neq \{r_2\}}} \sum_{n \in \mathbb{Z}} \left\{ \prod_{j=1}^2 \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\Pi_j(\mathcal{T}_J)) \\ n_{r_j} = n}} \prod_{a_j \in \Pi_j(\mathcal{T}_J)^\infty} |v_{n_{a_j}}|^2 \right) \right\}^{\frac{1}{2}}
 \end{aligned}$$

By Cauchy-Schwarz inequality in n ,

$$\begin{aligned}
 &\lesssim \frac{c_J}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \sup_{\substack{\mathcal{T}_J \in \mathfrak{B}\mathfrak{T}(2) \\ \Pi_2(\mathcal{T}_J) \neq \{r_2\}}} \prod_{j=1}^2 \left\{ \left(\sum_n \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\Pi_j(\mathcal{T}_J)) \\ n_{r_j} = n}} \prod_{a_j \in \Pi_j(\mathcal{T}_J)^\infty} |v_{n_{a_j}}|^2 \right) \right\}^{\frac{1}{2}} \\
 &\lesssim \frac{c_J}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \|v\|_{L^2}^{2J+2}.
 \end{aligned}$$

This proves the first estimate (4.13).

The second estimate (4.14) follows from (4.13) and $\ell_n^2 \subset \ell_n^6$, noting that, given $\mathcal{T}_J \in \mathfrak{B}\mathfrak{T}(J)$, we have $\#\{b : b \in \mathcal{T}_J^\infty\} = 2J + 2$. \square

Next, we treat $\mathcal{N}^{(J+1)}$ in (4.12). As before, we write

$$\mathcal{N}^{(J+1)} = \mathcal{N}_1^{(J+1)} + \mathcal{N}_2^{(J+1)}, \tag{4.16}$$

where $\mathcal{N}_1^{(J+1)}$ is the restriction of $\mathcal{N}^{(J+1)}$ onto C_J defined in (4.4) and $\mathcal{N}_2^{(J+1)} := \mathcal{N}^{(J+1)} - \mathcal{N}_1^{(J+1)}$. In the following lemma, we estimate the first term in (4.18):

$$\mathcal{N}_1^{(J+1)}(v) = - \sum_{\mathcal{T}_{J+1} \in \mathfrak{B}\mathfrak{T}(J+1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1})} \mathbf{1}_{(\cap_{j=1}^{J-1} C_j^c) \cap C_J} \frac{\langle n_r \rangle^{2s} e^{-i\tilde{\phi}_{J+1}t}}{\prod_{j=1}^J \tilde{\phi}_j} \prod_{a \in \mathcal{T}_{J+1}^\infty} v_{n_a}. \tag{4.17}$$

Then, we apply a normal form reduction once again to the second term $\mathcal{N}_2^{(J+1)}$ as in (4.12). In Subsection 4.4, we show that the error term $\mathcal{N}_2^{(J+1)}$ tends to 0 as $J \rightarrow \infty$.

Lemma 4.10. *Let $\mathcal{N}_1^{(J+1)}$ be as in (4.17). Then, for any $s < 1$, we have*

$$|\mathcal{N}_1^{(J+1)}(v)| \lesssim \frac{J^{\frac{3}{2}+} \cdot c_{J+1}}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \|v\|_{L^2}^{2J+4}. \tag{4.18}$$

Proof. We only discuss the case $\Pi_2(\mathcal{T}_{J+1}) = \{r_2\}$ since the modification for the case $\Pi_2(\mathcal{T}_{J+1}) \neq \{r_2\}$ is straightforward as in the proof of Lemma 4.9. By Cauchy-Schwarz

inequality and Lemma 4.6, we have

$$\begin{aligned}
|\mathcal{N}_1^{(J+1)}(v)| &\lesssim \|v\|_{L^2} \sum_{\substack{\mathcal{T}_{J+1} \in \mathfrak{B}\mathfrak{T}(J+1) \\ \Pi_2(\mathcal{T}_{J+1}) = \{r_2\}}} \left\{ \sum_{n \in \mathbb{Z}} \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1}) \\ n_r = n}} \mathbf{1}_{(\cap_{j=1}^{J-1} C_j^c) \cap C_J} \frac{\langle n \rangle^{4s}}{|\phi_1|^2} \prod_{j=2}^J \frac{1}{|\tilde{\phi}_j|^2} \right) \right. \\
&\quad \left. \times \left(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1}) \\ n_r = n}} \prod_{a \in \mathcal{T}_{J+1}^\infty \setminus \{r_2\}} |v_{n_a}|^2 \right) \right\}^{\frac{1}{2}} \\
&\lesssim \frac{J^{\frac{3}{2}+} \cdot c_{J+1}}{\prod_{j=2}^J (2j+2)^{\frac{3}{2}-}} \|v\|_{L^2}^{2J+4}.
\end{aligned}$$

This proves (4.18). \square

Remark 4.11. A notable difference from [15] appears in our definition of C_j in (4.4); on the one hand, the cutoff size on $\tilde{\phi}_{j+1}$ in [15] depended on $\tilde{\phi}_j$ and ϕ_1 . On the other hand, our choice of the cutoff size on $\tilde{\phi}_{j+1}$ in (4.4) is independent of $\tilde{\phi}_j$ or ϕ_1 , thus providing simplification of the argument.

Another difference appears in the first step of the normal form reductions. On the one hand, we simply applied the first normal form reduction in (4.11) without introducing a cutoff on the phase function ϕ_1 . On the other hand, in [15], a cutoff of the form¹⁶ $|\phi_1| > K$ was introduced to separate the first multilinear term into the nearly resonant and non-resonant parts. The use of this extra parameter $K = K(\|u_0\|_{L^2})$ allowed the authors to show that the local existence time can be given by $T \sim \|u_0\|_{L^2}^{-\alpha}$ for some $\alpha > 0$. See [15] for details. Since our argument only requires the summability (in J) of the multilinear forms, we do not need to introduce this extra parameter.

We conclude this subsection by briefly discussing how to justify all the formal steps performed in the normal form reductions. In particular, we need to justify

- (i) the application of the product rule and
- (ii) switching time derivatives and summations

Suppose that a solution v to (3.2) lies in $C(\mathbb{R}; H^{\frac{1}{6}}(\mathbb{T}))$. Then, from (3.2), we have

$$\|\partial_t v_n\|_{C_T \ell_n^\infty} \leq \|\mathcal{F}^{-1}(|\widehat{v}_n|)\|_{C_T L_x^3}^3 \lesssim \|\mathcal{F}^{-1}(|\widehat{v}_n|)\|_{C_T H_x^{\frac{1}{6}}}^3 = \|v\|_{C_T H_x^{\frac{1}{6}}}^3$$

for each $T > 0$, where $C_T B_x = C([-T, T]; B_x)$. Hence, $\partial_t v_n \in C([-T, T]; \ell_n^\infty)$, justifying (i) the application of the product rule. Note that given $N \in \mathbb{N}$, any solution v to (3.5) belongs to $C(\mathbb{R}; H^\infty(\mathbb{T}))$ and hence (i) is justified. Moreover, the summations over spatial frequencies in the normal form reductions applied to solutions to (3.5) are all finite and therefore, (ii) the switching time derivatives and summations over spatial frequencies trivially hold true for (3.5). In general, the proof of Lemma 4.9 shows that the summation defining $\mathcal{N}_0^{(j)}$ converges (absolutely and uniformly in time). Then, Lemma 5.1 in [15] allows us to switch the time derivative with the summations as temporal distributions, thus justifying differentiation by parts.

