L^q-SPECTRUM OF SELF-SIMILAR MEASURES WITH OVERLAPS IN THE ABSENCE OF SECOND-ORDER IDENTITIES

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Abstract. For a self-similar measure in $\mathbb{R}^d$ with overlaps but satisfies the so-called "bounded measure type condition" introduced by Tang and the authors, we set up a framework for deriving a closed formula for the L^q-spectrum of the measure for $q \geq 0$. The framework allows us to include iterated function systems that have different contraction ratios and those in higher dimension. For self-similar measures with overlaps, closed formulas for $\tau(q)$ have only been obtained earlier for measures satisfying Strichartz second-order identities. We illustrate how to use our results to prove the differentiability of the L^q-spectrum, obtain the multifractal dimension spectrum, and compute the Hausdorff dimension of the measure.

Contents

1. Introduction

Let μ be a positive finite Borel measure on $\mathbb{R}^d$ whose support $\operatorname{supp}(\mu)$ is compact. For $q \in \mathbb{R}$, the *L^q-spectrum* $\tau(q)$ of μ is defined as

$$\tau(q) := \varliminf_{\delta \to 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta},$$

where $B_\delta(x_i)$ is a disjoint family of δ-balls with center $x_i \in \operatorname{supp}(\mu)$ and the supremum is taken over all such families. The function $\tau(q)$ arises in the theory of multifractal decomposition of

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measures. A major goal of the theory is to compute the following *dimension spectrum*:

$$f(\alpha) := \dim_{\mathrm{H}} \left\{ x \in \operatorname{supp}(\mu) : \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\},$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. The multifractal formalism, a heuristic principle first proposed by physicists (see [7, 8] and the references therein), asserts that the dimension spectrum is equal to the Legendre transform of $\tau(q)$, i.e.,

$$f(\alpha) = \tau^*(\alpha) := \inf\{q\alpha - \tau(q) : \ q \in \mathbb{R}\}.$$

We are mainly interested in self-similar measures. For such measures, the multifractal formalism has been verified rigorously for those satisfying the separated open set condition [1, 3]. For self-similar measures defined by iterated function systems satisfying the weak separation condition, Lau and the first author [13] proved that if $\tau(q)$ is differentiable at $q \geq 0$, then the multifractal formulism at the corresponding point holds. Feng and Lau [5] removed the differentiability condition; they also studied the validity of the multiformal formalism in the region $q < 0$.

The L^q-spectrum also encodes other important information of the measure. For example, $\tau(0)$ is the negative of the box dimension of μ; if τ is differentiable at $q = 1$, then $\tau'(1)$ is equal to the Hausdorff dimension of μ (see [9, 13, 19, 23] and the references therein); for $q > 1$, $\tau(q)/(q-1)$ is the L^q-dimension of μ (see [24]).

The computation of L^q-spectrum thus plays a key role in the theory of multifractal measures. For self-similar and graph-directed self-similar measures satisfying the open set condition, $\tau(q)$ is computed by Cawley and Mauldin [1] and Edgar and Mauldin [3]. For self-similar measures with overlaps, the computation is much more difficult. Lau and the first author obtained $\tau(q)$, $q \geq 0$, for the infinite Bernoulli convolution associated with the golden ratio [12] and a class of convolutions of Cantor measures [14]. Feng [4] computed $\tau(q)$ for infinite Bernoulli convolutions associated with a class of Pisot numbers. The graph of $\tau(q)$ for $q < 0$ has been studied by Lau, Wang, Feng and Olivier [4, 6, 17].

The computation of $\tau(q)$ in [12] and [14] makes use of Strichartz second-order self-similar identities. Unfortunately, very few self-similar measures satisfy these identities. Thus, closed formulas for $\tau(q)$ have been obtained for only a few classes of measures that are defined by iterated function systems on $\mathbb{R}$ with the same contraction ratio. The main objective of this paper is to derive a closed formula for $\tau(q)$, $q \geq 0$, for self-similar measures satisfying the so-called *bounded measure type condition* (*Condition (B)*) introduced in [22]. It is worth mentioning that recently G. Deng and the first author [2] used a infinite matrix method to obtain the differentiability of the L^q-spectrum for a class if IFSs that includes some of those studies in this paper; however, the method does not yield a closed formula for $\tau(q)$.

Throughout this paper an *iterated function system (IFS)* refers to a finite family of contractions defined on a compact subset X of $\mathbb{R}^d$. The derivation of $\tau(q)$ in this paper is based on the following equivalent definition, which holds for $q \geq 0$:

$$\tau(q) = \inf\left\{\alpha : \overline{\lim_{\delta\to 0^+}} \frac{1}{\delta^{d+\alpha}} \int_X \mu(B_\delta(x))^q \, dx > 0\right\}. \tag{1.1}$$

(see [11, 12] and [13, Proposition 3.1]).

Let μ be a self-similar measure defined by a finite type IFS (see [10,15,21]) on $\mathbb{R}^d$. In Section 2, we define the set of all level-k islands $\mathfrak{I}_k$ (see Definition 2.6). Intuitively, each level-k island corresponds to a connected component of the level-k iterates of some fixed open set Ω; moreover, two islands $\mathcal{I}_1$ and $\mathcal{I}_2$ are of the same *measure type* (with respect to μ) if $\mu|_{I_2} = c\mu|_{I_1} \circ S^{-1}$ for some constant $c > 0$ and some similitude $S : I_1 \to I_2$, where I_j $(j = 1, 2)$ is the component corresponding to $\mathcal{I}_j$, and $\mu|_F$ denotes the restriction of the measure μ to $F \subseteq \mathbb{R}^d$.

To compute $\tau(q)$ we divide level-k iterates of some fixed bounded open set Ω under the IFS into connected components called *level-k islands*, and classify them into measure types. Our main assumption is Condition (B) introduced in [22], which, loosely speaking, holds if there exists some $k \geq 1$ such that there is a uniform bound on those level-m $(m > k)$ islands whose measure types, as well the measure types of their ancestors up to level $k+1$, are different from that of any level-k island. If $k =: k_b$ is the smallest integer satisfying this condition, we call the corresponding $\mathfrak{I}_{k_b} =: \mathbf{I}_b$ the *basic set of islands* (see Definition 2.15).

Assume $\{S_i\}_{i\in\Lambda}$ is a finite type IFS on $\mathbb{R}^d$ (see [15]) with Ω being a finite type condition set and assume that Condition (B) holds with $\mathbf{I}_b$ being the basic set of islands. Let $\mathbf{I} := \{\mathcal{I}_{1,\ell}\}_{\ell\in\Gamma} \subseteq \mathbf{I}_b$ be a minimal subset such that the measure type of any island in $\mathbf{I}_b$ equals that of some island in $\mathbf{I}$. Fix $q \geq 0$, define

$$\varphi_\ell(\delta) := \int_{I_{1,\ell}} \mu(B_\delta(x))^q \, dx \qquad \text{and} \qquad \Phi_\ell^{(\alpha)}(\delta) := \frac{1}{\delta^{d+\alpha}} \varphi_\ell(\delta) \qquad \text{for } \ell \in \Gamma,$$

where $I_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega)$. Then we can derive renewal equations for $\Phi_\ell^{(\alpha)}(\delta)$, and express them in vector form as:

$$\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z},$$

where $\alpha \in \mathbb{R}$, and

$$\begin{aligned}
&\mathbf{f} = \mathbf{f}^{(\alpha)}(x) = [f_\ell^{(\alpha)}(x)]_{\ell\in\Gamma}, \quad x \in \mathbb{R};\\
&f_\ell^{(\alpha)}(x) := \Phi_\ell^{(\alpha)}(e^{-x}) \quad \text{for } \ell \in \Gamma;\\
&\mathbf{M}_\alpha = [\mu_{m\ell}^{(\alpha)}]_{\ell,m\in\Gamma} \quad \text{is a finite matrix of Borel measures on } \mathbb{R};\\
&\mathbf{z} = \mathbf{z}^{(\alpha)}(x) = [z_\ell^{(\alpha)}(x)]_{\ell\in\Gamma} \quad \text{is a vector of error functions.}
\end{aligned} \tag{1.2}$$

Let

$$\mathbf{M}_\alpha(\infty) := \left[\mu_{m\ell}^{(\alpha)}(\mathbb{R})\right]_{\ell,m\in\Gamma}. \tag{1.3}$$

For each $\ell \in \Gamma$ and $\alpha \in \mathbb{R}$, define

$$F_\ell(\alpha) := \sum_{m\in\Gamma} \mu_{m\ell}^{(\alpha)}(\mathbb{R}) \quad \text{and} \quad D_\ell := \{\alpha \in \mathbb{R} : F_\ell(\alpha) < \infty\}. \tag{1.4}$$

If the error functions decay exponentially to 0 as $x\to\infty$, then the L^q-spectrum of μ is given by the unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1. The following is our main result.

Theorem 1.1. *Let μ be a self-similar measure defined by a finite type IFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$. Assume that μ satisfies Condition (B). Let $\mathbf{M}_\alpha(\infty)$ and $F_\ell(\alpha)$ be defined as in* (1.3) *and* (1.4)*.*

(a) *There exists a unique $\alpha\in\mathbb{R}$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to* 1*.*
(b) *If we assume, in addition, that for the unique α in (a), there exists $\epsilon>0$ such that for all $i\in\Gamma, z_\ell^{(\alpha)}(x)=o(e^{-\epsilon x})$ as $x\to\infty$. Then $\tau(q)=\alpha$ for $q\geq 0$.*

In Section 4, we illustrate Theorem 1.1 by the following family of IFSs on $\mathbb{R}$:

$$S_1(x)=\rho x,\qquad S_2(x)=rx+\rho(1-r),\qquad S_3(x)=rx+1-r, \tag{1.5}$$

where the contraction ratios $\rho,r\in(0,1)$ satisfy

$$\rho+2r-\rho r\leq 1, \tag{1.6}$$

i.e., $S_2(1)\leq S_3(0)$ (see Figure 1). This family of IFSs is first studied by Lau and Wang [16], and is used to illustrate the (general) finite type condition in [10, 15]. For a probability vector $(p_i)_{i=1}^3$, we define

$$w_1(k):=p_1\sum_{j=0}^k p_2^{k-j}p_3^j,\qquad k\geq 0. \tag{1.7}$$

Theorem 1.2. *Let μ be a self-similar measure defined by an IFS in* (1.5) *together with a probability vector $(p_i)_{i=1}^3$, and $w_1(k)$ be defined as in* (1.7)*. Then for $q\geq 0$, there exists a unique real number $\alpha:=\alpha(q)$ satisfying*

$$\rho^{-\alpha}(1-p_2^q r^{-\alpha})(1-p_3^q r^{-\alpha})\sum_{k=0}^\infty w_1(k)^q(r^{-\alpha})^k+r^{-\alpha}(p_2^q+p_3^q)=1. \tag{1.8}$$

Hence $\tau(q)=\alpha$. Moreover, τ is differentiable on $(0,\infty)$ and

$$\begin{aligned}&\dim_{\mathrm{H}}(\mu)=\tau'(1)=\\&\Big(\big(\sum_{i=2}^3 p_i\ln p_i-p_2p_3\sum_{i=2}^3\ln p_i\big)\sum_{k=0}^\infty w_1(k)-\Big(\prod_{i=2}^3(1-p_i)\Big)\sum_{k=0}^\infty w_1(k)\ln w_1(k)-\sum_{i=2}^3 p_i\ln p_i\Big)\\&\times\Big((p_2+p_3-2p_2p_3)\sum_{k=0}^\infty w_1(k)\ln r-\Big(\prod_{i=2}^3(1-p_i)\Big)\sum_{k=0}^\infty w_1(k)\ln(\rho r^k)-\sum_{i=2}^3 p_i\ln r\Big)^{-1}.\end{aligned}$$

Remark 1.3. *Substituting $q=0$ in* (1.8) *gives $\rho^{-\tau(0)}+2r^{-\tau(0)}-(\rho r)^{-\tau(0)}=1$. Hence $-\tau(0)$ is the Hausdorff dimension of the corresponding self-similar set (see [10, 15, 16]).*

In Section 4, we illustrate Theorem 1.1 by the following family of IFSs on $\mathbb{R}^2$:

$$\begin{aligned}S_1(\mathbf{x})&=\rho\mathbf{x}, & S_2(\mathbf{x})&=r\mathbf{x}+(\rho-\rho r,0),\\ S_3(\mathbf{x})&=r\mathbf{x}+(1-r,0), & S_4(\mathbf{x})&=r\mathbf{x}+(0,1-r),\end{aligned} \tag{1.9}$$

where the contraction ratios $\rho, r \in (0,1)$ satisfy

$$\rho + 2r - \rho r \le 1, \tag{1.10}$$

i.e., $S_2(1,0) \le S_3(0,0)$ (see Figure 2(a)). For any probability vector $(p_i)_{i=1}^4$, define

$$w_2(k) := p_1 \sum_{j=0}^{k} p_2^{k-j} p_3^j, \qquad k \ge 0. \tag{1.11}$$

Theorem 1.4. *Let μ be a self-similar measure defined by any IFS in* (1.9) *together with a probability vector $(p_i)_{i=1}^4$, and $w_2(k)$ be defined as in* (1.11). *For $q \ge 0$, there exists a unique real number $\alpha := \alpha(q)$ satisfying*

$$\rho^{-\alpha}\big(1 - p_2^q r^{-\alpha}\big)\big(1 - p_3^q r^{-\alpha}\big) \sum_{k=0}^{\infty} w_2(k)^q (r^{-\alpha})^k + r^{-\alpha} \sum_{i=2}^{4} p_i^q = 1. \tag{1.12}$$

Hence $\tau(q) = \alpha$. Moreover, τ is differentiable on $(0,\infty)$ and

$$\dim_{\mathrm{H}}(\mu) = \tau'(1) =$$

$$\Big(\big(\sum_{i=2}^{3} p_i \ln p_i - p_2 p_3 \sum_{i=2}^{3} \ln p_i \big) \sum_{k=0}^{\infty} w_2(k) - \Big(\prod_{i=2}^{3} (1-p_i) \Big) \sum_{k=0}^{\infty} w_2(k) \ln w_2(k) - \sum_{i=2}^{4} p_i \ln p_i \Big)$$

$$\times \Big(\big(p_2 + p_3 - 2p_2p_3\big) \sum_{k=0}^{\infty} w_2(k) \ln r - \Big(\prod_{i=2}^{3} (1-p_i) \Big) \sum_{k=0}^{\infty} w_2(k) \ln(\rho r^k) - \sum_{i=2}^{4} p_i \ln r \Big)^{-1}.$$

Remark 1.5. *Substituting $q = 0$ into* (1.12), *we get $\rho^{-\tau(0)} + 3r^{-\tau(0)} - (\rho r)^{-\tau(0)} = 1$. Again, $-\tau(0)$ is the Hausdorff dimension of the corresponding self-similar set (see [15, Example 5.2]).*

We use the vector-valued renewal theorem of Lau, Wang and Chu [18] to derive the stated formulas for $\tau(q)$; the classical renewal theorem used in [12] and [14] is not sufficient, as a finite number of renewal equations arise in our derivations. New techniques are also used in estimating the error terms and in proving the differentiability of $\tau(q)$.

This paper is organized as follows. In Section 2, we briefly recall the definition of Condition (B). In Section 3 we derive renewal equations and prove Theorem 1.1. Section 4 illustrates Theorem 1.1 by the class of one-dimensional IFSs (1.5) and prove Theorem 1.2. Section 5 studies IFSs in higher dimension and prove Theorem 1.4. Finally we state some comments and open questions in Section 6.

2. Self-similar measures of bounded measure type

In this section, we recall the definition of Condition (B) and then prove that it is satisfied by the self-similar measures defined by the IFSs in (1.9).

Let X be a compact subset of $\mathbb{R}^d$ with nonempty interior, and $\{S_i\}_{i\in\Lambda}$ be an IFS of contractive similitudes on X with attractor $K \subseteq \mathbb{R}^d$. To each probability vector $(p_i)_{i\in\Lambda}$ (i.e., $p_i > 0$ and $\sum_{i\in\Lambda} p_i = 1$), let μ be the associated self-similar measure, which satisfies the *self-similar identity*

$$\mu = \sum_{i\in\Lambda} p_i \mu \circ S_i^{-1}.$$

Moreover, $\operatorname{supp}(\mu) = K$.

2.1. **Finite type condition and measure type.** For $k \geq 1$, define

$$\Lambda^k := \{(i_1, \ldots, i_k) : i_j \in \Lambda \text{ for } j = 1, \ldots, k\},$$

where we call $\boldsymbol{i} \in \Lambda^k$ a word of length k, and denote its length by $|\boldsymbol{i}|$. If $k = 0$, we define $\Lambda^0 := \{\emptyset\}$. Also, we let $\Lambda^* := \bigcup_{k\geq 0} \Lambda^k$. We frequently write $\boldsymbol{i} := i_1 \cdots i_k$ instead of $\boldsymbol{i} = (i_1, \ldots, i_k)$ if no confusion is possible; in particular, we write $\boldsymbol{i} =: i_1^k$, if $i_j = i_1$ for all $j = 1, \ldots, k$. For $k \geq 0$ and $\boldsymbol{i} = i_1 \cdots i_k \in \Lambda^k$, we use the standard notation

$$S_{\boldsymbol{i}} := S_{i_1} \circ \cdots \circ S_{i_k}, \qquad r_{\boldsymbol{i}} := r_{i_1} \cdots r_{i_k}, \qquad p_{\boldsymbol{i}} := p_{i_1} \cdots p_{i_k},$$

with $S_\emptyset := id, r_\emptyset = p_\emptyset := 1$, where id is the identity map on $\mathbb{R}^d$.

For two indices $\boldsymbol{i}, \boldsymbol{j} \in \Lambda^*$, we write $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$ if $\boldsymbol{i}$ is a prefix of $\boldsymbol{j}$ or $\boldsymbol{i} = \boldsymbol{j}$, and denote by $\boldsymbol{i} \not\preccurlyeq \boldsymbol{j}$ if $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$ does not hold. Let $\{\mathcal{M}_k\}_{k=1}^\infty$ be a sequence of index sets, where $\mathcal{M}_k \subseteq \Lambda^*$. Let

$$\underline{m}_k = \underline{m}_k(\mathcal{M}_k) := \min\{|\boldsymbol{i}| : \boldsymbol{i} \in \mathcal{M}_k\} \quad \text{and} \quad \overline{m}_k = \overline{m}_k(\mathcal{M}_k) := \max\{|\boldsymbol{i}| : \boldsymbol{i} \in \mathcal{M}_k\}.$$

We also let $\mathcal{M}_0 := \{\emptyset\}$.

Definition 2.1. *We say that $\{\mathcal{M}_k\}_{k=0}^\infty$ is a* sequence of nested index sets *if it satisfies the following conditions:*

1. *both $\{\underline{m}_k\}$ and $\{\overline{m}_k\}$ are nondecreasing, and $\lim_{k\to\infty} \underline{m}_k = \lim_{k\to\infty} \overline{m}_k = \infty$;*
2. *for each $k \geq 1$, $\mathcal{M}_k$ is an antichain in Λ^*;*
3. *for each $\boldsymbol{j} \in \Lambda^*$ with $|\boldsymbol{j}| > \overline{m}_k$ or $\boldsymbol{j} \in \mathcal{M}_{k+1}$, there exists $\boldsymbol{i} \in \mathcal{M}_k$ such that $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$;*
4. *for each $\boldsymbol{j} \in \Lambda^*$ with $|\boldsymbol{j}| < \underline{m}_k$ or $\boldsymbol{j} \in \mathcal{M}_{k-1}$, there exists $\boldsymbol{i} \in \mathcal{M}_k$ such that $\boldsymbol{j} \preccurlyeq \boldsymbol{i}$;*
5. *there exists a positive integer L_0, independent of k, such that for all $\boldsymbol{i} \in \mathcal{M}_k$ and $\boldsymbol{j} \in \mathcal{M}_{k+1}$ with $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$, we have $|\boldsymbol{j}| - |\boldsymbol{i}| \leq L_0$.*

To define neighborhood types, we fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^\infty$.

Notation 2.2. (1) *For each integer $k \geq 0$, let $\mathcal{V}_k$ be the set of* level-k vertices *(with respect to $\{\mathcal{M}_k\}$) defined as*

$$\mathcal{V}_0 := \big\{(id, 0)\big\} \qquad \text{and} \qquad \mathcal{V}_k := \big\{(S_{\boldsymbol{i}}, k) : \boldsymbol{i} \in \mathcal{M}_k\big\} \quad \text{for all } k \geq 1.$$

We call $(id, 0)$ the root vertex *and denote it by $\boldsymbol{v}_{root}$.*

(2) *Let $\mathcal{V} := \bigcup_{k\geq 0} \mathcal{V}_k$ be the set of all vertices.*

(3) *For $\boldsymbol{v} = (S_{\boldsymbol{i}}, k) \in \mathcal{V}_k$, we use the convenient notation $S_{\boldsymbol{v}} := S_{\boldsymbol{i}}$ and $r_{\boldsymbol{v}} := r_{\boldsymbol{i}}$. It is possible to have $\boldsymbol{v} = (S_{\boldsymbol{i}}, k) = (S_{\boldsymbol{j}}, k)$ with $\boldsymbol{i} \neq \boldsymbol{j}$.*

(4) *More generally, for any $k \geq 0$ and any subset $\mathcal{A} \subset \mathcal{V}_k$, we use the notation*

$$S_{\mathcal{A}}(\Omega) := \bigcup_{\boldsymbol{v}\in\mathcal{A}} S_{\boldsymbol{v}}(\Omega). \tag{2.1}$$

Let $\Omega \subseteq X$ be a nonempty bounded open set which is *invariant* under $\{S_i\}_{i\in\Lambda}$, i.e., $\bigcup_{i\in\Lambda} S_i(\Omega) \subseteq \Omega$. Such an Ω exists by our assumption; in particular, X° is such a set.

