On the Free Surface Motion of Highly Subsonic Heat-conducting Inviscid Flows

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Abstract

For a free surface problem of a highly subsonic heat-conducting inviscid flow, under the Taylor sign condition, by adopting a geometric point of view of Christodoulou and Lindblad in the study of free surface problems of incompressible inviscid flows, the *a priori* estimates of Sobolev norms in 2-D and 3-D are proved by identifying a suitable higher order energy functional. The estimates for some geometric quantities such as the second fundamental form and the injectivity radius of the normal exponential map of the free surface are also given. The novelty in our analysis includes dealing with the strong coupling of large variation of temperature, heat-conduction, compressibility of fluids and the evolution of free surface, loss of symmetries of equations, and loss of derivatives in closing the argument which is a key feature compared with Christodoulou and Lindblad's work.

1 Introduction

Fluids free surface problems have been receiving much attentions due to their physical importance and challenge in the mathematical analysis. For incompressible inviscid flows, the local-in-time well-posedness in Soblev spaces was obtained first in [41, 42] for the irrotational case, and then in [2, 3, 6, 7, 12, 18, 24, 29, 35, 38, 46] for some extensions; the global or almost global existence was achieved recently in [13, 40, 43, 44]; and the singularity formation was proved in [5, 45]. One may refer to the survey [24] for more references. For compressible inviscid flows, the local-in-time well-posedness of smooth solutions was established for liquids in [27, 39]; while for gases with physical vacuum singularity (cf. [30, 31]), the related results can be found in [8, 9, 14, 20, 21, 32] for the local-in-time theories, and in [16, 17, 33, 47] for the global-in-time ones. Most of the above results are either for incompressible or isentropic fluids without taking the effect of heat-conductivity into account. In many physical situations, the heat-conductivity is an important driving force to motions of fluids free surfaces, for example, for a gaseous star. As noted in [25], the heat-conductivity plays an important role to driving the evolution of a star in the phase of secular evolution, while the viscosity plays much less role. In general, it is important and necessary to understand the role played by heat-conductivities to the dynamics of fluids free surfaces. However, as far as we know, there have been no results on the free surface problem of heat-conductive inviscid flows, though some results are available for viscous and heat-conductive flows, for example, in [34], where the viscosity plays an essential role to the regularity of solutions. If the effect of heat-conductivities is taken into account, the analysis will become difficult due to the strong coupling among the large variations of temperature, heat-conduction, entropy, velocity fields and evolutions of free surfaces, while no much difficulties will be created compared with isentropic flows if heat-conductivities are ignored, because entropy is transported along particle paths. This is analogue to the low Mach number limit problem of compressible fluids. As a step towards an understanding of the role played by heat-conductivity to the well-posedness of free surface problems for inviscid fluids, we consider in this paper the problem of a highly subsonic flow which is used to approximate general heat-conductive compressible inviscid flows when the Mach number is small (cf. [1]).

We consider the following problem: for $n = 2$ and $n = 3$

$$(\partial_t + v^k\partial_k)v_j + \mathcal{T}\partial_j p = 0, \quad j = 1, \cdots, n \qquad \text{in } \mathscr{D}, \tag{1.1a}$$

$$\text{div} v = \kappa\Delta\mathcal{T}, \quad (\partial_t + v^k\partial_k)\mathcal{T} = \kappa\mathcal{T}\Delta\mathcal{T}, \qquad \text{in } \mathscr{D}, \tag{1.1b}$$

describing the motion of a highly subsonic heat-conducting flow, where the velocity field $v = (v_1, \cdots, v_n)$, the temperature $\mathcal{T}$, the pressure p and the domain $\mathscr{D} \subset [0,T] \times \mathbb{R}^n$ are the unknowns to be determined, $\kappa > 0$ is the (scaled) heat-conductive coefficient. Here $v^k = \delta^{ki} v_i = v_k$, and we have used the Einstein summation convention. Given a connected bounded domain $\mathscr{D}_0 \subset \mathbb{R}^n$ and initial data $(v_0, \mathcal{T}_0)$ satisfying $\mathrm{div} v_0 = \kappa \Delta \mathcal{T}_0$, we want to find a set $\mathscr{D} \subset [0,T] \times \mathbb{R}^n$, a vector field v and scalar functions p and $\mathcal{T}$ solving (1.1) and satisfying the initial conditions:

$$\mathscr{D}_0 = \{x : \ (0,x) \in \mathscr{D}\} \ \text{ and } \ (v, \mathcal{T}) = (v_0, \mathcal{T}_0) \ \text{ on } \ \{0\} \times \mathscr{D}_0. \tag{1.2}$$

Let $\mathscr{D}_t = \{x \in \mathbb{R}^n : \ (t,x) \in \mathscr{D}\}$. We require the boundary conditions on the free surface $\partial \mathscr{D}_t$,

$$p = 0, \quad \mathcal{T} = \mathcal{T}_b \ \text{ and } \ v_{\mathcal{N}} = \varpi \ \text{ on } \ \partial \mathscr{D}_t \tag{1.3}$$

for each t, where $\mathcal{T}_b$ is a positive constant, $\mathcal{N}$ is the exterior unit normal to $\partial \mathscr{D}_t$, $v_{\mathcal{N}} = \mathcal{N}^i v_i$, and ϖ is the normal velocity of $\partial \mathscr{D}_t$. For the derivation and physical background of system (1.1), one may refer to [1].

We will prove *a priori* estimates for problem (1.1)-(1.3) in Sobolev spaces when the initial data satisfies

$$\min_{\partial \mathscr{D}_0} (-\partial_{\mathcal{N}} p) > 0, \tag{1.4}$$

which implies, as we will prove, that for some $T > 0$ and $0 \le t \le T$,

$$-\partial_{\mathcal{N}} p \ge \epsilon_b > 0 \ \text{ on } \ \partial \mathscr{D}_t, \tag{1.5}$$

where $\partial_{\mathcal{N}} = \mathcal{N}^j \partial_j$, and $\epsilon_b = 2^{-1} \min_{\partial \mathscr{D}_0} (-\partial_{\mathcal{N}} p)$. (1.5) is a natural stability condition called the *physical condition* or the *Taylor sign condition* for an incompressible inviscid fluid in literatures (cf. [4, 6, 7, 12, 23, 24, 29, 35, 41, 42, 46]), excluding the possibility of the Rayleigh-Taylor type instability (cf. [12]). Since system (1.1) keeps unchanged when a constant is added to p, the condition $p = 0$ on $\partial \mathscr{D}_t$ is equivalent to that of p being a constant on $\partial \mathscr{D}_t$. Therefore, the boundary conditions $p = 0$ and $\mathcal{T} = \mathcal{T}_b$ on $\partial \mathscr{D}_t$ is to match the exterior media with the constant pressure and temperature. The boundary condition $p = 0$ on $\partial \mathscr{D}_t$ is commonly used for incompressible flows without surface tensions in literatures (cf. [5, 6, 7, 12, 13, 15, 18, 26, 29, 35, 41, 44, 46] and references therein).

Another motivation of this article is to serve as a step to understand the behavior of solutions to free surface problems of inviscid heat-conducting flows when Mach numbers are small. The low Mach number limit was rigorously justified in [1] for the initial value or periodic problem of heat-conductive flows (see also [11, 22] for the related results). For fluids free surface problems, the only available result on the low Mach number limit is quite recent [28] for isentropic flows where the the low Mach number limit equations are incompressible. As observed in [1], the low Mach number limit equations are not incompressible anymore for heat-conducting flows, and limit problem becomes more complicated and subtle. Toward to this direction of low Mach number limit problems with free surfaces for heat-conducting flows, it is important to gain a good understanding of solutions to limiting flows since they may be used as the leading order approximation. This is one of motivations for us to study problem (1.1)-(1.3).

Note that system (1.1) is reduced to the usual incompressible Euler equation if the heat-conductivity coefficient $\kappa = 0$ or temperature $\mathcal{T}$ =constant, for which a geometric approach was introduced in [6] to study free surface problems without assuming that the flow is irrotational. We adopt this approach to study problem (1.1)-(1.3) for a highly subsonic heat-conductive flow. Several new and essential analytic difficulties occur in extending the analysis in [6] to heat-conductive flows, including loss of symmetries of equations, loss of derivatives in closing arguments, the strong coupling of the large variation of temperature, heat-conduction, compressibility of fluids due to the non-zero divergence of the velocity filed, and the evolution of free surfaces. These issues will be addressed in Section 1.2. We construct a higher order energy functionals from which the Sobolev norms of $H^s(\mathscr{D}_t)$ $(s = 0, 1, \cdots, n+2)$ of solutions can be derived. This energy functional involves space-time derivatives of the divergence of velocity fields, $\mathrm{div}v$, for which the estimates are quite involved. This is a key difference between the constructions of the higher order energy functional compared with that in [6]. Besides the *a priori* estimates of Sobolev norms for problem (1.1)-(1.3), estimates for some geometric quantities of free surfaces, for example, the L^∞-bound for the second fundamental form and lower bound for the injective radius of the normal exponential map are also given. The bounds for those geometric quantities are not only needed to bound Sobolev norms of solutions, but also vital to the understanding of the evolution of the geometry of free surfaces, for example, to the study of the formation of singularities, such as the curvature blow-up or the self-intersection. It should be noted that the singularities such as the splash singularity or splat singularity in [5, 10] and wave crests in [45] all occur on free surfaces.

Note that, besides in [6], the Riemannian geometry tools (parallel transports, vector fields and covariant differentiations) were intensively used in [26, 29, 28, 35, 36, 37] to study fluids free surface problems, of which one of advantages is to make full use of some intrinsic properties of the studied problems independent of choice of coordinates. The geometric approach used in [6] was also adopted to study free surface problems of incompressible MHD flows in [18] and incompressible Neo-Hookean elastodynamics in [19].

1.1 Main results

We will prove that the temporal derivative of the constructed higher order energy functional is controlled by itself. This higher order functional consists of a boundary part and an interior part. In order to define the boundary integral, we project the equations to the tangent space of the boundary as in [6]. The orthogonal projection Π to the tangent space of the boundary of a $(0, r)$ tensor α is defined to be the projection of each component along the normal as follows:

Definition 1.1 *The orthogonal projection Π to the tangent space of the boundary of a $(0, r)$ tensor α is defined to be the projection of each component along the normal:*

$$(\Pi\alpha)_{i_1\cdots i_r} = \Pi_{i_1}^{j_1} \cdots \Pi_{i_r}^{j_r} \alpha_{j_1\cdots j_r}, \quad \text{where} \quad \Pi_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j.$$

The tangential derivative of the boundary is defined by $\bar{\partial}_i = \Pi_i^j \partial_j$, and the second fundamental form of the boundary is defined by $\theta_{ij} = \bar{\partial}_i \mathcal{N}_j$.

As in [6], we also need a positive definite quadratic form $Q(\alpha, \beta)$ for tensors α and β of the same order which is the inner product of the tangential components when restricted to the boundary, and $Q(\alpha, \alpha)$ increases to the norm $|\alpha|^2$ in the interior. For this purpose, we extend the normals on the boundary to the interior as follows:

Definition 1.2 *Let ι_0 be the injectivity radius of the normal exponential map of $\partial\mathscr{D}_t$, i.e., the largest number such that the map*

$$\partial\mathscr{D}_t \times (-\iota_0, \iota_0) \ \to \ \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, \partial\mathscr{D}_t) < \iota_0\} : \quad (\bar{x}, t) \ \mapsto \ x = \bar{x} + \iota\mathcal{N}(\bar{x})$$

is an injection.

As in [6], we also need the definition of ι_1 as follows:

Definition 1.3 *Let $0 < \epsilon_1 \le 1/2$ be a fixed number, and let $\iota_1 = \iota_1(\epsilon_1)$ be the largest number such that*

$$|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \le \epsilon_1 \quad \text{whenever} \quad |\bar{x}_1 - \bar{x}_2| \le \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathscr{D}_t.$$

As shown in [6], ι_1 is equivalent to ι_0 in conjunction with a bound of the second fundamental form θ. We do not need it for the statement of the main theorem, but we will need it when we illustrate the main idea of the proof of our theorem, so we give its definition here, together with ι_0.

Definition 1.4 *Let d_0 be a fixed number such that $\iota_0/16 \le d_0 \le \iota_0/2$, and η be a smooth cutoff function on $[0,\infty)$ satisfying $0 \le \eta(s) \le 1$, $\eta(s) = 1$ when $s \le d_0/4$, $\eta(s) = 0$ when $s \ge d_0/2$, and $|\eta'(s)| \le 8/d_0$. Define*

$$\varrho^{ij}(t,x) = \delta^{ij} - \eta^2(d(t,x))\mathcal{N}^i(t,x)\mathcal{N}^j(t,x) \quad \text{in} \quad \mathscr{D}_t,$$

where

$$\mathcal{N}^j(t,x) = \delta^{ij}\mathcal{N}_i(t,x), \quad \mathcal{N}_i(t,x) = \partial_i d(t,x) \quad \text{and} \quad d(t,x) = \mathrm{dist}(x, \partial\mathscr{D}_t).$$

In particular, ϱ gives the the induced metric on the tangential space to the boundary:

$$\varrho^{ij} = \delta^{ij} - \mathcal{N}^i\mathcal{N}^j, \quad \varrho_{ij} = \delta_{ij} - \mathcal{N}_i\mathcal{N}_j \quad \text{on} \quad \partial\mathscr{D}_t.$$

With this setting, the above mentioned quadratic form $Q(\alpha,\beta)$ for $(0,r)$ tensors is defined by

$$Q(\alpha,\beta) = \varrho^{i_1j_1}\cdots\varrho^{i_rj_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r}. \tag{1.6}$$

We are concerned with the problem for the fixed κ in this paper, so we set $\kappa = 1$ from now on for the simplicity of the presentation. The energy functionals of each order are then defined by

$$E_0(t) = \int_{\mathscr{D}_t} \mathcal{T}^{-1}|v|^2 dx, \tag{1.7a}$$

$$\begin{aligned} E_r(t) = &\int_{\mathscr{D}_t} \mathcal{T}^{-1}\delta^{mn}Q(\partial^r v_m, \partial^r v_n)dx + \int_{\mathscr{D}_t} |\partial^{r-1}\mathrm{curl}v|^2 dx + \int_{\mathscr{D}_t} |\partial D_t^{r-1}\mathrm{div}v|^2 dx \\ &+ \int_{\partial\mathscr{D}_t} Q(\partial^r p, \partial^r p)(-\partial_{\mathcal{N}}p)^{-1}dS, \qquad r \ge 1, \end{aligned} \tag{1.7b}$$

where $D_t = \partial_t + v^k\partial_k$. The higher order energy functional is defined by $\sum_{r=0}^{n+2} E_r(t)$.

In order to state the main result of the paper, we set

$$\mathrm{Vol}\mathscr{D}_0 = \int_{\mathscr{D}_0} dx, \quad K_0 = \max_{x\in\partial\mathscr{D}_0}\{|\theta(0,x)| + |\iota_0^{-1}(0,x)|\}, \quad \underline{\epsilon}_0 = \min_{x\in\partial\mathscr{D}_0}(-\partial_{\mathcal{N}}p)(0,x), \tag{1.8a}$$

$$\underline{\mathcal{T}}_0 = \min_{x\in\mathscr{D}_0}\mathcal{T}(0,x), \quad \overline{\mathcal{T}}_0 = \max_{x\in\mathscr{D}_0}\mathcal{T}(0,x), \tag{1.8b}$$

$$M_0 = \max_{x\in\mathscr{D}_0}\{|\partial p(0,x)| + |\partial v(0,x)| + |\partial\mathcal{T}(0,x)|\}. \tag{1.8c}$$

The initial pressure $p_0(x) = p(0,x)$ is determined by the following Dirichelet problem:

$$\mathrm{div}(\mathcal{T}_0\partial p_0) = -\partial_i v_0^j \partial_j v_0^i - D_t\mathrm{div}v\big|_{t=0} \quad \text{in} \quad \mathscr{D}_0, \quad p_0 = 0 \quad \text{on} \quad \partial\mathscr{D}_0,$$

where $D_t\mathrm{div}v|_{t=0}$ can be given in terms of initial values v_0 and $\mathcal{T}_0$ via the equations $\mathrm{div}v = \Delta\mathcal{T}$ and $D_t\mathcal{T} = \mathcal{T}\Delta\mathcal{T}$. With these notations, the main theorem of the present work is stated as follows:

Theorem 1.5 *Let $n = 2, 3$. Suppose that*

$$0 < \mathrm{Vol}\mathscr{D}_0,\ \underline{\epsilon}_0,\ \underline{\mathcal{T}}_0,\ \overline{\mathcal{T}}_0 < \infty,\ K_0,\ M_0 < \infty.$$

Then there are continuous functions $\mathscr{T}_n$ such that if

$$T \le \mathscr{T}_n\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon}_0^{-1}, \underline{\mathcal{T}}_0^{-1}, \overline{\mathcal{T}}_0, M_0, E_0(0), \cdots, E_{n+2}(0)\right),$$

then any smooth solution of the free surface problem (1.1)-(1.3) *for $0 \le t \le T$ satisfies*

$$\sum_{s=0}^{n+2} E_s(t) \le 2\sum_{s=0}^{n+2} E_s(0), \qquad 0 \le t \le T, \tag{1.9a}$$

$$2^{-1}\mathrm{Vol}\mathscr{D}_0 \le \mathrm{Vol}\mathscr{D}_t \le 2\mathrm{Vol}\mathscr{D}_0, \qquad 0 \le t \le T, \tag{1.9b}$$

$$\underline{\mathcal{T}}_0 \le \mathcal{T} \le \overline{\mathcal{T}}_0 \ \text{ in } \ \mathscr{D}_t, \qquad 0 \le t \le T, \tag{1.9c}$$

$$|\theta| + |\iota_0^{-1}| \le CK_0 \ \text{ on } \ \partial\mathscr{D}_t, \qquad 0 \le t \le T, \tag{1.9d}$$

$$-\partial_{\mathcal{N}} p \ge 2^{-1}\underline{\epsilon}_0 \ \text{ on } \ \partial\mathscr{D}_t, \qquad 0 \le t \le T, \tag{1.9e}$$

for a certain constant C, where $\mathrm{Vol}\mathscr{D}_t = \int_{\mathscr{D}_t} dx$.

Remark 1.6 *The bound for $\|\partial(v,p)\|_{L^\infty(\mathscr{D}_0)}$ was not needed in [6] to show their result, because it could be controlled by initial values of their higher order energy functional, $\mathrm{Vol}\mathscr{D}_0$ and $\iota_1(0,x)$ via Sobolev lemmas and elliptic estimates. We need this bound to prove Theorem 1.5, which reflects the complicated coupling of variation of temperature, heat-conduction, and compressibility of the fluid in our analysis.*

We give some remarks on the choice of the higher order energy functional, and explain briefly the reason why we need $n+2$ derivatives in this functional, while only $n+1$ derivatives were needed in [6] when $n = 2, 3$. Let

$$\begin{aligned} E_r^a(t) = &\int_{\mathscr{D}_t} \mathcal{T}^{-1}\delta^{mn} Q(\partial^r v_m, \partial^r v_n) dx + \int_{\mathscr{D}_t} |\partial^{r-1}\mathrm{curl} v|^2 dx \\ &+ \int_{\partial\mathscr{D}_t} Q(\partial^r p, \partial^r p)(-\partial_{\mathcal{N}} p)^{-1} dS, \qquad r \ge 1. \end{aligned} \tag{1.10}$$

Note that E_r^a ($r \ge 1$) correspond to the energy functionals employed in [6] for the study of an incompressible flow when $\mathcal{T}$ is constant. In order to control the L^2-norm of $\partial^r v$ for compressible flows, one may attempt to use the following:

$$\widetilde{E}_r(t) = E_r^a(t) + \int_{\mathscr{D}_t} |\partial^{r-1}\mathrm{div} v|^2 dx, \qquad r \ge 1. \tag{1.11}$$

However, (1.11) does not work for the study the problem (1.1)-(1.3). In fact, $\mathrm{div} v$ satisfies the following parabolic type equation:

$$D_t \mathrm{div} v - \mathcal{T}\Delta \mathrm{div} v = \text{other terms}, \tag{1.12}$$

which requires a control of one more spatial derivative of $\mathrm{div} v$, besides $\partial^r v$. Here and thereafter, "other terms" means something that does not affect the terms we single out to discuss. So, one may try to include the L^2-norm of $\partial^r \mathrm{div} v$ into the r-th order energy functional:

$$\overline{E}_r(t) = E_r^a(t) + \int_{\mathscr{D}_t} |\partial^r \mathrm{div} v|^2 dx, \qquad r \ge 1. \tag{1.13}$$

Due to (1.12) and the boundary condition that $\text{div}v = 0$ on $\partial\mathscr{D}_t$, it is more convenient to include the L^2-norm of $\partial D_t^{r-1}\text{div}v$, instead of $\partial^r\text{div}v$, into the r-th order energy functional (1.7b). This is one of reasons why we choose such an energy functional. Indeed, one can see from the proof that it is not sufficient to study the problem (1.1)-(1.3) even adopting (1.13).