¹⁶In [15], this parameter was denoted by N . Here, we use K to avoid the confusion with the frequency truncation parameter $N \in \mathbb{N}$.

4.4. **On the error term $\mathcal{N}_2^{(J+1)}$.** In this subsection, we prove that $\mathcal{N}_2^{(J+1)}$ in (4.16) tends to 0 as $J \rightarrow \infty$ under some regularity assumption on v . From (4.12), we have

$$\mathcal{N}_2^{(J+1)}(v) = - \sum_{\mathcal{T}_{J+1} \in \mathfrak{B}\mathfrak{I}(J+1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1})} \mathbf{1}_{\cap_{j=1}^J C_j^c} \frac{\langle n_r \rangle^{2s} e^{-i\tilde{\phi}_{J+1}t}}{\prod_{j=1}^J \tilde{\phi}_j} \prod_{a \in \mathcal{T}_{J+1}^\infty} v_{n_a}. \quad (4.19)$$

Lemma 4.12. *Let $\mathcal{N}_2^{(J+1)}$ be as in (4.19). Then, given $v \in H^s(\mathbb{T})$, $s > \frac{1}{2}$, we have*

$$|\mathcal{N}_2^{(J+1)}(v)| \rightarrow 0, \quad (4.20)$$

as $J \rightarrow \infty$.

We point out that one can actually prove Lemma 4.12 under a weaker regularity assumption $s \geq \frac{1}{6}$. See [29]. For our purpose, however, we only need to prove the vanishing of the error term $\mathcal{N}_2^{(J+1)}$ for sufficiently regular functions; our main objective is to obtain an energy estimate (on the modified energy $\mathcal{E}_{N,t}$ defined in (3.7)) for solutions to the truncated equation (3.5). Given $N \in \mathbb{N}$, any solution v to (3.5) belongs to $C(\mathbb{R}; H^\infty(\mathbb{T}))$. Therefore, while the convergence speed in (4.20) depends on $N \in \mathbb{N}$, the final energy estimate (3.8) holds with an implicit constant *independent* of $N \in \mathbb{N}$.

Proof. Given $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1})$, it follows from Definition 4.4 and the triangle inequality that there exists $C_0 > 0$ such that

$$|n_r| \leq C_0^J |n_{b_k}| \quad (4.21)$$

for (at least) two terminal nodes $b_1, b_2 \in \mathcal{T}_{J+1}^\infty$. Then, by Young's inequality (placing $v_{n_{b_k}}$ in ℓ_n^2 , $k = 1, 2$, and the rest in ℓ_n^1) with (4.2), (4.4), and (4.21), we have

$$\begin{aligned} |\mathcal{N}_2^{(J+1)}(v)| &\lesssim \frac{C_0^{2sJ} \cdot c_J}{\prod_{j=2}^J (2j+2)^3} \sup_{\mathcal{T}_{J+1} \in \mathfrak{B}\mathfrak{I}(J+1)} \sum_{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{J+1})} \left(\prod_{k=1}^2 \langle n_{b_k} \rangle^s |v_{n_{b_k}}| \right) \left(\prod_{a \in \mathcal{T}_{J+1}^\infty \setminus \{b_1, b_2\}} |v_{n_a}| \right) \\ &\lesssim \frac{C_0^{2sJ} \cdot c_J}{\prod_{j=2}^J (2j+2)^3} \|v\|_{H^s}^{2J+4} \rightarrow 0, \end{aligned}$$

as $J \rightarrow \infty$. □

4.5. **Improved energy bound.** We are now ready to establish the improved energy estimate (3.8). Let v be a smooth global solution¹⁷ to (3.2). Then, by applying the normal form reduction J times, we obtain¹⁸

$$\partial_t \left(\frac{1}{2} \|v\|_{H^s}^2 \right) = \partial_t \sum_{j=2}^{J+1} \mathcal{N}_0^{(j)}(v) + \sum_{j=2}^{J+1} \mathcal{N}_1^{(j)}(v) + \sum_{j=2}^{J+1} \mathcal{R}^{(j)}(v) + \mathcal{N}_2^{(J+1)}.$$

Thanks to Lemma 4.12, by taking the limit as $J \rightarrow \infty$, we obtain

$$\partial_t \left(\frac{1}{2} \|v\|_{H^s}^2 \right) = \partial_t \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(v) + \sum_{j=2}^{\infty} \mathcal{N}_1^{(j)}(v) + \sum_{j=2}^{\infty} \mathcal{R}^{(j)}(v).$$

¹⁷In fact, it suffices to assume that $v \in C(\mathbb{R}; H^{\frac{1}{6}}(\mathbb{T}))$. See [15, 29].

¹⁸Once again, we are replacing ± 1 and $\pm i$ by 1 for simplicity since they play no role in our analysis.

In other words, defining the modified energy $\mathcal{E}_t(v)$ by

$$\mathcal{E}_t(v) := \frac{1}{2} \|v(t)\|_{H^s}^2 - \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(v)(t),$$

we have

$$\partial_t \mathcal{E}_t(v) = \sum_{j=2}^{\infty} \mathcal{N}_1^{(j)}(v)(t) + \sum_{j=2}^{\infty} \mathcal{R}^{(j)}(v)(t).$$

Suppose that $\|v\|_{C(\mathbb{R}; L^2)} \leq r$. Then, applying Lemmas 4.9 and 4.10 with (4.2), we obtain

$$\begin{aligned} |\partial_t \mathcal{E}_t(v)| &\lesssim \sum_{j=2}^{\infty} \frac{c_{j-1}}{\prod_{k=2}^{j-1} (2k+2)^{\frac{3}{2}-}} r^{2j} + \sum_{j=2}^{\infty} \frac{j^{\frac{3}{2}+} \cdot c_j}{\prod_{k=2}^{j-1} (2k+2)^{\frac{3}{2}-}} r^{2j+2} \\ &\quad + \sum_{j=2}^{\infty} \frac{j \cdot c_{j-1}}{\prod_{k=2}^{j-1} (2k+2)^{\frac{3}{2}-}} r^{2j+2} \\ &\leq C(r). \end{aligned}$$

In view of the boundedness of the frequency projections and noting that any solution to (3.5) is in $H^\infty(\mathbb{T})$, the same energy estimate holds for solutions to the truncated equation (3.5), uniformly in $N \in \mathbb{N}$.