Next, we recall the definitions of neighbors and neighborhoods.

Definition 2.3. *We say that two level-k vertices $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{V}_k$ (allowing $\boldsymbol{v} = \boldsymbol{v}'$) are* neighbors *(with respect to Ω and $\{\mathcal{M}_k\}$) if $S_{\boldsymbol{v}}(\Omega) \cap S_{\boldsymbol{v}'}(\Omega) \neq \emptyset$. We call the set of vertices*

$$\mathfrak{N}_\Omega(\boldsymbol{v}) := \{\boldsymbol{v}' : \boldsymbol{v}' \in \mathcal{V}_k \text{ is a neighbor of } \boldsymbol{v}\}$$

the neighborhood *of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$).*

Obviously $\boldsymbol{v} \in \mathfrak{N}_\Omega(\boldsymbol{v})$. If no confusion is possible, we omit the subscript Ω in $N_\Omega(\boldsymbol{v})$.

Let $\mathscr{S} := \{S_{\boldsymbol{j}} S_{\boldsymbol{i}}^{-1} : \boldsymbol{i}, \boldsymbol{j} \in \Lambda^*\}$. We define an equivalence relation on the set of vertices $\mathcal{V}$.

Definition 2.4. *Two vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ are said to be* equivalent*, denoted by $\boldsymbol{v} \sim_\sigma \boldsymbol{v}'$ (or simply $\boldsymbol{v} \sim \boldsymbol{v}'$), if for $\sigma := S_{\boldsymbol{v}'} S_{\boldsymbol{v}}^{-1} (\in \mathscr{S}) : \bigcup_{\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})} S_{\boldsymbol{u}}(X) \to X$, the following conditions hold:*

(1) $\{S_{\boldsymbol{u}'} : \boldsymbol{u}' \in \mathfrak{N}(\boldsymbol{v}')\} = \{\sigma S_{\boldsymbol{u}} : \boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})\}$*; in particular, $\sigma S_{\boldsymbol{u}}$ is defined for all $\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})$.*
(2) *for $\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})$ and $\boldsymbol{u}' \in \mathfrak{N}(\boldsymbol{v}')$ such that $S_{\boldsymbol{u}'} = \sigma S_{\boldsymbol{u}}$, and for any positive integer $\ell \geq 1$, an index $\boldsymbol{i} \in \Lambda^*$ satisfies $(S_{\boldsymbol{u}} S_{\boldsymbol{i}}, k+\ell) \in \mathcal{V}_{k+\ell}$ if and only if it satisfies $(S_{\boldsymbol{u}'} S_{\boldsymbol{i}}, k'+\ell) \in \mathcal{V}_{k'+\ell}$.*

It is direct to check that $\sim$ is an equivalence relation. We denote the equivalence class containing $\boldsymbol{v}$ by $[\boldsymbol{v}]$ and call it the *(neighborhood) type* of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$).

We define an infinite graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and directed edges defined as follows. Let $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{u} \in \mathcal{V}_{k+1}$. Suppose there exists $\boldsymbol{i} \in \mathcal{M}_k$, $\boldsymbol{j} \in \mathcal{M}_{k+1}$, and $\boldsymbol{l} \in \Lambda^*$ such that

$$\boldsymbol{v} = (S_{\boldsymbol{i}}, k), \qquad \boldsymbol{u} = (S_{\boldsymbol{j}}, k+1), \qquad \boldsymbol{j} = (\boldsymbol{i}, \boldsymbol{l}).$$

Then we connect a directed edge $\boldsymbol{l} : \boldsymbol{v} \to \boldsymbol{u}$. We call $\boldsymbol{v}$ a *parent* of $\boldsymbol{u}$ and $\boldsymbol{u}$ an *offspring* of $\boldsymbol{v}$. We write $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of all directed edges defined above.

Definition 2.5. *Let $\{S_i\}_{i\in\Lambda}$ be an IFS of contractive similitudes on a compact subset $X \subseteq \mathbb{R}^d$. We say that $\{S_i\}_{i\in\Lambda}$ is of* finite type *(or that it satisfies the* finite type condition*) if there exists a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^\infty$ and a nonempty bounded invariant open set $\Omega \subseteq X$ such that, with respect to Ω and $\{\mathcal{M}_k\}$, the set of equivalence classes $\mathcal{V}/_\sim := \{[\boldsymbol{v}] : \boldsymbol{v} \in \mathcal{V}\}$ is finite. We call such an Ω a* finite type condition set *(or FTC set).*

Definition 2.6. *A subset $\mathcal{I} \subseteq \mathcal{V}_k$ is called a* level-k island *(with respect to Ω and $\{\mathcal{M}_k\}$) if $S_{\mathcal{I}}(\Omega)$ is a connected component of $S_{\mathcal{V}_k}(\Omega)$.*

Remark 2.7. (1) *For each $\boldsymbol{v} \in \mathcal{V}_k$, there exists a unique island, denoted by $\mathcal{I}(\boldsymbol{v})$, containing $\boldsymbol{v}$ and, moreover, $\mathfrak{N}(\boldsymbol{v}) \subseteq \mathcal{I}(\boldsymbol{v})$.*
(2) *If $\{S_i\}_{i\in\Lambda}$ satisfies (OSC) with Ω being an OSC set, then $\mathcal{I}(\boldsymbol{v}) = \{\boldsymbol{v}\}$ for all $\boldsymbol{v} \in \mathcal{V}$.*

Notation 2.8. (1) *Let*

$$\mathfrak{I}_k := \{\mathcal{I} : \mathcal{I} \text{ is a level-}k \text{ island}\} \qquad \text{and} \qquad \mathfrak{I} := \bigcup_{k\geq 0} \mathfrak{I}_k$$

be the collection of all level-k islands and the collection of all islands, respectively.

(2) *Generalizing* (2.1), *for any* $k \geq 0$ *and any subset* $\mathcal{B} \subset \mathfrak{I}_k$, *we use the notation*

$$S_{\mathcal{B}}(\Omega) := \bigcup_{\mathcal{I} \in \mathcal{B}} S_{\mathcal{I}}(\Omega).$$

Definition 2.9. *We say that two islands* $\mathcal{I} \in \mathfrak{I}_k$ *and* $\mathcal{I}' \in \mathfrak{I}_{k'}$ *are* equivalent, *and denote it by* $\mathcal{I} \approx_\sigma \mathcal{I}'$ *(or simply,* $\mathcal{I} \approx \mathcal{I}'$*), if there exists some* $\sigma \in \mathscr{S}$ *such that* $\{S_{\boldsymbol{v}'} : \boldsymbol{v}' \in \mathcal{I}'\} = \{\sigma S_{\boldsymbol{v}} : \boldsymbol{v} \in \mathcal{I}\}$ *and, moreover,* $\boldsymbol{v} \sim_\sigma \boldsymbol{v}'$ *for any* $\boldsymbol{v} \in \mathcal{I}$ *and* $\boldsymbol{v}' \in \mathcal{I}'$ *satisfying* $S_{\boldsymbol{v}'} = \sigma S_{\boldsymbol{v}}$.

Notation 2.10. (1) *We denote the equivalence class of* $\mathcal{I}$ *by* $[\mathcal{I}]$ *and we call* $[\mathcal{I}]$ *the* (island) type *of* $\mathcal{I}$.

(2) *For* $\mathcal{I} \in \mathfrak{I}_k$, $\mathcal{I}' \in \mathfrak{I}_{k+1}$, $\mathcal{I}$ *is said to be a* parent *of* $\mathcal{I}'$ *and* $\mathcal{I}'$ *an* offspring *of* $\mathcal{I}$ *if for any* $\boldsymbol{v} \in \mathcal{I}'$, $\mathcal{I}$ *contains some parent of* $\boldsymbol{v}$. *For any* $k \geq 0$ *and* $\mathcal{I} \in \mathfrak{I}_k$, *let*

$$O(\mathcal{I}) := \{\mathcal{J} : \mathcal{J} \text{ is an offspring of } \mathcal{I}\} \tag{2.2}$$

be the collection of all offspring of $\mathcal{I}$.

Definition 2.11. *Let* μ *be a self-similar measure defined by an IFS* $\{S_i\}_{i \in \Lambda}$ *of finite type with* Ω *being an FTC set. Two equivalent vertices* $\boldsymbol{v} \in \mathcal{V}_k$ *and* $\boldsymbol{v}' \in \mathcal{V}_{k'}$ *are* μ-equivalent, *denoted by* $\boldsymbol{v} \sim_{\mu,\sigma,w} \boldsymbol{v}'$ *(or simply* $\boldsymbol{v} \sim_\mu \boldsymbol{v}'$*) if for* $\sigma = S_{\boldsymbol{v}'} \circ S_{\boldsymbol{v}}^{-1}$, *there exists a number* $w > 0$ *such that*

$$\mu|_{S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)} = w \cdot \mu|_{S_{\mathfrak{N}(\boldsymbol{v})}(\Omega)} \circ \sigma^{-1}.$$

As $\sim$ is an equivalence relation, so is $\sim_\mu$. Denote the μ-equivalence class of $\boldsymbol{v}$ by $[\boldsymbol{v}]_\mu$ and call it the *(neighborhood) measure type* of $\boldsymbol{v}$ (with respect to Ω, $\{\mathcal{M}_k\}$ and μ). Intuitively, $\boldsymbol{v} \sim_\mu \boldsymbol{v}'$ means that the measures $\mu|_{S_{\mathfrak{N}(\boldsymbol{v})}(\Omega)}$ and $\mu|_{S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)}$ have the same structure. The following proposition shows that μ-equivalent vertices generate the same number of offspring of each neighborhood measure type. The proof can be found in [22].

Proposition 2.12. *For two equivalent vertices* $\boldsymbol{v} \in \mathcal{V}_k$ *and* $\boldsymbol{v}' \in \mathcal{V}_{k'}$, *let* $\{\boldsymbol{u}_i\}_{i \in \Lambda_1}$ *and* $\{\boldsymbol{u}'_i\}_{i \in \Lambda'_1}$ *be the offspring of* $\boldsymbol{v}$ *and* $\boldsymbol{v}'$ *in* $\mathcal{G}$, *respectively. If* $[\boldsymbol{v}]_\mu = [\boldsymbol{v}']_\mu$, *then, counting multiplicity,* $\{[\boldsymbol{u}_i]_\mu : i \in \Lambda_1\} = \{[\boldsymbol{u}'_i]_\mu : i \in \Lambda'_1\}$.

Definition 2.13. *Let* μ *be a self-similar measure defined by a finite type IFS* $\{S_i\}_{i \in \Lambda}$ *on* $\mathbb{R}^d$ *with* Ω *being an FTC set. Two islands* $\mathcal{I} \in \mathfrak{I}_k$ *and* $\mathcal{I}' \in \mathfrak{I}_{k'}$ *are said to be* μ-equivalent, *denoted* $\mathcal{I} \approx_{\mu,\sigma,w} \mathcal{I}'$ *(or simply* $\mathcal{I} \approx_\mu \mathcal{I}'$*), if* $\mathcal{I} \approx_\sigma \mathcal{I}'$ *and there exists some* $w > 0$ *such that*

$$\mu|_{S_{\mathcal{I}'}(\Omega)} = w \cdot \mu|_{S_{\mathcal{I}}(\Omega)} \circ \sigma^{-1}. \tag{2.3}$$

We remark that (2.3) holds if and only if $\boldsymbol{v} \sim_{\mu,\sigma,w} \boldsymbol{v}'$ for any $\boldsymbol{v} \in \mathcal{I}$ and $\boldsymbol{v}' \in \mathcal{I}'$ satisfying $S_{\boldsymbol{v}'} = \sigma S_{\boldsymbol{v}}$. We note that $\approx_\mu$ is an equivalence relation. We denote the μ-equivalence class of $\mathcal{I}$ by $[\mathcal{I}]_\mu$, and call $[\mathcal{I}]_\mu$ the *(island) measure type* of $\mathcal{I}$ (with respect to Ω, $\{\mathcal{M}_k\}$ and μ). From the definition of $\approx_\mu$, we obtain an analog of Proposition 2.12 concerning $\approx_\mu$. That is, μ-equivalent islands generate the same number of offspring of each island measure type.

Definition 2.14. *Let* μ *be a self-similar measure defined by a finite type IFS. Let* $\mathbf{B} \subseteq \mathfrak{I}_k$ *for* $k \geq 0$ *and* $\mathcal{B}_\mu := \{[\mathcal{I}]_\mu : \mathcal{I} \in \mathbf{B}\}$. *We call* $\mathcal{I}$ *a* level-2 nonbasic island *with respect to* $\mathbf{B}$ *if*

$\mathcal{I} \in O(\mathcal{J})$ for some $\mathcal{J} \in \mathbf{B}$ and $[\mathcal{I}]_\mu \notin \mathcal{B}_\mu$. Inductively, for $\ell \geq 3$, we call $\mathcal{I}$ a level-ℓ nonbasic island *with respect to $\mathbf{B}$ if $\mathcal{I}$ is an offspring of some level-$(\ell-1)$ nonbasic island with respect to $\mathbf{B}$ and $[\mathcal{I}]_\mu \notin \mathcal{B}_\mu$.*

We remark that, by definition, for any $\ell \geq 2$, $\mathcal{I}$ is a level-ℓ nonbasic island with respect to $\mathbf{B}$ if and only if there exists a finite sequence of $\{\mathcal{I}_k\}_{k=1}^{\ell}$ such that $\mathcal{I}_1 \in \mathbf{B}$, $\mathcal{I}_\ell = \mathcal{I}$, $[\mathcal{I}_i]_\mu \notin \mathcal{B}_\mu$, and $\mathcal{I}_i$ is an offspring of $\mathcal{I}_{i-1}$ for all $i = 2, \ldots, \ell$. In particular, $\mathcal{I}_i$ is a level-i nonbasic island with respect to $\mathbf{B}$ for all $i = 2, \ldots, \ell$.

Definition 2.15. *Let μ be a self-similar measure defined by a finite type IFS on $\mathbb{R}^d$. We say that μ satisfies* Condition (B) *if there exists some $k \geq 1$ such that the number of level-ℓ nonbasic islands with respect to $\mathfrak{I}_k$ is uniformly bounded for all $\ell > k$. If $k =: k_b$ is the minimum non-negative integer satisfying this condition, then we call the corresponding $\mathfrak{I}_{k_b} =: \mathbf{I}_b$ the* basic set of islands.

The following two classes of examples for Condition (B) are proved in [22].

Example 2.16. *Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i\in\Lambda}$ in $\mathbb{R}^d$ satisfying (OSC). Then μ satisfies Condition (B).*

Let $\{S_i\}_{i=1}^3$ be defined as in (1.5) and μ be the self-similar measure associated with a probability vector $(p_i)_{i=1}^3$. Let $w_1(k)$, $k \geq 0$, be defined as in (1.7). We remark that for $k \geq 0$,

$$p_1 p_3^{k+1} + p_2 w_1(k) = p_1 p_2^{k+1} + p_3 w_1(k) = w_1(k+1) \quad \text{and} \quad w_1(k+1) \leq w_1(k) \leq p_1. \tag{2.4}$$

Example 2.17. *Let μ be the self-similar measure defined by any of the IFSs $\{S_i\}_{i=1}^3$ in* (1.5) *together with a probability vector $(p_i)_{i=1}^3$. Then μ satisfies Condition (B).*

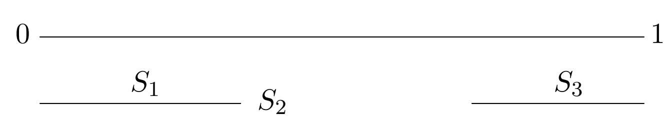


FIGURE 1. The first iteration of $\{S_i\}_{i=1}^3$ defined in (1.5). The figure is drawn with $\rho = 1/3$ and $r = 2/7$.

2.2. **Condition (B) for a class of IFSs on $\mathbb{R}^2$.** In this subsection, we prove that any self-similar measure defined by an IFS in (1.9) satisfies Condition (B).

Let $\{S_i\}_{i=1}^4$ be defined as in (1.9) and μ be the self-similar measure associated with a probability vector $(p_i)_{i=1}^4$. Let $w_2(k)$, $k \geq 0$, be defined as in (1.11). We remark that for $k \geq 0$,

$$p_1 p_3^{k+1} + p_2 w_2(k) = p_1 p_2^{k+1} + p_3 w_2(k) = w_2(k+1), \qquad w_2(k+1) \leq w_2(k) \leq p_1. \tag{2.5}$$

Throughout this subsection we let $X = [0,1] \times [0,1]$,

$$\Omega = X^\circ \quad \text{and} \quad W_n := \{2^{n-i}13^i : i = 0, 1, \ldots, n\}, \quad n \geq 1. \tag{2.6}$$

To simplify notation we let

$$\gamma_k := 1 - r^k, \qquad k \geq 0. \tag{2.7}$$

Define

$$\mathcal{I}_{1,1} := \{(S_1, 1), (S_2, 1)\}, \qquad \mathcal{I}_{1,2} := \{(S_3, 1)\}, \qquad \mathcal{I}_{1,3} := \{(S_4, 1)\} \tag{2.8}$$

(see Figure 2(a)) and

$$\begin{aligned}
I_{1,1} &:= S_{\mathcal{I}_{1,1}}(\Omega) = S_1(\Omega) \cup S_2(\Omega) = (0, \rho\gamma_1) \times (0, \rho) \cup (\rho\gamma_1, \rho\gamma_1 + r) \times (0, r), \\
I_{1,2} &:= S_{\mathcal{I}_{1,2}}(\Omega) = S_3(\Omega) = (\gamma_1, 1) \times (0, r), \\
I_{1,3} &:= S_{\mathcal{I}_{1,3}}(\Omega) = S_4(\Omega) = (0, r) \times (\gamma_1, 1),
\end{aligned} \tag{2.9}$$

where $\mathcal{I}_{1,i}$, $i = 1, 2, 3$, are defined in (2.8).

Example 2.18. *Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i=1}^4$ in* (1.9) *together with a probability vector $(p_i)_{i=1}^4$. Let Ω and W_n be as in* (2.6)*. Then μ satisfies Condition (B).*

To prove Example 2.18, we first summarize without proof some elementary properties.

Proposition 2.19. *Let $\{S_i\}_{i=1}^4$ be as in* (1.9) *and $\{\mathcal{I}_{1,i}\}_{i=1}^3$ be as in* (2.8)*. The following relations hold:*

(a) $S_{13} = S_{21}$. *Moreover, for any $\mathbf{i}, \mathbf{j} \in W_n$, $S_{\mathbf{i}} = S_{\mathbf{j}}$.*
(b) $\mathfrak{I}_1 = \{\mathcal{I}_{1,1}, \mathcal{I}_{1,2}, \mathcal{I}_{1,3}\}$.

Proposition 2.20. *Assume the hypotheses of Example 2.18 and $\{I_{1,i}\}_{i=1}^3$ defined as in* (2.9)*. Then (a)–(c) below hold, and (d)–(f) hold for all $k \geq 0$:*

(a)

$$\begin{aligned}
S_3(I_{1,1}) =& (\gamma_1, (1+\rho r)\gamma_1) \times (0, \rho r) \cup ((1+\rho r)\gamma_1, (1+\rho r)\gamma_1 + r^2) \times (0, r^2), \\
S_4(I_{1,1}) =& (0, \rho r\gamma_1) \times (\gamma_1, \gamma_1 + \rho r) \cup (\rho r\gamma_1, \rho r\gamma_1 + r^2) \times (\gamma_1, \gamma_1 + r^2).
\end{aligned}$$

(b)

$$\begin{aligned}
S_1(I_{1,2}) &= (\rho\gamma_1, \rho) \times (0, \rho r), \\
S_3(I_{1,2}) &= (\gamma_2, 1) \times (0, r^2), \\
S_4(I_{1,2}) &= (r\gamma_1, r) \times (\gamma_1, \gamma_1 + r^2).
\end{aligned}$$

(c) $S_3(I_{1,3}) = (\gamma_1, \gamma_1 + r^2) \times (r\gamma_1, r)$ *and* $S_4(I_{1,3}) = (0, r^2) \times (\gamma_2, 1)$.
(d)

$$\begin{aligned}
S_{2^k1}(I_{1,1}) =& (\rho\gamma_k, \rho\gamma_k + \rho^2 r^k\gamma_1) \times (0, \rho^2 r^k) \cup \\
& (\rho\gamma_k + \rho^2 r^k\gamma_1, \rho\gamma_k + \rho^2 r^k\gamma_1 + \rho r^{k+1}) \times (0, \rho r^{k+1}), \\
S_{2^k1}(I_{1,3}) =& (\rho\gamma_k, \rho\gamma_k + \rho r^{k+1}) \times (\rho r^k\gamma_1, \rho r^k).
\end{aligned}$$

(e)

$$\begin{aligned} S_{2^k}(I_{1,1}) =&(\rho\gamma_k, \rho\gamma_{k+1}) \times (0, \rho r^k) \cup (\rho\gamma_{k+1}, \rho\gamma_{k+1} + r^{k+1}) \times (0, r^{k+1}),\\ S_{2^k}(I_{1,2}) =&(r^k\gamma_1 + \rho\gamma_k, r^k + \rho\gamma_k) \times (0, r^{k+1}),\\ S_{2^k}(I_{1,3}) =&(\rho\gamma_k, \rho\gamma_k + r^{k+1}) \times (r^k\gamma_1, r^k). \end{aligned}$$

(f) $S_{2^k1}(\Omega) = (\rho\gamma_k, \rho) \times (0, \rho r^k)$ *and* $S_{2^k}(\Omega) = (\rho\gamma_k, \rho\gamma_k + r^k) \times (0, r^k)$.