The choice of the higher order energy functional $\sum_{r=0}^{n+2} E_r$ enables us to prove that the temporal derivative of it can be controlled by itself under the following *a priori* assumptions:

$$\underline{V} \le \text{Vol}\mathscr{D}_t(t) \le \overline{V} \quad \text{on } [0,T], \tag{1.14a}$$

$$|\theta| + 1/\iota_0 \le K, \quad -\partial_{\mathcal{N}} p \ge \epsilon_b > 0 \quad \text{on } \partial\mathscr{D}_t, \tag{1.14b}$$

$$\sum_{i=1}^{n-1}(|\partial_{\mathcal{N}} D_t^i p| + |\partial_{\mathcal{N}} D_t^i \text{div}v|) + |\partial^2 p| \le L \quad \text{on } \partial\mathscr{D}_t, \tag{1.14c}$$

$$|\partial p| + |\partial v| + |\partial \mathcal{T}| + |\partial \text{div}v| \le M \quad \text{in } \mathscr{D}_t, \tag{1.14d}$$

$$|D_t p| + |D_t \text{div}v| + |\partial^2 \mathcal{T}| \le \widetilde{M} \quad \text{in } \mathscr{D}_t. \tag{1.14e}$$

It should be noted that the *a priori* assumptions adopted in [6] for incompressible flows are the following:

$$|\theta| + 1/\iota_0 \le K, \quad -\partial_{\mathcal{N}} p \ge \epsilon_b > 0 \quad \text{on } \partial\mathscr{D}_t,$$

$$|\partial_{\mathcal{N}} D_t p| + |\partial^2 p| \le L \quad \text{on } \partial\mathscr{D}_t,$$

$$|\partial p| + |\partial v| \le M \quad \text{in } \mathscr{D}_t.$$

In closing the argument, the *a priori* assumptions, for example, on the L^∞-bounds for $\partial(v,p)$ in $\mathscr{D}_t$ and θ on $\partial\mathscr{D}_t$, need to be verified both in [6] and this article. In fact, these L^∞-bounds could be controlled in [6] by their higher order energy functional, $\text{Vol}\mathscr{D}_0$, $\min_{\partial\mathscr{D}_t}(-\partial_{\mathcal{N}} p)$ and $\max_{\partial\mathscr{D}_t}(\iota_1^{-1})$ via Sobolev lemmas, elliptic estimates and projection formulae. But this is not the case for the problem studied in this paper, that is, we do not have such simple and neat control of $\partial(v,p)$ and θ. This is a key feature for the problem studied here. Instead of using the method adopted in [6], we employ the evolution equations for $\partial(v,p)$ and θ, which causes the loss of derivatives. For example, in order to control $\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}$, we will need the control of $\|\partial^2 v\|_{L^\infty(\partial\mathscr{D}_t)}$, while a projection formula was used in [6] to control $\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}$ for which there is no need to control $\|\partial^2 v\|_{L^\infty(\partial\mathscr{D}_t)}$. This loss of derivative in the control of the L^∞-bound for θ forces us to use $n+2$ derivatives in the higher order functional, while only $n+1$ derivatives were needed in [6] for $n = 2, 3$. We will address these issues with more details in the next subsection.

1.2 Main issues and novelty in analysis

We first highlight the main issues in extending the analysis in [6] to problem (1.1)-(1.3) and then present the main strategy of the proof of Theorem 1.5. The big obstacle in the analysis lies in the strong coupling of large variation of temperature, heat-conduction and compressibility of fluids due to the non-zero divergence of the velocity filed, which creates essential and new challenges in the analysis. It should be noted that the analysis in this work for $\mathcal{T}$ =constant or $\text{div}v = 0$ reduces to that in [6]. Indeed, the sharp estimates in [6, 28] use all the symmetries of the incompressible or isentropic Euler equations, which are missing for (1.1) we consider here. The loss of symmetries of the equations we study is reflected by the following facts: for the problems of incompressible or isentropic Euler equations studied in [6, 28], the zero-th order energy functional is conserved in time, and the temporal derivative of the r-th ($r \ge 1$) order energy functional can be controlled by lower order functionals under some suitable a priori assumptions. However, in our case, the temporal derivatives of the zero-th and the first order energy functionals E_0 and E_1 depend on the higher order ones. The fact that E_0 is not conserved indicates some kind of loss of symmetries of the equations studied in this paper.

Another difficulty in our analysis is to deal with the problem of loss of derivatives when we work on evolution equations for some quantities in the *a priori* assumptions to obtain the bounds for them to close the argument. The first one is on the second fundamental form θ for free surfaces. The projection formula,

$$\Pi(\partial^2 p) = \theta \partial_{\mathcal{N}} p \quad \text{on} \quad \partial\mathscr{D}_t, \tag{1.15}$$

was used to estimate the L^∞-bound for θ in [6]. The reason that this can work in [6] is because one may obtain the L^∞-bound for $\partial^2 p$ on $\partial\mathscr{D}_t$ independent of the L^∞-bound for θ, which, together with the lower bound for $-\partial_{\mathcal{N}} p$ due to the Taylor sign condition, gives the L^∞-bound for θ. Indeed, it was proved in [6] that

$$\|\partial^2 p\|_{L^\infty(\partial\mathscr{D}_t)} \le C(K_1) \sum_{r=2}^{n+1} \|\partial^r p\|_{L^2(\partial\mathscr{D}_t)} \le C(K_1, \mathcal{E}_0, \cdots, \mathcal{E}_{n+1}, \mathrm{Vol}\mathscr{D}_t), \quad n = 2, 3, \tag{1.16}$$

where $\mathcal{E}_0 = E_0$ and $\mathcal{E}_r = E_r^a$ $(r \ge 1)$ with $\mathcal{T} = 1$, K_1 is the upper bound for $1/\iota_1$ on $\partial\mathscr{D}_t$ with ι_1 given in Definition 1.3. In the same spirit, the L^∞-bound for θ was obtained in [28] for isentropic Euler equations by replacing the pressure p in (1.15) by the enthalpy h. However, we can only obtain, for problem (1.1)-(1.3) that

$$\begin{aligned} \|\partial^2 p\|_{L^\infty(\partial\mathscr{D}_t)} \le & C(K_1, \mathrm{Vol}\mathscr{D}_t) \|\mathcal{T}^{-1}\|_{L^\infty(\mathscr{D}_t)} \|\partial p\|_{L^\infty(\mathscr{D}_t)} \|\theta\|_{L^\infty(\partial\mathscr{D}_t)} \|\partial^n \mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} \\ & + \text{other terms}, \quad n = 2, 3, \end{aligned} \tag{1.17}$$

from which it is clear that the projection formula used in [6] to give the L^∞-bound for θ cannot work directly for our problem. Indeed, (1.17) follows from Sobolev lemmas and the following estimates:

$$\|\partial^{n+1} p\|_{L^2(\partial\mathscr{D}_t)} \le C\|\Pi \partial^{n+1} p\|_{L^2(\partial\mathscr{D}_t)} + C(K_1, \mathrm{Vol}\mathscr{D}_t) \sum_{r=0}^{n} \|\partial^r \Delta p\|_{L^2(\mathscr{D}_t)}, \tag{1.18}$$

$$\|\partial^n \Delta p\|_{L^2(\mathscr{D}_t)} \le \|\mathcal{T}^{-1}\|_{L^\infty(\mathscr{D}_t)} \|\partial p\|_{L^\infty(\mathscr{D}_t)} \|\partial^{n+1} \mathcal{T}\|_{L^2(\mathscr{D}_t)} + \text{other terms}, \tag{1.19}$$

$$\begin{aligned} & \|\partial^{n+1} \mathcal{T}\|_{L^2(\mathscr{D}_t)} \le C\|\Pi \partial^{n+1} \mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} + \text{other terms} \\ & \le C\|\partial_{\mathcal{N}} \mathcal{T}\|_{L^\infty(\partial\mathscr{D}_t)} \|\bar{\partial}^{n-1} \theta\|_{L^2(\partial\mathscr{D}_t)} + C\|\theta\|_{L^\infty(\partial\mathscr{D}_t)} \|\partial^n \mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} + \text{other terms}. \end{aligned} \tag{1.20}$$

Here (1.18), (1.19) and (1.20) follow from elliptic estimates, the equation $\mathcal{T}\Delta p = -(\partial \mathcal{T}) \cdot \partial p +$other terms, and the projection formula, respectively.

Instead of using the projection formula, we need to use the evolution equations for θ. By doing so, we are led to the following estimate:

$$|D_t \theta| \le |\partial^2 v| + C|\theta||\partial v|,$$

from which it is clear that we need to get the L^∞-bounds for both ∂v and $\partial^2 v$ on $\partial\mathscr{D}_t$, while only the L^∞-bound for ∂v was sufficient in [6]. Thus, the L^∞-bound for one more derivative of the velocity field than that in [6] is needed. This causes the loss of one more derivative than that in [6]. Hence, we need to estimate $n+2$ derivatives in the energy functionals to close the argument, while only $n+1$ derivatives were needed in [6] for $n = 2, 3$. It should be noted that only ∂v enters equations (1.1), but not $\partial^2 v$, and thus one may think that the estimate of ∂v may be sufficient to close the argument as done in [6]. But the above argument suggests that this is not the case for the problem (1.1)-(1.3) which reflects the subtlety of this problem . It is extremely involved to bound $\partial^2 v$ before one obtains the L^∞-bound for θ in our case, due to the strong coupling of variation of temperature, heat-conduction, compressibility of the fluid and the evolution of the free surface.

In fact, even for the L^∞-bound for ∂v in $\mathscr{D}_t$, we will have to use the evolution equation of ∂v, while it was obtained by the Sobolev lemma in [6]:

$$\|\partial v\|^2_{L^\infty(\mathscr{D}_t)} \le C(K_1)\sum_{r=1}^{3}\|\partial^r v\|^2_{L^2(\mathscr{D}_t)} \le C(K_1)\sum_{r=1}^{3}\mathcal{E}_r \le C(K_1)\sum_{r=1}^{n+1}\mathcal{E}_r, \quad n=2,3. \tag{1.21}$$

For the problem considered in this paper, we do not have such a simple and neat estimate due to the complicated coupling as mentioned above. Indeed, if we try to use the Sobolev lemma as in [6], we can only get a bound depending on the L^∞-bound for θ that cannot be controlled by $n+1$ derivatives, as shown in the following:

$$\begin{aligned}\|\partial v\|^2_{L^\infty(\mathscr{D}_t)} \le & C(K_1, \mathrm{Vol}\mathscr{D}_t)\|\mathcal{T}^{-1}\|^2_{L^\infty(\mathscr{D}_t)}\|\partial\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|v\|_{L^\infty(\mathscr{D}_t)} \\ & \times\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}\|\partial^2\mathcal{T}\|_{L^2(\partial\mathscr{D}_t)}\|\partial^2 v\|_{L^2(\mathscr{D}_t)} + \text{other terms},\end{aligned} \tag{1.22}$$

which follows from Sobolev lemmas and the following estimates:

$$\|\partial^3 v\|^2_{L^\infty(\mathscr{D}_t)} \le C(E_3^a + \|\partial^2 \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}) \le CE_3^a + C(K_1, \mathrm{Vol}\mathscr{D}_t)\|\Delta \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}, \tag{1.23}$$

$$\|\Delta \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)} \le \|\mathcal{T}^{-1}\|^2_{L^\infty(\mathscr{D}_t)}\|\partial\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|v\|_{L^\infty(\mathscr{D}_t)}\sum_{k=1}^{3}\|\partial^k\mathcal{T}\|_{L^2(\mathscr{D}_t)}\sum_{j=0}^{2}\|\partial^j v\|_{L^2(\mathscr{D}_t)}, \tag{1.24}$$

$$\|\partial^3\mathcal{T}\|_{L^2(\mathscr{D}_t)} \le C\|\theta\|_{L^\infty(\partial\mathscr{D}_t)}\|\partial^2\mathcal{T}\|_{L^2(\partial\mathscr{D}_t)} + \text{other terms}. \tag{1.25}$$

Here (1.23), (1.24), and (1.25) follow from the divergence-curl decomposition, the equation $\mathcal{T}\Delta\mathrm{div}v = \partial^2\mathcal{T}\cdot\partial v$+other terms, and (1.20), respectively. The evolution equation $D_t\partial v = -\mathcal{T}\partial^2 p$+other terms and the Sobolev lemma lead to

$$\begin{aligned}\|D_t\partial v\|_{L^\infty(\mathscr{D}_t)} \le & \|\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|\partial^2 p\|_{L^\infty(\mathscr{D}_t)} + \text{other terms} \\ \le & C(K_1)\|\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\sum_{r=2}^{4}\|\partial^r p\|_{L^2(\mathscr{D}_t)} + \text{other terms}, \quad n=2,3,\end{aligned} \tag{1.26}$$

from which it is clear again that $n+2$ derivatives are needed to obtain the L^∞-bound for ∂v in the case of $n=2$.

1.3 The strategy of the proof

Next, we present the strategy of the proof. We want to prove that the temporal derivative of the higher order energy functional $\sum_{r=0}^{n+2} E_r$ can be bounded by itself under the *a priori* assumptions (1.14). For E_r^a given in (1.10), we can prove that

$$\frac{d}{dt}E_1^a(t) \le C_1(\cdot)\|\partial(v,\ p,\ \mathrm{div}v)\|^2_{L^2(\mathscr{D}_t)}, \tag{1.27a}$$

$$\begin{aligned}\frac{d}{dt}E_r^a(t) \le & C_r(\cdot)\Big(\|\partial^r(v,\ p,\ \mathcal{T},\ \mathrm{div}v)\|^2_{L^2(\mathscr{D}_t)} + \big\|\big(\partial^{r-1}v,\ \partial^r p,\ \Pi\partial^r D_t p\big)\big\|^2_{L^2(\partial\mathscr{D}_t)}\Big) \\ & + \text{other terms}, \quad r=2,\cdots,n+2,\end{aligned} \tag{1.27b}$$

where and thereafter $C_r(\cdot)$ stands for a constant depending continuously on the bounds in the a priori assumptions (1.14). We need to control the quantities on the right-hand side of (1.27) by the energy functionals. We will mainly discuss the estimates for the pressure p which appear also in [6] but require additional works in our problem due to the involvement of div$v$ in these estimates.

It follows from the definition of the energy functional E_r^a that for $r \geq 2$,

$$\|\Pi\partial^r p\|^2_{L^2(\partial\mathscr{D}_t)} \leq \|\partial_{\mathcal{N}} p\|_{L^\infty(\partial\mathscr{D}_t)} E_r^a.$$

Since $-\mathcal{T}\Delta p = D_t\mathrm{div}v$+other terms, one can use *elliptic estimates* to control all components of $\partial^r p$ from the tangential components $\Pi\partial^r p$ in the energy:

$$\begin{aligned}\|\partial^r p\|^2_{L^2(\partial\mathscr{D}_t)} &\leq C\|\Pi\partial^r p\|^2_{L^2(\partial\mathscr{D}_t)} + C(K, \mathrm{Vol}\mathscr{D}_t)\sum_{s=0}^{r-1}\|\partial^s\Delta p\|^2_{L^2(\mathscr{D}_t)} \\ &\leq C_r(\cdot)\left(E_r^a + \|\partial^{r-1}D_t\mathrm{div}v\|^2_{L^2(\mathscr{D}_t)}\right) + \text{other terms.}\end{aligned} \tag{1.28}$$

Under the physical condition $-\partial_{\mathcal{N}} p \geq \epsilon_b > 0$, we can use the higher order version of the projection formula to get

$$\begin{aligned}\|\bar{\partial}^{r-2}\theta\|^2_{L^2(\partial\mathscr{D}_t)} &\leq C_r(\cdot)\left(\|\Pi\partial^r p\|^2_{L^2(\partial\mathscr{D}_t)} + \|\partial^{r-1}p\|^2_{L^2(\partial\mathscr{D}_t)}\right) + \text{other terms} \\ &\leq C_r(\cdot)\left(E_r^a + E_{r-1}^a + \|\partial^{r-2}D_t\mathrm{div}v\|^2_{L^2(\mathscr{D}_t)}\right) + \text{other terms.}\end{aligned} \tag{1.29}$$

Once we have the bound for the second fundamental form, we can get estimates for solutions of any Dirichlet problem of elliptic equations. So, we can get estimates for $\mathcal{T}$, div$v$ and $D_t p$, which satisfy elliptic equations: $\Delta\mathcal{T} = \mathrm{div}v$, $\mathcal{T}\Delta\mathrm{div}v = D_t\mathrm{div}v$+other terms, and $\mathcal{T}\Delta D_t p = -D_t^2\mathrm{div}v$+other terms. Since the equation for $D_t p$ involves the highest order temporal derivative of div$v$, we show here how to control $D_t p$.

$$\begin{aligned}&\|\Pi\partial^r D_t p\|^2_{L^2(\partial\mathscr{D}_t)} \\ \leq &C\|\partial_{\mathcal{N}} D_t p\|^2_{L^\infty(\partial\mathscr{D}_t)}\|\bar{\partial}^{r-2}\theta\|^2_{L^2(\partial\mathscr{D}_t)} + C_r(\cdot)\|\partial^{r-1}D_t p\|^2_{L^2(\partial\mathscr{D}_t)} + \text{other terms} \\ \leq &C_r(\cdot)\left(\|\bar{\partial}^{r-2}\theta\|^2_{L^2(\partial\mathscr{D}_t)} + \|\partial^{r-2}\Delta D_t p\|^2_{L^2(\mathscr{D}_t)}\right) + \text{other terms} \\ \leq &C_r(\cdot)\left(E_r^a + E_{r-1}^a + \|\partial^{r-2}D_t\mathrm{div}v\|^2_{L^2(\mathscr{D}_t)} + \|\partial^{r-2}D_t^2\mathrm{div}v\|^2_{L^2(\mathscr{D}_t)}\right) + \text{other terms.}\end{aligned} \tag{1.30}$$

Since the terms involving div$v$ on the right-hand side of (1.28) and (1.30) cannot be controlled by $\sum_{s\leq r} E_s^a$, we introduce

$$E_r^d(t) = \int_{\mathscr{D}_t} |\partial D_t^{r-1}\mathrm{div}v|^2 dx, \quad r = 1, \cdots, n+2,$$

so that

$$E_r(t) = E_r^a(t) + E_r^d(t), \quad r = 1, \cdots, n+2.$$

It can be proven that the terms on the right-hand side of (1.27) can be controlled by $C_r(\cdot)\sum_{s\leq r} E_s$

when $r \geq 2$. For example, the idea of the estimates for $\partial^r \mathrm{div} v$ can be illustrated as follows:

$$
\begin{aligned}
&\|\partial^r \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}\\
\leq & C\|\Pi\partial^r \mathrm{div} v\|^2_{L^2(\partial\mathscr{D}_t)} + C(K, \mathrm{Vol}\mathscr{D}_t)\sum_{s=0}^{r-2}\|\partial^s\Delta \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}\\
\leq & C\|\partial_{\mathcal{N}}\mathrm{div} v\|^2_{L^2(\partial\mathscr{D}_t)}\|\bar{\partial}^{r-2}\theta\|^2_{L^2(\partial\mathscr{D}_t)} + C_r(\cdot)\left(\|\partial^{r-1}\mathrm{div} v\|^2_{L^2(\partial\mathscr{D}_t)} + \|\partial^{r-2}\Delta \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}\right)\\
& + \text{other terms}\\
\leq & C_r(\cdot)\left(\|\bar{\partial}^{r-2}\theta\|^2_{L^2(\partial\mathscr{D}_t)} + \|\partial^{r-2}\Delta \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}\right) + \text{other terms}\\
\leq & C_r(\cdot)\left(E^a_r + \|\partial^{r-2}D_t \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)} + \|\partial^{r-2}\Delta \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}\right) + \text{other terms},
\end{aligned}
$$

where (1.29) has been used to derive the last inequality. Using the equation $\mathcal{T}\Delta \mathrm{div} v = D_t \mathrm{div} v$+other terms, we may obtain

$$
\begin{aligned}
\|\partial^r \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)} \leq & C_r(\cdot)\left(E^a_r + \|\partial^{r-2}D_t \mathrm{div} v\|^2_{L^2(\mathscr{D}_t)}\right) + \text{other terms}\\
\leq & C_r(\cdot)\left(E^a_r + E^d_r + E^d_{r-1}\right) + \text{other terms}.
\end{aligned}
$$

Here the equation $\mathcal{T}\Delta D_t \mathrm{div} v = D_t^2 \mathrm{div} v$+other terms has been used to obtain the last inequality for $r = 4, 5$.