4.6. On the proof of Lemma 3.5. We conclude this section with a brief discussion on the proof of Lemma 3.5. First, note that Lemma 3.5 (ii) is an immediate corollary of Lemma 3.5 (i). Moreover, Lemma 3.5 (i) follows from Egoroff's theorem once we prove that

$$\mathfrak{S}_N(v) = \sum_{j=2}^{\infty} \mathcal{N}_{0,N}^{(j)}(v)$$

converges almost surely to

$$\mathfrak{S}_\infty(v) = \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(v).$$

See [26, Proposition 6.2]. In fact, one can show that $\mathfrak{S}_N(v)$ converges to $\mathfrak{S}_\infty(v)$ for any $v \in L^2(\mathbb{T})$.

Recall from (4.12) that $\mathcal{N}_0^{(j)}(v)$ consists of a sum of the multilinear forms associated with ordered bi-trees $\mathcal{T}_{j-1} \in \mathfrak{B}\mathfrak{T}(j-1)$. Given $N \in \mathbb{N}$, the multilinear form $\mathcal{N}_{0,N}^{(j)}(v)$ is obtained in a similar manner with the following modifications:

- (i) We set $v_n = 0$ for all $|n| > N$. This corresponds to setting $v_{n_a} = 0$ for all $|n| > N$ and all terminal nodes $a \in \mathcal{T}_{j-1}^\infty$.
- (ii) In view of (3.5), we also set $v_{n_a} = 0$ for all $|n| > N$ and all parental nodes in \mathcal{T}_{j-1} . This amounts to setting $v_{n_a} = 0$ for all $|n| > N$ and all non-terminal nodes $a \in \mathcal{T}_{j-1}^0$.

In particular, we have

$$\mathcal{N}_{0,N}^{(j)}(v) = \sum_{\mathcal{T}_{j-1} \in \mathfrak{B}\mathfrak{T}(j-1)} \sum_{\mathbf{n} \in \mathfrak{N}_N(\mathcal{T}_{j-1})} \mathbf{1}_{\cap_{k=1}^{j-2} C_k^c} \frac{\langle n_r \rangle^{2s} e^{-i\tilde{\phi}_{j-1} t}}{\prod_{k=1}^{j-1} \tilde{\phi}_k} \prod_{a \in \mathcal{T}_{j-1}^\infty} v_{n_a}, \quad (4.22)$$

where $\mathfrak{N}_N(\mathcal{T}_{j-1})$ is as in Definition 4.4. Namely, $\mathcal{N}_{0,N}^{(j)}(v)$ is obtained from $\mathcal{N}_0^{(j)}(v)$ by simply truncating all the frequencies (including the “hidden”¹⁹ parental frequencies) by $N \in \mathbb{N}$.

Write

$$\begin{aligned} \mathcal{N}_0^{(j)}(v) - \mathcal{N}_{0,N}^{(j)}(v) &= \left\{ \mathcal{N}_0^{(j)}(v) - \tilde{\mathcal{N}}_{0,N}^{(j)}(v) \right\} + \left\{ \tilde{\mathcal{N}}_{0,N}^{(j)}(v) - \mathcal{N}_{0,N}^{(j)}(v) \right\} \\ &=: \mathbf{I}_j + \mathbf{II}_j, \end{aligned} \tag{4.23}$$

where $\tilde{\mathcal{N}}_{0,N}^{(j)}(v)$ is obtained from $\mathcal{N}_{0,N}^{(j)}(v)$ by replacing $\mathbf{n} \in \mathfrak{N}_N(\mathcal{T}_{j-1})$ with $\mathbf{n} \in \mathfrak{N}_N^0(\mathcal{T}_{j-1})$. i.e. we are truncating only the parental frequencies at non-terminal nodes $a \in \mathcal{T}_{j-1}^0$. Then, by writing

$$\mathbf{II}_j = \tilde{\mathcal{N}}_{0,N}^{(j)}(v) - \tilde{\mathcal{N}}_{0,N}^{(j)}(\mathbf{P}_{\leq N}v),$$

it follows from the multilinearity and the boundedness in $L^2(\mathbb{T})$ (Lemma 4.9) that the second term \mathbf{II}_j tends to 0 as $N \rightarrow \infty$, by simply writing the difference in a telescoping sum. More precisely, we write \mathbf{II} as a telescoping sum, replacing $2j$ factors of v_{n_a} , $a \in \mathcal{T}_{j-1}^\infty$, into $2j$ factors of $(\mathbf{P}_{\leq N}v)_{n_a}$. This introduces $2j$ differences, each containing exactly one factor of $v - \mathbf{P}_{\leq N}v$ (tending to 0 as $N \rightarrow \infty$). We then simply apply Lemma 4.9 on each difference.

Similarly, we can show that \mathbf{I}_j in (4.23) tends to 0 as $N \rightarrow \infty$ by writing the difference in a telescoping sum. Namely, noting only the difference between $\mathcal{N}_0^{(j)}(v)$ and $\tilde{\mathcal{N}}_{0,N}^{(j)}(v)$ is the frequency cutoffs $\mathbf{1}_{|n_a| \leq N}$ at each non-terminal node $a \in \mathcal{T}_{j-1}^0$, we introduce $j-1$ differences by adding the frequency cutoff $\mathbf{1}_{|n_a| \leq N}$ at each non-terminal node in a sequential manner. By construction, each of the $j-1$ differences has one non-terminal node $a_* \in \mathcal{T}_{j-1}^0$ with the restriction $|n_{a_*}| > N$. Then, from Definition 4.4, we see that there exists at least one terminal node $b \in \mathcal{T}_{j-1}^\infty$ which is a descendants of a_* such that

$$|n_b| \geq C_0^{-J} |n_{a_*}| > C_0^{-J} N$$

for some $C_0 > 0$ (compare this with (4.21)). This forces each of the $j-1$ differences in the telescoping sum to tend to 0 as $N \rightarrow \infty$, and hence \mathbf{I}_j in (4.23) tends to 0 as $N \rightarrow \infty$.

Therefore, $\mathcal{N}_{0,N}^{(j)}(v)$ converges to $\mathcal{N}_0^{(j)}(v)$ as $n \rightarrow \infty$ for any $v \in L^2(\mathbb{T})$. Finally, in view of the fast decay in j in Lemma 4.9, the convergence of $\mathfrak{S}_N(v)$ to $\mathfrak{S}_\infty(v)$ follows from the dominated convergence theorem.