Proof. (a)–(c) follow from (2.9), and (d)–(f) can be proved directly by induction; we omit the details. □

Lemma 2.21. *Assume the hypotheses of Proposition 2.20. Then*

$$\mu(S_1(\Omega) \cap S_{2^k}(\Omega)) = \mu\Big(\bigcup_{i=1}^{3} S_1(I_{1,i}) \cap S_{2^k}(\Omega)\Big) = \mu(S_{2^k1}(\Omega)) \quad \text{for } k \geq 1. \tag{2.10}$$

Proof. First, we prove the first equality in (2.10). Since $\mu(S_1(\Omega)) = \mu(\cup_{i=1}^3 S_1(I_{1,i}))$, we have $\mu(S_1(\Omega) \cap A) = \mu((\cup_{i=1}^3 S_1(I_{1,i})) \cap A) = \mu\big(\cup_{i=1}^3 S_1(I_{1,i}) \cap S_{2^k}(\Omega)\big)$ for any $A \subseteq \Omega$.

Next, we show that

$$\bigcup_{i=1}^{3} S_1(I_{1,i}) \cap S_{2^k}(\Omega) = S_{2^k1}(\Omega) \quad \text{for all } k \geq 1. \tag{2.11}$$

By Proposition 2.20(b,d,f), we have

$$\begin{aligned} &S_1(I_{1,1}) = (0, \rho^2\gamma_1) \times (0, \rho^2) \cup (\rho^2\gamma_1, \rho^2\gamma_1 + \rho r) \times (0, \rho r),\\ &S_1(I_{1,2}) = (\rho\gamma_1, \rho) \times (0, \rho r), \qquad S_1(I_{1,3}) = (0, \rho r) \times (\rho\gamma_1, \rho), \end{aligned}$$

and $S_2(\Omega) = (\rho\gamma_1, \rho\gamma_1 + r) \times (0, r)$. It follows from (1.10) that $\rho r + \rho^2\gamma_1 \leq \rho\gamma_1$ and hence $S_1(I_{1,1}) \cap S_2(\Omega) = \emptyset$. Since $\rho < r + \rho\gamma_1$, we have $S_1(I_{1,2}) \cap S_2(\Omega) = (\rho\gamma_1, \rho) \times (0, \rho r) = S_{21}(\Omega)$, where in the last equality we use Proposition 2.20(f). Since $r < \gamma_1$, we have $\rho r < \rho\gamma_1$, and thus $S_1(I_{1,3}) \cap S_2(\Omega) = \emptyset$. Hence $\bigcup_{i=1}^3 S_1(I_{1,i}) \cap S_2(\Omega) = S_{21}(\Omega)$. Assume that the stated inequality holds for $k = m$, i.e., $\bigcup_{i=1}^3 S_1(I_{1,i}) \cap S_{2^m}(\Omega) = S_{2^m1}(\Omega)$. Then $S_1(I_{1,2}) \cap S_{2^m}(\Omega) = S_{2^m1}(\Omega)$ and $S_1(I_{1,i}) \cap S_{2^m}(\Omega) = \emptyset$ for $i = 1, 3$. For $k = m+1$, since $S_1(I_{1,i}) \cap S_{2^{m+1}}(\Omega) \subseteq S_1(I_{1,i}) \cap S_{2^m}(\Omega)$, we have $S_1(I_{1,i}) \cap S_{2^{m+1}}(\Omega) = \emptyset$ for $i = 1, 3$. By (2.9) and Proposition 2.19(a), we have

$$\begin{aligned} &S_1(I_{1,2}) \cap S_{2^{m+1}}(\Omega)\\ =\ & S_{13}(\Omega) \cap S_{2^{m+1}}(\Omega) = S_{21}(\Omega) \cap S_{2^{m+1}}(\Omega)\\ =\ & S_2(S_1(\Omega) \cap S_{2^m}(\Omega)) = S_2((\cup_{i=1}^3 S_1(I_{1,i})) \cap S_{2^m}(\Omega))\\ =\ & S_2\Big(\bigcup_{i=1}^{3} S_1(I_{1,i}) \cap S_{2^m}(\Omega)\Big) = S_2(S_{2^m1}(\Omega)) = S_{2^{m+1}1}(\Omega). \end{aligned}$$

This proves (2.11). Hence the second inequality in (2.10) holds. □

Part (a) of the following lemma explains the meaning of the factor $w_2(k)$.

Lemma 2.22. *Assume the hypotheses of Proposition* (2.20) *and let* $w_2(k)$ *be defined as in* (1.11). *Then*

(a) *for* $k \geq 0$ *and* $i = 1, 3$, $\mu|_{S_{2^k 1}(I_{1,i})} = w_2(k)\mu \circ S^{-1}_{2^k 1}$;
(b) *for* $k \geq 1$, $\mu|_{S_{2^k}(I_{1,1})} = w_2(k-1)\mu \circ S^{-1}_{2^{k-1}1} + p_2^k \mu \circ S^{-1}_{2^k}$;
(c) *for* $k \geq 1$ *and* $i = 2, 3$, $\mu|_{S_{2^k}(I_{1,i})} = p_2^k \mu \circ S^{-1}_{2^k}$;
(d) *for* $i = 1, 2, 3$ *and* $j = 3, 4$, $\mu|_{S_j(I_{1,i})} = p_j \circ \mu|_{I_{1,i}}$.

Proof. We only prove (a) for $i = 1$ as an example. By Proposition 2.20(d), we have

$$S_1(I_{1,1}) = (0, \rho^2\gamma_1) \times (0, \rho^2) \cup (\rho^2\gamma_1, \rho r + \rho^2\gamma_1) \times (0, \rho r).$$

Note that $S_2(\Omega) = (\rho\gamma_1, \rho\gamma_1 + r) \times (0, r)$. Moreover, $\rho\gamma_1 - (\rho^2\gamma_1 + \rho r) = \rho(1 - 2r - \rho + \rho r) \geq 0$, we have $S_1(I_{1,1}) \subseteq S_1(\Omega) \backslash S_2(\Omega)$. Hence $\mu(A) = p_1\mu \circ S_1^{-1}(A)$ for any $A \subseteq S_1(I_{1,1})$. Assume that the stated equality holds for $k = m$, i.e., $\mu|_{S_{2^m 1}(I_{1,1})} = w_2(m)\mu \circ S^{-1}_{2^m 1}$. For $k = m+1$, by Proposition 2.19(a), we have $S_{2^{m+1}1}(I_{1,1}) = S_{13^{m+1}}(I_{1,1})$. Then $S_1^{-1}(A) \subseteq S_{3^{m+1}}(I_{1,1})$ and $S_2^{-1}(A) \subseteq S_{2^m 1}(I_{1,1})$ for any $A \subseteq S_{2^{m+1}1}(I_{1,1})$. It follows that $\mu(S_1^{-1}(A)) = p_3^{m+1}\mu \circ S^{-1}_{3^{m+1}}(S_1^{-1}(A))$ and $\mu(S_2^{-1}(A)) = w_2(m)\mu \circ S^{-1}_{2^m 1}(S_2^{-1}(A))$. Thus

$$\begin{aligned}
\mu(A) &= p_1\mu \circ S_1^{-1}(A) + p_2\mu \circ S_2^{-1}(A) \\
&= p_1 p_3^{m+1}\mu \circ S^{-1}_{3^{m+1}}(S_1^{-1}(A)) + p_2 w_2(m)\mu \circ S^{-1}_{2^m 1}(S_2^{-1}(A)) \\
&= p_1 p_3^{m+1}\mu \circ S^{-1}_{13^{m+1}}(A) + p_2 w_2(m)\mu \circ S^{-1}_{2^{m+1}1}(A) \\
&= (p_1 p_3^{m+1} + p_2 w_2(m))\mu \circ S^{-1}_{2^{m+1}1}(A) \\
&= w_2(m+1)\mu \circ S^{-1}_{2^{m+1}1}(A).
\end{aligned}$$

The last equality follows from (2.5). This proves part (a) for $i = 1$. For the proof of part (c) in the case $i = 3$, we use Lemma 2.21. $\square$

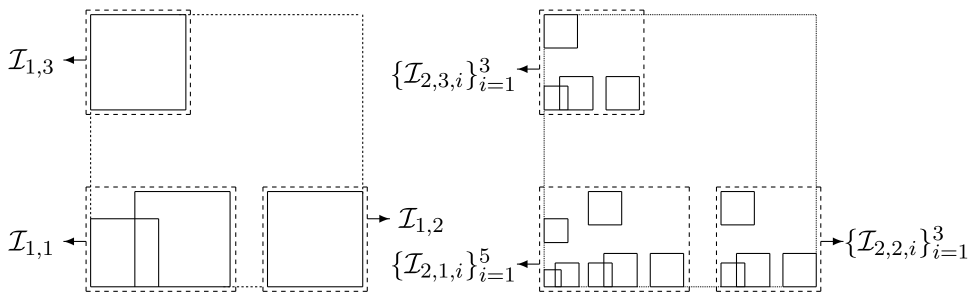
Proof of Example 2.18. By (2.6), we have $\Omega = (0, 1) \times (0, 1)$. For each $k \geq 0$, let $\mathcal{M}_k = \{1, 2, 3, 4\}^k$. We show that μ satisfies Condition (B) with $\mathbf{I}_b := \mathfrak{I}_1$ being the basic set of islands. Let $\mathcal{I}_{1,i}$ be defined as in (2.8). Thus $\mathbf{I}_b = \{\mathcal{I}_{1,1}, \mathcal{I}_{1,2}, \mathcal{I}_{1,3}\}$. Let $\mathbf{I}_{b,\mu} := \{[\mathcal{I}_{1,1}]_\mu, [\mathcal{I}_{1,2}]_\mu, [\mathcal{I}_{1,3}]_\mu\}$. It suffices to show that $\mathcal{I}_{k,1,3} := \{(S_{2^{k-1}1}, k), (S_{2^k}, k)\}$ is the only level-k nonbasic island with respect to $\mathbf{I}_b$ for any $k \geq 2$ (see Figure 2(b)). For $i = 2, 3$, since $[\mathcal{I}(\boldsymbol{v}_{\text{root}})]_\mu = [\mathcal{I}_{1,i}]_\mu$, none of the $\mathcal{I} \in O(\mathcal{I}_{1,i})$ is a nonbasic island with respect to $\mathbf{I}_b$ (see Figure 4). Upon iterating the IFS once, $\mathcal{I}_{1,1}$ generates the following five islands:

$$\begin{aligned}
&\mathcal{I}_{2,1,1} := \{(S_{11}, 2), (S_{12}, 2)\}, \quad \mathcal{I}_{2,1,2} := \{(S_{14}, 2)\}, \\
&\mathcal{I}_{2,1,3} := \{(S_{21}, 2), (S_{22}, 2)\}, \quad \mathcal{I}_{2,1,4} := \{(S_{23}, 2)\}, \quad \mathcal{I}_{2,1,5} := \{(S_{24}, 2)\}
\end{aligned}$$

(see Figure 3). Lemma 2.22 implies that $\mathcal{I}_{2,1,i} \in [\mathcal{I}_{1,i}]_\mu$ for $i = 1, 2, 4, 5$, and $[\mathcal{I}_{2,1,3}]_\mu \notin \mathbf{I}_{b,\mu}$. Thus $\mathcal{I}_{2,1,3}$ is the only level-2 nonbasic island with respect to $\mathbf{I}_b$. Assume that $\mathcal{I}_{k,1,3} := \{(S_{2^{k-1}1}, k), (S_{2^k}, k)\}$ is the only level-k nonbasic island with respect to $\mathbf{I}_b$. Similarly, $\mathcal{I}_{k,1,3}$ generates five islands, namely,

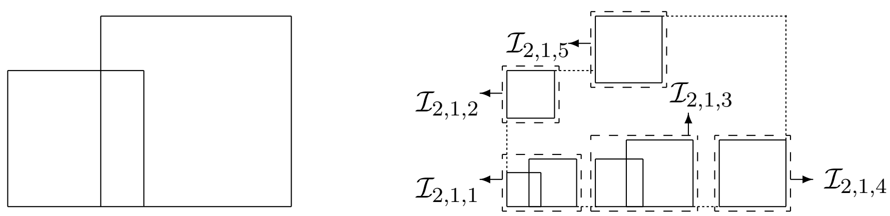
$$\begin{aligned}
&\mathcal{I}_{k+1,1,1} := \{(S_{2^{k-1}11}, k+1), (S_{2^{k-1}12}, k+1)\}, \qquad \mathcal{I}_{k+1,1,2} := \{(S_{2^{k-1}14}, k+1)\}, \\
&\mathcal{I}_{k+1,1,3} := \{(S_{2^k 1}, k+1), (S_{2^{k+1}}, k+1)\}, \qquad \mathcal{I}_{k+1,1,4} := \{(S_{2^k 3}, k+1)\}, \\
&\mathcal{I}_{k+1,1,5} := \{(S_{2^k 4}, k+1)\}.
\end{aligned}$$

Lemma 2.22 again implies that $\mathcal{I}_{k+1,1,i} \in [\mathcal{I}_{1,i}]_\mu$ for $i = 1, 2, 4, 5$, and $[\mathcal{I}_{k+1,1,3}]_\mu \notin \mathbf{I}_{b,\mu}$. Thus, $\mathcal{I}_{k+1,1,3}$ is the only level-$(k+1)$ nonbasic island with respect to $\mathbf{I}_b$, completing the proof. $\square$



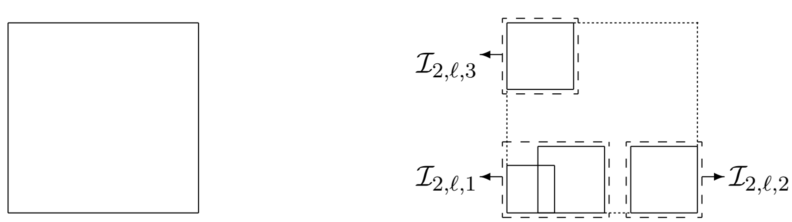
(a) level-1 islands $\{\mathcal{I}_{1,\ell}\}$ (b) level-2 islands $\{\mathcal{I}_{2,\ell,i}\}$

FIGURE 2. (a) First level iterations containing $\{\mathcal{I}_{1,\ell}\}_{\ell=1}^3$. (b) Second level iterations containing $\{\mathcal{I}_{2,1,i}\}_{i=1}^5$ and $\{\mathcal{I}_{2,\ell,i}\}_{i=1}^3$ for $\ell = 2, 3$. The figures are drawn with $\rho = 1/4$ and $r = 7/20$.



(a) $\mathcal{I}_{1,1}$ (b) $\{\mathcal{I}_{2,1,i}\}_{i=1}^5$

FIGURE 3. $\mathcal{I}_{1,1}$ and its offspring $\{\mathcal{I}_{2,1,i}\}_{i=1}^5$.



(a) $\mathcal{I}_{2,\ell}$ (b) $\{\mathcal{I}_{2,\ell,i}\}_{i=1}^3$

FIGURE 4. $\mathcal{I}_{2,\ell}$ and its offspring $\{\mathcal{I}_{2,\ell,i}\}_{i=1}^3$ for $\ell = 2, 3$.

3. Renewal equation and proof of Theorem 1.1

Let $\{S_i\}_{i\in\Lambda}$ be a finite type IFS on a compact subset $X\subseteq\mathbb{R}^d$ with FTC set $\Omega\subseteq X$ and let μ be the self-similar measure defined by $\{S_i\}_{i\in\Lambda}$ together with a probability vector $(p_i)_{i\in\Lambda}$. To compute $\tau(q)$ for $q\ge 0$, we will use the equivalent definition in (1.1). We show in this section that for the class of self-similar measures under consideration and for each $q\ge 0$, there exists $\alpha:=\alpha(q)$ such that

$$0<\overline{\lim_{\delta\to 0^+}}\frac{1}{\delta^{d+\alpha}}\int_X \mu(B_\delta(x))^q\,dx<\infty. \tag{3.1}$$

In the rest of this section, we assume that μ satisfies Condition (B). Let $\mathbf{I}_b:=\mathfrak{I}_{k_b}$ be the basic set of islands and $\mathbf{I}_{b,\mu}:=\{[\mathcal{I}]_\mu:\mathcal{I}\in\mathbf{I}_b\}$. We choose a subset $\mathbf{I}:=\{\mathcal{I}_{1,\ell}\}_{\ell\in\Gamma}\subseteq\mathbf{I}_b$ such that for any $\mathcal{I}\in\mathbf{I}_b$, there exists a unique $\ell\in\Gamma$ satisfying $\mathcal{I}\in[\mathcal{I}_{1,\ell}]_\mu$. Define

$$\varphi_\ell(\delta):=\int_{I_{1,\ell}}\mu(B_\delta(x))^q\,dx,\qquad \Phi_\ell^{(\alpha)}(\delta):=\frac{1}{\delta^{d+\alpha}}\varphi_\ell(\delta)\quad\text{for }\ell\in\Gamma, \tag{3.2}$$

where $I_{1,\ell}:=S_{\mathcal{I}_{1,\ell}}(\Omega)$.

Proposition 3.1. *Let $q\ge 0$. If*

$$0<\overline{\lim_{\delta\to 0^+}}\,\Phi_\ell^{(\alpha)}(\delta)<\infty\qquad\textit{for all }\ell\in\Gamma, \tag{3.3}$$

then $\tau(q)=\alpha$.

Proof. To find $\tau(q)$, it suffices to look for α such that (3.1) holds. Since $S_{\mathcal{I}_{1,\ell}}(\Omega)$ is a connected component of Ω, we have

$$\begin{aligned}\overline{\lim_{\delta\to 0^+}}\frac{1}{\delta^{d+\alpha}}\int_X\mu(B_\delta(x))^q\,dx &\le \sum_{\ell\in\Gamma}\overline{\lim_{\delta\to 0^+}}\frac{1}{\delta^{d+\alpha}}\int_{I_{1,\ell}}\mu(B_\delta(x))^q\,dx\\ &= \sum_{\ell\in\Gamma}\overline{\lim_{\delta\to 0^+}}\,\Phi_\ell^{(\alpha)}(\delta).\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) yields (3.1), and hence the proposition follows. □

For $\mathcal{I}\in\mathfrak{I}$, let $S_{\mathcal{I}}(\Omega)$ and $O(\mathcal{I})$ be defined as in (2.1) and (2.2), respectively. We denote the contraction ratio of a contractive similitude σ by r_σ. In view of Proposition 3.1, to find α satisfying (3.1), it suffices to study $\Phi_\ell^{(\alpha)}(\delta)$ for $\ell\in\Gamma$.

Step 1. Derivation of a functional equation for $\Phi_\ell^{(\alpha)}(\delta)$ for $\ell\in\Gamma$. For $\ell\in\Gamma$, define

$$\mathbf{I}_{2,\ell}:=\{\mathcal{I}\in O(\mathcal{I}_{1,\ell}):[\mathcal{I}]_\mu\in\mathbf{I}_{b,\mu}\}\qquad\text{and}\qquad \mathbf{I}'_{2,\ell}:=\{\mathcal{I}\in O(\mathcal{I}_{1,\ell}):[\mathcal{I}]_\mu\notin\mathbf{I}_{b,\mu}\}.$$

Thus $O(\mathcal{I}_{1,\ell})=\mathbf{I}_{2,\ell}\cup\mathbf{I}'_{2,\ell}$. For $k\ge 3$, if $\mathbf{I}'_{k-1,\ell}\ne\emptyset$, we define

$$\mathbf{I}_{k,\ell}:=\Big\{\mathcal{I}\in\bigcup_{\mathcal{J}\in\mathbf{I}'_{k-1,\ell}}O(\mathcal{J}):[\mathcal{I}]_\mu\in\mathbf{I}_{b,\mu}\Big\}\ \text{ and }\ \mathbf{I}'_{k,\ell}:=\Big\{\mathcal{I}\in\bigcup_{\mathcal{J}\in\mathbf{I}'_{k-1,\ell}}O(\mathcal{J}):[\mathcal{I}]_\mu\notin\mathbf{I}_{b,\mu}\Big\}.$$

We remark that for any $k\ge 2$, $\bigcup_{\ell\in\Gamma}\mathbf{I}'_{k,\ell}$ is the set of all level-k nonbasic islands with respect to $\mathbf{I}$.

Without loss of generality, we assume that Γ can be partitioned into two sub-collections, Γ_* and Γ'_*, defined as follows. For $\ell \in \Gamma$, we say $\ell \in \Gamma_*$ if there exists some $\kappa_\ell \geq 2$, depending on ℓ, such that κ_ℓ is the smallest number satisfying $\mathbf{I}'_{\kappa_\ell,\ell} = \emptyset$; otherwise, $\ell \in \Gamma'_*$. Define $\kappa_\ell := \infty$ for $\ell \in \Gamma'_*$.