We need to estimate the temporal derivative of E^d_r. One may get

$$
\begin{aligned}
\frac{d}{dt}E^d_r(t) \leq & C_r(\cdot)\left(\sum_{j=1}^{r-2}\left\|\partial^2 D_t^j(p,\ \mathrm{div} v)\right\|^2_{L^2(\mathscr{D}_t)} + \sum_{j=1}^{r-3}\left\|\partial^3 D_t^j p\right\|^2_{L^2(\mathscr{D}_t)}\right.\\
& \left. + \|\partial^r(v,\ p,\ \mathrm{div} v)\|^2_{L^2(\mathscr{D}_t)} + E^d_r\right) + \text{other terms}, \quad r = 1, \cdots, n+2.
\end{aligned}
$$

The task is then to control $\partial^2 D_t^{r-2}p$ ($r \geq 3$) and $\partial^3 D_t^{r-3}p$ ($r \geq 4$) by energy functionals. It follows from the equation $-\mathcal{T}\Delta D_t^{r-2}p = D_t^{r-1}\mathrm{div} v$+other terms ($r \geq 3$) that

$$
\begin{aligned}
\|\partial^2 D_t^{r-2}p\|^2_{L^2(\mathscr{D}_t)} \leq & C(K, \mathrm{Vol}\mathscr{D}_t)\|\Delta D_t^{r-2}p\|^2_{L^2(\mathscr{D}_t)}\\
\leq & C_r(\cdot)\|D_t^{r-1}\mathrm{div} v\|^2_{L^2(\mathscr{D}_t)} + \text{other terms}\\
\leq & C_r(\cdot)\|\partial D_t^{r-1}\mathrm{div} v\|^2_{L^2(\mathscr{D}_t)} + \text{other terms}\\
= & C_r(\cdot)E^d_r + \text{other terms}.
\end{aligned}
$$

It follows from the equation $-\mathcal{T}\Delta D_t^{r-3}p = D_t^{r-2}\mathrm{div} v$+other terms ($r \geq 4$) that

$$
\begin{aligned}
&\|\partial^3 D_t^{r-3}p\|^2_{L^2(\mathscr{D}_t)}\\
\leq & C\|\Pi\partial^3 D_t^{r-3}p\|^2_{L^2(\partial\mathscr{D}_t)} + C(K, \mathrm{Vol}\mathscr{D}_t)\sum_{s=0}^{1}\|\partial^s\Delta D_t^{r-3}p\|^2_{L^2(\mathscr{D}_t)}\\
\leq & C\|\partial_{\mathcal{N}}D_t^{r-3}p\|^2_{L^\infty(\partial\mathscr{D}_t)}\|\bar{\partial}\theta\|^2_{L^2(\partial\mathscr{D}_t)} + C_r(\cdot)\|\partial D_t^{r-2}\mathrm{div} v\|^2_{L^2(\mathscr{D}_t)} + \text{other terms}\\
\leq & C_r(\cdot)\left(E^a_{r-1} + E^d_{r-1}\right) + \text{other terms}.
\end{aligned}
$$

It should be pointed out that we need the bound of $\|\partial_{\mathcal{N}}D_t^{r-3}p\|_{L^\infty(\partial\mathscr{D}_t)}$ to control $\|\partial^3 D_t^{r-3}p\|_{L^2(\mathscr{D}_t)}$ by energy functionals. This is why the a priori assumptions (1.14) we made for the case of $n = 3$

include the bound of $\|\partial_{\mathcal{N}} D_t^2 p\|_{L^\infty(\partial\mathscr{D}_t)}$, which was not needed in [6]. Similarly, it can be seen from (1.30) that the a priori assumption on the bound of $\|\partial_{\mathcal{N}} D_t^2 \mathrm{div} v\|_{L^\infty(\partial\mathscr{D}_t)}$ is also needed when $n=3$. The verification of the a priori assumptions on these bounds is difficult, even on that for $\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)}$, which will be discussed later.

We may conclude, under the a priori assumptions (1.14), that there are continuous functions C_r $(0 \le r \le n+2)$ such that

$$
\begin{aligned}
&\frac{d}{dt}E_0(t) \le \|p\|^2_{L^2(\mathscr{D}_t)} + \|\mathrm{div} v\|^2_{L^2(\mathscr{D}_t)} \le C_0\left(\overline{V}, M, \underline{\mathcal{T}}_0^{-1}, \overline{\mathcal{T}}_0\right)\left(E_1(t)+E_2(t)\right),\\
&\frac{d}{dt}E_1(t) \le C_1\left(\overline{V}, K, M, \underline{\mathcal{T}}_0^{-1}, \overline{\mathcal{T}}_0\right)\left(E_1(t)+E_2(t)\right),\\
&\frac{d}{dt}E_2(t) \le C_2\left(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}_0^{-1}, \overline{\mathcal{T}}_0\right)\left(E_1(t)+E_2(t)\right),\\
&\frac{d}{dt}E_r(t) \le C_r\left(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}_0^{-1}, \overline{\mathcal{T}}_0, \sum_{s=1}^{r-1} E_s(t)\right)\sum_{s=1}^{r} E_s(t), \quad 3 \le r \le n+2.
\end{aligned}
$$

In order to close the arguments, we need to get the estimates for the a priori bounds in terms of the energy functionals E_r $(0 \le r \le n+2)$, for which the clear and detailed dependence of $C_r(\cdot)$ on the quantities in the a priori assumptions is crucial. The lower and upper bounds for $\mathrm{Vol}\mathscr{D}_t$, the L^∞-bound for θ and the lower bound for $\partial_{\mathcal{N}} p$ on $\partial\mathscr{D}_t$, and the L^∞-bound for $\partial(p, v, \mathcal{T})$ in $\mathscr{D}_t$ can be obtained by looking at the evolution of these quantities. The estimate for the lower bound for ι_0 follows from the same idea as in [6]. The estimates for other quantities in the a priori assumptions follow from Sobolev lemmas, the projection formula, and elliptic estimates. Here we point out some main differences compared with [6]. The estimate on $\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)}$ given by [6] cannot work for our problem. Indeed, the bound for $\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)}$ was obtained in [6] by use of the following fact: If $q=0$ on $\partial\mathscr{D}_t$, then

$$
\begin{aligned}
\|\partial_{\mathcal{N}} q\|_{L^\infty(\partial\mathscr{D}_t)} \le{}& C\|\partial^{n-1}\Delta q\|_{L^2(\mathscr{D}_t)} + C\left(K, \mathrm{Vol}\mathscr{D}_t, \|\theta\|_{L^2(\partial\mathscr{D}_t)}, \cdots,\right.\\
&\left.\|\bar\partial^{n-2}\theta\|_{L^2(\partial\mathscr{D}_t)}\right)\sum_{s=0}^{n-2}\|\partial^s\Delta q\|_{L^2(\mathscr{D}_t)}, \quad n=2,3.
\end{aligned}
\tag{1.31}
$$

This can be found in Proposition 5.10 of [6]. If we apply (1.31) to our problem, we get

$$
\begin{aligned}
\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)} \le{}& C\|\mathcal{T}^{-1}\|_{L^\infty(\mathscr{D}_t)}\|\partial\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|\bar\partial\theta\|_{L^2(\partial\mathscr{D}_t)}\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)}\\
&+ \text{other terms}, \quad n=3,
\end{aligned}
\tag{1.32}
$$

which cannot give the bound of $\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)}$ for our problem. In fact, (1.32) follows from the equation $\mathcal{T}\Delta D_t p = -(\partial\mathcal{T})\cdot\partial D_t p +$other terms, and

$$
\begin{aligned}
&\|\partial^2\Delta D_t p\|_{L^2(\mathscr{D}_t)} \le \|\mathcal{T}^{-1}\|_{L^\infty(\mathscr{D}_t)}\|\partial\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|\partial^3 D_t p\|_{L^2(\mathscr{D}_t)} + \text{other terms}\\
&\quad \le \|\mathcal{T}^{-1}\|_{L^\infty(\mathscr{D}_t)}\|\partial\mathcal{T}\|_{L^\infty(\mathscr{D}_t)}\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)}\|\bar\partial\theta\|_{L^2(\partial\mathscr{D}_t)} + \text{other terms}.
\end{aligned}
$$

To overcome the difficulty appearing in (1.32), we refine (1.31) to show: If $q=q_b$ on $\partial\mathscr{D}_t$ with q_b being a constant, then for any $\delta>0$,

$$
\begin{aligned}
\|\partial_{\mathcal{N}} q\|_{L^\infty(\partial\mathscr{D}_t)} \le{}& \delta\|\partial^{n-1}\Delta q\|_{L^2(\mathscr{D}_t)} + C\left(\delta^{-1}, K, \mathrm{Vol}\mathscr{D}_t, \|\theta\|_{L^2(\partial\mathscr{D}_t)}, \cdots,\right.\\
&\left.\|\bar\partial^{n-2}\theta\|_{L^2(\partial\mathscr{D}_t)}\right)\sum_{s=0}^{n-2}\|\partial^s\Delta q\|_{L^2(\mathscr{D}_t)}, \quad n=2,3.
\end{aligned}
\tag{1.33}
$$

Clearly, the bound for $\|\partial_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathscr{D}_t)}$ can be obtained by use of (1.33).

2 Preliminaries

In this section, we introduce Lagrangian transformation, the metric and covariant differentiation associated with it, the induced metric on the boundary, the geometry and regularity of the boundary, Sobolev lemmas, interpolation inequalities and estimates for the boundary. Those materials are basically from [6]. We list them here for the convenience of readers and the easier reference.

2.1 Lagrangian coordinates, the metric, and covariant differentiation in the interior

Let $x = x(t, y)$ be the change of variables given by

$$\partial_t x(t, y) = v\,(t, x(t, y)) \quad \text{and} \quad x(t, y) = x_0(y), \quad y \in \Omega. \tag{2.1}$$

Initially, when $t = 0$, we can start with either the Euclidean coordinates in $\Omega = \mathscr{D}_0$ or some other coordinates $x_0 : \Omega \to \mathscr{D}_0$ where x_0 is a diffeomorphism in which the domain Ω becomes simple. For each t we will then have a change of coordinates $x : \Omega \to \mathscr{D}_t$, taking $y \to x(t, y)$. The Euclidean metric δ_{ij} in $\mathscr{D}_t$ then induces a metric

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \tag{2.2}$$

on Ω for each fixed t.

The *covariant differentiation* of a $(0, r)$ tensor $w(t, y)$, is the $(0, r + 1)$ tensor given by

$$\nabla_a w_{a_1 \cdots a_r} = \frac{\partial w_{a_1 \cdots a_r}}{\partial y^a} - \Gamma^e_{a a_1} w_{e a_2 \cdots a_r} - \cdots - \Gamma^e_{a a_r} w_{a_1 \cdots a_{r-1} e},$$

where Γ^c_{ab} are the Christoffel symbols given by

$$\Gamma^c_{ab} = \frac{g^{cd}}{2} \left(\frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d} \right) = \frac{\partial y^c}{\partial x^i} \frac{\partial^2 x^i}{\partial y^a \partial y^b}$$

with g^{ab} being the inverse of g_{ab}. If $\omega(t, x)$ is the $(0, r)$ tensor expressed in the x-coordinates, then the same tensor $w(t, y)$ expressed in the y-coordinates is given by

$$w_{a_1 \cdots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \omega_{i_1 \cdots i_r}(t, x), \quad x = x(t, y),$$

and by the transformation properties for tensors,

$$\nabla_a w_{a_1 \cdots a_r}(t, y) = \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial \omega_{i_1 \cdots i_r}(t, x)}{\partial x^i}.$$

So that the norms of tensors are invariant under change of coordinates:

$$g^{a_1 b_1} \cdots g^{a_r b_r} w_{a_1 \cdots a_r} w_{b_1 \cdots b_r} = \delta^{i_1 j_1} \cdots \delta^{i_r j_r} \omega_{i_1 \cdots i_r} \omega_{j_1 \cdots j_r}. \tag{2.3}$$

Since the curvature vanishes in the x-coordinates, it must do so in the y-coordinates, and hence

$$[\nabla_a, \nabla_b] = 0.$$

Set

$$w_{a \cdots}{}^{b}{}_{\cdots c} = g^{bd} w_{a \cdots d \cdots c}.$$

The material derivative is defined as

$$D_t = \partial_t|_{x=const.} + v^k\partial_k = \partial_t|_{y=const.}, \quad \partial_k = \frac{\partial}{\partial x^k} = \frac{\partial y^a}{\partial x^k}\frac{\partial}{\partial y^a}.$$

Let α be a $(0, s)$ tensor and β be a $(0, r)$ tensor. Then $\alpha\widetilde{\otimes}\beta$ is used to denote some partial symmetrization of the tensor product $\alpha\otimes\beta$, i.e., a sum over some subset of the permutations of the indices divided by the number of permutations in that subset. Moreover $\alpha\tilde{\cdot}\beta$ is used to denote a partial symmetrization of the dot product $\alpha\cdot\beta$, which in turn is defined to be a contraction of the last index of α with the first index of β: $(\alpha\cdot\beta)_{i_1\cdots i_{r+s-2}} = g^{ij}\alpha_{i_1\cdots i_{s-1}i}\beta_{ji_s\cdots i_{r+s-2}}$.

The following lemmas are for temporal derivatives of the change of coordinates and commutators between temporal derivative and spatial derivatives, which are Lemmas 2.1 and 2.4 in [6], and will be used to calculate the higher order equations in Lagrangian coordinates.

Lemma 2.1 *Let $x = x(t, y)$ be the change of variables given by (2.1), and let g_{ab} be the metric given by (2.2). Let $v_i = \delta_{ij}v^j = v^i$ and $d\mu_g = \sqrt{\det g}dy$. Set*

$$u_a(t,y) = \frac{\partial x^j}{\partial y^a}v_j(t,x), \quad h_{ab} = (\nabla_a u_b + \nabla_b u_a)/2, \quad h^{ab} = g^{ac}g^{bd}h_{cd}, \quad \mathrm{div}u = g^{ab}\nabla_a u_b.$$

Then,

$$D_t\frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a}\frac{\partial v_i}{\partial x^k}, \quad D_t\frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^k}\frac{\partial v_k}{\partial x^i}, \tag{2.4}$$

$$D_t g_{ab} = 2h_{ab}, \quad D_t g^{ab} = -2h^{ab}, \quad D_t d\mu_g = g^{ab}h_{ab}d\mu_g = (\mathrm{div}u)d\mu_g. \tag{2.5}$$

Lemma 2.2 *Let $w_{a_1\cdots a_r}$ be a (0,r) tensor, q be a function, and $\Delta = g^{cd}\nabla_c\nabla_d$. Then,*

$$[D_t, \nabla_a]w_{a_1\cdots a_r} = -(\nabla_{a_1}\nabla_a u^e)w_{ea_2\cdots a_r} - \cdots - (\nabla_{a_r}\nabla_a u^e)w_{a_1\cdots a_{r-1}e}, \tag{2.6}$$

$$[D_t, \Delta]q = -2h^{ab}\nabla_a\nabla_b q - (\Delta u^e)\nabla_e q, \tag{2.7}$$

Furthermore,

$$[D_t, \nabla^r]q = -\sum_{s=1}^{r-1}\begin{pmatrix} r \\ s+1 \end{pmatrix}(\nabla^{s+1}u)\cdot\nabla^{r-s}q. \tag{2.8}$$

2.2 The geometry and regularity of the boundary

As in [6], we extend the normal to the boundary to the interior by a geodesic extension, which enables us to define a pseudo-Riemann metric in the whole domain whose restriction on the boundary is then the induced metric on the tangential space to the boundary. Using this induced metric, we can define the orthogonal projection of a tensor to the boundary, the covariant differentiation on the boundary, and the second fundamental form of the boundary as follows:

Definition 2.3 *Let $d(t, y) = \mathrm{dist}_g(y, \partial\Omega)$ be the geodesic distance to the boundary, which is the same as the Euclidean distance in the x-variables, and η be the smooth cut-off function given by Definition 1.4. Set $N_a(t, y) = \nabla_a d(t, y)$ and $N^a(t, y) = g^{ab}(t, y)N_b(t, y)$. Define*

$$\zeta^{ab}(t,y) = g^{ab}(t,y) - \widetilde{N}^a(t,y)\widetilde{N}^b(t,y), \quad \text{where} \quad \widetilde{N}^a(t,y) = \eta(d(t,y))N^a(t,y).$$

In particular, ζ gives the induced metric on the tangent space to the boundary:

$$\zeta_{ab} = g_{ab} - N_a N_b \quad \text{and} \quad \zeta^{ab} = g^{ab} - N^a N^b \quad \text{on} \quad \partial\Omega.$$

The orthogonal projection of a $(0, r)$ tensor $w(t, y)$ to the boundary is given by

$$(\Pi w)_{a_1 \cdots a_r} = \zeta_{a_1}^{c_1} \cdots \zeta_{a_r}^{c_r} w_{c_1 \cdots c_r}, \quad \text{where} \quad \zeta_a^c = \delta_a^c - N_a N^c.$$

The covariant differentiation on the boundary $\overline{\nabla}$ is given by $\overline{\nabla}_a = \zeta_a^c \nabla_c$. The second fundamental form of the boundary is given by $\theta_{ab}(t, y) = \overline{\nabla}_a N_b$.

It follows from Definitions 1.1, 1.4 and 2.3 that

$$N_a(t, y) = \frac{\partial x^j}{\partial y^a} \mathcal{N}_j(t, x) \quad \text{and} \quad \theta_{ab}(t, y) = \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \theta_{ij}(t, x).$$

Lemma 2.4 *Let N be the unit normal to $\partial\Omega$ and $d\mu_\zeta = \sqrt{\det g / (\sum N_a^2)} dS$ with dS being the Euclidean surface measure. On $[0, T] \times \partial\Omega$ we have*

$$D_t N_a = h_{NN} N_a, \quad D_t N^c = -2h_d^c N^d + h_{NN} N^c, \quad \text{where} \quad h_{NN} = N^a N^b h_{ab}, \tag{2.9}$$

$$D_t \zeta^{ab} = -2\zeta^{ac} \zeta^{bd} h_{cd}, \tag{2.10}$$

$$D_t d\mu_\zeta = \left(g^{ab} h_{ab} - h_{NN} \right) d\mu_\zeta = (\text{div} u - h_{NN}) \, d\mu_\zeta. \tag{2.11}$$

This is Lemma 3.9 in [6], where the proof can be found.

Definition 2.5 *For the multi-indices $I = (i_1, \cdots, i_r)$ and $J = (j_1, \cdots, j_r)$, set $g^{IJ} = g^{i_1 j_1} \cdots g^{i_r j_r}$ and $\zeta^{IJ} = \zeta^{i_1 j_1} \cdots \zeta^{i_r j_r}$. If α and β are $(0, r)$ tensors, let*

$$< \alpha, \beta >= g^{IJ} \alpha_I \beta_J \quad \text{and} \quad |\alpha|^2 =< \alpha, \alpha >= g^{IJ} \alpha_I \alpha_J.$$

Then for the projection $(\Pi\beta)_I = \zeta_I^J \beta_J$,

$$< \Pi\alpha, \Pi\beta >= \zeta^{IJ} \alpha_I \beta_J \quad \text{and} \quad |\Pi\alpha|^2 = \zeta^{IJ} \alpha_I \alpha_J \quad \text{on} \quad \partial\Omega.$$

Let

$$\|\alpha\| = \|\alpha\|_{L^2(\Omega)} = \left(\int_\Omega |\alpha|^2 d\mu_g \right)^{1/2} \quad \text{and} \quad |||\alpha||| = \|\alpha\|_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} |\alpha|^2 d\mu_\zeta \right)^{1/2}.$$

Moreover, we define the following notations:

$$\| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)} \quad \text{and} \quad ||| \cdot |||_{L^p} = ||| \cdot |||_{L^p(\partial\Omega)} \quad \text{for} \quad 1 \le p \le \infty.$$

Lemma 2.6 *With the notations in Definitions 1.2, 2.3 and 2.5, we have*

$$\|\nabla\zeta\|_{L^\infty} \le 512 \, (|||\theta|||_{L^\infty} + 1/\iota_0) \quad \text{and} \quad \|D_t\zeta\|_{L^\infty} \le 128\|h\|_{L^\infty}. \tag{2.12}$$

This is Lemma 3.11 in [6], where the proof can be found.

The following Lemma which is Lemma 3.6 in [6] shows that ι_1 given in Definition (1.3) is equivalent to ι_0 in conjunction with a bound of the second fundamental form.