5. PROOF OF THEOREM 1.6: NON QUASI-INVARIANCE UNDER THE DISPERSIONLESS MODEL

In this section, we present the proof of Theorem 1.6. The basic ingredients are the Fourier series representation of the (fractional) Brownian loops, the law of the iterated logarithm, and the solution formula (1.19) to the dispersionless model (1.18). More precisely, we show that, while the Gaussian random initial data distributed according to μ_s satisfies the law of the iterated logarithm, the solution given by (1.19) does not satisfy the law of the iterated logarithm for any non-zero time. We divide the argument into three cases: (i) $s = 1$ corresponding to the Brownian/Ornstein-Uhlenbeck loop, (ii) $\frac{1}{2} < s < \frac{3}{2}$, corresponding to the fractional Brownian loop (and $s > \frac{3}{2}$ with $s \notin \frac{1}{2} + \mathbb{N}$), and (iii) $s \in \frac{1}{2} + \mathbb{N}$: the critical

¹⁹Namely, the parental frequencies at the non-terminal nodes do not appear explicitly in the sum in (4.22) but they implicitly appear through the relation in Definition 4.4.

case. For simplicity, we set $t = 1$ in the following. The proof for non-zero $t \neq 1$ follows in a similar manner.

5.1. Brownian/Ornstein-Uhlenbeck loop. We first consider the $s = 1$ case. Under the law of the random Fourier series

$$u(x) = u(x; \omega) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx} \quad (5.1)$$

corresponding to the Gaussian measure μ_1 , $\operatorname{Re} u$ and $\operatorname{Im} u$ are independent stationary Ornstein-Uhlenbeck (OU) processes (in x) on $[0, 2\pi)$. Recall that the law of this process can be written as

$$u(x) \stackrel{d}{=} \mathbf{P}_{\neq 0} w(x) + g_0 = w(x) - \int_0^{2\pi} w(y) dy + g_0, \quad (5.2)$$

where w is a complex OU bridge with $w(0) = w(2\pi) = 0$ and g_0 is a standard complex-valued Gaussian random variable (independent from w).

We now recall the law of the iterated logarithm for the Brownian motion (see [32, I.16.1]):

Proposition 5.1. *Let $B(t)$ be a standard Brownian motion on \mathbb{R}_+ . Then, for each $t \geq 0$,*

$$\limsup_{h \downarrow 0} \frac{B(t+h) - B(t)}{\sqrt{2h \log \log \frac{1}{h}}} = 1, \quad (5.3)$$

almost surely.

It follows from the representation (5.2), the absolute continuity²⁰ of the Brownian bridge with respect to the Brownian motion on any interval $[0, \gamma)$, $\gamma < 2\pi$, and the absolute continuity of the OU bridge with respect to the Brownian bridge also on any interval $[0, \gamma)$, that the limit (5.3) also holds for $\operatorname{Re} u$ and $\operatorname{Im} u$ on $[0, 2\pi)$.

Define ψ by

$$\psi(h) = \sqrt{2h \log \log \frac{1}{h}}, \quad 0 < h < 1.$$

Let $0 \leq x < 2\pi$. As a corollary to Proposition 5.1, we have

$$\limsup_{h \downarrow 0} \frac{\operatorname{Re} u(x+h) - \operatorname{Re} u(x)}{\psi(h)} = 1 \quad (5.4)$$

almost surely.

In the following, by a direct calculation, we show that $\operatorname{Re}[e^{-i|u|^2} u]$ does not satisfy (5.4) with a positive probability. This will show that the pushforward measure $\tilde{\Phi}(t)_* \mu_s$ under the dynamics of the dispersionless model (1.18) is not absolutely continuous with respect to the Gaussian measure μ_s .

²⁰The absolute continuity property claimed here can be easily seen by the Fourier series representations of the Brownian motion/bridge (with (5.1) and (5.2)) and Kakutani's theorem (Lemma 5.3 below). For example, the Brownian motion $B(t)$ on $[0, 2\pi)$ has the following Fourier-Wiener series

$$B(t) = g_0 t + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{g_n}{n} e^{int}.$$

On the one hand, we have

$$\begin{aligned} & \operatorname{Re}[e^{-i|u(y)|^2}u(y)] - \operatorname{Re}[e^{-i|u(x)|^2}u(x)] \\ &= (\operatorname{Re} u(y) - \operatorname{Re} u(x)) \cos |u(y)|^2 + (\cos |u(y)|^2 - \cos |u(x)|^2) \operatorname{Re} u(x) \\ & \quad + (\operatorname{Im} u(y) - \operatorname{Im} u(x)) \sin |u(y)|^2 + (\sin |u(y)|^2 - \sin |u(x)|^2) \operatorname{Im} u(x). \end{aligned}$$

On the other hand, by the Taylor expansion with $\eta(x, y) = |u(y)|^2 - |u(x)|^2$, we have

$$\begin{aligned} \cos |u(y)|^2 &= \cos |u(x)|^2 - \sin |u(x)|^2 \cdot \eta(x, y) + O(\eta^2(x, y)), \\ \sin |u(y)|^2 &= \sin |u(x)|^2 + \cos |u(x)|^2 \cdot \eta(x, y) + O(\eta^2(x, y)). \end{aligned}$$

Putting together, we obtain

$$\begin{aligned} & \operatorname{Re}[e^{-i|u(y)|^2}u(y)] - \operatorname{Re}[e^{-i|u(x)|^2}u(x)] \\ &= (\operatorname{Re} u(y) - \operatorname{Re} u(x)) \cos |u(y)|^2 - \sin |u(x)|^2 \cdot \eta(x, y) \operatorname{Re} u(x) \\ & \quad + (\operatorname{Im} u(y) - \operatorname{Im} u(x)) \sin |u(y)|^2 + \cos |u(x)|^2 \cdot \eta(x, y) \operatorname{Im} u(x) \\ & \quad + O(\eta^2(x, y)) \cdot (|\operatorname{Re} u(x)| + |\operatorname{Im} u(x)|) \\ &= (\operatorname{Re} u(y) - \operatorname{Re} u(x)) \cos |u(y)|^2 \\ & \quad - \sin |u(x)|^2 \cdot \{(\operatorname{Re} u(y) - \operatorname{Re} u(x))(\operatorname{Re} u(y) + \operatorname{Re} u(x))\} \operatorname{Re} u(x) \\ & \quad - \sin |u(x)|^2 \cdot \{(\operatorname{Im} u(y) - \operatorname{Im} u(x))(\operatorname{Im} u(y) + \operatorname{Im} u(x))\} \operatorname{Re} u(x) \\ & \quad + (\operatorname{Im} u(y) - \operatorname{Im} u(x)) \sin |u(y)|^2 \\ & \quad + \cos |u(x)|^2 \cdot \{(\operatorname{Re} u(y) - \operatorname{Re} u(x))(\operatorname{Re} u(y) + \operatorname{Re} u(x))\} \operatorname{Im} u(x) \\ & \quad + \cos |u(x)|^2 \cdot \{(\operatorname{Im} u(y) - \operatorname{Im} u(x))(\operatorname{Im} u(y) + \operatorname{Im} u(x))\} \operatorname{Im} u(x) \\ & \quad + O(\eta^2(x, y)) \cdot (|\operatorname{Re} u(x)| + |\operatorname{Im} u(x)|). \end{aligned} \tag{5.5}$$