Condition (B) implies that $\sum_{\ell\in\Gamma} \#\mathbf{I}'_{k,\ell}$ is uniformly bounded for any $k \geq 2$. Fix $\ell \in \Gamma$. Then for any $2 \leq k \leq \kappa_\ell$, we use two finite disjoint subsets $G_{k,\ell}, G'_{k,\ell} \subseteq \mathbb{Z}$ to label the elements of $\mathbf{I}_{k,\ell}$ and $\mathbf{I}'_{k,\ell}$; more precisely,

$$\mathbf{I}_{k,\ell} = \{\mathcal{I}_{k,\ell,i} : i \in G_{k,\ell}\} \qquad \text{and} \qquad \mathbf{I}'_{k,\ell} = \{\mathcal{I}_{k,\ell,i} : i \in G'_{k,\ell}\}.$$

Condition (B) implies that $0 \leq \#G'_{k,\ell} \leq M$, where $M > 0$ is a constant. We remark that $G'_{\kappa_\ell,\ell} = \emptyset$. Define

$$I_{k,\ell,i} := S_{\mathcal{I}_{k,\ell,i}}(\Omega) \qquad \text{for } 2 \leq k \leq \kappa_\ell \text{ and } i \in G_{k,\ell} \cup G'_{k,\ell}.$$

Then for all $\ell \in \Gamma_*$ we have

$$\varphi_\ell(\delta) \;=\; \sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} \int_{I_{j,\ell,i}} \mu(B_\delta(x))^q \, dx, \tag{3.5}$$

while for all $\ell \in \Gamma'_*$ and $n \geq 2$,

$$\varphi_\ell(\delta) = \sum_{j=2}^{n} \sum_{i\in G_{j,\ell}} \int_{I_{j,\ell,i}} \mu(B_\delta(x))^q \, dx + \sum_{\ell\in G'_{n,\ell}} \int_{I_{n,\ell,i}} \mu(B_\delta(x))^q \, dx. \tag{3.6}$$

For $\ell \in \Gamma, 2 \leq k \leq \kappa_\ell, i \in G_{k,\ell}$ and $\delta > 0$, let $\widetilde{I}_{k,\ell,i}(\delta)$ be the largest subset of $I_{k,\ell,i}$ satisfying $B_\delta(x) \subseteq I_{k,\ell,i}$ for any $x \in \widetilde{I}_{k,\ell,i}(\delta)$. We denote $\widehat{I}_{k,\ell,i}(\delta) := I_{k,\ell,i} \backslash \widetilde{I}_{k,\ell,i}(\delta)$. So for $\ell \in \Gamma_*$ (3.5) can be written as

$$\varphi_\ell(\delta) = \sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} \int_{\widetilde{I}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx + \sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} \int_{\widehat{I}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx,$$

while for $\ell \in \Gamma'_*$ and $n \geq 2$, (3.6) can be expressed as

$$\begin{aligned}\varphi_\ell(\delta) \;=\;& \sum_{j=2}^{n} \sum_{i\in G_{j,\ell}} \int_{\widetilde{I}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx + \sum_{j=2}^{n} \sum_{i\in G_{j,\ell}} \int_{\widehat{I}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx \\ &+ \sum_{i\in G'_{n,\ell}} \int_{I_{n,\ell,i}} \mu(B_\delta(x))^q \, dx.\end{aligned}$$

For $\ell \in \Gamma, 2 \leq k \leq \kappa_\ell$ and $i \in G_{k,\ell}$, there exist unique $\sigma(k,\ell,i) \in \mathscr{S}, w(k,\ell,i) > 0$ and $c(k,\ell,i) \in \Gamma$ such that $\mathcal{I}_{1,c(k,\ell,i)} \approx_{\mu,\sigma(k,\ell,i),w(k,\ell,i)} \mathcal{I}_{k,\ell,i}$. By Definition 2.13, we have

$$\mu|_{S_{\mathcal{I}_{k,\ell,i}}(\Omega)} = w(k,\ell,i) \circ \mu|_{S_{\mathcal{I}_{1,c(k,\ell,i)}}(\Omega)} \circ \sigma(k,\ell,i)^{-1}.$$

For $\widetilde{I}_{k,\ell,i}(\delta) \subseteq I_{k,\ell,i}$, let $\widetilde{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ be the largest subset of $I_{1,c(k,\ell,i)}$ satisfying $B_{\delta/r_{\sigma(k,\ell,i)}}(x) \subseteq I_{1,c(k,\ell,i)}$ for any $x \in \widetilde{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$. Thus

$$\mu|_{\widetilde{I}_{k,\ell,i}(\delta)} = w(k,\ell,i) \circ \mu|_{\widetilde{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})} \circ \sigma(k,\ell,i)^{-1}.$$

We denote $\widehat{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)}) = I_{1,c(k,\ell,i)}\backslash\widetilde{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$. Hence for $\ell\in\Gamma_*$,

$$\varphi_\ell(\delta) = \sum_{j=2}^{\kappa_\ell}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q {r_{\sigma(j,\ell,i)}}^d \int_{I_{1,c(j,\ell,i)}} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q\,dx \;+\sum_{j=2}^{\kappa_\ell}(e_j^\ell(\delta)-\tilde{e}_j^\ell(\delta)), \tag{3.7}$$

where

$$\begin{aligned} e_j^\ell(\delta) &= \sum_{i\in G_{j,\ell}} \int_{\widehat{I}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q\,dx,\\ \tilde{e}_j^\ell(\delta) &= \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q {r_{\sigma(j,\ell,i)}}^d \int_{\widehat{I}_{1,c(j,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q\,dx, \end{aligned}$$

while for $\ell\in\Gamma_*'$ and $n\ge 2$,

$$\begin{aligned} \varphi_\ell(\delta) &= \sum_{j=2}^{n}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q {r_{\sigma(j,\ell,i)}}^d \int_{I_{1,c(j,\ell,i)}} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q\,dx\\ &\quad+\sum_{j=2}^{n}(e_j^\ell(\delta)-\tilde{e}_j^\ell(\delta)) + \sum_{i\in G'_{n,\ell}} \int_{I_{n,\ell,i}} \mu(B_\delta(x))^q\,dx, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} e_j^\ell(\delta) &= \sum_{i\in G_{j,\ell}} \int_{\widehat{I}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q\,dx,\\ \tilde{e}_j^\ell(\delta) &= \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q {r_{\sigma(j,\ell,i)}}^d \int_{\widehat{I}_{1,c(j,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q\,dx. \end{aligned}$$

Multiply both sides of (3.7) and (3.8) by $\delta^{-(d+\alpha)}$, and using (3.2), we have for $\ell\in\Gamma_*$,

$$\Phi_\ell^{(\alpha)}(\delta) := \sum_{j=2}^{\kappa_\ell}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q {r_{\sigma(j,\ell,i)}}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)}) + E_\ell^{(\alpha)}(\delta), \tag{3.9}$$

where

$$E_\ell^{(\alpha)}(\delta) = \sum_{j=2}^{\kappa_\ell} \delta^{-(d+\alpha)}(e_j^\ell(\delta)-\tilde{e}_j^\ell(\delta))$$

and

$$\begin{aligned} \Phi_\ell^{(\alpha)}(\delta) &:= \sum_{j=2}^{n}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q {r_{\sigma(j,\ell,i)}}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)}) + \sum_{j=2}^{n}\delta^{-(d+\alpha)}(e_j^\ell(\delta)-\tilde{e}_j^\ell(\delta))\\ &\quad+\delta^{-(d+\alpha)}\sum_{i\in G'_{n,\ell}}\int_{I_{n,\ell,i}} \mu(B_\delta(x))^q\,dx \qquad \text{for } \ell\in\Gamma_*' \text{ and } n\ge 2. \end{aligned} \tag{3.10}$$

Let N be the largest number of n satisfying $\delta\le r_{\sigma(n,\ell,i)}$ for $\ell\in\Gamma_*', i\in G_{n,\ell}$. Taking $n:=N$ in (3.10), we have

$$\begin{aligned} \Phi_\ell^{(\alpha)}(\delta) &:= \sum_{j=2}^{\infty}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q {r_{\sigma(j,\ell,i)}}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)})\\ &\quad+E_\ell^{(\alpha)}(\delta) - E_{\ell,\infty}^{(\alpha)}(\delta) \quad \text{for } \ell\in\Gamma_*' \text{ and } N\ge 2, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} E_\ell^{(\alpha)}(\delta) &= \sum_{j=2}^{N} \delta^{-(d+\alpha)}(e_j^\ell(\delta) - \tilde{e}_j^\ell(\delta)) + \delta^{-(d+\alpha)} \sum_{i\in G'_{N,\ell}} \int_{I_{N,\ell,i}} \mu(B_\delta(x))^q\,dx \\ E_{\ell,\infty}^{(\alpha)}(\delta) &= \sum_{j=N+1}^{\infty} \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}{}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)}). \end{aligned}$$

Step 2. Derivation of the vector-valued equation. For each $\ell \in \Gamma$, define

$$f_\ell(x) = f_\ell^{(\alpha)}(x) := \Phi_\ell^{(\alpha)}(e^{-x}).$$

If we let $\delta = e^{-x}$, then $\Phi_\ell^{(\alpha)}(\beta\delta) = f_\ell(x - \ln\beta)$ for any $\beta > 0$. Combining (3.9) and (3.11), we have, for $\ell \in \Gamma_*$,

$$f_\ell(x) = \sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} f_{c(j,\ell,i)}(x + \ln(r_{\sigma(j,\ell,i)})) + z_\ell^{(\alpha)}(x), \tag{3.12}$$

where $z_\ell^{(\alpha)}(x) = E_\ell^{(\alpha)}(e^{-x})$ and for $\ell \in \Gamma'_*$ and $N \ge 2$,

$$f_\ell(x) = \sum_{j=2}^{\infty} \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} f_{c(j,\ell,i)}(x + \ln(r_{\sigma(j,\ell,i)})) + z_\ell^{(\alpha)}(x), \tag{3.13}$$

where $z_\ell^{(\alpha)}(x) = E_\ell^{(\alpha)}(e^{-x}) - E_{\ell,\infty}^{(\alpha)}(e^{-x})$.

For $\ell, m \in \Gamma$, let $\mu_{m\ell}^{(\alpha)}$ be the discrete measure such that

$$\mu_{m\ell}^{(\alpha)}(-\ln(r_{\sigma(j,\ell,i)})) := w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} \qquad \text{for } 2 \le j \le \kappa_\ell,\ i \in G_{j,\ell},\ c(j,\ell,i) = m. \tag{3.14}$$

We summarize the above derivations in the following theorem.

Theorem 3.2. *Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i\in\Lambda}$ of finite type. Assume that μ satisfies Condition (B). Let $\mathbf{f}$, $\mathbf{M}_\alpha$, and $\mathbf{z}$ be defined as in* (1.2). *Then $\mathbf{f}$ satisfies the vector-valued renewal equation $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$.*

Proof of Theorem 1.1. We use a similar argument as that in [20, Theorem 1.1]. (a) We observe that each $F_i(\alpha)$ is a strictly increasing continuous positive function of α and

$$F_\ell(-\infty) = 0 \qquad \text{and} \qquad F_\ell(\infty) = \infty. \tag{3.15}$$

Thus there exists a unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is 1.

(b) Let α be the unique number in part (a). Let $\mathbf{m} := [m_{k\ell}^{(\alpha)}] = [\int_0^\infty x\,d\mu_{k\ell}^{(\alpha)}]$ be the moment matrix. Following the proof of [20, Theorem 1.1(b)], we need to show that some moment condition holds, and it suffices to show that

$$0 < \sum_{k\in\Gamma} m_{k\ell}^{(\alpha)} < \infty.$$

It is easy to check that for $\ell \in \Gamma$, $\sum_{k\in\Gamma} m_{k\ell}^{(\alpha)}$ takes the following values:

$$\sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} \big|\ln(r_{\sigma(j,\ell,i)})\big|.$$

(3.15) implies that there exists $\epsilon > 0$, such that $0 < F_i(\alpha+\epsilon) < \infty$. Then

$$\begin{aligned} 0 &< \sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} \big|\ln(r_{\sigma(j,\ell,i)})\big| \\ &= \sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-(\alpha+\epsilon)} r_{\sigma(j,\ell,i)}^{\epsilon} \big|\ln(r_{\sigma(j,\ell,i)})\big| < \infty. \end{aligned}$$

Moreover, it follows from (3.14), we have $\sum_{m\in\Gamma} \mu_{m\ell}^{(\alpha)}(0) = 0 < \sum_{m\in\Gamma} \mu_{m\ell}^{(\alpha)}(\infty)$, i.e., each column of $\mathbf{M}_\alpha$ is nondegenerate at 0. From Theorem 3.2, we have $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$. By (3.14), $\mu_{ji}^{(\alpha)}(\mathbb{R}) > 0$ and hence $\mathbf{M}_\alpha(\infty)$ is irreducible. It follows from [20, Theorem 4.1] that there exist positive constants C_1, C_2 such that $C_1 \le f_\ell^{(\alpha)}(x) \le C_2$ for all x. Proposition 3.1 now implies that $\tau(q) = \alpha$. $\square$

4. A class of finite IFSs with overlaps on $\mathbb{R}$

In this section, we derive renewal equations and compute the L^q-spectrum of self-similar measures μ defined by the IFSs in (1.5). Let $X := [0,1]$ and $\Omega = (0,1)$. Define

$$\mathcal{I}_{1,1} = \{(S_1,1),(S_2,1)\}, \qquad \mathcal{I}_{1,2} = \{(S_3,1)\}.$$

It follows from Example 2.17 that μ satisfies Condition (B) with $\mathbf{I}_b = \{\mathcal{I}_{1,1}, \mathcal{I}_{1,2}\}$ being the basic set of islands. Moreover, $\mathbf{I} = \mathbf{I}_{b,\mu} = \{[\mathcal{I}_{1,1}]_\mu, [\mathcal{I}_{1,2}]_\mu\}, \Gamma = \{1,2\}, \Gamma_* = \{2\}, \Gamma'_* = \{1\}, \kappa_1 = \infty$ and $\kappa_2 = 2$. For $\ell \in \Gamma$ and $2 \le k \le \kappa_\ell$, let $\mathbf{I}_{k,\ell}, \mathbf{I}'_{k,\ell}, G_{k,\ell}, G'_{k,\ell}$ be defined as in Section 3. Define

$$\begin{aligned} &\mathcal{I}_{k,1,1} := \{(S_{2^{k-2}11},k),(S_{2^{k-2}12},k)\}, \\ &\mathcal{I}_{k,1,2} := \{(S_{2^{k-1}1},k),(S_{2^k},k)\}, \qquad \mathcal{I}_{k,1,3} := \{(S_{2^{k-1}3},k)\} \end{aligned}$$

for $k \ge 2$. Let $\mathcal{I}_{2,2,1} := \{(S_{31},2),(S_{32},2)\}$ and $\mathcal{I}_{2,2,2} := \{(S_{33},2)\}$ (see Figure 5). Using [22, Lemma 2.14], we have $\mathbf{I}_{k,1} = \{\mathcal{I}_{k,1,1}, \mathcal{I}_{k,1,3}\}, \mathbf{I}'_{k,1} = \{\mathcal{I}_{k,1,2}\}, \mathbf{I}_{2,2} = \{\mathcal{I}_{2,2,1}, \mathcal{I}_{2,2,2}\}, \mathbf{I}'_{2,2} = \emptyset$. Hence $G_{k,1} = \{1,3\}, G'_{k,1} = \{2\}, G_{2,2} = \{1,2\}, G'_{2,2} = \emptyset$. Define $I_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega)$ for $\ell \in \Gamma$. Let

$$\begin{aligned} &I_{k,1,1} := S_{\mathcal{I}_{k,1,1}}(\Omega) = S_{2^{k-2}1}(I_{1,1}), \qquad I_{k,1,2} := S_{\mathcal{I}_{k,1,2}}(\Omega) = S_{2^{k-1}}(I_{1,1}), \\ &I_{k,1,3} := S_{\mathcal{I}_{k,1,3}}(\Omega) = S_{2^{k-1}}(I_{1,2}), \end{aligned} \tag{4.1}$$

for $k \ge 2$ and

$$I_{2,2,i} := S_{\mathcal{I}_{2,2,i}}(\Omega) = S_3(I_{1,i}), \qquad \text{for } i = 1,2. \tag{4.2}$$

In the rest of this section, fix $q \ge 0$ and let $w_1(k)$ be defined as in (1.7).

First, we derive functional equations for $\Phi_\ell^{(\alpha)}(\delta)$ for $\ell = 1,2$. Combining (3.5), (3.6), (4.1) and (4.2), we have

$$\varphi_1(\delta) = \sum_{j=2}^{n} \Big(\int_{I_{j,1,1}} + \int_{I_{j,1,3}}\Big) \mu(B_\delta(x))^q\,dx + \int_{I_{n,1,2}} \mu(B_\delta(x))^q\,dx$$

and

$$\varphi_2(\delta) = \Big(\int_{I_{2,2,1}} + \int_{I_{2,2,2}}\Big) \mu(B_\delta(x))^q\,dx.$$

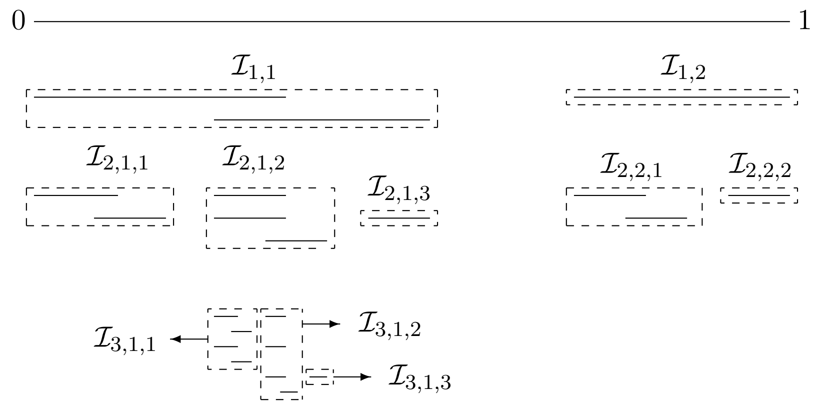


FIGURE 5. First, second and third levels of iterations containing $\{\mathcal{I}_{1,\ell}\}$, $\{\mathcal{I}_{2,\ell,i}\}$ and $\{\mathcal{I}_{3,1,i}\}$. The figure is drawn with $\rho = 1/3$ and $r = 2/7$.

For $\ell \in \Gamma$, $2 \le k \le \kappa_\ell$, $i \in G_{k,\ell}$ and $\delta > 0$, let $\widetilde{I}_{k,\ell,i}(\delta), \widehat{I}_{k,\ell,i}(\delta), \widetilde{I}_{1,c(k,\ell,,i)}(\delta/r_{\sigma(k,\ell,i)})$ and $\widehat{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ be defined as in Section 3. Combining (4.1) and (4.2), we have for $j \ge 2$,

$$\begin{aligned}
\widetilde{I}_{j,1,1}(\delta) &= (S_{2^{j-2}11}(0) + \delta, S_{2^{j-2}12}(1) - \delta),\\
\widehat{I}_{j,1,1}(\delta) &= (S_{2^{j-2}11}(0), S_{2^{j-2}11}(0) + \delta) \cup (S_{2^{j-2}12}(1) - \delta, S_{2^{j-2}12}(1)),\\
\widetilde{I}_{j,1,3}(\delta) &= (S_{2^{j-1}3}(0) + \delta, S_{2^{j-1}3}(1) - \delta),\\
\widehat{I}_{j,1,3}(\delta) &= (S_{2^{j-1}3}(0), S_{2^{j-1}3}(0) + \delta) \cup (S_{2^{j-1}3}(1) - \delta, S_{2^{j-1}3}(1)),\\
\widetilde{I}_{2,2,1}(\delta) &= (S_{31}(0) + \delta, S_{32}(1) - \delta),\\
\widehat{I}_{2,2,1}(\delta) &= (S_{31}(0), S_{31}(0) + \delta) \cup (S_{32}(1) - \delta, S_{32}(1)),\\
\widetilde{I}_{2,2,2}(\delta) &= (S_{33}(0) + \delta, S_{33}(1) - \delta),\\
\widehat{I}_{2,2,2}(\delta) &= (S_{33}(0), S_{33}(0) + \delta) \cup (S_{33}(1) - \delta, S_{33}(1)).
\end{aligned} \tag{4.3}$$

(See Figures 6 and 7.)

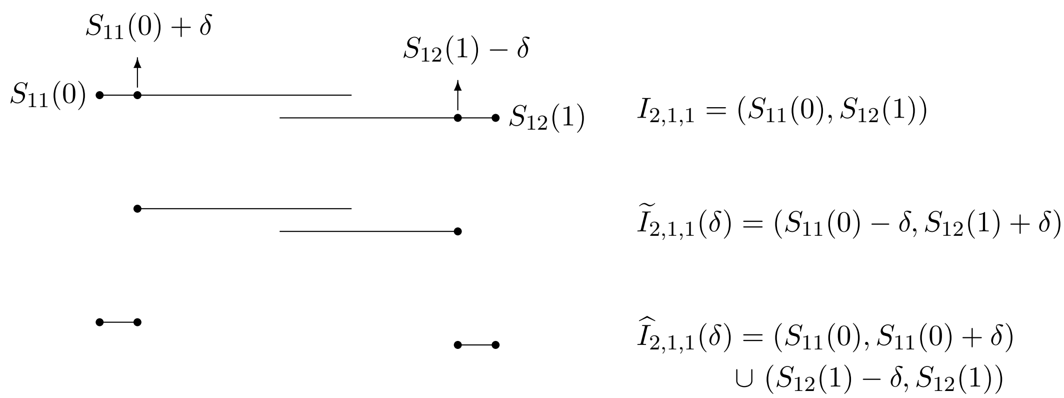
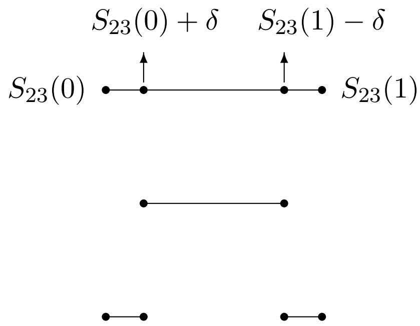


FIGURE 6. Figure showing the sets $I_{2,1,1}, \widetilde{I}_{2,1,1}(\delta)$ and $\widehat{I}_{2,1,1}(\delta)$.