Lemma 2.7 *Suppose that $|\theta| \le K$, and let ι_0 and ι_1 be as in Definitions 1.2 and 1.3. Then*

$$\iota_0 \ge \min\{\iota_1/2, \ 1/K\} \quad \text{and} \quad \iota_1 \ge \min\{2\iota_0, \ \epsilon_1/K\}.$$

The advantage to using ι_1, instead of ι_0, is that it is easier to control the evolution off.

2.3 Sobolev lemmas, interpolation inequalities and estimates for the boundary

Lemma 2.8 (Lemmas A.1-A.4 in [6]) *Let α be a $(0,r)$ tensor and $\iota_1 \geq 1/K_1$. Assume k, m are positive integers, and $p \geq 1$. We have*
(i) if $2 \leq p \leq s \leq q \leq \infty$ and $m/s = k/p + (m-k)/q$,

$$|||\overline{\nabla}^k \alpha|||_{L^s}^m \leq C(k,m,n,s)|||\alpha|||_{L^q}^{m-k}|||\overline{\nabla}^m \alpha|||_{L^p}^k, \tag{2.13}$$

$$\left(\sum_{i=0}^k \|\nabla^i \alpha\|_{L^s}\right)^m \leq C(k,m,n,s)\|\alpha\|_{L^q}^{m-k}\left(\sum_{i=0}^m K_1^{m-i}\|\nabla^i \alpha\|_{L^p}\right)^k; \tag{2.14}$$

(ii) for any $\delta > 0$,

$$|||\alpha|||_{L^{(n-1)p/(n-1-kp)}} \leq C(k,n,p)\sum_{i=0}^k K_1^{k-i}|||\nabla^i \alpha|||_{L^p}, \quad 1 \leq p < (n-1)/k, \tag{2.15}$$

$$|||\alpha|||_{L^\infty} \leq \delta|||\nabla^k \alpha|||_{L^p} + C(\delta^{-1}, K_1, k, n, p)\sum_{i=0}^{k-1}|||\nabla^i \alpha|||_{L^p}, \quad p > (n-1)/k, \tag{2.16}$$

$$\|\alpha\|_{L^{np/(n-kp)}} \leq C(k,n,p)\sum_{i=0}^k K_1^{k-i}\|\nabla^i \alpha\|_{L^p}, \quad 1 \leq p < n/k, \tag{2.17}$$

$$\|\alpha\|_{L^\infty} \leq C(k,n,p)\sum_{i=0}^k K_1^{k-i}\|\nabla^i \alpha\|_{L^p}, \quad p > n/k. \tag{2.18}$$

Lemma 2.9 *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant, then for $r = 2,3,4$,*

$$\begin{aligned} |||\Pi\nabla^r q||| \leq & 2|||\nabla_N q|||_{L^\infty}|||\overline{\nabla}^{r-2}\theta||| + C\sum_{k=1}^{r-1}|||\theta|||_{L^\infty}^k|||\nabla^{r-k}q||| \\ & + C\sum_{k=1}^{r-3}|||\theta|||_{L^\infty}|||\nabla_N q|||_{L^\infty}|||\overline{\nabla}^k\theta|||, \end{aligned} \tag{2.19}$$

where C is a positive number. If, in addition, $|\nabla_N q| \geq \epsilon$ and $|\nabla_N q| \geq 2\epsilon|||\nabla_N q|||_{L^\infty}$ on $\partial\Omega$ for a certain positive constant ϵ, then there exists a positive number C such that

$$\begin{aligned} |||\overline{\nabla}^{r-2}\theta||| \leq & \epsilon^{-2}|||\Pi\nabla^r q||| + C\epsilon^{-3}\sum_{k=1}^{r-1}|||\theta|||_{L^\infty}^k|||\nabla^{r-k}q||| \\ & + C\epsilon^{-2}\sum_{k=1}^{r-3}|||\theta|||_{L^\infty}|||\nabla_N q|||_{L^\infty}|||\overline{\nabla}^k\theta|||, \quad r = 2,3,4. \end{aligned} \tag{2.20}$$

Proof. Simple calculations give that on $\partial\Omega$,

$$\Pi\nabla^2 q = (\nabla_N q)\theta, \quad \Pi\nabla^3 q = (\nabla_N q)\overline{\nabla}\theta + 3(\overline{\nabla}\nabla_N q)\widetilde{\otimes}\theta + 2(\theta\,\tilde{\cdot}\,\theta)\widetilde{\otimes}\nabla q, \tag{2.21}$$

$$\begin{aligned} \Pi\nabla^4 q = & (\nabla_N q)\overline{\nabla}^2\theta + 4(\overline{\nabla}\theta)\widetilde{\otimes}\overline{\nabla}\nabla_N q + 6\theta\widetilde{\otimes}\overline{\nabla}^2\nabla_N q + 3(\theta\widetilde{\otimes}\theta)\nabla_N^2 q \\ & + 7((\overline{\nabla}\theta)\,\tilde{\cdot}\,\theta)\widetilde{\otimes}\nabla q - \theta\widetilde{\otimes}(\theta\,\tilde{\cdot}\,\theta)\nabla_N q + 3((\theta\,\tilde{\cdot}\,\theta)\,\tilde{\cdot}\,\theta)\widetilde{\otimes}N\widetilde{\otimes}\nabla q \\ & + 8(\theta\,\tilde{\cdot}\,\theta)\widetilde{\otimes}(\overline{\nabla}\nabla_N q)\widetilde{\otimes}N + 3\theta\widetilde{\otimes}(\theta\,\tilde{\cdot}\,\overline{\nabla}\nabla_N q)\widetilde{\otimes}N, \end{aligned}$$

where we have used the facts that $\overline{\nabla} q = 0$, $\theta \cdot N = 0$, $(\overline{\nabla}\theta) \cdot N = -\theta \cdot \theta$ and $(\overline{\nabla}^2 \theta) \cdot N = -3(\overline{\nabla}\theta) \tilde{\cdot}\, \theta$. Clearly, (2.19) and (2.20) hold for $r = 2, 3$, because of $\overline{\nabla}\nabla_N q = N^e \overline{\nabla}\nabla_e q$ and

$$|||\Pi\nabla^3 q - (\nabla_N q)\overline{\nabla}\theta||| \le 3|||\theta|||_{L^\infty} |||\nabla^2 q||| + 2|||\theta|||^2_{L^\infty} |||\nabla q|||. \tag{2.22}$$

For $r = 4$, we first derive from (2.13), Hölder's inequality and Young's inequality that for any positive constant δ,

$$\begin{aligned} ||||\overline{\nabla}\theta||\overline{\nabla}\nabla_N q|||| \le& |||\overline{\nabla}\theta|||_{L^4} |||\overline{\nabla}\nabla_N q|||_{L^4} \\ \le& C|||\theta|||^{1/2}_{L^\infty} |||\overline{\nabla}^2\theta|||^{1/2} |||\nabla_N q|||^{1/2}_{L^\infty} |||\overline{\nabla}^2 \nabla_N q|||^{1/2} \\ \le& (\delta/4)|||\nabla_N q|||_{L^\infty} |||\overline{\nabla}^2\theta||| + C\delta^{-1} |||\theta|||_{L^\infty} |||\overline{\nabla}^2 \nabla_N q|||; \end{aligned}$$

which, together with $\nabla^2_N q = N^e \nabla_N \nabla_e q$ and $|\overline{\nabla}^2 \nabla_N q| \le |\nabla^3 q| + 3|\theta||\nabla^2 q|$, gives that for any positive constant δ,

$$\begin{aligned} |||\Pi\nabla^4 q - (\nabla_N q)\overline{\nabla}^2\theta||| \le& \delta |||\nabla_N q|||_{L^\infty} |||\overline{\nabla}^2\theta||| \\ &+ 7|||\theta|||_{L^\infty} |||\nabla_N q|||_{L^\infty} |||\overline{\nabla}\theta||| + C\delta^{-1} \sum_{k=1}^{3} |||\theta|||^k_{L^\infty} |||\nabla^{4-k} q|||. \end{aligned} \tag{2.23}$$

Here C is a positive number. Clearly, choose $\delta = 1$ in (2.23) to prove (2.19). Note that

$$2\epsilon |||\nabla_N q|||_{L^\infty} |||\overline{\nabla}^2\theta||| \le |||(\nabla_N q)\overline{\nabla}^2\theta||| \le |||\Pi\nabla^4 q||| + |||\Pi\nabla^4 q - (\nabla_N q)\overline{\nabla}^2\theta|||,$$

we then prove (2.20) by choosing $\delta = \epsilon$ in (2.23). □

Lemma 2.10 *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant, then*

$$\begin{aligned} |||\Pi\nabla^5 q||| \le& 2|||\nabla_N q|||_{L^\infty} |||\overline{\nabla}^3\theta||| + C\sum_{k=1}^{4} |||\theta|||^k_{L^\infty} |||\nabla^{5-k} q||| \\ &+ C(K_1, |||\theta|||_{L^\infty})(|||\overline{\nabla}^2\theta||| + |||\overline{\nabla}\theta|||) \sum_{k=1}^{4} |||\nabla^k q|||. \end{aligned} \tag{2.24}$$

If, in addition, $|\nabla_N q| \ge \epsilon$ and $|\nabla_N q| \ge 2\epsilon |||\nabla_N q|||_{L^\infty}$ on $\partial\Omega$ for a certain positive constant ϵ,

$$\begin{aligned} |||\overline{\nabla}^3\theta||| \le& \epsilon^{-2} |||\Pi\nabla^5 q||| + \epsilon^{-3} C \sum_{k=1}^{4} |||\theta|||^k_{L^\infty} |||\nabla^{5-k} q||| \\ &+ \epsilon^{-3} C(K_1, |||\theta|||_{L^\infty})(|||\overline{\nabla}^2\theta||| + |||\overline{\nabla}\theta|||) \sum_{k=1}^{4} |||\nabla^k q|||. \end{aligned} \tag{2.25}$$

Proof. This lemma can be shown in a similar way to proving Lemma 2.9 by noticing the following fact:

$$\begin{aligned} &||||\overline{\nabla}\theta||\overline{\nabla}^2 \nabla_N q|||| + ||||\overline{\nabla}^2\theta||\overline{\nabla}\nabla_N q|||| \\ \le& |||\overline{\nabla}\theta|||_{L^6} |||\overline{\nabla}^2 \nabla_N q|||_{L^3} + |||\overline{\nabla}^2\theta|||_{L^3} |||\overline{\nabla}\nabla_N q|||_{L^6} \\ \le& C|||\theta|||^{2/3}_{L^\infty} |||\overline{\nabla}^3\theta|||^{1/3} |||\nabla_N q|||^{1/3}_{L^\infty} |||\overline{\nabla}^3 \nabla_N q|||^{2/3} \\ &+ C|||\theta|||^{1/3}_{L^\infty} |||\overline{\nabla}^3\theta|||^{2/3} |||\nabla_N q|||^{2/3}_{L^\infty} |||\overline{\nabla}^3 \nabla_N q|||^{1/3} \\ \le& \delta |||\nabla_N q|||_{L^\infty} |||\overline{\nabla}^3\theta||| + C\delta^{-1} |||\theta|||_{L^\infty} |||\overline{\nabla}^3 \nabla_N q||| \end{aligned} \tag{2.26}$$

for any positive constant δ, and

$$|||\nabla^2 q||\overline{\nabla}\theta||| \le |||\nabla^2 q|||_{L^\infty}|||\overline{\nabla}\theta||| \le C(K_1)|||\overline{\nabla}\theta\| \sum_{k=2}^{4} |||\nabla^k q|||. \tag{2.27}$$

Here (2.26) follows from (2.13), Hölder's inequality and Young's inequality, and (2.27) follows from (2.16). □

Remark 2.11 *ϵ appearing on the right-hand side of (2.20) and (2.25) can be chosen as*

$$\epsilon = |||(\nabla_N q)^{-1}|||_{L^\infty}^{-1} \min\left\{1,\ 2^{-1}|||\nabla_N q|||_{L^\infty}^{-1}\right\}. \tag{2.28}$$

In particular, it follows from (2.21) and (2.22) that

$$|||\theta|||_{L^s} \le |||(\nabla_N q)^{-1}|||_{L^\infty}|||\Pi\nabla^2 q|||_{L^s}, \quad 2 \le s \le \infty, \tag{2.29}$$

$$|||\overline{\nabla}\theta||| \le |||(\nabla_N q)^{-1}|||_{L^\infty}\left(|||\Pi\nabla^3 q||| + 3\sum_{k=1}^{2}|||\theta|||_{L^\infty}^k|||\nabla^{3-k}q|||\right). \tag{2.30}$$

Lemma 2.12 (Lemmas 5.5-5.6 in [6]) *Let w be a $(0,1)$ tensor and define a scalar $\mathrm{div}w = g^{ab}\nabla_a w_b$ and a $(0,2)$ tensor $\mathrm{curl}w_{ab} = \nabla_a w_b - \nabla_b w_a$. If $|\theta| + 1/\iota_0 \le K$, then for any nonnegative integer r,*

$$|\nabla^{r+1}w|^2 \le C\left(g^{ij}\zeta^{kl}\zeta^{IJ}(\nabla_k\nabla_I^r w_i)\nabla_l\nabla_J^r w_j + |\nabla^r \mathrm{div}w|^2 + |\nabla^r \mathrm{curl}w|^2\right), \tag{2.31}$$

$$\begin{aligned}\|\nabla^{r+1}w\|^2 \le\ & C\int_\Omega \widetilde{N}^i\widetilde{N}^j g^{kl}\zeta^{IJ}(\nabla_k\nabla_I^r w_i)\nabla_l\nabla_J^r w_j d\mu_g \\ & + C\left(\|\nabla^r\mathrm{div}w\|^2 + \|\nabla^r\mathrm{curl}w\|^2 + K^2\|\nabla^r w\|^2\right),\end{aligned} \tag{2.32}$$

$$|||\nabla^r w|||^2 \le C\left(\|\nabla^{r+1}w\| + K\|\nabla^r w\|\right)\|\nabla^r w\|, \tag{2.33}$$

$$|||\nabla^r w|||^2 \le C|||\Pi\nabla^r w|||^2 + C\left(\|\nabla^r\mathrm{div}w\| + \|\nabla^r\mathrm{curl}w\| + K\|\nabla^r w\|\right)\|\nabla^r w\|, \tag{2.34}$$

$$\|\nabla^{r+1}w\|^2 \le C|||\nabla^{r+1}w||||||\nabla^r w||| + C\left(\|\nabla^r\mathrm{div}w\|^2 + \|\nabla^r\mathrm{curl}w\|^2\right), \tag{2.35}$$

$$\begin{aligned}\|\nabla^{r+1}w\|^2 \le\ & C|||\Pi\nabla^{r+1}w||||||\Pi(N^i\nabla^r w_i)||| \\ & + C\left(\|\nabla^r\mathrm{div}w\|^2 + \|\nabla^r\mathrm{curl}w\|^2 + K^2\|\nabla^r w\|^2\right),\end{aligned} \tag{2.36}$$

$$\begin{aligned}\|\nabla^{r+1}w\|^2 \le\ & C|||\Pi(N^i\nabla^{r+1}w_i)||||||\Pi\nabla^r w||| \\ & + C\left(\|\nabla^r\mathrm{div}w\|^2 + \|\nabla^r\mathrm{curl}w\|^2 + K^2\|\nabla^r w\|^2\right).\end{aligned} \tag{2.37}$$

Indeed, the proof of (2.31)-(2.32) can also be found in [26]. The proof of (2.33)-(2.37) are based on the divergence theorem, and (2.36)-(2.37) are based additionally on (2.32).

Lemma 2.13 (Lemma A.5 in [6]) *Suppose that $q = 0$ on $\partial\Omega$. Then*

$$\|q\| \le C(\mathrm{Vol}\Omega)^{1/n}\|\nabla q\| \quad \text{and} \quad \|\nabla q\| \le C(\mathrm{Vol}\Omega)^{1/n}\|\Delta q\|. \tag{2.38}$$

As a consequence of Lemmas 2.12 and 2.13, we have

Corollary 2.14 *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant. If $|\theta| + 1/\iota_0 \le K$, we have for any $r \ge 2$ and $\delta > 0$,*

$$\|q - q_b\| \le C(\mathrm{Vol}\Omega)^{1/n}\|\nabla q\|, \quad \|\nabla q\| + \|\nabla^2 q\| \le C(K, \mathrm{Vol}\Omega)\|\Delta q\|, \tag{2.39}$$

$$\|\nabla^r q\| + |||\nabla^r q||| \le C|||\Pi\nabla^r q||| + C(K, \mathrm{Vol}\Omega)\sum_{s=0}^{r-1}\|\nabla^s\Delta q\|, \tag{2.40}$$

$$\|\nabla^r q\| + |||\nabla^{r-1} q||| \le \delta|||\Pi\nabla^r q||| + C(\delta^{-1}, K, \mathrm{Vol}\Omega)\sum_{s=0}^{r-2}\|\nabla^s\Delta q\|. \tag{2.41}$$

Clearly, (2.39) is a consequence of (2.38) and (2.37). The proof of (2.40) and (2.41) can be found in Proposition 5.8, [6]. (Indeed, (2.40) follows from (2.33)-(2.35) and (2.38), and (2.41) follows from (2.33), (2.35), (2.36) and (2.38).)

Lemma 2.15 *Let $q = q_b$ on $\partial\Omega$ with q_b being a constant. If $|\theta| + 1/\iota_0 \le K$ and $\iota_1 \ge 1/K_1$, then for any $\delta > 0$,*

$$|||\nabla q|||_{L^\infty} \le \begin{cases} \delta\|\nabla\Delta q\| + C(\delta^{-1}, K, K_1, |||\theta|||, \mathrm{Vol}\Omega)\|\Delta q\|, & n = 2; \\ \delta\|\nabla^2\Delta q\| + C(\delta^{-1}, K, K_1, |||\overline{\nabla}\theta|||, \mathrm{Vol}\Omega)(\|\nabla\Delta q\| + \|\Delta q\|), & n = 3. \end{cases} \tag{2.42}$$

Proof. When $n = 3$, it follows from (2.22) and (2.16) that for any $\delta > 0$,

$$|||\Pi\nabla^3 q||| \le \delta|||\nabla^3 q||| + C(\delta^{-1}, K, K_1, |||\overline{\nabla}\theta|||)(|||\nabla^2 q||| + |||\nabla q|||). \tag{2.43}$$

In view of (2.41), (2.33) and (2.39), we see that for any $\delta_1 > 0$,

$$|||\nabla^2 q||| \le \delta_1|||\Pi\nabla^3 q||| + C(\delta_1^{-1}, K, \mathrm{Vol}\Omega)(\|\nabla\Delta q\| + \|\Delta q\|), \tag{2.44}$$

$$|||\nabla q||| \le C(K)(\|\nabla^2 q\| + \|\nabla q\|) \le C(K, \mathrm{Vol}\Omega)\|\Delta q\|. \tag{2.45}$$

Substitute (2.44) and (2.45) into (2.43) and choose suitable small δ_1 to obtain for any $\delta > 0$,

$$2^{-1}|||\Pi\nabla^3 q||| \le \delta|||\nabla^3 q||| + C(\delta^{-1}, K, K_1, |||\overline{\nabla}\theta|||, \mathrm{Vol}\Omega)(\|\nabla\Delta q\| + \|\Delta q\|).$$

This, together with (2.40), gives

$$|||\nabla^3 q||| \le C(K, \mathrm{Vol}\Omega)\|\nabla^2\Delta q\| + C(K, K_1, |||\overline{\nabla}\theta|||, \mathrm{Vol}\Omega)(\|\nabla\Delta q\| + \|\Delta q\|). \tag{2.46}$$

So, (2.42) follows from (2.16), (2.46) (2.44) and (2.45) in the case of $n = 3$. Similarly, (2.42) can be shown when $n = 2$. □

3 Higher Order Equations

Let $u(t, y)$ be the same tensor of the velocity $v(t, x)$ expressed in the y-coordinates, i.e.,

$$u_a(t, y) = \frac{\partial x^j}{\partial y^a} v_j(t, x).$$

Then, system (1.1) can be rewritten as

$$D_t u_a + \mathcal{T}\nabla_a p = (\nabla_a u_c)u^c, \tag{3.1a}$$

$$\mathrm{div} u = \Delta\mathcal{T}, \quad D_t\mathcal{T} = \mathcal{T}\Delta\mathcal{T}. \tag{3.1b}$$

It follows from (3.1a) and (2.6) that

$$D_t\nabla_b u_a + \mathcal{T}\nabla_b\nabla_a p = -(\nabla_b\mathcal{T})\nabla_a p + (\nabla_a u_e)\nabla_b u^e, \tag{3.2}$$

which implies

$$D_t \mathrm{curl} u_{ab} = (\nabla_b\mathcal{T})\nabla_a p - (\nabla_a\mathcal{T})\nabla_b p, \tag{3.3}$$

$$D_t \mathrm{div} u + \mathcal{T}\Delta p = -(\nabla\mathcal{T}) \cdot \nabla p - (\nabla_e u) \cdot \nabla u^e. \tag{3.4}$$