Fix $0 \leq x < 2\pi$. Let $\{h_n = h_n(\omega)\}_{n \in \mathbb{N}}$ be a (random) sequence achieving the limit supremum in (5.4) almost surely. Then, for this sequence $\{h_n\}_{n \in \mathbb{N}}$, we have

$$\limsup_{n \rightarrow \infty} \frac{|\operatorname{Im} u(x + h_n) - \operatorname{Im} u(x)|}{\psi(h_n)} \leq 1 \tag{5.6}$$

almost surely. Divide the expression in (5.5) by $\psi(h_n)$, after replacing y by $x + h_n$. Then, by taking the limit as $n \rightarrow \infty$ and applying (5.4) and (5.6), we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{\operatorname{Re}[e^{-i|u(x+h_n)|^2}u(x+h_n)] - \operatorname{Re}[e^{-i|u(x)|^2}u(x)]}{\psi(h_n)} \\ & \geq -2 \sin |u(x)|^2 \cdot (\operatorname{Re} u(x))^2 \\ & \quad - |\cos |u(x)|^2| - |\sin |u(x)|^2| - 2 \cdot |\sin |u(x)|^2 \cdot \operatorname{Im} u(x) \operatorname{Re} u(x)| \\ & \quad - 2 \cdot |\cos |u(x)|^2 \cdot \operatorname{Re} u(x) \operatorname{Im} u(x)| - 2 \cdot |\cos |u(x)|^2| (\operatorname{Im} u(x))^2, \end{aligned} \tag{5.7}$$

almost surely.

Fix $M \gg 1$ by

$$M^2 = -\frac{\pi}{2} + 2k\pi, \tag{5.8}$$

for some large $k \in \mathbb{N}$ (to be chosen later). Given $\varepsilon > 0$, define the set

$$A = \{\omega \in \Omega : |\operatorname{Re} u(x; \omega) - M| \leq \varepsilon, |\operatorname{Im} u(x; \omega)| \leq \varepsilon\}.$$

Noting that under the law of the OU loop, $\operatorname{Re} u(x)$ and $\operatorname{Im} u(x)$ are independent Gaussian random variables, we have

$$P(A) \geq \delta(M, \epsilon) > 0$$

for any $\epsilon > 0$. By choosing $\epsilon > 0$ sufficiently small such that $\epsilon M \ll 1$, we have

$$||u(x)|^2 - M^2| \leq 2\epsilon(M + \epsilon) = o(1). \quad (5.9)$$

Then, from (5.8) and (5.9), we have

$$\begin{aligned} \text{RHS of (5.7)} &\geq 2|\sin |u(x)|^2| \cdot M(M - 3\epsilon) - 2(1 + \epsilon(M + 2\epsilon)) \\ &\geq M^2 \end{aligned}$$

on A . By choosing $M \gg 1$, we see that the set

$$A_1 = \left\{ \limsup_{h \downarrow 0} \frac{\operatorname{Re}[e^{-i|u(x+h)|^2} u(x+h)] - \operatorname{Re}[e^{-i|u(x)|^2} u(x)]}{\psi(h)} = 1 \right\}$$

does not have probability 1 under the law of u . Therefore, μ_1 is not quasi-invariant under the flow of the dispersionless model (1.18).

5.2. Fractional Brownian motion. In this subsection, we extend the previous result to the distribution of the random Fourier series

$$u_s(x) = \sum_{n \in \mathbb{Z}} \frac{g_n}{\langle n \rangle^s} e^{inx}, \quad (5.10)$$

corresponding to μ_s . For $\frac{1}{2} < s < \frac{3}{2}$, the series (5.10) is related to a *fractional Brownian motion*. Recall that a fractional Brownian motion with Hurst parameter H , $0 < H < 1$, is the Gaussian process $B^H(t), t \geq 0$ with stationary increments and covariance

$$\mathbb{E}[B^H(t_1)B^H(t_2)] = \frac{\rho(H)}{2}(t_1^{2H} + t_2^{2H} - |t_2 - t_1|^{2H}), \quad (5.11)$$

where

$$\rho(H) = \mathbb{E}[(B^H(1))^2] = -2 \frac{\cos(\pi H)}{\pi} \Gamma(-2H) \quad \text{when } H \neq \frac{1}{2} \quad \text{and} \quad \rho\left(\frac{1}{2}\right) = 1.$$

When $H = \frac{1}{2}$, a fractional Brownian motion becomes the standard Brownian motion. In the following, we only consider the case $H \neq \frac{1}{2}$.

It is known that there is a subtle issue on building a series representation for a fractional Brownian motion B^H . Instead, we consider the following series²¹

$$\widehat{B}^H(t) = \widetilde{g}_0 t + \sqrt{2} \sum_{n \geq 1} \left(\widetilde{g}_n \frac{\cos(nt) - 1}{n^{H+\frac{1}{2}}} + \widetilde{g}'_n \frac{\sin(nt)}{n^{H+\frac{1}{2}}} \right),$$

for $t \in [0, 2\pi]$, where \widetilde{g}_n and \widetilde{g}'_n are now independent *real-valued* standard Gaussians. Then, Picard [30] showed the following result on the relation between B^H and \widehat{B}^H .

Lemma 5.2 (Theorems 24 and 27 in Section 6 of [30]). *The processes $B^H(t)$ and $\widehat{B}^H(t)$ can be coupled in such a way that*

$$B^H(t) - \widehat{B}^H(t)$$

²¹As mentioned in Section 1, we drop the factor of 2π .

is a C^∞ -function on $(0, 2\pi]$. Moreover, if $H \neq \frac{1}{2}$, then the laws of B^H and \widehat{B}^H are equivalent on $[0, T]$ for $T < 2\pi$ (and mutually singular if $T = 2\pi$).

Recall Kakutani's criterion [19] in the Gaussian case.