$I_{2,1,3} = (S_{11}(0), S_{12}(1))$

$\widetilde{I}_{2,1,3}(\delta) = (S_{23}(0) - \delta, S_{23}(1) + \delta)$

$$\widehat{I}_{2,1,3}(\delta) = (S_{23}(0), S_{23}(0) + \delta) \cup (S_{23}(1) - \delta, S_{23}(1))$$

FIGURE 7. Figure showing the sets $I_{2,1,3}, \widetilde{I}_{2,1,3}(\delta)$ and $\widehat{I}_{2,1,3}(\delta)$.

It follows from (4.1), (4.2) and in [22, Lemma 2.14] that for $j \geq 2$,

$$\mu(I_{j,1,1}) = w_1(j-2)\mu(I_{1,1}) \qquad \text{and} \qquad \mu(I_{j,1,2}) = p_2^{j-1}\mu(I_{1,2}).$$

Thus

$$\begin{aligned} &w(j,1,1) = w_1(j-2), && c(j,1,1) = 1, && \sigma(j,1,1) = S_{2^{j-2}1}, && r_{\sigma(j,1,1)} = \rho r^{j-2}, \\ &w(j,1,2) = p_2^{j-1}, && c(j,1,2) = 2, && \sigma(j,1,2) = S_{2^{j-1}}, && r_{\sigma(j,1,2)} = r^{j-1}, \end{aligned}$$

and

$$\begin{aligned} \widetilde{I}_{1,1}(\delta/\rho r^{j-2}) &= (S_1(0) + \delta/\rho r^{j-2}, S_2(1) - \delta/\rho r^{j-2}), \\ \widetilde{I}_{1,2}(\delta/r^{j-1}) &= (S_3(0) + \delta/r^{j-1}, S_3(1) - \delta/r^{j-1}), \\ \widehat{I}_{1,1}(\delta/\rho r^{j-2}) &= (S_1(0), S_1(0) + \delta/\rho r^{j-2}) \cup (S_2(1) - \delta/\rho r^{j-2}, S_2(1)), \\ \widehat{I}_{1,2}(\delta/r^{j-1}) &= (S_3(0), S_3(0) + \delta/r^{j-1}) \cup (S_3(1) - \delta/r^{j-1}, S_3(1)). \end{aligned}$$

Since $\mu|_{S_3(I_{1,i})} = p_3\mu \circ S_3^{-1}$ on $S_3(I_{1,i})$ for $i = 1, 2$, by using (4.2), we have $\mu(I_{2,2,i}) = p_3\mu(I_{1,i})$. Hence $w(2,2,i) = p_3, c(2,2,i) = i, \sigma(2,2,i) = S_3, r_{\sigma(2,2,i)} = r$ and

$$\begin{aligned} \widetilde{I}_{1,1}(\delta/r) &= (S_1(0) + \delta/r, S_2(1) - \delta/r), \\ \widetilde{I}_{1,2}(\delta/r) &= (S_3(0) + \delta/r, S_3(1) - \delta/r), \\ \widehat{I}_{1,1}(\delta/r) &= (S_1(0), S_1(0) + \delta/r) \cup (S_2(1) - \delta/r, S_2(1)), \\ \widehat{I}_{1,2}(\delta/r) &= (S_3(0), S_3(0) + \delta/r) \cup (S_3(1) - \delta/r, S_3(1)). \end{aligned}$$

By (3.7) and (3.8), we have

$$\begin{aligned} \varphi_1(\delta) \;=\; &\sum_{j=2}^{n} \Big(w_1(j-2)^q \rho r^{j-2} \int_{I_{1,1}} \mu(B_{\delta/\rho r^{j-2}}(x))^q \, dx + (p_2^q r)^{j-1} \int_{I_{1,2}} \mu(B_{\delta/r^{j-1}}(x))^q \, dx \Big) \\ &+ \sum_{j=2}^{n} (e_j^1(\delta) - \tilde{e}_j^1(\delta)) + \int_{I_{n,1,2}} \mu(B_\delta(x))^q \, dx, \end{aligned} \tag{4.4}$$

and

$$\varphi_2(\delta) = p_3^q r \Big(\int_{I_{1,1}} + \int_{I_{1,2}} \Big) \mu(B_{\delta/r}(x))^q \, dx + e_2^2(\delta) - \tilde{e}_2^2(\delta), \tag{4.5}$$

where

$$\begin{aligned}
e_j^1(\delta) =& \Big(\int_{\widehat{I}_{j,1,1}(\delta)} + \int_{\widehat{I}_{j,1,3}(\delta)} \Big) \mu(B_\delta(x))^q \, dx, \\
\tilde{e}_j^1(\delta) =& w_1(j-2)^q \rho r^{j-2} \int_{\widehat{I}_{1,1}(\delta/\rho r^{j-2})} \mu(B_{\delta/\rho r^{j-2}}(x))^q \, dx \\
&+ (p_2^q r)^{j-1} \int_{\widehat{I}_{1,2}(\delta/r^{j-1})} \mu(B_{\delta/r^{j-1}}(x))^q \, dx, \qquad (4.6) \\
e_2^2(\delta) =& \Big(\int_{\widehat{I}_{2,2,1}(\delta)} + \int_{\widehat{I}_{2,2,2}(\delta)} \Big) \mu(B_\delta(x))^q \, dx, \\
\tilde{e}_2^2(\delta) =& p_3^q r \Big(\int_{\widehat{I}_{1,1}(\delta/r)} + \int_{\widehat{I}_{1,2}(\delta/r)} \Big) \mu(B_{\delta/r}(x))^q \, dx.
\end{aligned}$$

Multiplying both sides of (4.4) and (4.5) by $\delta^{-(1+\alpha)}$ and using (3.2), we have

$$\begin{aligned}
\Phi_1^{(\alpha)}(\delta) \quad = \quad & \sum_{j=2}^{n} \big(w_1(j-2)^q (\rho r^{j-2})^{-\alpha} \Phi_1^{(\alpha)}(\delta/\rho r^{j-2}) + (p_2^q r^{-\alpha})^{j-1} \Phi_2^{(\alpha)}(\delta/r^{j-1}) \big) \\
& + \sum_{j=2}^{n} \delta^{-1-\alpha} (e_j^1(\delta) - \tilde{e}_j^1(\delta)) + \delta^{-1-\alpha} \int_{I_{n,1,2}} \mu(B_\delta(x))^q \, dx \qquad (4.7)
\end{aligned}$$

and

$$\Phi_2^{(\alpha)}(\delta) = p_3^q r^{-\alpha} \big(\Phi_1^{(\alpha)}(\delta/r) + \Phi_2^{(\alpha)}(\delta/r) \big) + \delta^{-1-\alpha} (e_2^2(\delta) - \tilde{e}_2^2(\delta)).$$

Let N be the largest integer satisfying $\delta \le \min\{\rho r^{N-2}, r^{N-1}\}$. Taking $n = N$ in (4.7), we have

$$\begin{aligned}
\Phi_1^{(\alpha)}(\delta) \quad = \quad & \sum_{j=2}^{\infty} \big(w_1(j-2)^q (\rho r^{j-2})^{-\alpha} \Phi_1^{(\alpha)}(\delta/\rho r^{j-2}) + (p_2^q r^{-\alpha})^{j-1} \Phi_2^{(\alpha)}(\delta/r^{j-1}) \big) \\
& + E_1^{(\alpha)}(\delta) - E_{1,\infty}^{(\alpha)}(\delta), \qquad (4.8)
\end{aligned}$$

where

$$\begin{aligned}
E_1^{(\alpha)}(\delta) &= \sum_{j=2}^{N} \delta^{-1-\alpha} (e_j^1(\delta) - \tilde{e}_j^1(\delta)) + \delta^{-1-\alpha} \int_{I_{N,1,2}} \mu(B_\delta(x))^q \, dx, \\
E_{1,\infty}^{(\alpha)}(\delta) &= \sum_{j=N+1}^{\infty} \big(w_1(j-2)^q (\rho r^{j-2})^{-\alpha} \Phi_1^{(\alpha)}(\delta/\rho r^{j-2}) + (p_2^q r^{-\alpha})^{j-1} \Phi_2^{(\alpha)}(\delta/r^{j-1}) \big).
\end{aligned}$$

Let

$$\Phi_2^{(\alpha)}(\delta) = p_3^q r^{-\alpha} \big(\Phi_1^{(\alpha)}(\delta/r) + \Phi_2^{(\alpha)}(\delta/r) \big) + E_2^{(\alpha)}(\delta), \qquad (4.9)$$

where

$$E_2^{(\alpha)}(\delta) = \delta^{-1-\alpha} (e_2^2(\delta) - \tilde{e}_2^2(\delta)).$$

Next, we derive a vector-valued equation. It follows from (3.12), (3.13), (4.8) and (4.9) that

$$\begin{aligned}
f_1(x) = \sum_{j=2}^{\infty} \Big(& w_1(j-2)^q (\rho r^{j-2})^{-\alpha} f_1\big(x + \ln(\rho r^{j-2})\big) \\
& + (p_2^q r^{-\alpha})^{j-1} f_2(x + \ln(r^{j-1})) \Big) + z_1^{(\alpha)}(x)
\end{aligned}$$

and

$$f_2(x) = p_3^q r^{-\alpha} \sum_{i=1}^{2} f_i\big(x + \ln r\big) + z_2^{(\alpha)}(x),$$

where $z_1^{(\alpha)}(x) = E_1^{(\alpha)}(e^{-x}) - E_{1,\infty}^{(\alpha)}(e^{-x}), z_2^{(\alpha)}(x) = E_2^{(\alpha)}(e^{-x})$. For $\ell, m = 1, 2$, let $\mu_{m\ell}^{(\alpha)}$ be the discrete measures such that for $j \geq 2$,

$$\begin{aligned}
&\mu_{11}^{(\alpha)}(-\ln(\rho r^{j-2})) = (w_1(j-2))^q(\rho r^{j-2})^{-\alpha},\\
&\mu_{21}^{(\alpha)}(-\ln(r^{j-1})) = (p_2^q r^{-\alpha})^{j-1},\\
&\mu_{12}^{(\alpha)}(-\ln r) = \mu_{22}^{(\alpha)}(-\ln r) = p_3^q r^{-\alpha}.
\end{aligned}$$

Then

$$\begin{aligned}
&\mu_{11}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty} w_1(j-2)^q(\rho r^{j-2})^{-\alpha}, \qquad \mu_{21}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1},\\
&\mu_{12}^{(\alpha)}(\mathbb{R}) = \mu_{22}^{(\alpha)}(\mathbb{R}) = p_3^q r^{-\alpha}.
\end{aligned} \tag{4.10}$$

For fixed $q \geq 0$, let

$$\begin{aligned}
&F_1(\alpha) := \sum_{j=2}^{\infty} w_1(j-2)^q(\rho r^{j-2})^{-\alpha} + \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1}, \qquad F_2(\alpha) := 2p_3^q r^{-\alpha}\\
&D_\ell := \{\alpha \in \mathbb{R} : F_\ell(\alpha) < \infty\}, \qquad \ell = 1, 2.
\end{aligned} \tag{4.11}$$

and

$$\mathbf{M}_\alpha(\infty) = \begin{pmatrix} \sum_{j=2}^{\infty} w_1(j-2)^q(\rho r^{j-2})^{-\alpha} & p_3^q r^{-\alpha} \\ \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1} & p_3^q r^{-\alpha} \end{pmatrix}.$$

Finally, we show that the error terms $z_\ell^{(\alpha)}(x) = o(e^{-\epsilon x})$ as $x \to \infty$, i.e., $E_\ell^{(\alpha)}(\delta) = o(\delta^\epsilon)$ and $E_{1,\infty}^{(\alpha)}(\delta) = o(\delta^\epsilon)$ as $\delta \to 0$ for some $\epsilon > 0$ and $\ell = 1, 2$.

Proposition 4.1. (a) *$\Phi_1^{(\alpha)}(\delta/\rho r^k) \leq 1$ for any $k \geq N-1$.*
(b) *$\Phi_2^{(\alpha)}(\delta/r^k) \leq 1$ for any $k \geq N$.*

Proof. (a) It follows from the definition of N that $\delta \geq \rho r^k$ for any $k \geq N-1$. Hence

$$\Phi_1^{(\alpha)}(\delta/\rho r^k) = \frac{1}{(\delta/\rho r^k)^{1+\alpha}} \int_{I_{1,1}} \mu\Big(B_{\delta/\rho r^k}(x)\Big)^q\, dx \leq (\rho r^k/\delta)^{1+\alpha} \leq 1.$$

Hence $\Phi_1^{(\alpha)}(\delta/\rho r^k) \leq 1$ for any $k \geq N$.

(b) The proof is similar to that of (a). □

The following proposition can be proved directly by using induction; we omit the details.

Proposition 4.2. (a) *$S_{2^k}(1) = r^k + \rho(1 - r^k)$ for any $k \geq 1$.*
(b) *$S_{2^{k-1}1}(0) = \rho(1 - r^{k-1})$ for any $k \geq 1$.*

Proposition 4.3. *For $q \geq 0$, let $F_1(\alpha)$ and D_1 be defined as in (4.11). Then D_1 is open.*

Proof. Let $p := \max\{p_2, p_3\}$. In view of (1.7), we consider the following two cases for $w_1(k)$.

Case 1. $p_2 = p_3$. Then $w_1(k) = (k+1)p_1 p_2^k$; moreover,

$$p_1 p^k \le w_1(k) = (k+1)p_1 p^k. \tag{4.12}$$

Thus,

$$\begin{aligned}\lim_{k\to\infty} \sqrt[k]{w_1(k)^q(\rho r^k)^{-\alpha}} &= \lim_{k\to\infty} \sqrt[k]{((k+1)p_1p^k)^q(\rho r^k)^{-\alpha}}\\ &= \lim_{k\to\infty} \sqrt[k]{(k+1)^q p_1^q \rho^{-\alpha}}\cdot p^q/r^\alpha\\ &= p^q/r^\alpha. \end{aligned}\tag{4.13}$$

Case 2. $p_2 \ne p_3$. Assume $p_2 > p_3$. Then

$$w_1(k) = p_1 p_2^k \sum_{j=0}^{k} (p_3/p_2)^j = p_1 p_2^k \frac{1-(p_3/p_2)^{k+1}}{1-p_3/p_2}.$$

Note that

$$1 \le \frac{1-(p_3/p_2)^{k+1}}{1-p_3/p_2} < \frac{1}{1-p_3/p_2} = \frac{p_2}{p_2-p_3} =: c.$$

Thus $p_1p_2^k \le w_1(k) \le cp_1p_2^k$. Similarly, if $p_3 > p_2$, we have $p_1p_3^k \le w_1(k) \le cp_1p_3^k$. So if $p_2 \ne p_3$, we have

$$p_1 p^k \le w_1(k) \le cp_1p^k. \tag{4.14}$$

Hence

$$\lim_{k\to\infty} \sqrt[k]{w_1(k)^q(\rho r^k)^{-\alpha}} = p^q/r^\alpha \qquad \text{if } p_2 \ne p_3. \tag{4.15}$$

Combining (4.13) and (4.15), we have $\lim_{k\to\infty} \sqrt[k]{w_1(k)^q(\rho r^k)^{-\alpha}} = p^q/r^\alpha$. By the root test, the series $\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha}$ is convergent if $p^q/r^\alpha < 1$, i.e., $\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha}$ and $\sum_{k=0}^\infty (p^q/r^\alpha)^k$ have the same radius of convergence. If $p^q/r^\alpha = 1$, then $\sum_{k=0}^\infty (p^q/r^\alpha)^k = \infty$. It follows from (4.12) and (4.14) that $(p_1p^k)^q \le w_1(k)^q$ for $q \ge 0$. For $k \ge 0$, we have $(p_1p^k)^q(\rho r^k)^{-\alpha} \le w_1(k)^q(\rho r^k)^{-\alpha}$. Thus

$$\infty = p_1^q \rho^{-\alpha} \sum_{k=0}^\infty (p^q/r^\alpha)^k \le \sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha}.$$

Hence D_1 is open. $\square$

Proposition 4.4. *For $q \ge 0$, assume that $\alpha \in D_\ell$ for $\ell = 1, 2$. Then there exists $\epsilon > 0$ such that*

(a) $\sum_{j=N+1}^\infty w_1(j-2)^q(\rho r^{j-2})^{-\alpha}\Phi_1^{(\alpha)}(\delta/\rho r^{j-2}) = o(\delta^\epsilon)$;

(b) $\sum_{j=N+1}^\infty (p_2^q r^{-\alpha})^{j-1}\Phi_2^{(\alpha)}(\delta/r^{j-1}) = o(\delta^\epsilon)$;

(c) $\sum_{j=2}^N \delta^{-1-\alpha}(e_j^1(\delta) - \tilde{e}_j^1(\delta)) = o(\delta^\epsilon)$;

(d) $\delta^{-1-\alpha}\int_{I_{N,1,2}} \mu(B_\delta(x))^q\,dx = o(\delta^\epsilon)$;

(e) $\delta^{-1-\alpha}(e_2^2(\delta) - \tilde{e}_2^2(\delta)) = o(\delta^\epsilon)$.

Proof. (a) By Proposition 4.3, $D_1 = \{\alpha \in \mathbb{R} : F_1(\alpha) < \infty\}$ is open. Thus there exists $\epsilon > 0$ sufficiently small such that $F_1(\alpha+\epsilon) \in D_1$. So there exists a positive constant C such that

$$\sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha-\epsilon} + \sum_{j=N+1}^{\infty} (p_2^q r^{-\alpha-\epsilon})^{j-1} \le C.$$

Since

$$(\rho r^{N-1})^{-\epsilon} \sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha} \le \sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha-\epsilon},$$

we have $\sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha} \le C(\rho r^{N-1})^{\epsilon} \le C\delta^{\epsilon}$, where the last inequality follows from the definition of N. Combining these with Proposition 4.1(a), we have $\sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha} \Phi_1^{(\alpha)}(\delta/\rho r^{j-2}) = o(\delta^{\epsilon})$.

(b) The proof is similar to that of (a).

(c) It suffices to show that $e_j^1(\delta) = o(\delta^{1+\alpha+\epsilon})$ and $\tilde{e}_j^1(\delta) = o(\delta^{1+\alpha+\epsilon})$ for $2 \le j \le N$. It follows from (4.6) and (4.3) that

$$e_j^1(\delta) = \Big(\int_{S_{2^{j-2}1_1}(0)}^{S_{2^{j-2}1_1}(0)+\delta} + \int_{S_{2^{j-2}1_2}(1)-\delta}^{S_{2^{j-2}1_2}(1)} + \int_{S_{2^{j-1}3}(0)}^{S_{2^{j-1}3}(0)+\delta} + \int_{S_{2^{j-1}3}(1)-\delta}^{S_{2^{j-1}3}(1)} \Big) \mu(B_\delta(x))^q \, dx.$$

As an example we only prove $\int_{S_{2^{j-2}1_1}(0)}^{S_{2^{j-2}1_1}(0)+\delta} \mu(B_\delta(x))^q \, dx = o(\delta^{1+\alpha+\epsilon})$. It follows from (a) and (b) that

$$w_1(N-1)^q \le C\delta^{\alpha+\epsilon} \qquad \text{and} \qquad p_2^{Nq} \le C\delta^{\alpha+\epsilon}. \tag{4.16}$$

Since $B_\delta(x) \subseteq B_{2\delta}(S_{2^{j-2}1_1}(0))$ for any $x \in (S_{2^{j-2}1_1}(0), S_{2^{j-2}1_1}(0)+\delta)$ and

$$\mu(B_{2\delta}(S_{2^{j-2}1_1}(0))) = p_1 w_1(j-2) \mu(B_{2\delta/\rho^2 r^{j-2}}(0)) \le p_1 w_1(j-2),$$

we have

$$\begin{aligned} \int_{S_{2^{j-2}1_1}(0)}^{S_{2^{j-2}1_1}(0)+\delta} \mu(B_\delta(x))^q \, dx &\le (\mu(B_{2\delta}(S_{2^{j-2}1_1}(0))))^q \delta \le p_1^q w_1(j-2)^q \delta \\ &\le (p_1 p_2^{1-N})^q w_1(N-1)^q \delta \le C(p_1 p_2^{1-N})^q \delta^{1+\alpha+\epsilon}, \end{aligned}$$

where the third inequality holds because for $0 \le k \le N-2$

$$\begin{aligned} w_1(k) &= \frac{p_1(p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1})(p_2^k + p_2^{k-1}p_3 + \cdots + p_3^k)}{(p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1})} \\ &\le \frac{w_1(N-1)(p_2+p_3)^k}{p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1}} \le p_2^{1-N} w_1(N-1), \end{aligned} \tag{4.17}$$

and the last inequality follows from (4.16). The estimate $\tilde{e}_2^2(\delta) = o(\delta^{1+\alpha+\epsilon})$ can be established as that for $e_2^2(\delta) = o(\delta^{1+\alpha+\epsilon})$.