Moreover, we can derive from (3.1b) and (2.7) that

$$\begin{aligned} &D_t \mathrm{div} u - \mathcal{T}\Delta \mathrm{div} u \\ &= -(\Delta u^e)\nabla_e\mathcal{T} - 2g^{ab}(\nabla_e\nabla_a\mathcal{T})\nabla_b u^e + (\mathrm{div} u)^2 + 2g^{ab}(\nabla_a\mathcal{T})\nabla_b \mathrm{div} u. \end{aligned} \tag{3.5}$$

Lemma 3.1 *Let q be any given function. Then for any integer $r \geq 2$,*

$$\begin{aligned} &\left|D_t\nabla^r u + \mathcal{T}\nabla^{r+1}p\right| + \left|D_t\nabla^{r-1}\mathrm{curl}u\right| + \left|\mathcal{T}\nabla^{r-1}\Delta p + \nabla^{r-1}D_t\mathrm{div}u\right| \\ &\qquad \leq C\sum_{s=0}^{r-1}\left(\left|\nabla^{1+s}u\right|\left|\nabla^{r-s}u\right| + \left|\nabla^{1+s}\mathcal{T}\right|\left|\nabla^{r-s}p\right|\right), \end{aligned} \tag{3.6}$$

$$|D_t\nabla^r q + (\nabla^r u)\cdot\nabla q - \nabla^r D_t q| \leq C\sum_{s=1}^{r-2}\left|(\nabla^{1+s}u)\cdot\nabla^{r-s}q\right|, \tag{3.7}$$

$$|\Pi(D_t\nabla^r q + (\nabla^r u)\cdot\nabla q - \nabla^r D_t q)| \leq C\sum_{s=1}^{r-2}\left|\Pi((\nabla^{1+s}u)\cdot\nabla^{r-s}q)\right|. \tag{3.8}$$

Proof. This lemma can be proved in a similar way to deriving Lemma 6.1 in [6], so we sketch the proof and omit the details. First, we can apply the following fact

$$[D_t, \partial_i] = -(\partial_i v^k)\partial_k \quad \text{and} \quad [D_t, \partial^r] = -\sum_{s=0}^{r-1}\begin{pmatrix} r \\ s+1 \end{pmatrix}(\partial^{1+s}v)\cdot\partial^{r-s}$$

to (1.1) and change coordinates to obtain

$$D_t\nabla^r u_a + \nabla^r\left(\mathcal{T}\nabla_a p\right) = (\nabla_a u_c - \nabla_c u_a)\nabla^r u^c - \sum_{s=1}^{r-2}\begin{pmatrix} r \\ s+1 \end{pmatrix}\left(\nabla^{1+s}u\right)\cdot\nabla^{r-s}u_a. \tag{3.9}$$

The estimate for curl$u$ can be shown similarly. For any $r \geq 0$, take ∇^r of (3.4) to get

$$\mathcal{T}\nabla^r\Delta p + \nabla^r D_t\mathrm{div}u = -\left(\nabla^r(\mathcal{T}\Delta p) - \mathcal{T}\nabla^r\Delta p\right) - \nabla^r\left((\nabla\mathcal{T})\cdot\nabla p + (\nabla_e u)\cdot\nabla u^e\right).$$

This proves (3.6). Clearly, (3.7) and (3.8) follow directly from (2.8). □

Lemma 3.2 *For $r \geq 0$ and $s = 0, 1, 2$, we have*

$$|\nabla^r D_t\mathcal{T}| + |D_t\nabla^r\mathcal{T}| \leq C|\mathcal{T}\nabla^r\mathrm{div}u| + C\sum_{s=0}^{r-1}|\nabla^{s+1}u||\nabla^{r-s}\mathcal{T}|, \tag{3.10}$$

$$|\nabla^r D_t^2\mathcal{T}| \leq C\sum_{s_1=0}^{r}|\nabla^{s_1}\mathcal{T}|\left(|\nabla^{r-s_1}D_t\mathrm{div}u| + \sum_{s_2=0}^{r-s_1}|\nabla^{r-s_1-s_2}\mathrm{div}u||\nabla^{s_2}\mathrm{div}u|\right), \tag{3.11}$$

$$|\nabla^s D_t^3\mathcal{T}| \leq C\sum_{i=0}^{s}|\nabla^i\mathcal{T}|H_{s-i}, \tag{3.12}$$

where

$$\begin{aligned} H_0 &= |D_t^2\mathrm{div}u| + (|D_t\mathrm{div}u| + |\nabla u|^2)|\nabla u|, \\ H_1 &= |\nabla D_t^2\mathrm{div}u| + |\nabla D_t\mathrm{div}u||\nabla u| + (|D_t\mathrm{div}u| + |\nabla u|^2)|\nabla\mathrm{div}u|, \\ H_2 &= |\nabla^2 D_t^2\mathrm{div}u| + |\nabla^2 D_t\mathrm{div}u||\nabla u| + |\nabla D_t\mathrm{div}u||\nabla\mathrm{div}u| \\ &\quad + |\nabla^2\mathrm{div}u|(|D_t\mathrm{div}u| + |\nabla u|^2) + |\nabla\mathrm{div}u|^2|\nabla u|. \end{aligned}$$

Proof. It follows from (3.1b) that

$$D_t\mathcal{T} = \mathcal{T}\text{div}u, \quad D_t^2\mathcal{T} = \mathcal{T}\left((\text{div}u)^2 + D_t\text{div}u\right),$$
$$D_t^3\mathcal{T} = \mathcal{T}\left((\text{div}u)^3 + 3(\text{div}u)D_t\text{div}u + D_t^2\text{div}u\right), \tag{3.13}$$

which, together with (3.7), proves (3.10)-(3.12). □

Lemma 3.3 *For any integer $r \geq 0$, we have*

$$|\nabla^r D_t\nabla u| \leq |\mathcal{T}||\nabla^{r+2}p| + C\sum_{s=0}^{r}\left(|\nabla^{s+1}\mathcal{T}||\nabla^{r+1-s}p| + |\nabla^{s+1}u||\nabla^{r+1-s}u|\right), \tag{3.14}$$

$$|D_t^2\nabla u| \leq C\sum_{r=1}^{2}|\nabla^{2-r}\mathcal{T}|\left(|\nabla^r D_t p| + \sum_{s=1}^{r}|\nabla^s p||\nabla^{r+1-s}u|\right) + C|\nabla u|^3, \tag{3.15}$$

$$|D_t^2\nabla^2 u| \leq C\sum_{r=1}^{3}|\nabla^{3-r}\mathcal{T}|\left(|\nabla^r D_t p| + \sum_{s=1}^{r}|\nabla^s p||\nabla^{r+1-s}u|\right) + C|\nabla^2 u||\nabla u|^2, \tag{3.16}$$

$$|D_t^3\nabla u| \leq C\{|\mathcal{T}|(|\nabla^2 D_t^2 p| + |\nabla^2 D_t p||\nabla u|) + \mathcal{T}^2|\nabla^3 p||\nabla p| + |\nabla\mathcal{T}||\nabla D_t^2 p| + \mathfrak{L}_1\}, \tag{3.17}$$

where

$$\mathfrak{L}_1 = |\nabla D_t p|(|\mathcal{T}\nabla^2 u| + |\nabla\mathcal{T}||\nabla u|) + (|\mathcal{T}\nabla^2 p| + |\nabla\mathcal{T}||\nabla p| + |\nabla u|^2)(|D_t\text{div}u|$$
$$+ |\mathcal{T}\nabla^2 p| + |\nabla\mathcal{T}||\nabla p| + |\nabla u|^2) + |\mathcal{T}\nabla p|(|\nabla D_t\text{div}u| + |\nabla^2\mathcal{T}||\nabla p| + |\nabla^2 u||\nabla u|). \tag{3.18}$$

Proof. Clearly, (3.14) follows from (3.2). It follows from (3.2) and (2.8) that

$$D_t^2\nabla_b u_a = -\mathcal{T}\left(\nabla_b\nabla_a D_t p - (\nabla_a\nabla_b u^e)\nabla_e p\right) - (D_t\mathcal{T})\nabla_b\nabla_a p$$
$$- (\nabla_b D_t\mathcal{T})\nabla_a p - (\nabla_b\mathcal{T})\nabla_a D_t p + D_t\left((\nabla_a u_e)\nabla_b u^e\right)$$

and

$$D_t^3\nabla_b u_a = -\mathcal{T}\left(\nabla_b\nabla_a D_t^2 p - (\nabla_a\nabla_b u^e)\nabla_e D_t p\right) + \mathcal{T}D_t\left((\nabla_a\nabla_b u^e)\nabla_e p\right)$$
$$- 2(D_t\mathcal{T})\left(\nabla_b\nabla_a D_t p - (\nabla_a\nabla_b u^e)\nabla_e p\right) - (D_t^2\mathcal{T})\nabla_b\nabla_a p - (\nabla_b D_t^2\mathcal{T})\nabla_a p$$
$$- 2(\nabla_b D_t\mathcal{T})\nabla_a D_t p - (\nabla_b\mathcal{T})\nabla_a D_t^2 p + D_t^2\left((\nabla_a u_e)\nabla_b u^e\right),$$

which, together with (2.5), (3.10), (3.11) and (3.6), imply (3.15) and (3.17). Choose $r = 2$ in (3.9) to get

$$D_t\nabla^2 u_a + \mathcal{T}\nabla^2\nabla_a p = -(\nabla^2\mathcal{T})\nabla_a p - 2(\nabla\mathcal{T})\widetilde{\otimes}\nabla\nabla_a p + (\nabla^2 u^e)\text{curl}u_{ae}. \tag{3.19}$$

Taking D_t of (3.19) and noticing (2.5), (3.3), (3.6), (3.7) and (3.10), we prove (3.16). □

Lemma 3.4 *We have*

$$|D_t^2\nabla^2\mathcal{T}| \leq C\{|\nabla^2\mathcal{T}|(|D_t\text{div}u| + |\nabla u|^2 + |\nabla\mathcal{T}||\nabla p|) + |\nabla\mathcal{T}|(|\nabla D_t\text{div}u| + |\nabla\mathcal{T}||\nabla^2 p|$$
$$+ |\nabla^2 u||\nabla u|) + \mathcal{T}(|\nabla^2 D_t\text{div}u| + |\nabla\mathcal{T}||\nabla^3 p| + |\nabla u||\nabla^3 u| + |\nabla\text{div}u||\nabla^2 u|)\}, \tag{3.20}$$

$$|D_t^3\nabla^2\mathcal{T}| \leq C(|\mathcal{T}\nabla^2 D_t^2\text{div}u| + \mathfrak{L}_2), \tag{3.21}$$

where

$$\mathfrak{L}_2 = \mathcal{T}\{|\nabla^2 D_t \mathrm{div} u||\nabla u| + (|\nabla D_t \mathrm{div} u| + |\nabla \mathrm{div} u||\nabla u|)|\nabla^2 u| + |\nabla^2 \mathrm{div} u|(|D_t \mathrm{div} u| \\ + |\nabla u|^2) + |\nabla \mathrm{div} u||\mathcal{T}\nabla^3 p|\} + |\nabla \mathcal{T}|\{|\nabla D_t^2 \mathrm{div} u| + |\nabla D_t \mathrm{div} u||\nabla u| + |\nabla^2 u|(|\nabla u|^2 \\ + |D_t \mathrm{div} u|)\} + |\nabla^2 \mathcal{T}|\{|D_t^2 \mathrm{div} u| + (|D_t \mathrm{div} u| + |\nabla u|^2)|\nabla u| + |\mathcal{T}\nabla \mathrm{div} u||\nabla p|\} \\ + |\nabla \mathcal{T}| \sum_{r=1}^{3} |\nabla^{3-r}\mathcal{T}| \left(|\nabla^r D_t p| + \sum_{s=1}^{r} |\nabla^s p||\nabla^{r+1-s} u| \right). \tag{3.22}$$

Proof. By repeat use of (2.6), we have for any given function q,

$$D_t^2 \nabla^2 q = -D_t((\nabla^2 u) \cdot \nabla q) - (\nabla^2 u) \cdot \nabla D_t q + \nabla^2 D_t^2 q,$$
$$D_t^3 \nabla^2 q = \nabla^2 D_t^3 q - (\nabla^2 u) \cdot \nabla D_t^2 q - D_t \left((\nabla^2 u) \cdot \nabla D_t q \right) - D_t^2 \left((\nabla^2 u) \cdot \nabla q \right),$$

which, together with (2.5), implies

$$|D_t^2 \nabla^2 q| \le |\nabla^2 D_t^2 q| + C(|\nabla^2 u||\nabla D_t q| + |D_t \nabla^2 u||\nabla q| + |\nabla^2 u||\nabla u||\nabla q|), \tag{3.23}$$
$$|D_t^3 \nabla^2 q| \le |\nabla^2 D_t^3 q| + C\{|\nabla^2 u||\nabla D_t^2 q| + |D_t \nabla^2 u|(|\nabla D_t q| + |\nabla u||\nabla q|) \\ + |D_t^2 \nabla^2 u||\nabla q| + (|D_t \nabla u| + |\nabla u|^2)|\nabla^2 u||\nabla q| + |\nabla^2 u||\nabla D_t q||\nabla u|\}.$$

With this fact, we can use (3.6), (3.10)-(3.12) and (3.16) to show (3.20) and (3.21).

Lemma 3.5 *For $r \ge 0$, we have*

$$|\nabla^r D_t \Delta u| \le |\nabla^{r+1} D_t \mathrm{div} u| + C \sum_{s=0}^{r+1} \left(|\nabla^{1+s} u||\nabla^{r+2-s} u| + |\nabla^{1+s}\mathcal{T}||\nabla^{r+2-s} p| \right), \tag{3.24}$$

$$|D_t^2 \Delta u| \le |\nabla D_t^2 \mathrm{div} u| + C \sum_{r=1}^{3} |\nabla^{3-r}\mathcal{T}| \sum_{s=1}^{r} |\nabla^s p||\nabla^{r+1-s} u| + C \sum_{r=1}^{2} |\nabla^{3-r}\mathcal{T}||\nabla^r D_t p| \\ + C|\nabla^2 u||\nabla u|^2, \tag{3.25}$$

$$|D_t^3 \Delta u| \le |\nabla D_t^3 \mathrm{div} u| + C(|\nabla u||\mathcal{T}\nabla^3 D_t p| + |\nabla \mathcal{T}||\nabla^2 D_t^2 p| + |\nabla^2 \mathcal{T}||\nabla D_t^2 p| + \mathfrak{L}_3), \tag{3.26}$$

where

$$\mathfrak{L}_3 = (|\mathcal{T}\nabla^2 p| + |\nabla \mathcal{T}||\nabla p| + |\nabla u|^2)(|\nabla \mathcal{T}||\nabla^2 p| + |\nabla^2 \mathcal{T}||\nabla p| + |\nabla^2 u||\nabla u|) \\ + |\nabla^2 p|(|\mathcal{T}\nabla D_t \mathrm{div} u| + |\nabla \mathcal{T}||D_t \mathrm{div} u|) + |\nabla p|\{|\mathcal{T}\nabla^2 D_t \mathrm{div} u| \\ + |\nabla \mathcal{T}|(|\mathcal{T}\nabla^3 p| + |\nabla D_t \mathrm{div} u|) + |\nabla^2 \mathcal{T}||D_t \mathrm{div} u|\} + |\nabla D_t p||\mathcal{T}\nabla^2 \mathrm{div} u| \\ + |\nabla^2 u| \sum_{r=1}^{2} |\nabla^{2-r}\mathcal{T}| \left(|\nabla^r D_t p| + \sum_{s=1}^{r} |\nabla^s p||\nabla^{r+1-s} u| \right) \\ + |\nabla u| \left\{ \sum_{r=1}^{3} |\nabla^{3-r}\mathcal{T}| \left(|\nabla^r D_t p| + \sum_{s=1}^{r} |\nabla^s p||\nabla^{r+1-s} u| \right) - |\mathcal{T}\nabla^3 D_t p| \right\}. \tag{3.27}$$

Proof. Take D_t^r $(r = 1, 2, 3)$ of (3.3) and use (3.10) and (3.11) to get

$$|D_t \mathrm{curl} u| \le C|\nabla \mathcal{T}||\nabla p|, \tag{3.28}$$
$$|D_t^2 \mathrm{curl} u| \le C|\nabla \mathcal{T}|(|\nabla D_t p| + |\nabla p||\nabla u|) + C\mathcal{T}|\nabla p||\nabla \mathrm{div} u|. \tag{3.29}$$

It follows from (3.3) and (2.6) that

$$D_t\nabla_c \text{curl}u_{ab} =(\nabla_c\nabla_b\mathcal{T})\nabla_a p+(\nabla_b\mathcal{T})\nabla_c\nabla_a p-(\nabla_c\nabla_a\mathcal{T})\nabla_b p-(\nabla_a\mathcal{T})\nabla_c\nabla_b p \\ -(\nabla_a\nabla_c u^e)\text{curl}u_{eb}-(\nabla_b\nabla_c u^e)\text{curl}u_{ae}, \tag{3.30}$$

which, together with (2.5), (3.10), (3.6), (3.7) and (3.28), implies

$$|D_t^2\nabla\text{curl}u|\le C|\mathcal{T}|(|\nabla^3 p||\nabla u|+|\nabla^2 p||\nabla^2 u|+|\nabla p||\nabla^2\text{div}u|)+C|\nabla\mathcal{T}|(|\nabla^2 D_t p| \\ +|\nabla^2 p||\nabla u|+|\nabla p||\nabla^2 u|)+C|\nabla^2\mathcal{T}|(|\nabla D_t p|+|\nabla p||\nabla u|)+C|\nabla^2 u||\nabla u|^2. \tag{3.31}$$

Take D_t^2 of (3.30) and use (2.5), (3.20), (3.23), (3.10), (3.11), (3.6), (3.7), (3.16), (3.28) and (3.29) to obtain

$$|D_t^3\nabla\text{curl}u|\le C\left(|\mathcal{T}||\nabla u||\nabla^3 D_t p|+|\nabla\mathcal{T}||\nabla^2 D_t^2 p|+|\nabla^2\mathcal{T}||\nabla D_t^2 p|+\mathfrak{L}_3\right), \tag{3.32}$$

where $\mathfrak{L}_3$ is given by (3.27). It follows from the definition of Δ that

$$\Delta u_a=g^{ce}\nabla_c\nabla_e u_a=\nabla_a\text{div}u+g^{ce}\nabla_c(\nabla_e u_a-\nabla_a u_e)=\nabla_a\text{div}u+g^{ce}\nabla_c\text{curl}u_{ea},$$

which, together with (2.5), implies

$$D_t\Delta u_a=\nabla_a D_t\text{div}u+(D_t g^{ce})\nabla_c\text{curl}u_{ea}+g^{ce}D_t\nabla_c\text{curl}u_{ea}$$
$$|D_t^2\Delta u|\le|\nabla D_t^2\text{div}u|+C(|D_t\nabla u|+|\nabla u|^2)|\nabla^2 u| \\ +C(|\nabla u||D_t\nabla\text{curl}u|+|D_t^2\nabla\text{curl}u|),$$
$$|D_t^3\Delta u|\le|\nabla D_t^3\text{div}u|+C(|D_t^2\nabla u|+|D_t\nabla u||\nabla u|+|\nabla u|^3)|\nabla^2 u| \\ +C((|D_t\nabla u|+|\nabla u|^2)|D_t\nabla\text{curl}u|+|\nabla u||D_t^2\nabla\text{curl}u|+|D_t^3\nabla\text{curl}u|).$$