Lemma 5.3. *Let $\{g_n\}_{n \in \mathbb{N}}$ and $\{\tilde{g}_n\}_{n \in \mathbb{N}}$ be two sequences of centered Gaussian random variables with variances $\mathbb{E}[g_n^2] = \sigma_n^2 > 0$ and $\mathbb{E}[\tilde{g}_n^2] = \tilde{\sigma}_n^2 > 0$. Then, the laws of the sequences $\{g_n\}_{n \in \mathbb{N}}$ and $\{\tilde{g}_n\}_{n \in \mathbb{N}}$ are equivalent if and only if*

$$\sum_{n \in \mathbb{N}} \left(\frac{\tilde{\sigma}_n^2}{\sigma_n^2} - 1 \right)^2 < \infty.$$

From (5.10), we have

$$\operatorname{Re} u_s(x) = \operatorname{Re} g_0 + \sum_{n \geq 1} \left(\frac{\operatorname{Re} g_n + \operatorname{Re} g_{-n}}{\langle n \rangle^s} \cos(nx) + \frac{-\operatorname{Im} g_n + \operatorname{Im} g_{-n}}{\langle n \rangle^s} \sin(nx) \right). \quad (5.12)$$

Then, applying Lemma 5.3 to the sequences

$$\left\{ \tilde{g}_0 t, \sqrt{2} \frac{\tilde{g}_n}{n^{H+\frac{1}{2}}}, \sqrt{2} \frac{\tilde{g}'_n}{n^{H+\frac{1}{2}}} \right\}$$

and

$$\left\{ \operatorname{Re} g_0, \frac{\operatorname{Re} g_n + \operatorname{Re} g_{-n}}{\langle n \rangle^s}, \frac{-\operatorname{Im} g_n + \operatorname{Im} g_{-n}}{\langle n \rangle^s} \right\},$$

we see that if $s = H + \frac{1}{2}$, then the series (5.12) for $\operatorname{Re} u_s$ and

$$\tilde{B}^H := \widehat{B}^H - \int_0^{2\pi} (\widehat{B}^H(\alpha) - \tilde{g}_0 \alpha) d\alpha \quad (5.13)$$

have laws that are mutually absolutely continuous. Therefore, in view of the computation above and Lemma 5.2 with [30, Theorem 35], we see that the laws of B^H and $\operatorname{Re} u_s$ are equivalent on $[0, T]$ for $T < 2\pi$. The same holds for $\operatorname{Im} u_s$.

We use the following version of the law of the iterated logarithm for Gaussian processes with stationary increments [25, Theorem 7.2.15]. First, recall the following definition. We say that a function f is called a normalized regularly varying function at zero with index $\alpha > 0$ if it can be written in the form

$$f(x) = Cx^\alpha \exp \left(\int_1^x \frac{\varepsilon(u)}{u} du \right)$$

for some constant $C \neq 0$ and $\lim_{u \rightarrow 0} \varepsilon(u) = 0$.

Proposition 5.4. *Let $G = \{G(x), x \in [0, 2\pi]\}$ be a Gaussian process with stationary increments and let*

$$\sigma^2(h) = \mathbb{E}[|G(h) - G(0)|^2].$$

If $\sigma^2(h)$ is a normalized regularly varying function at zero with index $0 < \alpha < 2$, then

$$\limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{|G(h) - G(0)|}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} = 1$$

almost surely.

Using the covariance (5.11), we have

$$\sigma^2(h) = \mathbb{E}[|B^H(h) - B^H(0)|^2] = Ch^{2H}. \quad (5.14)$$

Hence, Proposition 5.4 holds for $G(x) = B^H(x)$, $H < 1$. Then, by the absolute continuity, the conclusion of Proposition 5.4 with 0 replaced by any $x \in (0, 2\pi)$ also holds for $\operatorname{Re} u_s$ and $\operatorname{Im} u_s$; for any $\frac{1}{2} < s < \frac{3}{2}$, we have

$$\limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{|\operatorname{Re} u_s(x+h) - \operatorname{Re} u_s(x)|}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} = 1 \quad (5.15)$$

almost surely. Applying the law of the iterated logarithm conditionally on the set where (5.15) holds, we also have

$$\limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{|\operatorname{Im} u_s(x+h) - \operatorname{Im} u_s(x)|}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} \leq 1$$

almost surely. We can now reproduce exactly the proof in the previous subsection for $\frac{1}{2} < s < \frac{3}{2}$. This proves Theorem 1.6 for $\frac{1}{2} < s < \frac{3}{2}$.

Next, we consider the case $s \geq \frac{3}{2}$ such that $s \notin \frac{1}{2} + \mathbb{N}$. We consider the critical case $s \in \frac{1}{2} + \mathbb{N}$ in the next subsection. The main point is to note that u_s in (5.10) has a C^r -version for each integer $r < \lfloor s - \frac{1}{2} \rfloor$. Indeed, we have the following:

Lemma 5.5. *Let $X(t)$, $t \in \mathbb{R}$, be a stationary Gaussian process with the covariance function*

$$\rho(t) = \int e^{i\alpha t} \nu(d\alpha).$$

If $\int |\alpha|^{2+\varepsilon} \nu(d\alpha) < \infty$ for some $\varepsilon > 0$, then there is a version of the process $X(t)$ such that $\partial_t X(t)$ exists and is continuous. Moreover, $\partial_t X(t)$ is a stationary Gaussian process with covariance

$$\int e^{i\alpha t} \alpha^2 \nu(d\alpha).$$

Since we work on \mathbb{T} , the spectral measure $\nu(d\alpha)$ is the counting measure on \mathbb{Z} and

$$\rho(x) = 2 \sum_{n \in \mathbb{Z}} \frac{e^{inx}}{\langle n \rangle^{2s}}.$$

Note that when $s > \frac{3}{2}$, we have

$$2 \sum_{n \in \mathbb{Z}} \frac{|n|^{2+\varepsilon}}{\langle n \rangle^{2s}} < \infty$$

for sufficiently small $\varepsilon > 0$ and thus we can apply Lemma 5.5. Differentiating u_s in (5.10) $\lfloor s - \frac{1}{2} \rfloor$ times, we obtain a process

$$\partial_x^r u_s(x) \stackrel{d}{=} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{g_n}{|n|^{-r} \langle n \rangle^s} e^{inx}$$

with $r = \lfloor s - \frac{1}{2} \rfloor$. Given $s \in (\frac{1}{2} + j, \frac{3}{2} + j)$ for some $j \in \mathbb{N}$, we have $s - r = s - j \in (\frac{1}{2}, \frac{3}{2})$. Noting that $|n|^{-r} \langle n \rangle^s \sim \langle n \rangle^{s-r}$, we can apply Lemma 5.3 to show the laws of \tilde{B}^H defined

in (5.13), $\operatorname{Re} \partial_x^r u_s$, and $\operatorname{Im} \partial_x^r u_s$ are equivalent. Hence, proceeding as before, we can apply Proposition 5.4 to $\operatorname{Re} \partial_x^r u_s$ and $\operatorname{Im} \partial_x^r u_s$. Namely, we obtain

$$\limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{|\operatorname{Re} \partial_x^r u_s(x+h) - \operatorname{Re} \partial_x^r u_s(x)|}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} = 1 \quad (5.16)$$

almost surely. Applying the law of the iterated logarithm conditionally on the set where (5.16) holds, we also have

$$\limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{|\operatorname{Im} \partial_x^r u_s(x+h) - \operatorname{Im} \partial_x^r u_s(x)|}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} \leq 1 \quad (5.17)$$

almost surely.