(d) By (4.1), we have

$$\begin{aligned} \int_{I_{N,1,2}} \mu(B_\delta(x))^q \, dx &= \Big(\int_{S_{2^{N-1}1}(0)}^{S_{2^{N-1}1}(0)+\delta} + \int_{S_{2^{N-1}1}(0)+\delta}^{S_{2^N}(1)-\delta} + \int_{S_{2^N}(1)-\delta}^{S_{2^N}(1)} \Big) \mu(B_\delta(x))^q \, dx \\ &=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned}$$

We first show that $\delta^{-1-\alpha}(\mathrm{I}) = o(\delta^{\epsilon/2})$. For any $x \in (S_{2^{N-1}1}(0), S_{2^{N-1}1}(0)+\delta)$, we have $B_\delta(x) \subseteq B_{2\delta}(S_{2^{N-1}1}(0))$ and $\mu(B_{2\delta}(S_{2^{N-1}1}(0))) = w_1(N-1)\mu(B_{2\delta/\rho r^{N-1}}(0)) \le w_1(N-1)$. Combining these with (4.16) we have

$$(\mathrm{I}) \le \mu(B_{2\delta}(S_{2^{N-1}1}(0)))^q\delta \le w_1(N-1)^q\delta \le C\delta^{1+\alpha+\epsilon}.$$

It follows that $\delta^{-1-\alpha}(\mathrm{I}) = o(\delta^{\epsilon/2})$.

Next, we show that $\delta^{-1-\alpha}(\mathrm{II}) = o(\delta^{\epsilon/2})$. It follows from [22, Lemma 2.14] that

$$\mu|_{S_{2^{N-1}}(I_{1,1})} = w_1(N-1)\mu\circ S_{2^{N-1}1}^{-1} + p_2^N\mu\circ S_{2^N}^{-1} \quad \text{on } S_{2^{N-1}}(I_{1,1}).$$

Thus $\mu(B_\delta(x)) \le w_1(N-1) + p_2^N$ for $x \in (S_{2^{N-1}1}(0)+\delta, S_{2^N}(0)-\delta)$. Combining Proposition 4.2, (4.16) and (1.6), we have

$$\begin{aligned}
(\mathrm{II}) &\le (S_{2^N}(1) - S_{2^{N-1}1}(0) - 2\delta)(w_1(N-1)+p_2^N)^q \\
&\le r^{N-1}(2r+\rho(1-r))((C\delta^{\alpha+\epsilon})^{1/q} + (C\delta^{\alpha+\epsilon})^{1/q})^q \\
&\le C' r^{N-1}\delta^{\alpha+\epsilon} \le C' r^{-1}\delta^{1+\alpha+\epsilon},
\end{aligned}$$

i.e., $\delta^{-1-\alpha}(\mathrm{II}) = o(\delta^{\epsilon/2})$.

The proof of $\delta^{-1-\alpha}(\mathrm{III}) = o(\delta^{\epsilon/2})$ is similar to that for $\delta^{-1-\alpha}(\mathrm{I}) = o(\delta^{\epsilon/2})$. Hence $\delta^{-1-\alpha}\int_{S_{2^{N-1}}(I_{1,1})}\mu(B_\delta(x))^q\,dx =$ $o(\delta^{\epsilon/2})$.

(e) The proof is similar to that of (c). $\square$

Proof of Theorem 1.2. Combining Theorem 1.1 and Proposition 4.4 yields $\tau(q) = \alpha$. Let

$$G(q,\alpha) := (1-p_2^q r^{-\alpha})(1-p_3^q r^{-\alpha})\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha} + r^{-\alpha}(p_2^q+p_3^q) - 1. \tag{4.18}$$

We show that $G(q,\alpha)$ is C^1. It follows from Proposition 4.3 that $\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha} < \infty$ for any $(q,\alpha) \in (0,\infty)\times D_1$. Since $w_1(k) \le p_1 < 1$, $\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha}$ is strictly decreasing in q and strictly increasing in α. Thus for any $(q_0,\alpha_0) \in (0,\infty)\times D_1$, the series converges uniformly on $\{(q,\alpha) : q \ge q_0, \alpha \le \alpha_0\}$. Moreover, it follows from (4.12) and (4.14) that

$$\lim_{k\to\infty} w_1(k) = 0. \tag{4.19}$$

Hence, for any $(q,\alpha) \in (0,\infty)\times D_1$,

$$\begin{aligned}
G_q(q,\alpha) =& \big(-p_2^q r^{-\alpha}(1-p_3^q r^{-\alpha})\ln p_2 - p_3^q r^{-\alpha}(1-p_2^q r^{-\alpha})\ln p_3\big)\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha} \\
&+ (1-p_2^q r^{-\alpha})(1-p_3^q r^{-\alpha})\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha}\ln w_1(k) + r^{-\alpha}\sum_{i=2}^3 p_i^q\ln p_i
\end{aligned}$$

and

$$\begin{aligned}
G_\alpha(q,\alpha) =& \big(p_2^q(1-p_3^q r^{-\alpha}) + p_3^q(1-p_2^q r^{-\alpha})\big)r^{-\alpha}\ln r\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha} \\
&+ (1-p_2^q r^{-\alpha})(1-p_3^q r^{-\alpha})\sum_{k=0}^\infty w_1(k)^q(\rho r^k)^{-\alpha}\ln(\rho r^k)^{-1} + r^{-\alpha}\sum_{i=2}^3 p_i^q\ln r^{-1}.
\end{aligned}$$

A similar argument as above shows that $G(q,\alpha)$ is C^1.

We now show that $G_\alpha(\tilde q,\tilde\alpha)\neq 0$ for any $(\tilde q,\tilde\alpha)\in(0,\infty)\times D_1$ satisfying $G(\tilde q,\tilde\alpha)=0$. Since $\tau(q)$ is convex, we can let $\{q_n\}$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} q_n=\tilde q$ and that τ is differentible at each q_n. Then (1.8) implies that

$$G_q(q_n,\alpha_n)+G_\alpha(q_n,\alpha_n)\cdot\alpha'(q_n)=0 \quad \text{for all } n, \text{ and thus} \quad G_q(\tilde q,\tilde\alpha)+G_\alpha(\tilde q,\tilde\alpha)\cdot\alpha'_-(\tilde q)=0,$$

where $\alpha'_-(\tilde q)$ denotes left-hand derivative of $\alpha(q)(=\tau(q))$ at $\tilde q$.

Suppose, on the contrary, that $G_\alpha(\tilde q,\tilde\alpha)=0$. Then $G_q(\tilde q,\tilde\alpha)=0$. So $G_\alpha(\tilde q,\tilde\alpha)-G_q(\tilde q,\tilde\alpha)=0$. It follows from $G(\tilde q,\tilde\alpha)=0$ that

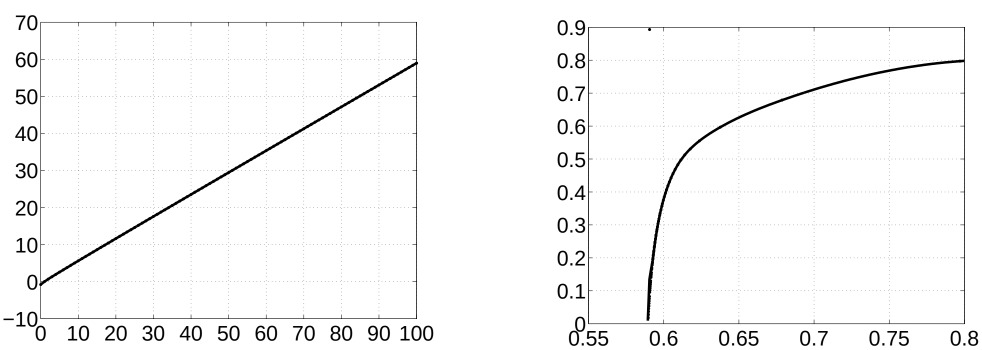
$$\sum_{k=0}^{\infty} w_1(k)^{\tilde q}(\rho r^k)^{-\tilde\alpha}=\frac{1-(p_2^{\tilde q}+p_3^{\tilde q})r^{-\tilde\alpha}}{(1-p_2^{\tilde q}r^{-\tilde\alpha})(1-p_3^{\tilde q}r^{-\tilde\alpha})}. \tag{4.20}$$

Substituting (4.20) into the above expressions for G_q and G_α, simplifying the result, and using the fact that $0<p_i^{\tilde q}r^{-\tilde\alpha}<1$ for $i=2,3$, we get

$$\begin{aligned}
0&=G_\alpha(\tilde q,\tilde\alpha)-G_q(\tilde q,\tilde\alpha)\\
&=p_2^{\tilde q}r^{-\tilde\alpha}(\ln r^{-1}-\ln p_2)\frac{p_3^{\tilde q}r^{-\tilde\alpha}}{1-p_2^{\tilde q}r^{-\tilde\alpha}}+p_3^{\tilde q}r^{-\tilde\alpha}(\ln r^{-1}-\ln p_3)\frac{p_2^{\tilde q}r^{-\tilde\alpha}}{1-p_3^{\tilde q}r^{-\tilde\alpha}}\\
&\quad+(1-p_2^{\tilde q}r^{-\tilde\alpha})(1-p_3^{\tilde q}r^{-\tilde\alpha})\sum_{k=0}^{\infty}w_1(k)^{\tilde q}(\rho r^k)^{-\tilde\alpha}\big(\ln(\rho r^k)^{-1}-\ln w_1(k)\big)>0,
\end{aligned}$$

a contradiction. Hence $G_\alpha(q,\alpha)\neq 0$ for any $(q,\alpha)\in(0,\infty)\times D_1$ satisfying $G(q,\alpha)=0$. The implicit function theorem now implies that τ is differentiable on $(0,\infty)$ and the stated formula for $\dim_{\mathrm H}(\mu)$ follows by computing $\tau'(1)=-G_q(1,0)G_\alpha(1,0)^{-1}$ (see [9,19]). This completes the proof. $\square$

Figure 8 shows the graphs of $\tau(q)$ and $f(\alpha)$, $q\geq 0$, for some measure in the family. For this example, we have $\dim_{\mathrm H}(\mu)=\tau'(1)\approx 0.720268$ and $\dim_{\mathrm H}(K)=-\tau(0)\approx 0.797012$, where K is the self-similar set corresponding to the IFS in (1.5).



(a) $\tau(q)$ (b) $f(\alpha)$

FIGURE 8. Graphs of $\tau(q)$ and $f(\alpha)$ for the self-similar measure in Example 2.17 with $\rho=1/3$, $r=2/7$, $p_1=1/2$, $p_2=1/4$, and $p_3=1/4$.

5. A class of finite IFSs with overlaps on $\mathbb{R}^2$

In this section, we derive renewal equations and compute the L^q-spectrum of self-similar measure μ defined by the IFSs in (1.9) together with a probability vector $(p_i)_{i=1}^4$. Let $X := [0,1]\times[0,1], \Omega = (0,1)\times(0,1)$. Define

$$\mathcal{I}_{1,1} = \{(S_1,1),(S_2,1)\}, \qquad \mathcal{I}_{1,2} = \{(S_3,1) \qquad \mathcal{I}_{1,3} = \{(S_4,1)\}.$$

It follows from Example 2.18 that μ satisfies Condition (B) with $\mathbf{I}_b = \{\mathcal{I}_{1,1},\mathcal{I}_{1,2},\mathcal{I}_{1,3}\}$ being the basic set of islands. Moreover, $\mathbf{I} = \mathbf{I}_{b,\mu} = \{[\mathcal{I}_{1,1}]_\mu, [\mathcal{I}_{1,2}]_\mu, [\mathcal{I}_{1,3}]_\mu\}, \Gamma = \{1,2,3\}, \Gamma_* = \{2,3\}, \Gamma'_* = \{1\}, \kappa_1 = \infty$ and $\kappa_2 = \kappa_3 = 2$. Let $\mathbf{I}_{k,\ell}, \mathbf{I}'_{k,\ell}, G_{k,\ell}, G'_{k,\ell}$ be defined as in Section 3 for $\ell \in \Gamma$ and $2 \le k \le \kappa_\ell$. Define

$$\begin{aligned}
&\mathcal{I}_{k,1,1} := \{(S_{2^{k-2}11},k),(S_{2^{k-2}12},k)\}, && \mathcal{I}_{k,1,2} := \{(S_{2^{k-2}14},k)\},\\
&\mathcal{I}_{k,1,3} := \{(S_{2^{k-1}1},k),(S_{2^k},k)\}, && \mathcal{I}_{k,1,4} := \{(S_{2^{k-1}3},k)\},\\
&\mathcal{I}_{k,1,5} := \{(S_{2^{k-1}4},k)\},
\end{aligned}$$

for $k \ge 2$, and

$$\mathcal{I}_{2,\ell,1} = \{(S_{(\ell+1)1},2),(S_{\ell+12},2)\}, \qquad \mathcal{I}_{2,\ell,2} = \{(S_{(\ell+1)3},2)\}, \qquad \mathcal{I}_{2,\ell,3} = \{(S_{(\ell+1)4},2)\},$$

for $\ell = 2,3$ (see Figure 2). It follows from Lemma 2.22 that $\mathbf{I}_{k,1} = \{\mathcal{I}_{k,1,i} : i = 1,2,4,5\}$, $\mathbf{I}'_{k,1} = \{\mathcal{I}_{k,1,3}\}$, $\mathbf{I}_{2,\ell} = \{\mathcal{I}_{2,\ell,i} : i = 1,2,3\}$, and $\mathbf{I}'_{2,\ell} = \emptyset$. Hence $G_{k,1} = \{1,2,4,5\}$, $G'_{k,1} = \{3\}$, $G_{2,\ell} = \{1,2,3\}$, and $G'_{2,\ell} = \emptyset$ for $k \ge 2$ and $\ell = 2,3$. Define $I_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega)$ for $\ell \in \Gamma$. Let

$$\begin{aligned}
&I_{k,1,1} := S_{\mathcal{I}_{k,1,1}}(\Omega) = S_{2^{k-2}1}(I_{1,1}), && I_{k,1,2} := S_{\mathcal{I}_{k,1,2}}(\Omega) = S_{2^{k-2}1}(I_{1,3}),\\
&I_{k,1,3} := S_{\mathcal{I}_{k,1,3}}(\Omega) = S_{2^{k-1}}(I_{1,1}), && I_{k,1,4} := S_{\mathcal{I}_{k,1,4}}(\Omega) = S_{2^{k-1}}(I_{1,2}),\\
&I_{k,1,5} := S_{\mathcal{I}_{k,1,5}}(\Omega) = S_{2^{k-1}}(I_{1,3}),
\end{aligned} \tag{5.1}$$

for $k \ge 2$ and

$$I_{2,\ell,i} := S_{\mathcal{I}_{2,\ell,i}}(\Omega) = S_{\ell+1}(I_{1,i}) \quad \text{ for } \ell = 2,3 \text{ and } i = 1,2,3. \tag{5.2}$$

In the rest of this section, let $w_2(k)$ be defined as in (1.11).

First, we derive functional equations for $\Phi_\ell^{(\alpha)}(\delta)$ for $\ell = 1,2,3$. Combining (3.5), (3.6), (5.1) and (5.2), we have

$$\varphi_1(\delta) \;=\; \Big(\sum_{j=2}^{n}\Big(\int_{I_{j,1,1}} + \int_{I_{j,1,2}} + \int_{I_{j,1,4}} + \int_{I_{j,1,5}}\Big) + \int_{I_{n,1,3}}\Big)\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x},$$

and

$$\varphi_\ell(\delta) = \sum_{i=1}^{3}\int_{I_{2,\ell,i}} \mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x} \qquad \text{for } \ell = 2,3.$$

For $\ell \in \Gamma$, $2 \le k \le \kappa_\ell, i \in G_{k,\ell}$, and $\delta > 0$, let $\widetilde{I}_{k,\ell,i}(\delta)$, $\widehat{I}_{k,\ell,i}(\delta)$, $\widetilde{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ and $\widehat{I}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ be defined as in Section 3. Recall from (2.7) that $\gamma_k := 1 - r^k$. Combining

(5.1), (5.2) and Proposition 2.20, we have for $j \geq 2$,

$$\begin{aligned}
\widetilde{I}_{j,1,1}(\delta) =&(\rho\gamma_{j-2}+\delta, \rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\delta)\times(\delta,\rho^2 r^{j-2}-\delta)\cup\\
&(\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\delta, \rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho r^{j-1}-\delta)\times(\delta,\rho r^{j-1}-\delta),\\
\widehat{I}_{j,1,1}(\delta) =&(\rho\gamma_{j-2}, \rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho\gamma_{j-1})\times(0,\delta)\cup\\
&(\rho\gamma_{j-2}, \rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\delta)\times(\rho^2 r^{j-2}-\delta,\rho^2 r^{j-2})\cup\\
&(\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1, \rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho r^{j-1})\times(\rho r^{j-1}-\delta,\rho r^{j-1})\cup\\
&(\rho\gamma_{j-2}, \rho\gamma_{j-2}+\delta)\times(\delta,\rho^2 r^{j-2}-\delta)\cup\\
&(\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1, \rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\delta)\times(\rho^2 r^{j-2},\rho r^{j-1}-\delta)\cup\\
&(\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho r^{j-1}-\delta, \rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho r^{j-1})\times(\delta,\rho r^{j-1}-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{j,1,2}(\delta) =&(\rho\gamma_{j-2}+\delta, \rho\gamma_{j-2}+\rho r^{j-1}-\delta)\times(\rho r^{j-2}\gamma_1+\delta,\rho r^{j-2}-\delta),\\
\widehat{I}_{j,1,2}(\delta) =&(\rho\gamma_{j-2}, \rho\gamma_{j-2}+\rho r^{j-1})\times\big((\rho r^{j-2}\gamma_1,\rho r^{j-2}\gamma_1+\delta)\cup(\rho r^{j-2}-\delta,\rho r^{j-2})\big)\cup\\
&\big((\rho\gamma_{j-2}, \rho\gamma_{j-2}+\delta)\cup(\rho\gamma_{j-2}+\rho r^{j-1}-\delta, \rho\gamma_{j-2}+\rho r^{j-1})\big)\\
&\times(\rho r^{j-1}\gamma_1+\delta,\rho r^{j-2}-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{j,1,4}(\delta) =&(r^{j-1}\gamma_1+\rho\gamma_{j-1}+\delta, r^{j-1}+\rho\gamma_{j-1}+\delta)\times(\delta,r^j-\delta),\\
\widehat{I}_{j,1,4}(\delta) =&(r^{j-1}\gamma_1+\rho\gamma_{j-1}, r^{j-1}+\rho\gamma_{j-1})\times\big((0,\delta)\cup(r^j-\delta,r^j)\big)\cup\\
&\big((r^{j-1}\gamma_1+\rho\gamma_{j-1}, r^{j-1}\gamma_1+\rho\gamma_{j-1}+\delta)\cup(r^{j-1}+\rho\gamma_{j-1}-\delta, r^{j-1}+\rho\gamma_{j-1})\big)\\
&\times(\delta,r^k-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{j,1,5}(\delta) =&(\rho\gamma_{j-1}+\delta, r^j+\rho\gamma_{j-1}-\delta)\times(r^{j-1}\gamma_1+\delta,r^{j-1}-\delta),\\
\widehat{I}_{j,1,5}(\delta) =&(\rho\gamma_{j-1}, r^j+\rho\gamma_{j-1})\times\big((r^{j-1}\gamma_1,r^{j-1}\gamma_1+\delta)\times(r^{j-1}-\delta,r^{j-1})\big)\cup\\
&\big((\rho\gamma_{j-1}, \rho\gamma_{j-1}+\delta)\cup(r^{j-1}+\rho\gamma_{j-1}-\delta, r^j+\rho\gamma_{j-1})\big)\times(r^{j-1}\gamma_1+\delta,r^{j-1}-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{2,2,1}(\delta) =&(\gamma_1+\delta, (1+\rho r)\gamma_1+\delta)\times(\delta,\rho r-\delta)\cup\\
&((1+\rho r)\gamma_1+\delta, (1+\rho r)\gamma_1+r^2-\delta)\times(\delta,r^2-\delta),\\
\widehat{I}_{2,2,1}(\delta) =&(\gamma_1, (1+\rho r)\gamma_1+r^2)\times(0,\delta)\cup(\gamma_1,(1+\rho r)\gamma_1+\delta)\times(\rho r-\delta,\rho r)\cup\\
&((1+\rho r)\gamma_1, (1+\rho r)\gamma_1+r^2)\times(r^2-\delta,r^2)\cup(\gamma_1,\gamma_1+\delta)\times(\delta,\rho r-\delta)\cup\\
&((1+\rho r)\gamma_1, (1+\rho r)\gamma_1+\delta)\times(\rho r,r^2-\delta)\cup\\
&((1+\rho r)\gamma_1+r^2-\delta, (1+\rho r)\gamma_1)\times(\delta,r^2-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{2,2,2}(\delta) =&(\gamma_2+\delta, 1-\delta)\times(\delta,r^2-\delta),\\
\widehat{I}_{2,2,2}(\delta) =&(\gamma_2, 1)\times\big((0,\delta)\cup(r^2-\delta,r^2)\big)\cup((\gamma_2,\gamma_2+\delta)\cup(1-\delta,1))\times(\delta,r^2-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{2,2,3}(\delta) =&(\gamma_1+\delta, r^2+\gamma_1-\delta)\times(r\gamma_1+\delta,r-\delta),\\
\widehat{I}_{2,2,3}(\delta) =&\big((\gamma_1,\gamma_1+\delta)\cup(r^2+\gamma_1-\delta,r^2+\gamma_1)\big)\times(r\gamma_1+\delta,r-\delta)\cup(\gamma_1,r^2+\gamma_1)\times\\
&\big((r\gamma_1,r\gamma_+\delta)\cup(r-\delta,r)\big),
\end{aligned}$$

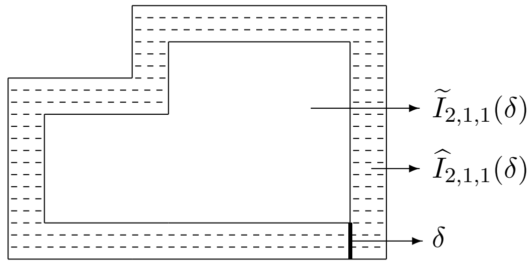


FIGURE 9. The middle part and shaded region are $\widetilde{I}_{2,1,1}(\delta)$ and $\widehat{I}_{2,1,1}(\delta)$, respectively, the union is $I_{2,1,1}$.