With this, the lemma can be proved by noting (3.6), (3.15), (3.31) and (3.32). □

Lemma 3.6 *We have for $r\ge 0$,*

$$|\mathcal{T}\Delta D_t p+(\nabla\mathcal{T})\cdot\nabla D_t p|\le|D_t^2\text{div}u|+C\{|\mathcal{T}|(|\nabla^2 p||\nabla u|+|\nabla p||\nabla^2 u|) \\ +|\nabla\mathcal{T}||\nabla p||\nabla u|+|\nabla u|^3\}, \tag{3.33}$$

$$|\mathcal{T}\nabla^r\Delta D_t p|\le|\nabla^r D_t^2\text{div}u|+C\sum_{s_1=0}^{r+1}|\nabla^{s_1}\mathcal{T}|\sum_{s_2=0}^{r+1-s_1}|\nabla^{s_2+1}u||\nabla^{r+2-s_1-s_2}p| \\ +C\sum_{s=0}^{r}|\nabla^{s+1}\mathcal{T}||\nabla^{r+1-s}D_t p|+C\sum_{s_1+s_2+s_3=r}|\nabla^{1+s_1}u||\nabla^{1+s_2}u||\nabla^{1+s_3}u|, \tag{3.34}$$

$$|\mathcal{T}\Delta D_t^2 p+(\nabla\mathcal{T})\cdot\nabla D_t^2 p|\le|D_t^3\text{div}u|+C(|\nabla u||\mathcal{T}\nabla^2 D_t p|+\mathfrak{L}_1), \tag{3.35}$$

$$|\mathcal{T}\nabla^r\Delta D_t^2 p|\le|\nabla^r D_t^3\text{div}u|+C\sum_{i=1}^{r+1}|\nabla^i\mathcal{T}||\nabla^{r+2-i}D_t^2 p|+C\mathfrak{L}_4, \tag{3.36}$$

$$|\mathcal{T}\Delta D_t^3 p+(\nabla\mathcal{T})\cdot\nabla D_t^3 p|\le|D_t^4\text{div}u|+C(|\nabla u||\mathcal{T}\nabla^2 D_t^2 p|+|\nabla u|\mathfrak{L}_1+\mathfrak{L}_5), \tag{3.37}$$

where $\mathfrak{L}_1$ is defined by (3.18),

$$\mathfrak{L}_4=|\nabla^{r+1}(\mathcal{T}(\nabla u)\nabla D_t p)|+|\nabla^{r+1}(\mathcal{T}(\nabla p)D_t\text{div}u)|+|\nabla^{r+1}(\mathcal{T}(\nabla p)(\nabla\mathcal{T})\nabla p)| \\ +|\nabla^{r+1}(\mathcal{T}(\nabla p)(\nabla u)\nabla u)|+|\nabla^r(\mathcal{T}^2(\nabla^2 p)\nabla^2 p)|+|\nabla^r((\nabla u)(\nabla u)(\nabla u)\nabla u)|,$$

and

$$\mathfrak{L}_5 = |\nabla D_t^2 p|(|\mathcal{T}\nabla^2 u| + |\nabla\mathcal{T}||\nabla u|) + (|\mathcal{T}\nabla^2 D_t p| + |\nabla\mathcal{T}||\nabla D_t p| + |\nabla^2 u||\mathcal{T}\nabla p|)(|D_t \mathrm{div} u| + |\mathcal{T}\nabla^2 p| + |\nabla\mathcal{T}||\nabla p| + |\nabla u|^2) + |\mathcal{T}\nabla D_t p|(|\nabla D_t \mathrm{div} u| + |\nabla^2\mathcal{T}||\nabla p|) + |\mathcal{T}\nabla p||\nabla D_t^2 \mathrm{div} u| + (|\mathcal{T}\nabla^2 p| + |\nabla\mathcal{T}||\nabla p|)|D_t^2 \mathrm{div} u| + \mathcal{T}|\nabla p|^2|\nabla^2 \mathrm{div} u| + |\mathcal{T}\nabla p||\nabla u||\mathcal{T}\nabla^3 p|.$$

Proof. In view of (2.7) and (3.4), we see that

$$\begin{aligned} &\Delta D_t p = D_t \Delta p + 2(\nabla_e \nabla p)\cdot\nabla u^e + (\Delta u)\cdot\nabla p, \\ &\mathcal{T} D_t \Delta p = -D_t^2 \mathrm{div} u - (D_t\mathcal{T})\Delta p - D_t\left((\nabla\mathcal{T})\cdot\nabla p + (\nabla_e u)\cdot\nabla u^e\right), \end{aligned} \tag{3.38}$$

which means

$$\begin{aligned} \mathcal{T}\Delta D_t p = &- D_t^2 \mathrm{div} u - (D_t\mathcal{T})\Delta p - \{(\nabla D_t\mathcal{T})\cdot\nabla p + (\nabla\mathcal{T})\cdot\nabla D_t p \\ &+ (D_t g^{ab})(\nabla_a\mathcal{T})\nabla_b p\} - \{g^{ab}g^{ef}((D_t\nabla_e u_a)\nabla_b u_f + (\nabla_e u_a) D_t \nabla_b u_f) \\ &+ (D_t(g^{ab}g^{ef}))(\nabla_e u_a)\nabla_b u_f\} + 2\mathcal{T}(\nabla_e\nabla p)\cdot\nabla u^e + \mathcal{T}(\Delta u)\cdot\nabla p. \end{aligned} \tag{3.39}$$

Clearly, (3.33) can be derived directly from (3.39). (3.34) can be proved by taking ∇^r of (3.39) and using (3.14) and (3.10).

It follows from (2.7) that

$$\begin{aligned} \mathcal{T}\Delta D_t^2 p &= \mathcal{T} D_t \Delta D_t p + 2\mathcal{T}(\nabla_e\nabla D_t p)\cdot\nabla u^e + \mathcal{T}(\Delta u)\cdot\nabla D_t p \\ &= D_t(\mathcal{T}\Delta D_t p) - (D_t\mathcal{T})\Delta D_t p + 2\mathcal{T}(\nabla_e\nabla D_t p)\cdot\nabla u^e + \mathcal{T}(\Delta u)\cdot\nabla D_t p, \end{aligned} \tag{3.40}$$

which, together with (3.13), implies

$$\begin{aligned} |\mathcal{T}\Delta D_t^2 p + (\nabla\mathcal{T})\cdot\nabla D_t^2 p| \le &|D_t(\mathcal{T}\Delta D_t p) + (\nabla\mathcal{T})\cdot\nabla D_t^2 p| \\ &+ C|\mathcal{T}|(|\nabla^2 D_t p||\nabla u| + |\nabla D_t p||\nabla^2 u|). \end{aligned} \tag{3.41}$$

Take D_t of (3.39) and use (2.5), (3.7), (3.10), (3.11), (3.14), (3.15), (3.24) and (3.38) to get

$$|D_t(\mathcal{T}\Delta D_t p) + (\nabla\mathcal{T})\cdot\nabla D_t^2 p| \le |D_t^3 \mathrm{div} u| + C(|\mathcal{T}||\nabla u||\nabla^2 D_t p| + \mathfrak{L}_1),$$

where $\mathfrak{L}_1$ is given by (3.18). With (3.41), we then prove (3.35). (3.36) can be proved in a similar way to deriving (3.34).

(3.37) can be shown in a similar way to deriving (3.35). □

Lemma 3.7 *For $r \ge 0$ and $i = 0, 1, 2$, we have*

$$|\nabla^r D_t \mathrm{div} u - \mathcal{T}\nabla^r \Delta \mathrm{div} u| \le C\sum_{s=0}^{r+1} |\nabla^{s+1}\mathcal{T}||\nabla^{r-s+2} u| + C\sum_{s=0}^{r} |\nabla^s \mathrm{div} u||\nabla^{r-s} \mathrm{div} u| \tag{3.42}$$

and

$$\begin{aligned} |\nabla^i D_t^2 \mathrm{div} u - \mathcal{T}\nabla^i \Delta D_t \mathrm{div} u| \le &\, C\sum_{r=-1}^{i-1} |\mathcal{T}\nabla^{r+3} p||\nabla^{i+1-r}\mathcal{T}| \\ &+ C\sum_{r=1}^{i+2}\left(|\mathcal{T}\nabla^r \mathrm{div} u| + \sum_{s=0}^{r-1}|\nabla^{s+1} u||\nabla^{r-s}\mathcal{T}|\right)|\nabla^{i+3-r} u| \\ &+ C\sum_{r=0}^{i}\left(|\nabla^r D_t \mathrm{div} u||\nabla^{i+1-r} u| + |\nabla^{r+1} D_t \mathrm{div} u||\nabla^{i+1-r}\mathcal{T}|\right) \\ &+ C\sum_{r=-1}^{i}\sum_{s=0}^{r+1}\left(|\nabla^{1+s} u||\nabla^{r+2-s} u| + |\nabla^{1+s}\mathcal{T}||\nabla^{r+2-s} p|\right)|\nabla^{i-r+1}\mathcal{T}|. \end{aligned} \tag{3.43}$$

Proof. Clearly, (3.42) follows directly from (3.5). Take D_t of (3.5) and use (2.7) to get

$$D_t^2\text{div}u - \mathcal{T}\Delta D_t\text{div}u = (D_t\mathcal{T})\Delta\text{div}u - \mathcal{T}\{(\Delta u)\cdot\nabla\text{div}u + 2(\nabla_e\nabla\text{div}u)\cdot\nabla u^e\}$$
$$- D_t\{(\Delta u)\cdot\nabla\mathcal{T} + 2(\nabla_e\nabla\mathcal{T})\cdot\nabla u^e - (\text{div}u)^2 - 2(\nabla\mathcal{T})\cdot\nabla\text{div}u\}. \tag{3.44}$$

Taking ∇^i $(i = 1, 2)$ of (3.44) and noticing (2.5), (3.10), (3.6), (3.14) and (3.24), we can obtain (3.43). □

Lemma 3.8 *We have*

$$|D_t^3\text{div}u - \mathcal{T}\Delta D_t^2\text{div}u| \le C(|\nabla\mathcal{T}||\nabla D_t^2\text{div}u| + |\nabla u||D_t^2\text{div}u| + \mathfrak{L}_6), \tag{3.45}$$
$$|D_t^4\text{div}u - \mathcal{T}\Delta D_t^3\text{div}u| \le C(|\nabla\mathcal{T}||\nabla D_t^3\text{div}u| + |\nabla u||D_t^3\text{div}u| + \mathfrak{L}_{71} + \mathfrak{L}_{72}), \tag{3.46}$$

where

$$\begin{aligned}\mathfrak{L}_6 = {}& |\nabla\mathcal{T}|\left\{\sum_{r=1}^{3}|\nabla^{3-r}\mathcal{T}|\sum_{s=1}^{r}|\nabla^s p||\nabla^{r+1-s}u| + \sum_{r=1}^{2}|\nabla^{3-r}\mathcal{T}||\nabla^r D_t p|\right\}\\
&+ |\nabla^2\mathcal{T}|\left\{\sum_{r=1}^{2}|\nabla^{2-r}\mathcal{T}|\left(|\nabla^r D_t p| + \sum_{s=1}^{r}|\nabla^s p||\nabla^{r+1-s}u|\right) + |\nabla u|^3\right\}\\
&+ |\nabla u|\left\{\sum_{s=0}^{2}|\nabla^s\mathcal{T}||\nabla^{2-s}D_t\text{div}u| + |\mathcal{T}\nabla^3 u||\nabla u|\right\} + |D_t\text{div}u|^2\\
&+ |\nabla^2 u|\{|\mathcal{T}\nabla D_t\text{div}u| + |\nabla u||\mathcal{T}\nabla\text{div}u| + |\nabla\mathcal{T}|(|D_t\text{div}u| + |\nabla u|^2)\}\\
&+ |\mathcal{T}\nabla^2\text{div}u|(|\mathcal{T}\nabla^2 p| + |\nabla\mathcal{T}||\nabla p| + |\nabla u|^2 + |D_t\text{div}u|),\end{aligned} \tag{3.47}$$

$$\begin{aligned}\mathfrak{L}_{71} = {}& |D_t^3\nabla u||\nabla^2\mathcal{T}| + |D_t^3\Delta u||\nabla\mathcal{T}| + |D_t^3\nabla^2\mathcal{T}||\nabla u| + |D_t^2\Delta u|(|\nabla u||\nabla\mathcal{T}|\\
&+ |\mathcal{T}\nabla\text{div}u|) + |D_t^2\nabla u|(|\mathcal{T}\nabla^2\text{div}u| + |\nabla^2\mathcal{T}||\nabla u| + |\nabla\mathcal{T}||\nabla^2 u|)\\
&+ |D_t\nabla^2 u||\nabla u||\mathcal{T}\nabla\text{div}u| + |D_t\Delta u|(|D_t\nabla u||\nabla\mathcal{T}| + |\nabla D_t^2\mathcal{T}| + |\mathcal{T}\nabla D_t\text{div}u|\\
&+ |\nabla\mathcal{T}||\nabla u|^2 + |\mathcal{T}\nabla\text{div}u||\nabla u|) + (|D_t\nabla u| + |\nabla u|^2)\{|\mathcal{T}\nabla^2 D_t\text{div}u|\\
&+ |D_t^2\nabla^2\mathcal{T}| + |D_t\nabla^2\mathcal{T}||\nabla u| + (|D_t\nabla u| + |\nabla u|^2)|\nabla^2\mathcal{T}| + |\mathcal{T}\nabla^2\text{div}u||\nabla u|\\
&+ |\nabla^2 u|(|\mathcal{T}\nabla\text{div}u| + |\nabla\mathcal{T}||\nabla u|) + |\nabla D_t\text{div}u||\nabla\mathcal{T}|\}\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}\mathfrak{L}_{72} = {}& |\nabla D_t\text{div}u||\nabla D_t^2\mathcal{T}| + |\mathcal{T}\nabla^2 D_t^2\text{div}u||\nabla u| + |\nabla D_t^2\text{div}u|(|\mathcal{T}\nabla\text{div}u| + |\nabla\mathcal{T}||\nabla u|\\
&+ |\mathcal{T}\Delta u|) + (|\nabla D_t^3\mathcal{T}| + |\nabla D_t^2\mathcal{T}||\nabla u|)|\nabla\text{div}u| + |D_t^2\text{div}u||D_t\text{div}u|\\
&+ |D_t^3\mathcal{T}||\nabla^2\text{div}u| + (|\nabla^2 u||\nabla\text{div}u| + |\nabla u||\nabla^2\text{div}u| + |\nabla^2 D_t\text{div}u|)|\mathcal{T}D_t\text{div}u|\\
&+ |\nabla^2 u|(|\nabla u||\nabla\mathcal{T}||D_t\text{div}u| + |\nabla u||\mathcal{T}D_t\text{div}u|) + |\Delta u||\nabla D_t^3\mathcal{T}|.\end{aligned} \tag{3.49}$$

Proof. Take D_t^r $(r = 1, 2)$ of (3.44) and use (2.7) to get

$$\begin{aligned}&D_t^3\text{div}u - \mathcal{T}\Delta D_t^2\text{div}u\\
={}&(D_t\mathcal{T})\Delta D_t\text{div}u - \mathcal{T}((\Delta u)\cdot\nabla D_t\text{div}u + 2(\nabla_e\nabla D_t\text{div}u)\cdot\nabla u^e)\\
&+ D_t\{(D_t\mathcal{T})\Delta\text{div}u - \mathcal{T}((\Delta u)\cdot\nabla\text{div}u + 2(\nabla_e\nabla\text{div}u)\cdot\nabla u^e)\}\\
&- D_t^2\{(\Delta u)\cdot\nabla\mathcal{T} + 2(\nabla_e\nabla\mathcal{T})\cdot\nabla u^e - (\text{div}u)^2 - 2(\nabla\mathcal{T})\cdot\nabla\text{div}u\}\end{aligned} \tag{3.50}$$

and

$$
\begin{aligned}
&D_t^4\mathrm{div}u-\mathcal{T}\Delta D_t^3\mathrm{div}u\\
=&(D_t\mathcal{T})\Delta D_t^2\mathrm{div}u-\mathcal{T}((\Delta u)\cdot\nabla D_t^2\mathrm{div}u+2(\nabla_e\nabla D_t^2\mathrm{div}u)\cdot\nabla u^e)\\
&+D_t\{(D_t\mathcal{T})\Delta D_t\mathrm{div}u-\mathcal{T}((\Delta u)\cdot\nabla D_t\mathrm{div}u+2(\nabla_e\nabla D_t\mathrm{div}u)\cdot\nabla u^e)\}\\
&+D_t^2\{(D_t\mathcal{T})\Delta\mathrm{div}u-\mathcal{T}((\Delta u)\cdot\nabla\mathrm{div}u+2(\nabla_e\nabla\mathrm{div}u)\cdot\nabla u^e)\}\\
&-D_t^3\{(\Delta u)\cdot\nabla\mathcal{T}+2(\nabla_e\nabla\mathcal{T})\cdot\nabla u^e-(\mathrm{div}u)^2-2(\nabla\mathcal{T})\cdot\nabla\mathrm{div}u\}. \qquad (3.51)
\end{aligned}
$$

With these two equations, we can obtain (3.45) and (3.46) by simple calculations and noticing (2.5), (2.7), (3.7), (3.23), (3.10), (3.11), (3.14), (3.15), (3.20), (3.24) and (3.25). □

4 Proof of Theorem 1.5

Let

$$
\begin{aligned}
&E_0(t)=\int_\Omega\mathcal{T}^{-1}|u|^2d\mu_g \qquad (4.1a)\\
&E_r(t)=\int_\Omega\mathcal{T}^{-1}g^{ab}\zeta^{cd}Q(\nabla^{r-1}\nabla_c u_a,\nabla^{r-1}\nabla_d u_b)d\mu_g+\int_\Omega|\nabla^{r-1}\mathrm{curl}u|^2d\mu_g\\
&+\int_\Omega|\nabla D_t^{r-1}\mathrm{div}u|^2d\mu_g+\int_{\partial\Omega}\zeta^{cd}Q(\nabla^{r-1}\nabla_c p,\nabla^{r-1}\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta,\quad r\ge 1, \qquad (4.1b)
\end{aligned}
$$

where $Q(\alpha,\beta)=\varsigma^{IJ}\alpha_I\beta_J$. Suppose that the following *a priori* assumptions are true:

$$
\begin{aligned}
&\underline{V}\le\mathrm{Vol}\Omega(t)\le\overline{V} && \text{on }\ [0,T], && (4.2a)\\
&|\theta|+1/\iota_0\le K && \text{on }\ [0,T]\times\partial\Omega, && (4.2b)\\
&-\nabla_N p\ge\epsilon_b>0 && \text{on }\ [0,T]\times\partial\Omega, && (4.2c)\\
&\sum_{i=1}^{n-1}(|\nabla_N D_t^i p|+|\nabla_N D_t^i\mathrm{div}u|)+|\nabla^2 p|\le L && \text{on }\ [0,T]\times\partial\Omega, && (4.2d)\\
&|\nabla p|+|\nabla u|+|\nabla\mathcal{T}|+|\nabla\mathrm{div}u|\le M && \text{in }\ [0,T]\times\Omega, && (4.2e)\\
&|D_t p|+|D_t\mathrm{div}u|+|\nabla^2\mathcal{T}|\le\widetilde{M} && \text{in }\ [0,T]\times\Omega, && (4.2f)
\end{aligned}
$$

where $\mathrm{Vol}\Omega(t)=\int_\Omega d\mu_g$. Let ι_0 and ι_1 be as in Definitions 1.2 and 1.3. Then we have, due to Lemma 2.7 and (4.2b), that

$$
\iota_1^{-1}\le\max\{\epsilon_1^{-1}\||\theta|\|_{L^\infty},\ (2\iota_0)^{-1}\}\le\epsilon_1^{-1}K,
$$

which means

$$
\iota_1^{-1}(t)\le K_1\quad\text{on}\quad[0,T],\quad\text{where}\quad K_1=\epsilon_1^{-1}K. \qquad (4.3)
$$

Before stating the result, let us notice the boundary conditions and maximum principle, which are due to (1.3) and (3.1b), as follows:

$$
\begin{aligned}
&p=0,\quad\mathcal{T}=\mathcal{T}_b,\quad D_t\mathcal{T}=0\ \text{ and }\ \mathrm{div}u=0\ \text{ on }\ [0,T]\times\partial\Omega, \qquad (4.4)\\
&\underline{\mathcal{T}}\le\mathcal{T}\le\overline{\mathcal{T}}\ \text{ in }\ [0,T]\times\overline{\Omega}, \qquad (4.5)
\end{aligned}
$$

where

$$
\underline{\mathcal{T}}=\min\left\{\min_{y\in\Omega}\mathcal{T}(0,y),\ \mathcal{T}_b\right\}\quad\text{and}\quad\overline{\mathcal{T}}=\max\left\{\max_{y\in\Omega}\mathcal{T}(0,y),\ \mathcal{T}_b\right\}.
$$

Proposition 4.1 *Let $n = 2, 3$ and $1 \le r \le n + 2$. Then there are continuous functions $\mathcal{F}_r$ with $\mathcal{F}_r\big|_{t=0} = 1$, such that for any smooth solutions of (1.1)-(1.3) for $0 \le t \le T$ satisfying (4.2), we have*

$$E_0(t) \le E_0(0) + \left[\mathcal{F}_1\left(t, \overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right) - 1\right] \sum_{s=1}^{2} E_s(0), \tag{4.6a}$$

$$\sum_{s=1}^{2} E_s(t) \le \mathcal{F}_2\left(t, \overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right) \sum_{s=1}^{2} E_s(0), \tag{4.6b}$$

$$\sum_{s=1}^{r} E_s(t) \le \mathcal{F}_r\left(t, \overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_1(0), \cdots, E_{r-1}(0)\right) \sum_{s=1}^{r} E_s(0), \ r \ge 3. \tag{4.6c}$$

Proposition 4.2 *Let $\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}$ and M_0 be defined by (1.8), and $n = 2, 3$. Then there are continuous functions $\mathscr{T}_n$ such that if*

$$T \le \mathscr{T}_n\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0)\right), \tag{4.7}$$

then any smooth solutions of the free surface problem (1.1)-(1.3) for $0 \le t \le T$ satisfies

$$\sum_{s=0}^{n+2} E_s(t) \le 2 \sum_{s=0}^{n+2} E_s(0), \quad 0 \le t \le T, \tag{4.8}$$

$$2^{-1}\mathrm{Vol}\mathscr{D}_0 \le \mathrm{Vol}\Omega(t) \le 2\mathrm{Vol}\mathscr{D}_0, \quad 0 \le t \le T, \tag{4.9}$$

$$|||\theta(t, \cdot)|||_{L^\infty} + \iota_0^{-1}(t) \le 18K_0, \quad 0 \le t \le T, \tag{4.10}$$

$$-\nabla_N p(t, y) \ge 2^{-1}\underline{\epsilon_0} \quad \text{for} \quad y \in \partial\Omega, \quad 0 \le t \le T, \tag{4.11}$$

$$\|\nabla p(t, \cdot)\|_{L^\infty} + \|\nabla u(t, \cdot)\|_{L^\infty} + \|\nabla \mathcal{T}(t, \cdot)\|_{L^\infty} \le 2M_0, \quad 0 \le t \le T, \tag{4.12}$$

$$\begin{aligned} &\sum_{i=1}^{n-1}\left(|||\nabla D_t^i p(t, \cdot)|||_{L^\infty} + |||\nabla D_t^i \mathrm{div} u(t, \cdot)|||_{L^\infty}\right) + |||\nabla^2 p(t, \cdot)|||_{L^\infty} \\ &+ \|\nabla \mathrm{div} u(t, \cdot)\|_{L^\infty} + \|D_t p(t, \cdot)\|_{L^\infty} + \|D_t \mathrm{div} u(t, \cdot)\|_{L^\infty} + \|\nabla^2 \mathcal{T}(t, \cdot)\|_{L^\infty} \\ &\le C\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0)\right), \quad 0 \le t \le T. \end{aligned} \tag{4.13}$$

Clearly, Theorem 1.5 is a conclusion of this proposition. (Indeed, the compatibility condition $\mathcal{T}(0, y) = \mathcal{T}_b$ on $\partial\Omega$ implies that $\underline{\mathcal{T}} = \min_{y\in\Omega} \mathcal{T}(0, y)$ and $\overline{\mathcal{T}} = \max_{y\in\Omega} \mathcal{T}(0, y)$.)