With (5.16) and (5.17) at hand, we can basically repeat the proof in Subsection 5.1 by differentiating (5.5) and applying (5.16) and (5.17). A straightforward application of the product rule to compute $\partial_x^r (e^{-i|u_s(x)|^2} u_s(x))$ would be computationally cumbersome. Thus, we perform some simplification before taking derivatives. First, note that from (5.16) and (5.17) with (5.14), we have

$$\limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{|\operatorname{Re} \partial_x^j u_s(x+h) - \operatorname{Re} \partial_x^j u_s(x)|}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} = 0, \quad (5.18)$$

$$\limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{|\operatorname{Im} \partial_x^j u_s(x+h) - \operatorname{Im} \partial_x^j u_s(x)|}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} = 0 \quad (5.19)$$

almost surely, for $j = 0, 1, \dots, r-1$.

In the following, we will take r derivatives (in x) of both sides of (5.5) by setting $y = x+h$. In view of (5.18) and (5.19), we see that, after taking r derivatives, dividing by $\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}$, and taking $\lim_{\delta \rightarrow 0} \sup_{|h| \leq \delta}$, the only terms in (5.5) that survive are those terms where all the r derivatives falls only on $\operatorname{Re} u_s(x+h) - \operatorname{Re} u_s(x)$ (or $\operatorname{Im} u_s(x+h) - \operatorname{Im} u_s(x)$) to which we can apply (5.16) and (5.17). Therefore, we obtain

$$\begin{aligned} & \limsup_{\delta \rightarrow 0} \sup_{|h| \leq \delta} \frac{\operatorname{Re}[\partial_x^r (e^{-i|u_s(x+h)|^2} u_s(x+h))] - \operatorname{Re}[\partial_x^r (e^{-i|u_s(x)|^2} u_s(x))]}{\sqrt{2\sigma^2(|h|) \log \log \frac{1}{|h|}}} \\ & \geq -2 \sin |u_s(x)|^2 (\operatorname{Re} u_s(x))^2 \\ & \quad - |\cos |u_s(x)|^2| - |\sin |u_s(x)|^2| - 2 \cdot |\sin |u_s(x)|^2 \operatorname{Im} u_s(x) \operatorname{Re} u_s(x)| \\ & \quad - 2 \cdot |\cos |u_s(x)|^2 \operatorname{Im} u_s(x) \operatorname{Re} u_s(x)| - 2 \cdot |\cos |u_s(x)|^2| (\operatorname{Im} u_s(x))^2, \end{aligned}$$

which is exactly the right-hand side of (5.7). The rest follows as in Subsection 5.1.

5.3. Critical case: $s \in \frac{1}{2} + \mathbb{N}$. In this case, we cannot simply apply Proposition 5.4 directly, because, taking $s = \frac{3}{2}$ for example, we have²²

$$\begin{aligned} \sigma_{\frac{3}{2}}(x) &= \mathbb{E}[|u_{\frac{3}{2}}(x) - u_{\frac{3}{2}}(0)|^2] = 2 \sum_{n \in \mathbb{Z}} \frac{|1 - e^{inx}|^2}{(1 + n^2)^{\frac{3}{2}}} \\ &= 2 \sum_{n=1}^{\infty} \frac{(1 - \cos(nx))^2 + \sin^2(nx)}{(1 + n^2)^{\frac{3}{2}}} \\ &\sim x^2 \cdot \log \frac{1}{|x|}, \end{aligned} \tag{5.20}$$

as $x \rightarrow 0$. In particular, $\sigma_{\frac{3}{2}}(x)$ is not a normalized regularly varying function at zero with index $0 < \alpha < 2$. Hence, Proposition 5.4 is not applicable.

In [2], the authors considered the Gaussian process on \mathbb{R}^n with covariance function given by the kernel of the inverse of a quite general elliptic pseudodifferential operator and studied the precise regularity of the process. In particular, they obtained a result generalizing Proposition 5.4 by very different methods from those in [25].

For us, the relevant operator is $2^{-1}(\text{Id} - \partial_x^2)^s$ on \mathbb{T} . In this case, which the authors of [2] call “critical” owing to the behavior (5.20) of the increments, the relevant result from [2, Theorem 1.3 (ii)] reads as follows.

Proposition 5.6. *Let $X_{\frac{3}{2}}$ be the stationary Gaussian process on \mathbb{R} with the covariance operator $2(\text{Id} - \partial_x^2)^{-\frac{3}{2}}$. Then, $X_{\frac{3}{2}}(x)$ has continuous sample paths. Moreover, there exists a constant $c_{\frac{3}{2}} > 0$ such that for each $y \in \mathbb{R}$, we have*

$$\limsup_{x \rightarrow y} \frac{|X_{\frac{3}{2}}(x) - X_{\frac{3}{2}}(y)|}{|x - y| \sqrt{\log \frac{1}{|x-y|} \log \log \log \frac{1}{|x-y|}}} = c_{\frac{3}{2}}$$

almost surely.

The log log from the classical law of the iterated logarithm and Proposition 5.4 is now replaced by a factor involving the triply iterated logarithm log log log. In the following, we state and prove an analogue of Proposition 5.6 on \mathbb{T} in a direct manner. See Proposition 5.7 below. Using this almost sure constancy of the modulus of continuity (Proposition 5.7), we can once again repeat the argument presented in Subsection 5.1.

The results in [2] are much more general than Proposition 5.6. In particular, they apply to operators with variable coefficients. In that case, the local modulus of continuity of the process can change from point to point (although it is constant across different realizations of the sample path). In our specific case, it is possible to give a more elementary proof, using the classical Khintchine’s law of the iterated logarithm for independent sums, that the process $u_{\frac{3}{2}}$ has an exact modulus of continuity almost surely. In terms of the setting in [2], this simplified proof comes as no surprise since our process $u_{\frac{3}{2}}$ has a particularly simple representation as a sum of independent terms with respect to which the covariance operator is diagonal.

²²This computation follows from the computations in the proof of Proposition 5.7 below.

Proposition 5.7. *Let $u_{\frac{3}{2}}$ be given by the random Fourier series in (5.10) with $s = \frac{3}{2}$. Then, for each $y \in \mathbb{T}$, we have*

$$\limsup_{x \rightarrow y} \frac{|u_{\frac{3}{2}}(x) - u_{\frac{3}{2}}(y)|}{2^{\frac{3}{2}} |x - y| \sqrt{\log \frac{1}{|x-y|} \log \log \log \frac{1}{|x-y|}}} = 1 \quad (5.21)$$

almost surely.

Once we prove Proposition 5.7, we can proceed as in Subsection 5.1 when $s = \frac{3}{2}$. For $s \in \frac{3}{2} + \mathbb{N}$, the modification is straightforward following the second half of Subsection 5.2 and thus we omit details.