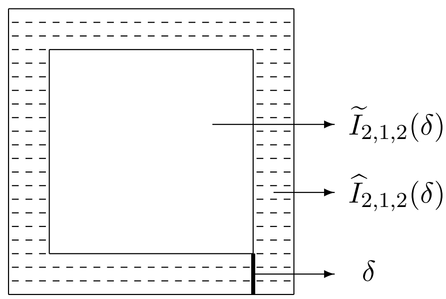


FIGURE 10. The middle part and shaded region are $\widetilde{I}_{2,1,2}(\delta)$ and $\widehat{I}_{2,1,2}(\delta)$, respectively, the union is $I_{2,1,2}$.

$$\begin{aligned}
\widetilde{I}_{2,3,1}(\delta) =&(\delta, \rho r\gamma_1+\delta)\times(\gamma_1+\delta, \rho r+\gamma_1-\delta)\cup\\
&(\rho r\gamma_1+\delta, \rho r\gamma_1+r^2-\delta)\times(\gamma_1+\delta, r^2+\gamma_1-\delta),\\
\widehat{I}_{2,3,1}(\delta) =&(0, \rho r\gamma_1+r^2)\times(\gamma_1, \gamma_1+\delta)\cup(0, \rho r\gamma_1+\delta)\times(\rho r+\gamma_1-\delta, \rho r+\gamma_1)\cup\\
&(\rho r\gamma_1, \rho r\gamma_1+r^2)\times(r^2+\gamma_1-\delta, r^2+\gamma_1)\cup(0,\delta)\times(\gamma_1+\delta, \rho r+\gamma_1-\delta)\cup\\
&(\rho r\gamma_1, \rho r\gamma_1+\delta)\times(\rho r+\gamma_1, r^2+\gamma_1-\delta)\cup\\
&(\rho r\gamma_1+r^2-\delta, \rho r\gamma_1+r^2)\times(\gamma_1+\delta, r^2+\gamma_1-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{2,3,2}(\delta) =&(r\gamma_1+\delta, r-\delta)\times(\gamma_1+\delta, r^2+\gamma_1-\delta),\\
\widehat{I}_{2,3,2}(\delta) =&(r\gamma_1, r)\times\big((\gamma_1, \gamma_1+\delta)\cup(r^2+\gamma_1-\delta, r^2+\gamma_1)\big)\cup(\gamma_1+\delta, r^2+\gamma_1-\delta)\times\\
&\big((r\gamma_1, r\gamma_1+\delta)\cup(r-\delta, r)\big),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_{2,3,3}(\delta) =&(\delta, r^2-\delta)\times(\gamma_2+\delta, 1-\delta),\\
\widehat{I}_{2,3,3}(\delta) =&(0, r^2)\times((\gamma_2, \gamma_2+\delta)\cup(1-\delta, 1))\cup\big((0,\delta)\cup(r^2-\delta, r^2)\big)\times(\gamma_2+\delta, 1-\delta).
\end{aligned}$$

(See Figures 9 and 10)

It follows from (5.1), (5.2) and Lemma 2.22 that for $i=1,2,3$ and $j\geq 2$,

$$\begin{aligned} &\mu(I_{j,1,1})=w_2(j-2)\mu(I_{1,1}), &&\mu(I_{j,1,2})=w_2(j-2)\mu(I_{1,3}),\\ &\mu(I_{j,1,4})=p_2(j-1)\mu(I_{1,2}), &&\mu(I_{j,1,5})=p_2(j-1)\mu(I_{1,3}),\\ &\mu(I_{2,2,i})=p_3\mu(I_{1,i}), &&\mu(I_{2,3,i})=p_4\mu(I_{1,i}). \end{aligned}$$

Thus

$$\begin{aligned} &w(j,1,1)=w_2(j-2), && c(j,1,1)=1, && \sigma(j,1,1)=S_{2^{j-2}1}, && r_{\sigma(j,1,1)}=\rho r^{j-2},\\ &w(j,1,2)=w_2(j-2), && c(j,1,2)=3, && \sigma(j,1,2)=S_{2^{j-2}1}, && r_{\sigma(j,1,2)}=\rho r^{j-2},\\ &w(j,1,4)=p_2^{j-1}, && c(j,1,4)=2, && \sigma(j,1,4)=S_{2^{j-1}}, && r_{\sigma(j,1,1)}=r^{j-1},\\ &w(j,1,5)=p_2^{j-1}, && c(j,1,5)=3, && \sigma(j,1,5)=S_{2^{j-1}}, && r_{\sigma(j,1,2)}=r^{j-1},\\ &w(2,2,i)=p_3, && c(2,2,i)=i, && \sigma(2,2,i)=S_3, && r_{\sigma(2,2,i)}=r,\\ &w(2,3,i)=p_4, && c(2,3,i)=i, && \sigma(2,3,i)=S_4, && r_{\sigma(2,3,i)}=r, \end{aligned}$$

$$\begin{aligned} \widetilde{I}_{1,1}(\delta/\rho r^{j-2}) =&(\delta/\rho r^{j-2},\rho\gamma_1+\delta/\rho r^{j-2})\times(\delta/\rho r^{j-2},\rho-\delta/\rho r^{j-2})\cup\\ &(\rho\gamma_1+\delta/\rho r^{j-2},\rho\gamma_1+r-\delta/\rho r^{j-2})\times(\delta/\rho r^{j-2},r-\delta/\rho r^{j-2}),\\ \widehat{I}_{1,1}(\delta/\rho r^{j-2}) =&(0,\rho\gamma_1+r)\times(0,\delta/\rho r^{j-2})\cup\\ &(0,\rho\gamma_1+\delta/\rho r^{j-2})\times(\rho-\delta/\rho r^{j-2},\rho)\cup\\ &(\rho\gamma_1,\rho\gamma_1+r)\times(r-\delta/\rho r^{j-2},r)\cup\\ &(0,\delta/\rho r^{j-2})\times(\delta/\rho r^{j-2},\rho-\delta/\rho r^{j-2})\cup\\ &(\rho\gamma_1,\rho\gamma_1+\delta/\rho r^{j-2})\times(\rho,r-\delta/\rho r^{j-2})\cup\\ &(\rho\gamma_1+r-\delta/\rho r^{j-2},\rho\gamma_1+r)\times(\delta/\rho r^{j-2},r-\delta/\rho r^{j-2}),\\ \widetilde{I}_{1,3}(\delta/\rho r^{j-2}) =&(\delta/\rho r^{j-2},r-\delta/\rho r^{j-2})\times(1-r+\delta/\rho r^{j-2},1-\delta/\rho r^{j-2}),\\ \widehat{I}_{1,3}(\delta/\rho r^{j-2}) =&(0,r)\times\big((1-r+\delta/\rho r^{j-2})\cup(1-\delta/\rho r^{j-2},1)\big)\cup\\ &\big((0,\delta/\rho r^{j-2})\cup(r-\delta/\rho r^{j-2},r)\big)\times(1-r+\delta/\rho r^{j-2},1-\delta/\rho r^{j-2}),\\ \widetilde{I}_{1,2}(\delta/r^{j-1}) =&(\gamma_1+\delta/r^{j-1},1-\delta/r^{j-1})\times(\delta/r^{j-1},r-\delta/r^{j-1}),\\ \widehat{I}_{1,2}(\delta/r^{j-1}) =&(\gamma_1,1)\times\big((0,\delta/r^{j-1})\cup(r-\delta/r^{j-1},r)\big)\cup\\ &\big((\gamma_1,\gamma_1+\delta/r^{j-1})\cup(1-\delta/r^{j-1},1)\big)\times(\delta/r^{j-1},r-\delta/r^{j-1}),\\ \widetilde{I}_{1,3}(\delta/r^{j-1}) =&(\delta/r^{j-1},r-\delta/r^{j-1})\times(1-r+\delta/r^{j-1},1-\delta/r^{j-1}),\\ \widehat{I}_{1,3}(\delta/r^{j-1}) =&(0,r)\times\big((1-r+\delta/r^{j-1})\cup(1-\delta/r^{j-1},1)\big)\cup\\ &\big((0,\delta/r^{j-1})\cup(r-\delta/r^{j-1},r)\big)\times(1-r+\delta/r^{j-1},1-\delta/r^{j-1}),\\ \widetilde{I}_{1,1}(\delta/r) =&(\delta/r,\rho\gamma_1+\delta/r)\times(\delta/r,\rho-\delta/r)\cup\\ &(\rho\gamma_1+\delta/r,\rho\gamma_1+r-\delta/r)\times(\delta/r,r-\delta/r),\\ \widehat{I}_{1,1}(\delta/r) =&(0,\rho\gamma_1+r)\times(0,\delta/r)\cup(0,\rho\gamma_1+\delta/r)\times(\rho-\delta/r,\rho)\cup\\ &(\rho\gamma_1,\rho\gamma_1+r)\times(r-\delta/r,r)\cup(0,\delta/r)\times(\delta/r,\rho-\delta/r)\cup\\ &(\rho\gamma_1,\rho\gamma_1+\delta/r)\times(\rho,r-\delta/r)\cup\\ &(\rho\gamma_1+r-\delta/r,\rho\gamma_1+r)\times(\delta/r,r-\delta/r), \end{aligned}$$

$$
\begin{aligned}
\widetilde{I}_{1,2}(\delta/r) =&(\gamma_1+\delta/r, 1-\delta/r)\times(\delta/r, r-\delta/r),\\
\widehat{I}_{1,2}(\delta/r) =&(\gamma_1, 1)\times\big((0,\delta/r)\cup(r-\delta/r, r)\big)\cup\\
&\big((\gamma_1,\gamma_1+\delta/r)\cup(1-\delta/r,1)\big)\times(\delta/r, r-\delta/r),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{I}_{1,3}(\delta/r) =&(\delta/r, r-\delta/r)\times(1-r+\delta/r, 1-\delta/r),\\
\widehat{I}_{1,3}(\delta/r) =&(0,r)\times\big((1-r+\delta/r)\cup(1-\delta/r,1)\big)\cup\\
&\big((0,\delta/r)\cup(r-\delta/r,r)\big)\times(1-r+\delta/r, 1-\delta/r).
\end{aligned}
$$

By (3.7) and (3.8), we have

$$
\begin{aligned}
\varphi_1(\delta) \;=\;& \sum_{j=2}^{n} w_2(j-2)^q(\rho r^{j-2})^2\Big(\int_{I_{1,1}}+\int_{I_{1,3}}\Big)\mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q\,d\mathbf{x}\\
&+\sum_{j=2}^{n}(p_2^q r^2)^{j-1}\Big(\int_{I_{1,2}}+\int_{I_{1,3}}\Big)\mu(B_{\delta/r^{j-1}}(\mathbf{x}))^q\,d\mathbf{x}\\
&+\sum_{j=2}^{n}(e_j^1(\delta)-\tilde{e}_j^1(\delta))+\int_{I_{n,1,3}}\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x}
\end{aligned}
\tag{5.3}
$$

and

$$
\varphi_\ell(\delta)=p_{\ell+1}^q r^2\sum_{i=1}^{3}\int_{I_{1,i}}\mu(B_{\delta/r}(\mathbf{x}))^q\,d\mathbf{x}+e_2^\ell(\delta)-\tilde{e}_2^\ell(\delta)\qquad\text{for } \ell=2,3,
\tag{5.4}
$$

where

$$
\begin{aligned}
e_j^1(\delta) =&\Big(\int_{\widehat{I}_{j,1,1}(\delta)}+\int_{\widehat{I}_{j,1,2}(\delta)}+\int_{\widehat{I}_{j,1,4}(\delta)}+\int_{\widehat{I}_{j,1,5}(\delta)}\Big)\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x},\\
\tilde{e}_j^1(\delta) =&w_2(j-2)^q(\rho r^{j-2})^2\Big(\int_{\widehat{I}_{1,1}(\delta/\rho r^{j-2})}+\int_{\widehat{I}_{1,3}(\delta/\rho r^{j-2})}\Big)\mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q\,d\mathbf{x}\\
&+(p_2^q r^2)^{j-1}\Big(\int_{\widehat{I}_{1,2}(\delta/r^{j-1})}+\int_{\widehat{I}_{1,3}(\delta/r^{j-1})}\Big)\mu(B_{\delta/r^{j-1}}(\mathbf{x}))^q\,d\mathbf{x},\\
e_2^\ell(\delta) =&\sum_{i=1}^{3}\int_{\widehat{I}_{2,\ell,i}(\delta)}\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x},\\
\tilde{e}_2^\ell(\delta) =&p_{\ell+1}^q r^2\sum_{i=1}^{3}\int_{\widehat{I}_{1,i}(\delta/r)}\mu(B_{\delta/r}(\mathbf{x}))^q\,d\mathbf{x}\qquad\text{for } \ell=2,3.
\end{aligned}
\tag{5.5}
$$

Multiplying both sides of (5.3) and (5.4) by $\delta^{-(2+\alpha)}$, and using (3.2), we have

$$
\begin{aligned}
\Phi_1^{(\alpha)}(\delta) \;=\;& \sum_{j=2}^{n} w_2(j-2)^q(\rho r^{j-2})^{-\alpha}\sum_{i=1,3}\Phi_i^{(\alpha)}(\delta/\rho r^{j-2})\\
&+\sum_{j=2}^{n}(p_2^q r^{-\alpha})^{j-1}\sum_{i=2,3}\Phi_i^{(\alpha)}(\delta/r^{j-1})\\
&+\sum_{j=2}^{n}\delta^{-2-\alpha}(e_j^1(\delta)-\tilde{e}_j^1(\delta))+\delta^{-2-\alpha}\int_{I_{n,1,3}}\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x},
\end{aligned}
\tag{5.6}
$$

and

$$\Phi_\ell^{(\alpha)}(\delta)=p_{\ell+1}^q r^{-\alpha}\sum_{i=1}^3\Phi_i^{(\alpha)}(\delta/r)+\delta^{-2-\alpha}(e_2^\ell(\delta)-\tilde e_2^\ell(\delta))\qquad\text{for }\ell=2,3.$$

Let N be the largest integer such that $\delta\le\min\{\rho r^{N-2},r^{N-1}\}$. Taking $n=N$ in (5.6), we have

$$\begin{aligned}\Phi_1^{(\alpha)}(\delta)&=\sum_{j=2}^\infty w_2(j-2)^q(\rho r^{j-2})^{-\alpha}\sum_{i=1,3}\Phi_i^{(\alpha)}(\delta/\rho r^{j-2})\\&\quad+\sum_{j=2}^\infty(p_2^q r^{-\alpha})^{j-1}\sum_{i=2,3}\Phi_i^{(\alpha)}(\delta/r^{j-1})+E_1^{(\alpha)}(\delta)-E_{1,\infty}^{(\alpha)}(\delta),\end{aligned}\tag{5.7}$$

where

$$\begin{aligned}E_1^{(\alpha)}(\delta)&:=\sum_{j=2}^N\delta^{-2-\alpha}(e_j^1(\delta)-\tilde e_j^1(\delta))+\delta^{-2-\alpha}\int_{I_{N,1,3}}\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x},\\E_{1,\infty}^{(\alpha)}(\delta)&:=\sum_{j=N+1}^\infty w_2(j-2)^q(\rho r^{j-2})^{-\alpha}\sum_{i=1,3}\Phi_i^{(\alpha)}(\delta/\rho r^{j-2})\\&\quad+\sum_{j=N+1}^\infty(p_2^q r^{-\alpha})^{j-1}\sum_{i=2,3}\Phi_i^{(\alpha)}(\delta/r^{j-1}).\end{aligned}$$

Let

$$\Phi_\ell^{(\alpha)}(\delta)=p_{\ell+1}^q r^{-\alpha}\sum_{i=1}^3\Phi_i^{(\alpha)}(\delta/r)+E_\ell^{(\alpha)}(\delta)\qquad\text{for }\ell=2,3,\tag{5.8}$$

where

$$E_\ell^{(\alpha)}(\delta):=\delta^{-2-\alpha}(e_2^\ell(\delta)-\tilde e_2^\ell(\delta)).$$

Next, we derive a vector-valued renewal equation. It follows from (3.12), (3.13), (5.7) and (5.8) that

$$\begin{aligned}f_1(x)&=\sum_{j=2}^\infty w_2(j-2)^q(\rho r^{j-2})^{-\alpha}\sum_{i=1,3}f_i(x+\ln(\rho r^{j-2}))\\&\quad+\sum_{j=2}^\infty(p_2^q r^{-\alpha})^{j-1}\sum_{i=2,3}f_i(x+\ln(r^{j-1}))+z_1^{(\alpha)}(x),\end{aligned}$$

and

$$f_\ell(x)=p_{\ell+1}^q r^{-\alpha}\sum_{i=1}^3 f_i(x+\ln(r))+z_\ell^{(\alpha)}(x)\qquad\text{for }\ell=2,3,$$

where

$$z_1^{(\alpha)}(x)=E_1^{(\alpha)}(e^{-x})-E_{1,\infty}^{(\alpha)}(e^{-x}),\qquad z_\ell^{(\alpha)}(x)=E_\ell^{(\alpha)}(e^{-x}).$$

For $\ell,m=1,2$, let $\mu_{m\ell}^{(\alpha)}$ be the discrete measures such that for $j\ge2$,

$$\begin{aligned}&\mu_{m1}^{(\alpha)}(-\ln(\rho r^{j-2}))=w_2(j-2)^q(\rho r^{j-2})^{-\alpha}&&\text{for }m=1,3,\\&\mu_{m1}^{(\alpha)}(-\ln(r^{j-1}))=(p_2^q r^{-\alpha})^{j-1}&&\text{for }m=2,3,\\&\mu_{m\ell}^{(\alpha)}(-\ln(r))=p_{\ell+1}^q r^{-\alpha},&&\text{for }m=1,2,3\text{ and }\ell=2,3.\end{aligned}$$

Then

$$\mu_{11}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha}, \qquad \mu_{21}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1},$$

$$\mu_{31}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} + \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1},$$

$$\mu_{m\ell}^{(\alpha)}(\mathbb{R}) = p_{\ell+1}^q r^{-\alpha}, \qquad \text{for } m = 1,2,3 \text{ and } \ell = 2,3.$$

For fixed $q \geq 0$,

$$\begin{aligned} F_1(\alpha) &= 2\Big(\sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} + \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1}\Big), \\ F_\ell(\alpha) &= 3p_{\ell+1}^q r^{-\alpha} \qquad \text{for } \ell = 2,3, \\ D_\ell &= \{\alpha \in \mathbb{R} : F_\ell(\alpha) < \infty\} \quad \text{for } \ell = 1,2,3, \end{aligned} \tag{5.9}$$

and

$$\mathbf{M}_\alpha(\infty) = \begin{pmatrix} a & p_3^q r^{-\alpha} & p_4^q r^{-\alpha} \\ b & p_3^q r^{-\alpha} & p_4^q r^{-\alpha} \\ a+b & p_3^q r^{-\alpha} & p_4^q r^{-\alpha} \end{pmatrix},$$

where $a := \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha}$ and $b := \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1}$.

Finally, we want to show that the error terms $z_\ell^{(\alpha)}(x) = o(e^{-\epsilon x})$ as $x\to\infty$, i.e., $E_\ell^{(\alpha)}(\delta) = o(\delta^\epsilon)$ and $E_{1,\infty}^{(\alpha)}(\delta) = o(\delta^\epsilon)$ as $\delta\to 0$ for some $\epsilon > 0$ and $\ell = 1,2,3$.

Proposition 5.1. (a) *$\Phi_i^{(\alpha)}(\delta/\rho r^k) \leq 1$ for $i = 1,3$ and any $k \geq N-1$.*
(b) *$\Phi_i^{(\alpha)}(\delta/r^k) \leq 1$ for $i = 2,3$ and any $k \geq N$.*

Proof. (a) It follows from the definition of N that $\delta \geq \rho r^k$ for any $k \geq N-1$. Thus

$$\Phi_i^{(\alpha)}(\delta/\rho r^k) = \frac{\varphi_i(\delta/\rho r^k)}{(\delta/\rho r^k)^{2+\alpha}} \leq \int_{I_{1,i}} \mu(B_{\delta/\rho r^k}(\mathbf{x}))^q\, d\mathbf{x} \leq \int_{I_{1,i}} d\mathbf{x} \leq 1 \quad \text{for } i = 1,3.$$

This proves part (a).