4.1 Energy estimates

In the proof we make use of a fact, which follows from (2.5), that for a function $f = f(t, y)$,

$$\frac{d}{dt}\int_\Omega f d\mu_g = \int_\Omega (D_t f + f \mathrm{div} u)\, d\mu_g. \tag{4.14}$$

First, we deal with the temporal derivatives of $\int_\Omega |\nabla D_t^r \mathrm{div} u|^2 d\mu_g$, which can be bounded as follows:

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\nabla D_t^r \mathrm{div} u|^2 d\mu_g \le C_r(\cdots) &\left(\sum_{i=0}^{1} \|\nabla^i D_t^r \mathrm{div} u\|^2 + \sum_{i=0}^{2}\sum_{j=1}^{r-1} \|\nabla^i D_t^j \mathrm{div} u\|^2\right. \\ &\left. + \sum_{i=1}^{2} \|\nabla^i D_t^{r-1} p\|^2 + \sum_{i=1}^{3}\sum_{j=1}^{r-2} \|\nabla^i D_t^j p\|^2 + \sum_{i=1}^{r+1}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2\right) + \sigma(r)\|\nabla^{r+1} \mathrm{div} u\|^2\right) \end{aligned} \tag{4.15}$$

for $r = 1,2,3,4$, where $\sigma(r) = 1$ for $r = 1,2$, and $\sigma(r) = 0$ for $r = 3,4$. However, various quantities that the constants C_r in (4.15) depend are quite different for different values of r. Identifying clearly the quantities that the constants C_r depend will be important to closing the arguments.

Lemma 4.3 *We have*

$$\frac{d}{dt}\int_\Omega |\nabla \mathrm{div} u|^2 d\mu_g \leq C\left(\|\mathcal{T}^{-1}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}\right)\left(\|\nabla^2\mathcal{T}\|^2 + \|\nabla^2 u\|^2 + \|\nabla u\|^2\right), \quad (4.16)$$

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla D_t \mathrm{div} u|^2 d\mu_g \leq\; & C(\|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla^2\mathcal{T}\|_{L^\infty}, \\ & \|\nabla \mathrm{div} u\|_{L^\infty})\left(\sum_{i=0}^{1}\|\nabla^i D_t \mathrm{div} u\|^2 + \|\nabla^2 \mathrm{div} u\|^2 + \sum_{i=1}^{2}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2\right)\right),\end{aligned} \quad (4.17)$$

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla D_t^2 \mathrm{div} u|^2 d\mu_g \leq\; & C(K_1, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \\ & \|\nabla^2\mathcal{T}\|_{L^\infty}, \|\nabla \mathrm{div} u\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty})\Bigg(\sum_{i=0}^{1}\|\nabla^i D_t^2 \mathrm{div} u\|^2 + \sum_{i=0}^{2}\|\nabla^i D_t \mathrm{div} u\|^2 \\ & + \|\nabla^3 \mathrm{div} u\|^2 + \sum_{i=1}^{2}\|\nabla^i D_t p\|^2 + \sum_{i=1}^{3}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2\right)\Bigg),\end{aligned} \quad (4.18)$$

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla D_t^3 \mathrm{div} u|^2 d\mu_g \leq\; & C(K_1, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \\ & \|\nabla^2\mathcal{T}\|_{L^\infty}, \|\nabla \mathrm{div} u\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, \|\nabla D_t p\|_{L^\infty}, \|\nabla D_t^2 \mathrm{div} u\|, \|D_t^2 \mathrm{div} u\|, \\ & \|\nabla^2 D_t \mathrm{div} u\|, \|\nabla D_t \mathrm{div} u\|, \|\nabla^2 \mathrm{div} u\|)\Bigg(\sum_{i=0}^{1}\|\nabla^i D_t^3 \mathrm{div} u\|^2 + \sum_{i=0}^{2}\sum_{j=1}^{2}\|\nabla^i D_t^j \mathrm{div} u\|^2 \\ & + \sum_{i=1}^{2}\|\nabla^i D_t^2 p\|^2 + \sum_{i=1}^{3}\|\nabla^i D_t p\|^2 + \sum_{i=1}^{4}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2\right)\Bigg),\end{aligned} \quad (4.19)$$

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla D_t^4 \mathrm{div} u|^2 d\mu_g \leq\; & C(K_1, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \\ & \|\nabla^2\mathcal{T}\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, \|\nabla D_t p\|_{L^\infty}, \|\nabla^2 u\|_{L^\infty}, \|\nabla^2 p\|_{L^\infty}, \|\nabla^2 \mathrm{div} u\|_{L^\infty}, \\ & \|D_t^2 \mathrm{div} u\|_{L^\infty}, \|\nabla^2 D_t p\|_{L^\infty})\Bigg(\sum_{i=0}^{1}\|\nabla^i D_t^4 \mathrm{div} u\|^2 + \sum_{i=0}^{2}\sum_{j=1}^{3}\|\nabla^i D_t^j \mathrm{div} u\|^2 \\ & + \sum_{i=1}^{2}\|\nabla^i D_t^3 p\|^2 + \sum_{i=1}^{3}\sum_{j=1}^{2}\|\nabla^i D_t^j p\|^2 + \sum_{i=1}^{5}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2\right)\Bigg).\end{aligned} \quad (4.20)$$

Proof. Notice the following identity: for $r \geq 0$,

$$\begin{aligned}\frac{1}{2}D_t|\nabla D_t^r \mathrm{div} u|^2 + \mathcal{T}|\Delta D_t^r \mathrm{div} u|^2 =\; & (D_t^{r+1}\mathrm{div} u - \mathcal{T}\Delta D_t^r \mathrm{div} u)(-\Delta D_t^r \mathrm{div} u) \\ & + \mathrm{div}\left((D_t^{r+1}\mathrm{div} u)\nabla D_t^r \mathrm{div} u\right) + \frac{1}{2}(D_t g^{ab})(\nabla_a D_t^r \mathrm{div} u)\nabla_b D_t^r \mathrm{div} u.\end{aligned}$$

This, together with (4.14), (4.4) and (2.5), implies that, for $r \geq 0$,

$$\frac{d}{dt}\int_\Omega |\nabla D_t^r \text{div}u|^2 d\mu_g + \int_\Omega \mathcal{T}|\Delta D_t^r \text{div}u|^2 d\mu_g \leq 3\|\nabla u\|_{L^\infty}\int_\Omega |\nabla D_t^r \text{div}u|^2 d\mu_g + \|\mathcal{T}^{-1}\|_{L^\infty}\int_\Omega |D_t^{r+1}\text{div}u - \mathcal{T}\Delta D_t^r \text{div}u|^2 d\mu_g. \tag{4.21}$$

By virtue of (3.5), we have

$$\|D_t \text{div}u - \mathcal{T}\Delta \text{div}u\|^2 \leq C\left(\|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}\right)\left(\|\nabla^2 \mathcal{T}\|^2 + \|\nabla^2 u\|^2 + \|\nabla u\|^2\right), \tag{4.22}$$

which, together with (4.21), yields (4.16).

It follows from (3.43) that

$$\|D_t^2 \text{div}u - \mathcal{T}\Delta D_t \text{div}u\|^2 \leq C(\|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla^2 \mathcal{T}\|_{L^\infty}, \|\nabla \text{div}u\|_{L^\infty})\left(\sum_{i=0}^{1}\|\nabla^i D_t \text{div}u\|^2 + \sum_{i=1}^{2}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2\right) + \|\nabla^2 \text{div}u\|^2\right), \tag{4.23}$$

which, together with (4.21), gives (4.17).

It follows from (3.45), (2.14), Hölder's inequality and Young's inequality that

$$\|D_t^3 \text{div}u - \mathcal{T}\Delta D_t^2 \text{div}u\|^2 \leq C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla^2 \mathcal{T}\|_{L^\infty}, \|\nabla \text{div}u\|_{L^\infty}, \|D_t \text{div}u\|_{L^\infty})\left(\sum_{i=0}^{1}\|\nabla^i D_t^2 \text{div}u\|^2 + \sum_{i=0}^{2}\|\nabla^i D_t \text{div}u\|^2 + \sum_{i=1}^{2}\|\nabla^i D_t p\|^2 + \sum_{i=1}^{3}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2 + \|\nabla^i \text{div}u\|^2\right)\right), \tag{4.24}$$

which, together with (4.21), yields (4.18). Indeed, the following type of estimates have been used to derive (4.24).

$$\begin{aligned}
&\|(\nabla^2 p)\cdot \nabla^2 u\|^2 \leq \|\nabla^2 p\|_{L^4}^2 \|\nabla^2 u\|_{L^4}^2 \\
\leq & C(K_1)\|\nabla p\|_{L^\infty}\sum_{i=1}^{3}\|\nabla^i p\|\|\nabla u\|_{L^\infty}\sum_{i=1}^{3}\|\nabla^i u\| \\
\leq & C(K_1, \|\nabla p\|_{L^\infty}, \|\nabla u\|_{L^\infty})\sum_{i=1}^{3}\left(\|\nabla^i p\|^2 + \|\nabla^i u\|^2\right),
\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}
&\|(\nabla D_t \text{div}u)\cdot \nabla^2 u\|^2 \leq \|\nabla D_t \text{div}u\|_{L^4}^2 \|\nabla^2 u\|_{L^4}^2 \\
\leq & C(K_1)\|D_t \text{div}u\|_{L^\infty}\sum_{i=0}^{2}\|\nabla^i D_t \text{div}u\|\|\nabla u\|_{L^\infty}\sum_{i=1}^{3}\|\nabla^i u\| \\
\leq & C(K_1, \|D_t \text{div}u\|_{L^\infty}, \|\nabla u\|_{L^\infty})\sum_{i=1}^{3}\left(\|\nabla^{i-1} D_t \text{div}u\|^2 + \|\nabla^i u\|^2\right).
\end{aligned} \tag{4.26}$$

By virtue of (3.45), (2.18), (2.14), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
&\|D_t^4\mathrm{div}u-\mathcal{T}\Delta D_t^3\mathrm{div}u\|^2\le C(K_1,\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla p\|_{L^\infty},\|\nabla^2\mathcal{T}\|_{L^\infty},\\
&\quad\|\nabla\mathrm{div}u\|_{L^\infty},\|D_t\mathrm{div}u\|_{L^\infty},\|\nabla D_tp\|_{L^\infty},\|\nabla D_t^2\mathrm{div}u\|,\|D_t^2\mathrm{div}u\|,\|\nabla^2D_t\mathrm{div}u\|,\\
&\quad\|\nabla D_t\mathrm{div}u\|,\|\nabla^2\mathrm{div}u\|)\left(\sum_{i=0}^{1}\|\nabla^iD_t^3\mathrm{div}u\|^2+\sum_{i=0}^{2}\left(\|\nabla^iD_t^2\mathrm{div}u\|^2+\|\nabla^iD_t\mathrm{div}u\|^2\right)\right.\\
&\quad\left.+\sum_{i=1}^{2}\|\nabla^iD_t^2p\|^2+\sum_{i=1}^{3}\|\nabla^iD_tp\|^2+\sum_{i=1}^{4}\left(\|\nabla^iu\|^2+\|\nabla^ip\|^2\right)\right),
\end{aligned}\tag{4.27}$$

which, together with (4.21), gives (4.19). In addition to (4.25) and (4.26), the following type of estimates have been used to derive (4.27).

$$\|(\nabla D_t^2\mathrm{div}u)\cdot\nabla^2u\|\le\|\nabla D_t^2\mathrm{div}u\|\|\nabla^2u\|_{L^\infty}\le C(K_1)\|\nabla D_t^2\mathrm{div}u\|\sum_{i=2}^{4}\|\nabla^iu\|,\tag{4.28}$$

$$\|(\nabla^2\mathrm{div}u)D_t^2\mathrm{div}u\|\le\|\nabla^2\mathrm{div}u\|\|D_t^2\mathrm{div}u\|_{L^\infty}\le C(K_1)\|\nabla^2\mathrm{div}u\|\sum_{i=0}^{2}\|\nabla^iD_t^2\mathrm{div}u\|,\tag{4.29}$$

and

$$\begin{aligned}
&\|(\nabla^2p)\cdot\nabla^3u\|\le\|\nabla^2p\|_{L^6}\|\nabla^3u\|_{L^3}\\
\le&C(K_1)\|\nabla p\|_{L^\infty}^{2/3}\left(\sum_{i=1}^{4}\|\nabla^ip\|\right)^{1/3}\|\nabla u\|_{L^\infty}^{1/3}\left(\sum_{i=1}^{4}\|\nabla^iu\|\right)^{2/3}\\
\le&C(K_1,\|\nabla p\|_{L^\infty},\|\nabla u\|_{L^\infty})\sum_{i=1}^{4}\left(\|\nabla^ip\|+\|\nabla^iu\|\right).
\end{aligned}\tag{4.30}$$

Similarly, we can obtain (4.20). □

Lemma 4.4 *It holds that*

$$\frac{d}{dt}\int_\Omega\mathcal{T}^{-1}|u|^2d\mu_g\le\|p\|^2+\|\mathrm{div}u\|^2,\tag{4.31}$$

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\mathcal{T}^{-1}g^{ac}\zeta^{bd}(\nabla_bu_a)\nabla_du_cd\mu_g+\frac{d}{dt}\int_\Omega|\mathrm{curl}u|^2d\mu_g\\
&\quad\le C(K,\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})\left(\|\nabla u\|^2+\|\nabla\mathrm{div}u\|^2+\|\nabla p\|^2\right).
\end{aligned}\tag{4.32}$$

Proof. It follows from (3.1a), (3.1b) and (3.2) that

$$\frac{1}{2}D_t\left(\mathcal{T}^{-1}|u|^2\right)+\frac{1}{2}\mathcal{T}^{-1}|u|^2\mathrm{div}u=-\mathrm{div}(pu)+p\mathrm{div}u$$

and

$$\begin{aligned}
&\frac{1}{2}D_t\left(\mathcal{T}^{-1}g^{ac}\zeta^{bd}(\nabla_bu_a)\nabla_du_c\right)+\frac{1}{2}\mathcal{T}^{-1}g^{ac}\zeta^{bd}(\nabla_bu_a)(\nabla_du_c)\mathrm{div}u\\
=&-\mathrm{div}\left(\zeta^{bd}(\nabla_bp)\nabla_du\right)+\frac{1}{2}\mathcal{T}^{-1}\left(D_t\left(g^{ac}\zeta^{bd}\right)\right)(\nabla_bu_a)\nabla_du_c+(\nabla_du)\cdot(\nabla\zeta^{bd})\nabla_bp\\
&+\zeta^{bd}(\nabla_bp)\nabla_d\mathrm{div}u+\zeta^{bd}\mathcal{T}^{-1}\left((\nabla_du)\cdot(\nabla u_e)\nabla_bu^e-(\nabla_b\mathcal{T})(\nabla_du)\cdot\nabla p\right),
\end{aligned}$$

which, together with (4.14), (4.4), (2.5) and (2.12), imply that

$$\frac{d}{dt}\int_\Omega \mathcal{T}^{-1}|u|^2 d\mu_g = 2\int_\Omega p\text{div}u d\mu_g \le \int_\Omega (p^2+|\text{div}u|^2)d\mu_g \tag{4.33}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathcal{T}^{-1}g^{ac}\zeta^{bd}(\nabla_b u_a)\nabla_d u_c d\mu_g \\
\le& C(K,\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})\int_\Omega \left(|\nabla u|^2+(|\nabla u|+|\nabla \text{div}u|)|\nabla p|\right)d\mu_g \\
\le& C(K,\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})\int_\Omega \left(|\nabla u|^2+|\nabla \text{div}u|^2+|\nabla p|^2\right)d\mu_g.
\end{aligned} \tag{4.34}$$

It follows from (4.14), (2.5) and (3.3) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\text{curl}u|^2 d\mu_g \le& C(\|\nabla u\|_{L^\infty})\int_\Omega \left(|\text{curl}u||D_t\text{curl}u|+|\text{curl}u|^2\right)d\mu_g \\
\le& C(\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})\int_\Omega \left(|\nabla u|^2+|\nabla p|^2\right)d\mu_g.
\end{aligned}$$

This, together with (4.33) and (4.34), gives (4.31) and (4.32). □

Lemma 4.5 *For $r\ge 2$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c u_a)\nabla_J^{r-1}\nabla_d u_b d\mu_g + \frac{d}{dt}\int_\Omega |\nabla^{r-1}\text{curl}u|^2 d\mu_g \\
&+\frac{d}{dt}\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c p)(\nabla_J^{r-1}\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta \\
\le& C(K,\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty},|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty},|||\nabla_N D_t p|||_{L^\infty},|||\nabla u|||_{L^\infty}) \\
&\times\left(\|\nabla^r u\|^2+\|\nabla^r\text{div}u\|^2+\|D_t\nabla^r u+\mathcal{T}\nabla^{r+1}p\|^2+\|D_t\nabla^{r-1}\text{curl}u\|^2+\|\nabla^r p\|^2\right. \\
&\left.+|||\Pi\nabla^r p|||^2+|||\Pi(D_t\nabla^r p+(\nabla^r u)\cdot\nabla p)|||^2\right).
\end{aligned} \tag{4.35}$$

Proof. When $r\ge 1$, simple calculations give that

$$\begin{aligned}
&\frac{1}{2}D_t\left(\mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c u_a)\nabla_J^r\nabla_d u_b\right)+\frac{1}{2}\mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c u_a)(\nabla_J^r\nabla_d u_b)\text{div}u \\
=& -\text{div}\left(\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)\nabla_J^r\nabla_d u\right)+\frac{1}{2}\mathcal{T}^{-1}\left(D_t\left(g^{ab}\zeta^{cd}\zeta^{IJ}\right)\right)(\nabla_I^r\nabla_c u_a)\nabla_J^r\nabla_d u_b \\
&+\mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(D_t\nabla_I^r\nabla_c u_a+\mathcal{T}\nabla_I^r\nabla_c\nabla_a p)\nabla_J^r\nabla_d u_b \\
&+(\nabla_J^r\nabla_d u)\cdot\left(\nabla(\zeta^{cd}\zeta^{IJ})\right)(\nabla_I^r\nabla_c p)+\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)\nabla_J^r\nabla_d\text{div}u.
\end{aligned} \tag{4.36}$$