Proof. Without loss of generality, set $y = 0$. By writing

$$\begin{aligned} u_{\frac{3}{2}}(x) - u_{\frac{3}{2}}(0) &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{inx} - 1}{\langle n \rangle^{\frac{3}{2}}} g_n \\ &= \sum_{n=1}^{\infty} \frac{\cos(nx) - 1}{\langle n \rangle^{\frac{3}{2}}} (g_n + g_{-n}) + i \sum_{n=1}^{\infty} \frac{\sin(nx)}{\langle n \rangle^{\frac{3}{2}}} (g_n - g_{-n}), \end{aligned} \quad (5.22)$$

we first show that the first term on the right-hand side of (5.22) does not contribute to the limit in (5.21). Then, we break up the second sum into $\log \frac{1}{|x|}$ pieces, each with variance of order 1, plus a small remainder, and then apply the classical law of the iterated logarithm for a sum of i.i.d. random variables. As we see below, the leading order contribution comes from the sum

$$\sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{\sin(nx)}{\langle n \rangle^{\frac{3}{2}}} (g_n - g_{-n}). \quad (5.23)$$

We split the first sum on the right-hand side of (5.22) into $\{n > \lfloor \frac{1}{|x|} \rfloor\}$ and $\{1 \leq n \leq \lfloor \frac{1}{|x|} \rfloor\}$. The contribution from $\{n > \lfloor \frac{1}{|x|} \rfloor\}$ is a mean-zero Gaussian random variable with variance

$$\sigma_L^2 := 4 \sum_{n=L}^{\infty} \frac{(\cos(nx) - 1)^2}{\langle n \rangle^3} \lesssim L^{-2}. \quad (5.24)$$

In particular, when $L = \lfloor \frac{1}{|x|} \rfloor + 1$, we have $\sigma_L^2 = O(x^2)$. Then, for $\lambda > 0$, we have

$$P \left(\left| \sum_{n=L}^{\infty} \frac{\cos(nx) - 1}{\langle n \rangle^{\frac{3}{2}}} (g_n + g_{-n}) \right| \geq \sigma_L \lambda \right) \lesssim e^{-c\lambda^2}.$$

Taking $\lambda = c\sqrt{\log L}$ for sufficiently large $c > 1$, the right-hand side is summable in L . Hence, by the Borel-Cantelli lemma and the variance bound (5.24), there exists $C > 0$ such that

$$\sup_{L \geq L_0} \frac{\left| \sum_{n=L}^{\infty} \frac{\cos(nx) - 1}{\langle n \rangle^{\frac{3}{2}}} (g_n + g_{-n}) \right|}{CL^{-1} \sqrt{\log L}} \leq 1$$

for some $L_0 = L_0(\omega) < \infty$ with probability 1. As a consequence, we obtain

$$\limsup_{x \rightarrow 0} \frac{\left| \sum_{n > \lfloor \frac{1}{|x|} \rfloor} \frac{\cos(nx) - 1}{\langle n \rangle^{\frac{3}{2}}} (g_n + g_{-n}) \right|}{|x| \sqrt{\log \frac{1}{|x|} \log \log \log \frac{1}{|x|}}} = 0$$

almost surely.

The contribution from $\{1 \leq n \leq \lfloor \frac{1}{|x|} \rfloor\}$ to the first sum on the right-hand side of (5.22) can be estimated in a similar manner by noticing that it is a mean-zero Gaussian random variable with variance

$$\sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{(\cos(nx) - 1)^2}{\langle n \rangle^3} \lesssim x^4 \sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{n^4}{\langle n \rangle^3} \lesssim x^2.$$

This shows that the contribution from the first sum on the right-hand side of (5.22) to the limit (5.21) is 0.

Next, we consider the second sum on the right-hand side of (5.22). The contribution from $\{n > \lfloor \frac{1}{|x|} \rfloor\}$ can be estimated as above. We split the main term in (5.23) as follows. Write

$$\sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{\sin(nx)}{\langle n \rangle^{\frac{3}{2}}} (g_n - g_{-n}) = x \sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{n}{\langle n \rangle^{\frac{3}{2}}} (g_n - g_{-n}) + \sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{h(nx)}{\langle n \rangle^{\frac{3}{2}}} (g_n - g_{-n}), \quad (5.25)$$

where

$$h(z) = \sin z - z \sim z^3(1 + o(1)).$$

The second term in (5.25) can be treated as a remainder by noticing that that it is a mean-zero Gaussian random variable with variance

$$4 \sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{h^2(nx)}{\langle n \rangle^3} \lesssim x^6 \sum_{n=1}^{\lfloor \frac{1}{|x|} \rfloor} \frac{n^6}{\langle n \rangle^3} \lesssim x^2.$$

It remains to consider the first term in (5.25). First, define a sequence $\{N(k)\}_{k=0}^{\infty} \subset \mathbb{N}$ by setting $N(0) = 0$ and

$$N(k) = \min \left\{ N > N(k-1) : \sum_{n=N(k-1)+1}^N \frac{n^2}{\langle n \rangle^3} \geq 1 \right\}$$

for $k \in \mathbb{N}$. Noting that

$$\sum_{n=M}^N \frac{n^2}{\langle n \rangle^3} = \sum_{n=M}^N \frac{1}{n} + O\left(\sum_{n=M}^N \frac{1}{n^3}\right) = \log N - \log M + O\left(\frac{1}{M}\right) \quad (5.26)$$

for $N > M \geq 1$, we first see that $N(k) \geq C_1 e^k$. Using (5.26) once again,

$$\log N(k) + O(1) = \sum_{n=1}^{N(k)} \frac{n^2}{\langle n \rangle^3} \leq k + \sum_{n=1}^k \frac{1}{N(n) + 1} \leq k + O(1),$$

giving $N(k) \leq C_2 e^k$. Putting together, we have

$$C_1 e^k \leq N(k) \leq C_2 e^k. \quad (5.27)$$

Now, we define a sequence $\{X_k\}_{k \in \mathbb{N}}$ of independent Gaussian random variables by setting

$$X_k = x \sum_{n=N(k-1)+1}^{N(k)} \frac{n}{\langle n \rangle^{\frac{3}{2}}} (g_n - g_{-n}).$$

Then, we have

$$\mathbb{E}[|\operatorname{Re} X_k|^2] = \mathbb{E}[|\operatorname{Im} X_k|^2] = x^2(1 + O(N(k)^{-1})).$$

Finally, define $L(|x|)$ by

$$L(|x|) = \inf \left\{ k : N(k) \geq \left\lfloor \frac{1}{|x|} \right\rfloor \right\}.$$

Then, from (5.27), we have

$$L(|x|) = \log \frac{1}{|x|} \cdot (1 + o(1)).$$

Applying Khintchine’s law of the iterated logarithm to the sum

$$S(x) = \sum_{k=1}^{L(|x|)} X_k,$$

we find

$$\limsup_{x \rightarrow 0} \frac{S(x)}{2^{\frac{3}{2}} |x| \sqrt{\log \frac{1}{|x|} \log \log \log \frac{1}{|x|}}} = 1$$

almost surely. This completes the proof of Proposition 5.7. □

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