(b) The proof is similar to that of (a). □

Proposition 5.2. *For $q \geq 0$, let $F_1(\alpha), D_1$ be defined as in* (5.9). *Then D_1 is open.*

Proof. The proof is similar to that of Proposition 4.3. □

Proposition 5.3. *For $q \geq 0$, assume that $\alpha \in D_\ell$ for $\ell = 1,2,3$. Then there exists $\epsilon > 0$ such that*

(a) $\sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \sum_{i=1,3} \Phi_i^{(\alpha)}(\delta/\rho r^{j-2}) = o(h^\epsilon)$;

(b) $\sum_{j=N+1}^{\infty}(p_2^q r^{-\alpha})^{j-1} \sum_{i=2,3} \Phi_i^{(\alpha)}(\delta/r^{j-1}) = o(\delta^\epsilon)$;

(c) $\sum_{j=2}^{N} \delta^{-2-\alpha}(e_j^1(\delta) - \tilde{e}_j^1(\delta)) = o(\delta^\epsilon)$;

(d) $\delta^{-2-\alpha} \int_{I_{N,1,3}} \mu(B_\delta(\mathbf{x}))^q\, d\mathbf{x} = o(\delta^\epsilon)$;

(e) $\delta^{-2-\alpha}(e_2^\ell(\delta) - \tilde{e}_2^\ell(\delta)) = o(\delta^\epsilon)$ *for $\ell = 2,3$.*

Proof. (a) By Proposition 5.2, we have $D_1 = \{\alpha \in \mathbb{R} : F_1(\alpha) < \infty\}$ is open. Thus there exists $\epsilon > 0$ such that $F_1(\alpha+\epsilon) \in D_1$. So there exists a positive constant C such that

$$\sum_{j=N+1}^{\infty} w_2(j-2)^q (\rho r^{j-2})^{-\alpha-\epsilon} + \sum_{j=N+1}^{\infty} (p_2^q r^{-\alpha-\epsilon})^{j-1} \le C.$$

Since

$$(\rho r^{N-1})^{-\epsilon} \sum_{j=N+1}^{\infty} w_2(j-2)^q (\rho r^{j-2})^{-\alpha} \le \sum_{j=N+1}^{\infty} w_2(j-2)^q (\rho r^{j-2})^{-\alpha-\epsilon},$$

we have $\sum_{j=N+1}^{\infty} w_2(j-2)^q (\rho r^{j-2})^{-\alpha} \le C(\rho r^{N-1})^{\epsilon} \le C\delta^{\epsilon}$, where the last inequality follows from the definition of N. Combining this with Proposition 5.1(a), we have

$$\sum_{j=N+1}^{\infty} w_2(j-2)^q (\rho r^{j-2})^{-\alpha} \sum_{i=1,3} \Phi_i^{(\alpha)}(\delta/\rho r^{j-2}) \le 2C\delta^{\epsilon}.$$

This proves part (a).

(b) The proof is similar to that of (a).

(c) It suffices to show that $e_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$ and $\tilde{e}_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$ for $2 \le j \le N$. In order to estimate the remaining error terms, we will need the following facts. It follows from (a) and (b) that

$$w_2(N-1)^q \le 2C\delta^{\alpha+\epsilon} \qquad \text{and} \qquad p_2^{Nq} \le 2C\delta^{\alpha+\epsilon}. \tag{5.10}$$

By (5.5), we have

$$e_j^1(\delta) = \sum_{i=1,2,4,5} \int_{\widehat{I}_{j,1,i}(\delta)} \mu(B_\delta(x,y))^q \, dx\, dy.$$

As an example we only prove that $\int_{\widehat{I}_{j,1,1}(\delta)} \mu(B_\delta(x,y))^q \, dx\, dy = o(\delta^{2+\alpha+\epsilon})$. Note that

$$\begin{aligned}
&\int_{\widehat{I}_{j,1,1}(\delta)} \mu(B_\delta(x,y))^q \, dx\, dy \\
= \;& \Big(\int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho\gamma_{j-1}} \int_0^{\delta} + \int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\delta} \int_{\rho^2 r^{j-2}-\delta}^{\rho^2 r^{j-2}} \\
& + \int_{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1}^{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho r^{j-1}} \int_{\rho r^{j-1}-\delta}^{\rho r^{j-1}} + \int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\delta} \int_{\delta}^{\rho^2 r^{j-2}-\delta} \\
& + \int_{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1}^{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\delta} \int_{\rho^2 r^{j-2}}^{\rho r^{j-1}-\delta} + \int_{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho r^{j-1}-\delta}^{\rho\gamma_{j-2}+\rho^2 r^{j-2}\gamma_1+\rho r^{j-1}} \int_{\delta}^{\rho r^{j-1}-\delta} \Big) \mu(B_\delta(x,y))^q \, dx\, dy \\
=: \;& \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6.
\end{aligned}$$

For $\mathcal{E}_1$, since $\sqrt{(\rho^2 r^{j-2}\gamma_1 + \rho\gamma_{j-1})^2 + \delta^2} \le 2\rho$, we have $B_\delta(x,y) \subseteq B_{2\rho+\delta}(S_{2^{j-2}11}(0,0))$ for $(x,y) \in (\rho\gamma_{j-2}, \rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1 + \rho\gamma_{j-1}) \times (0,\delta)$. Note that

$$\mu(B_{2\rho+\delta}(S_{2^{j-2}11}(0,0))) = p_1 w_2(j-2)\mu(B_{(2\rho+\delta)/\rho^2 r^{j-2}}(0,0)) \le p_1 w_2(j-2)$$

and for $0 \le k \le N-2$,

$$\begin{aligned} w_2(k) &= \frac{p_1(p_2^{N-1}+p_2^{N-2}p_3+\cdots+p_3^{N-1})(p_2^k+p_2^{k-1}p_3+\cdots+p_3^k)}{(p_2^{N-1}+p_2^{N-2}p_3+\cdots+p_3^{N-1})} \\ &\le \frac{w_1(N-1)(p_2+p_3)^k}{p_2^{N-1}+p_2^{N-2}p_3+\cdots+p_3^{N-1}} \le p_2^{1-N}w_2(N-1). \end{aligned} \tag{5.11}$$

Combining these with the definition of N, we have

$$\begin{aligned} \mathcal{E}_1 &\le (p_1 w_2(j-2))^q(\rho^2 r^{j-2}\gamma_1+\rho\gamma_{j-1})\delta \le 2\rho p_1^q w_2(j-2)^q\delta \\ &\le 2p_1^q p_2^{(1-N)q} w_2(N-1)^q \rho r^{N-1} r^{1-N} \le 2(p_1p_2^{1-N})^q r^{1-N}\delta^{2+\alpha+\epsilon}. \end{aligned}$$

The proofs for $\mathcal{E}_2 \le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_3 \le C\delta^{2+\alpha+\epsilon}$ are similar.

For $\mathcal{E}_4$, since

$$\mathcal{E}_4 \le \int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\delta}\int_0^{\rho^2r^{j-2}-\delta} \mu(B_\delta(x,y))^q\,dx\,dy$$

and $\sqrt{(\delta^2+(\rho^2r^{j-2}-\delta)^2} \le \rho^2 r^{j-2} \le \rho^2$, we have $B_\delta(x,y) \subseteq B_{\rho^2+\delta}(S_{2^{j-2}12}(0,0))$ for $(x,y) \in (\rho\gamma_{j-2}, \rho\gamma_{j-2}+\delta)\times(0,\rho^2r^{j-2}-\delta)$. Note that $\mu(B_{\rho^2+\delta}(S_{2^{j-2}12}(0,0))) \le p_2 w_2(j-2)$. Combining these with (5.10), (5.11) and the definition of N, we have

$$\begin{aligned} \mathcal{E}_4 &\le (p_2w_2(j-2))^q(\rho^2r^{j-2}-\delta)\delta \le p_2^q w_2(j-2)^q\rho^2\delta \\ &\le p_2^{(2-N)q}w_2(N-1)^q\rho r^{1-N}\rho r^{N-1}\delta \\ &\le 2C\rho r^{1-N}p_2^{(2-N)q}\delta^{2+\alpha+\epsilon}. \end{aligned}$$

The proofs for $\mathcal{E}_5 \le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_6 \le C\delta^{2+\alpha+\epsilon}$ are similar.

Combining the estimates for $\mathcal{E}_1,\dots,\mathcal{E}_6$, we have $\int_{\widehat{I}_{j,1,1}(\delta)} \mu(B_\delta(x,y))^q\,dx\,dy \le C\delta^{2+\alpha+\epsilon}$.

Next, we will show that $\tilde{e}_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$. By (5.5), we have

$$\begin{aligned} \tilde{e}_j^1(\delta) &= w_2(j-2)^q(\rho r^{j-2})^2\Big(\int_{\widehat{I}_{1,1}(\delta/\rho r^{j-2})}+\int_{\widehat{I}_{1,3}(\delta/\rho r^{j-2})}\Big)\mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q\,d\mathbf{x} \\ &\quad +(p_2^q r^2)^{j-1}\Big(\int_{\widehat{I}_{1,2}(\delta/r^{j-1})}+\int_{\widehat{I}_{1,3}(\delta/r^{j-1})}\Big)\mu(B_{\delta/r^{j-1}}(\mathbf{x}))^q\,d\mathbf{x}. \end{aligned}$$

As an example we only prove

$$w_2(j-2)^q(\rho r^{j-2})^2\int_{\widehat{I}_{1,1}(\delta/\rho r^{j-2})}\mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q\,d\mathbf{x} = o(\delta^{2+\alpha+\epsilon}).$$

Note that

$$
\begin{aligned}
& w_2(j-2)^q(\rho r^{j-2})^2 \int_{\widehat{I}_{1,1}(\delta/\rho r^{j-2})} \mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q \, d\mathbf{x} \\
= \;& w_2(j-2)^q(\rho r^{j-2})^2 \Big(\int_0^{\rho\gamma_1 + r} \int_0^{\delta/\rho r^{j-2}} + \int_0^{\rho\gamma_1+\delta/\rho r^{j-2}} \int_{\rho-\delta/\rho r^{j-2}}^{\rho} \\
& + \int_{\rho\gamma_1}^{\rho\gamma_1+r} \int_{r-\delta/\rho r^{j-2}}^{r} + \int_0^{\delta/\rho r^{j-2}} \int_{\delta/\rho r^{j-2}}^{\rho-\delta/\rho r^{j-2}} + \int_{\rho\gamma_1}^{\rho\gamma_1+\delta/\rho r^{j-2}} \int_{\rho}^{r-\delta/\rho r^{j-2}} \\
& + \int_{\rho\gamma_1+r-\delta/\rho r^{j-2}}^{\rho\gamma_1+r} \int_{\delta/\rho r^{j-2}}^{r-\delta/\rho r^{j-2}} \Big) \mu(B_{\delta/\rho r^{j-2}}(x,y)^q \, dx \, dy \\
=: \;& \widetilde{\mathcal{E}}_1 + \widetilde{\mathcal{E}}_2 + \widetilde{\mathcal{E}}_3 + \widetilde{\mathcal{E}}_4 + \widetilde{\mathcal{E}}_5 + \widetilde{\mathcal{E}}_6.
\end{aligned}
$$

Since $\sqrt{(\rho\gamma_1 + r)^2 + (\delta/\rho r^{j-2})^2} \le \rho + r + \delta/\rho r^{j-2}$, we have

$$
B_{\delta/\rho r^{j-2}}(x,y) \subseteq B_{\rho+r+2\delta/\rho r^{j-2}}(S_1(0,0)).
$$

Note that $\mu(B_{\rho+r+2\delta/\rho r^{j-2}})(S_1(0,0)) \le p_1$. Thus

$$
\begin{aligned}
\widetilde{\mathcal{E}}_1 \;&\le\; w_2(j-2)^q(\rho r^{j-2})^2 p_1^q (\rho\gamma_1 + r)\delta/\rho r^{j-2} \le w_2(j-2)^q \rho r^{j-2} p_1^q (\rho + r)\delta \\
&\le\; p_1^q p_2^{(1-N)q} w_2(N-1)^q \rho(\rho+r)\delta \le r^{1-N}(p_1 p_2^{1-N})^q w_2(N-1)^q \rho(\rho r^{N-1} + r^N)\delta \\
&\le\; 2C\rho r^{1-N}(p_1 p_2^{1-N})^q \delta^{2+\alpha+\epsilon}.
\end{aligned}
$$

The proofs for $\widetilde{\mathcal{E}}_2 \le C\delta^{2+\alpha+\epsilon}$ and $\widetilde{\mathcal{E}}_3 \le C\delta^{2+\alpha+\epsilon}$ are similar. For $\widetilde{\mathcal{E}}_4$, we have

$$
\begin{aligned}
\widetilde{\mathcal{E}}_4 \;&\le\; w_2(j-2)^q(\rho r^{j-2})^2 \int_0^{\delta/\rho r^{j-2}} \int_0^{\rho-\delta/\rho r^{j-2}} \mu(B_{\delta/\rho r^{j-2}}(x,y)^q \, dx \, dy \\
&\le\; w_2(j-2)^q(\rho r^{j-2})^2 \mu(B_{\rho+\delta/\rho r^{j-2}}(S_1(0,0)))^q (\rho - \delta/\rho r^{j-2})\delta/\rho r^{j-2} \\
&\le\; p_1^q p_2^{(1-N)q} w_2(N-1)^q (\rho^2 r^{j-2} - \delta)\delta \\
&\le\; (p_1 p_2^{1-N})^q w_2(N-1)^q \rho r^{N-1} \rho r^{1-N} \delta \\
&\le\; 2C\rho r^{1-N}(p_1 p_2^{1-N})^q \delta^{2+\alpha+\epsilon}.
\end{aligned}
$$

The proofs for $\widetilde{\mathcal{E}}_5 \le C\delta^{2+\alpha+\epsilon}$ and $\widetilde{\mathcal{E}}_6 \le C\delta^{2+\alpha+\epsilon}$ are similar. Hence,

$$
w_2(j-2)^q(\rho r^{j-2})^2 \int_{\widehat{I}_{1,1}(\delta/\rho r^{j-2})} \mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q \, d\mathbf{x} = o(\delta^{2+\alpha+\epsilon}).
$$

Similarly, we can derive analogous results for the second, third and fourth terms of $\tilde{e}_j^1(\delta)$. Thus $\tilde{e}_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$. This proves part (c).

(d) It suffices to show that $\int_{I_{N,1,3}} \mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x} \le C\delta^{2+\alpha+\epsilon}$. It follows from (5.1) and Proposition 2.20(d) that

$$\begin{aligned}
&\int_{I_{N,1,3}} \mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x}\\
=\;& \Big(\int_{\rho\gamma_{N-1}}^{\rho\gamma_N}\int_0^{\rho r^{N-1}} + \int_{\rho\gamma_N}^{\rho\gamma_N+r^N}\int_0^{r^N}\Big)\mu(B_\delta(x,y))^q\,dx\,dy\\
=\;& \Big(\int_{\rho\gamma_{N-1}+\delta}^{\rho\gamma_N+\delta}\int_\delta^{\rho r^{N-1}-\delta} + \int_{\rho\gamma_N+\delta}^{\rho\gamma_N+r^N-\delta}\int_\delta^{r^N-\delta}\\
&+\int_{\rho\gamma_{N-1}}^{\rho\gamma_N+r^N}\int_0^{\delta} + \int_{\rho\gamma_{N-1}}^{\rho\gamma_N+\delta}\int_{\rho r^{N-1}-\delta}^{\rho r^{N-1}}\\
&+\int_{\rho\gamma_N}^{\rho\gamma_N+r^N}\int_{r^N-\delta}^{r^N} + \int_{\rho\gamma_{N-1}}^{\rho\gamma_{N-1}+\delta}\int_\delta^{\rho r^{N-1}-\delta}\\
&+\int_{\rho\gamma_N}^{\rho\gamma_N+\delta}\int_{\rho r^{N-1}}^{r^N-\delta} + \int_{\rho\gamma_N+r^N-\delta}^{\rho\gamma_N+r^N}\int_\delta^{r^N-\delta}\Big)\mu(B_\delta(x,y))^q\,dx\,dy\\
=:\;& \mathcal{E}_1^N+\mathcal{E}_2^N+\mathcal{E}_3^N+\mathcal{E}_4^N+\mathcal{E}_5^N+\mathcal{E}_6^N+\mathcal{E}_7^N+\mathcal{E}_8^N.
\end{aligned}$$

By Lemma 2.22(c), we have $\mu|_{S_{2^{N-1}}(I_{1,1})} = w_2(N-2)\mu\circ S^{-1}_{2^{N-2}1} + p_2^{N-1}\mu\circ S^{-1}_{2^{N-1}}$, and hence $\mu(S_{2^{N-1}}(I_{1,1})) \le w_2(N-2)+p_2^{N-1}$. Since $B_\delta(x,y)\subseteq S_{2^{N-1}}(I_{1,1})$ for $(x,y)\in(\rho\gamma_{N-1}+\delta,\rho\gamma_N+\delta)\times(\delta,\rho r^{N-1}-\delta)\cup(\rho\gamma_N+\delta,\rho\gamma_N+r^N-\delta)\times(\delta,r^N-\delta)$, (5.10) implies

$$\begin{aligned}
\mathcal{E}_1^N+\mathcal{E}_2^N &\le (w_2(N-2)+p_2^{N-1})^q\big((\rho\gamma_N-\rho\gamma_{N-1})(\rho r^{N-1}-2\delta)+(r^N-2\delta)^2\big)\\
&\le (p_2^{1-N}w_2(N-1)+p_2^{-1}p_2^N)((\rho r^{N-1})^2+r^{2N})\\
&\le 2C(p_2^{1-N}+p_2^{-1})\delta^{2+\alpha+\epsilon}.
\end{aligned}$$

For the other six terms, we have

$$\begin{aligned}
\mathcal{E}_3^N &\le \mu(B_{\rho r+2\delta}(S_{2^{N-1}1}(0,0)))^q(\rho\gamma_N+r^N-\rho\gamma_{N-1})\delta\\
&\le w_2(N-1)^q(r^N+\rho r^{N-1})\delta \le 2C\delta^{2+\alpha+\epsilon}.
\end{aligned}$$

The proofs for $\mathcal{E}_4^N\le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_5^N\le C\delta^{2+\alpha+\epsilon}$ are similar. For $\mathcal{E}_6^N$, we have

$$\begin{aligned}
\mathcal{E}_6^N &\le \int_{\rho\gamma_{N-1}}^{\rho\gamma_{N-1}+\delta}\int_0^{\rho r^{N-1}-\delta}\mu(B_\delta(x,y))^q\,dx\,dy\\
&\le \mu(B_{\rho r+\delta}(S_1(0,0)))^q(\rho r^{N-1}-\delta)\delta \le p_1^q\rho r^{N-1}\delta\\
&\le p_1^q p_2^{-Nq}p_2^{Nq}\delta^2 \le 2C(p_1p_2^{-N})^q\delta^{2+\alpha+\epsilon}.
\end{aligned}$$

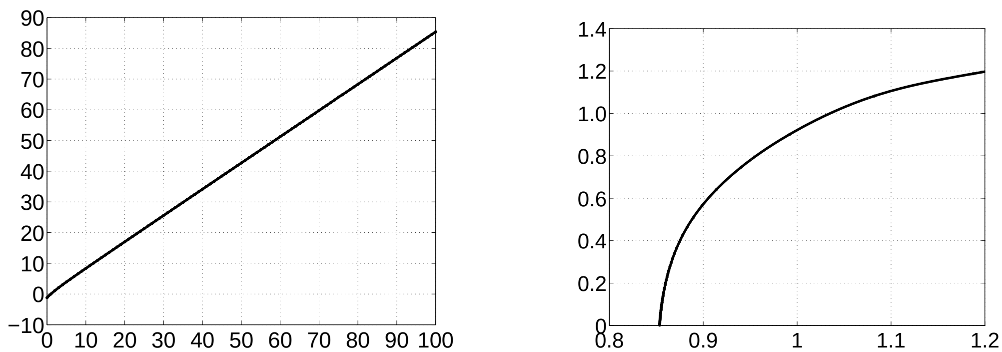
The proofs for $\mathcal{E}_7^N\le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_8^N\le C\delta^{2+\alpha+\epsilon}$ are similar. This proves part (d); part (e) can be proved similarly. □

Proof of Theorem 1.4. Combining Theorem 1.1 and Proposition 5.3, we have $\tau(q)=\alpha$. Let

$$G(q,\alpha) := \big(1-p_2^q r^{-\alpha}\big)\big(1-p_3^q r^{-\alpha}\big)\sum_{k=0}^{\infty} w_2(k)^q(\rho r^k)^{-\alpha} + r^{-\alpha}\sum_{i=2}^{4} p_i^q - 1.$$

Similar to the proof of Theorem 1.2, we can show that $G(q,\alpha)$ is C^1 and that $G_\alpha(q,\alpha) \neq 0$ for any (q,α) satisfying $G(a,\alpha)=0$. The implicit function theorem now implies that τ is differentiable on $(0,\infty)$ and the formula for $\dim_{\mathrm{H}}(\mu)$ follows by computing $\tau'(1) = -G_q(1,0)G_\alpha(1,0)^{-1}$. This completes the proof. $\square$

Figure 11 shows graphs of $\tau(q)$ and $f(\alpha)$ for one of the measures. For this example, $\dim_{\mathrm{H}}(\mu) = \tau'(1) \approx 1.13748$ and $\dim_{\mathrm{H}}(K) = -\tau(0) \approx 1.18726$, where K is the self-similar set.



(a) $\tau(q)$ (b) $f(\alpha)$

FIGURE 11. Graphs of $\tau(q)$ and $f(\alpha)$ for a self-similar measure in Example 2.18, with $r = 7/20$ and $\rho = p_i = 1/4$ for $i = 1,2,3,4$.

6. Comments and questions

The spectral dimension of certain infinite IFSs has been computed in [22]. The method in this paper can be applied to those IFSs to obtain $\tau(q)$.

It is interesting to compute $\tau(q)$ for $q < 0$ and see whether there is any phase transition. Our method cannot be applied to this case.

Infinite Bernoulli convolutions associated with Pisot numbers (and have overlaps) do not satisfy Condition (B), and second-order identities are satisfied only by the one associated with the golden ratio. It is of interest to compute the spectral dimension of infinite Bernoulli convolutions associated with other Pisot numbers; new techniques are perhaps needed.

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