Due to (4.4), we have $\nabla_a p=\overline{\nabla}_a p+N_a\nabla_N p=N_a\nabla_N p$ on $\partial\Omega$ and

$$\begin{aligned}
&\int_\Omega \text{div}\left(\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)\nabla_J^r\nabla_d u\right)d\mu_g=\int_{\partial\Omega}N_a\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d u^a)d\mu_\zeta \\
=& -\int_{\partial\Omega}N_a(\nabla_N p)\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d u^a)(-\nabla_N p)^{-1}d\mu_\zeta \\
=& -\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(D_t\nabla_J^r\nabla_d p+(\nabla_J^r\nabla_d u^a)\nabla_a p)(-\nabla_N p)^{-1}d\mu_\zeta \\
&+\int_{\partial\Omega}\zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(D_t\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta.
\end{aligned} \tag{4.37}$$

Moreover, it follows from (2.11) that

$$
\begin{aligned}
&\int_{\partial\Omega} \zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(D_t\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta \\
=&\frac{1}{2}\frac{d}{dt}\int_{\partial\Omega} \zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta \\
&-\frac{1}{2}\int_{\partial\Omega}\Big(D_t(\zeta^{cd}\zeta^{IJ}(-\nabla_N p)^{-1})\Big)(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d p)d\mu_\zeta \\
&-\frac{1}{2}\int_{\partial\Omega} \zeta^{cd}\zeta^{IJ}(\nabla_I^r\nabla_c p)(\nabla_J^r\nabla_d p)(-\nabla_N p)^{-1}(\mathrm{div}u-h_{NN})d\mu_\zeta. \qquad (4.38)
\end{aligned}
$$

Notice that on $\partial\Omega$,

$$
\begin{aligned}
|D_t(-\nabla_N p)^{-1}| \leq&|(-\nabla_N p)^{-1}|^2|D_t\nabla_N p| \leq |(-\nabla_N p)^{-1}|^2(|(D_tN^c)\nabla_c p| + |\nabla_N D_t p|) \\
=&|(-\nabla_N p)^{-1}|^2(|\Big(-2h_d^cN^d + h_{NN}N^c\Big)\nabla_c p| + |\nabla_N D_t p|) \\
=&|(-\nabla_N p)^{-1}|^2(|h_{NN}\nabla_N p| + |\nabla_N D_t p|), \qquad (4.39)
\end{aligned}
$$

because of (2.9) and (4.4). We then obtain, with the help of (4.36)-(4.38), (4.14), (2.5), (2.10) and (2.12), that for $r\geq 2$,

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c u_a)\nabla_J^{r-1}\nabla_d u_b d\mu_g \\
&+\frac{d}{dt}\int_{\partial\Omega} \zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c p)(\nabla_J^{r-1}\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta \\
\leq&C(K,\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty})\int_\Omega\{|\nabla^r u|(|\nabla^r u| + |D_t\nabla^r u + \mathcal{T}\nabla^{r+1}p| + |\nabla^r p|) \\
&+|\nabla^r p||\nabla^r \mathrm{div}u|\}d\mu_g + C(|||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, |||\nabla u|||_{L^\infty}) \\
&\times\int_{\partial\Omega}(|\Pi\nabla^r p||\Pi(D_t\nabla^r p + (\nabla^r u)\cdot\nabla p)| + |\Pi\nabla^r p|^2)d\mu_\zeta. \qquad (4.40)
\end{aligned}
$$

It follows from (4.14) and (2.5) that for $r\geq 2$

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla^{r-1}\mathrm{curl}u|^2 d\mu_g \\
\leq&C(\|\nabla u\|_{L^\infty})\int_\Omega\left(|\nabla^{r-1}\mathrm{curl}u|^2 + |D_t\nabla^{r-1}\mathrm{curl}u||\nabla^{r-1}\mathrm{curl}u|\right)d\mu_g,
\end{aligned}
$$

which, together with (4.40), gives (4.35). □

Lemma 4.6 *For $r=2,3,4,5$, we have*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathcal{T}^{-1}g^{ab}\zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c u_a)\nabla_J^{r-1}\nabla_d u_b d\mu_g + \frac{d}{dt}\int_\Omega |\nabla^{r-1}\mathrm{curl}u|^2 d\mu_g \\
&+\frac{d}{dt}\int_{\partial\Omega} \zeta^{cd}\zeta^{IJ}(\nabla_I^{r-1}\nabla_c p)(\nabla_J^{r-1}\nabla_d p)(-\nabla_N p)^{-1}d\mu_\zeta \\
&\leq C(K,K_1,\|\mathcal{T}^{-1}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla p\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty}, \\
&|||\nabla_N D_t p|||_{L^\infty},|||\nabla u|||_{L^\infty},|||\nabla^2 p|||_{L^\infty})\left(\sum_{i=1}^r\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2 + \|\nabla^i\mathcal{T}\|^2\right) + \|\nabla^r\mathrm{div}u\|^2\right. \\
&\left.+|||\Pi\nabla^r p|||^2 + |||\Pi\nabla^r D_t p|||^2 + \sum_{i=2}^{r-1}|||\nabla^i u|||^2 + \sum_{i=3}^{r}|||\nabla^i p|||^2 + \sum_{i=2}^{r-3}|||\overline{\nabla}^i\theta|||^2\right). \qquad (4.41)
\end{aligned}
$$

Proof. It follows from (3.6) and Hölder's inequality that

$$
\begin{aligned}
&\|D_t\nabla^r u+\mathcal{T}\nabla^{r+1}p\|+\|D_t\nabla^{r-1}\mathrm{curl}u\|\le C(\|\nabla u\|_{L^\infty},\|\nabla p\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})\\
&\times\begin{cases}
\|\nabla^2u\|+\|\nabla^2p\|+\|\nabla^2\mathcal{T}\|, & r=2,\\
\|\nabla^3u\|+\|\nabla^3p\|+\|\nabla^3\mathcal{T}\|+\|\nabla^2u\|_{L^4}^2+\|\nabla^2p\|_{L^4}\|\nabla^2\mathcal{T}\|_{L^4}, & r=3,\\
\|\nabla^4u\|+\|\nabla^4p\|+\|\nabla^4\mathcal{T}\|+\|\nabla^2u\|_{L^6}\|\nabla^3u\|_{L^3} & \\
\quad+\|\nabla^2p\|_{L^6}\|\nabla^3\mathcal{T}\|_{L^3}+\|\nabla^3p\|_{L^3}\|\nabla^2\mathcal{T}\|_{L^6}, & r=4,\\
\|\nabla^5u\|+\|\nabla^5p\|+\|\nabla^5\mathcal{T}\|+\|\nabla^2u\|_{L^8}\|\nabla^4u\|_{L^{\frac{8}{3}}}+\|\nabla^3u\|_{L^4}^2 & \\
\quad+\|\nabla^2p\|_{L^8}\|\nabla^4\mathcal{T}\|_{L^{\frac{8}{3}}}+\|\nabla^3p\|_{L^4}\|\nabla^3\mathcal{T}\|_{L^4}+\|\nabla^4p\|_{L^{\frac{8}{3}}}\|\nabla^2\mathcal{T}\|_{L^8}, & r=5,
\end{cases}
\end{aligned}
$$

which, together with (2.14) and Young's inequality, implies for $r=2,3,4,5$,

$$
\begin{aligned}
&\|D_t\nabla^r u+\mathcal{T}\nabla^{r+1}p\|+\|D_t\nabla^{r-1}\mathrm{curl}u\|\\
\le&C(K_1,\|\nabla u\|_{L^\infty},\|\nabla p\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})\sum_{i=1}^r\left(\|\nabla^iu\|+\|\nabla^ip\|+\|\nabla^i\mathcal{T}\|\right).
\end{aligned}\tag{4.42}
$$

It follows from (3.8) and (4.4) that

$$
\begin{aligned}
&|||\Pi(D_t\nabla^rp+(\nabla^ru)\cdot\nabla p)|||\le|||\Pi\nabla^rD_tp|||\\
&+\begin{cases}
0, & r=2,\\
C|||\nabla^2u|||\,|||\nabla^2p|||_{L^\infty}, & r=3,\\
C(|||\nabla^3u|||\,|||\nabla^2p|||_{L^\infty}+|||\Pi((\nabla^2u)\cdot\nabla^3p)|||), & r=4,\\
C(|||\nabla^4u|||\,|||\nabla^2p|||_{L^\infty}+|||\Pi((\nabla^3u)\cdot\nabla^3p)|||+|||\Pi((\nabla^2u)\cdot\nabla^4p)|||), & r=5.
\end{cases}
\end{aligned}\tag{4.43}
$$

Notice that on $\partial\Omega$,

$$
\begin{aligned}
&|\Pi((\nabla^2u)\cdot\nabla^3p)|\le|\overline{\nabla}\nabla u||\overline{\nabla}\nabla^2p|,\\
&|\Pi((\nabla^3u)\cdot\nabla^3p)|\le|\overline{\nabla}^2\nabla u||\overline{\nabla}\nabla^2p|+CK|\overline{\nabla}\nabla u||\overline{\nabla}\nabla^2p|,\\
&|\Pi((\nabla^2u)\cdot\nabla^4p)|\le|\overline{\nabla}\nabla u||\overline{\nabla}^2\nabla^2p|+CK|\overline{\nabla}\nabla u||\overline{\nabla}\nabla^2p|.
\end{aligned}
$$

We then obtain, with the help of Hölder's inequality, (2.13) and Young's inequality, that

$$
\begin{aligned}
&|||\Pi((\nabla^2u)\cdot\nabla^3p)|||\le|||\overline{\nabla}\nabla u|||_{L^4}|||\overline{\nabla}\nabla^2p|||_{L^4}\\
\le&C|||\nabla u|||_{L^\infty}^{\frac12}|||\overline{\nabla}^2\nabla u|||^{\frac12}|||\nabla^2p|||_{L^\infty}^{\frac12}|||\overline{\nabla}^2\nabla^2p|||^{\frac12}\\
\le&C\left(|||\nabla u|||_{L^\infty},|||\nabla^2p|||_{L^\infty}\right)\left(|||\overline{\nabla}^2\nabla u|||+|||\overline{\nabla}^2\nabla^2p|||\right),
\end{aligned}\tag{4.44}
$$

$$
\begin{aligned}
&|||\Pi((\nabla^3u)\cdot\nabla^3p)|||+|||\Pi((\nabla^2u)\cdot\nabla^4p)|||\\
\le&|||\overline{\nabla}^2\nabla u|||_{L^3}|||\overline{\nabla}\nabla^2p|||_{L^6}+|||\overline{\nabla}\nabla u|||_{L^6}|||\overline{\nabla}^2\nabla^2p|||_{L^3}+CK|||\overline{\nabla}\nabla u|||_{L^4}|||\overline{\nabla}\nabla^2p|||_{L^4}\\
\le&C|||\nabla u|||_{L^\infty}^{\frac13}|||\overline{\nabla}^3\nabla u|||^{\frac23}|||\nabla^2p|||_{L^\infty}^{\frac23}|||\overline{\nabla}^3\nabla^2p|||^{\frac13}+CK|||\overline{\nabla}\nabla u|||_{L^4}|||\overline{\nabla}\nabla^2p|||_{L^4}\\
&+C|||\nabla u|||_{L^\infty}^{\frac23}|||\overline{\nabla}^3\nabla u|||^{\frac13}|||\nabla^2p|||_{L^\infty}^{\frac13}|||\overline{\nabla}^3\nabla^2p|||^{\frac23}\\
\le&C(K,|||\nabla u|||_{L^\infty},|||\nabla^2p|||_{L^\infty})\left(\sum_{i=2}^3|||\overline{\nabla}^i\nabla u|||+\sum_{i=2}^3|||\overline{\nabla}^i\nabla^2p|||\right).
\end{aligned}\tag{4.45}
$$

For a $(0,2)$ tensor α, we have on $\partial\Omega$,

$$|\overline{\nabla}^2\alpha| \le |\nabla^2\alpha| + CK|\nabla\alpha| \quad\text{and}\quad |\overline{\nabla}^3\alpha| \le |\nabla^3\alpha| + C\left(K|\nabla^2\alpha| + K^2|\nabla\alpha| + |\overline{\nabla}\theta||\overline{\nabla}\alpha|\right).$$

This, together with (4.43), (4.44) and (4.45), implies that

$$|||\Pi(D_t\nabla^r p + (\nabla^r u)\cdot\nabla p)||| \le |||\Pi\nabla^r D_t p|||$$
$$+ \begin{cases} C\left(|||\nabla u|||_{L^\infty}, |||\nabla^2 p|||_{L^\infty}\right)\left(\sum_{i=2}^{3}|||\nabla^i u||| + \sum_{i=3}^{4}|||\nabla^i p|||\right), & r=4, \\ C(K, |||\nabla u|||_{L^\infty}, |||\nabla^2 p|||_{L^\infty})\left(\sum_{i=2}^{4}|||\nabla^i u||| + \sum_{i=3}^{5}|||\nabla^i p||| + |||\overline{\nabla}^2\theta|||\right), & r=5, \end{cases}$$

which implies, using (4.43), that for $r = 2,3,4,5$,

$$|||\Pi(D_t\nabla^r p + (\nabla^r u)\cdot\nabla p)||| \le |||\Pi\nabla^r D_t p|||$$
$$+ C(K, |||\nabla u|||_{L^\infty}, |||\nabla^2 p|||_{L^\infty})\left(\sum_{i=2}^{r-1}|||\nabla^i u||| + \sum_{i=3}^{r}|||\nabla^i p||| + \sum_{i=2}^{r-3}|||\overline{\nabla}^i\theta|||\right). \tag{4.46}$$

So, (4.41) follows from (4.35), (4.42) and (4.46). □

4.2 Elliptic estimates

Before starting with the proof of (4.6), let us first see what a bound for the energy (4.1) implies.

Lemma 4.7 *We have*

$$\|D_t^{r-1}\mathrm{div}u\|^2 \le C(\mathrm{Vol}\Omega)\|\nabla D_t^{r-1}\mathrm{div}u\|^2 \le C(\mathrm{Vol}\Omega)E_r(t), \quad r\ge 1, \tag{4.47}$$

$$\|\mathcal{T}-\mathcal{T}_b\|^2 + \|\nabla\mathcal{T}\|^2 + \|\nabla^2\mathcal{T}\|^2 \le C(K,\mathrm{Vol}\Omega)E_1, \tag{4.48}$$

$$\|u\|^2 \le \|\mathcal{T}\|_{L^\infty}E_0, \quad \|\nabla u\|^2 \le C(\mathrm{Vol}\Omega, \|\mathcal{T}\|_{L^\infty})E_1, \tag{4.49}$$

$$|||u|||^2 \le C(K,\mathrm{Vol}\Omega,\|\mathcal{T}\|_{L^\infty})(E_0+E_1), \quad |||\nabla\mathcal{T}|||^2 \le C(K,\mathrm{Vol}\Omega)E_1. \tag{4.50}$$

Proof. (4.47) follows from (4.4) and (2.38). (4.48) follows from (4.4), (2.39), (3.1b) and (4.47). (4.49) follows from (2.31) and (4.47). (4.50) follows from (2.33), (4.49) and (4.48). □

Lemma 4.8 *We have*

$$\|\nabla^2 u\|^2 + |||\nabla u|||^2 \le C(\mathrm{Vol}\Omega, \|\mathcal{T}\|_{L^\infty})\left(E_1+E_2\right), \tag{4.51}$$

$$\|p\|^2 + \|\nabla p\|^2 \le C(\mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty})\left(E_1+E_2\right), \tag{4.52}$$

$$\begin{aligned}&\|\nabla^2\mathrm{div}u\|^2 + |||\nabla\mathrm{div}u|||^2 + \|\nabla^2 p\|^2 + |||\nabla_N p|||^2 \\ &\quad \le C(K,\mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty})\left(E_1+E_2\right),\end{aligned} \tag{4.53}$$

$$|||\Pi\nabla^r p|||^2 \le |||\nabla_N p|||_{L^\infty}E_r, \quad r\ge 2, \tag{4.54}$$

$$\begin{aligned}|||\nabla^2 p|||^2 \le& C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \\ & |||\nabla_N p|||_{L^\infty})\left(E_1+E_2\right),\end{aligned} \tag{4.55}$$

$$|||\theta|||^2 \le |||(\nabla_N p)^{-1}|||^2_{L^\infty}|||\nabla_N p|||_{L^\infty}E_2, \tag{4.56}$$

$$|||\nabla^2\mathcal{T}|||^2 \le C(K,\mathrm{Vol}\Omega, |||\nabla_N p|||_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N\mathcal{T}|||_{L^\infty})\left(E_1+E_2\right), \tag{4.57}$$

$$|||\Pi\nabla^2 D_t p|||^2 \le C(|||\nabla_N D_t p|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty})E_2. \tag{4.58}$$

Proof. The bound for $\|\nabla^2 u\|$ in (4.51) follows from (2.31). The bound for $|\|\nabla u\||$ in (4.51) follows from (2.33), (4.49) and the bound just obtained for $\|\nabla^2 u\|$. The bound for $\|\nabla^2 \mathrm{div} u\|$ in (4.53) follows from (4.22), (2.39), (4.51) and Lemma 4.7. The bound for $|\|\nabla \mathrm{div} u\||$ in (4.53) follows from (2.33) and the bound just obtained for $\|\nabla^2 \mathrm{div} u\|$. Let q be a function satisfying $q = 0$ on $\partial\Omega$, we have for any $\delta > 0$,

$$\begin{aligned}\int_\Omega \mathcal{T}|\nabla q|^2 d\mu_g =& \int_\Omega \mathrm{div}(\mathcal{T} q \nabla q) d\mu_g - \int_\Omega (\mathcal{T}\Delta q + (\nabla\mathcal{T})\cdot\nabla q) q d\mu_g \\ \leq& \frac{1}{2\delta}\int_\Omega |\mathcal{T}\Delta q + (\nabla\mathcal{T})\cdot\nabla q|^2 d\mu_g + \frac{\delta}{2}\int_\Omega q^2 d\mu_g.\end{aligned}$$

Due to (2.38), we can then choose a suitably small δ to obtain

$$\|\nabla q\|^2 \leq C(\mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty})\|\mathcal{T}\Delta q + (\nabla\mathcal{T})\cdot\nabla q\|^2. \tag{4.59}$$

It follows from (3.4) that

$$\|\mathcal{T}\Delta p + (\nabla\mathcal{T})\cdot\nabla p\| \leq \|D_t \mathrm{div} u\| + \|\nabla u\|_{L^\infty}\|\nabla u\|, \tag{4.60}$$

which, together with (4.59), (4.47) and (4.51), gives the bound for $\|\nabla p\|$ in (4.52). The bound for $\|p\|$ in (4.52) follows from (2.38) and the bound just obtained for $\|\nabla p\|$. The bound for $\|\nabla^2 p\|$ in (4.53) follows from (2.39), (4.60), (4.52) and Lemma 4.7. The bound for $|\|\nabla_N p\||$ in (4.53) follows from (2.33), (4.52) and the bound just obtained for $\|\nabla^2 p\|$. Clearly, (4.54) holds. Due to (3.6), (2.14), Hölder's inequality and Young's inequality, we have for $r = 1, 2, 3, 4$,

$$\begin{aligned}\|\mathcal{T}\nabla^r \Delta p\|^2 \leq& C\|\nabla^r D_t \mathrm{div} u\|^2 + C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}) \\ &\times \sum_{i=1}^{r+1}\left(\|\nabla^i u\|^2 + \|\nabla^i \mathcal{T}\|^2 + \|\nabla^i p\|^2\right).\end{aligned} \tag{4.61}$$

(4.55) follows from (2.40), (4.54), (4.61), (4.53), (4.48) and (4.51). (4.56) follows from (2.29) and (4.54). (4.57) follows from (2.40), (2.21), (3.1b), (4.56) and (4.47). (4.58) follows from (2.21) and (4.56). □

Lemma 4.9 *We have*

$$\|\nabla^3 u\|^2 + |\|\nabla^2 u\||^2 \leq C(K, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty})\sum_{i=1}^3 E_i, \tag{4.62}$$

$$\begin{aligned}&\|D_t p\|^2 + \|\nabla D_t p\|^2 + \|\nabla^2 D_t p\|^2 + |\|\nabla D_t p\||^2 \\ &\quad \leq C(K, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty})\sum_{i=1}^3 E_i,\end{aligned} \tag{4.63}$$

$$\begin{aligned}\|\nabla^3 p\|^2 \leq& C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \\ &\|\nabla p\|_{L^\infty}, |\|\nabla_N p\||_{L^\infty})\sum_{i=1}^3 E_i,\end{aligned} \tag{4.64}$$

$$\begin{aligned}|\|\overline{\nabla}\theta\||^2 \leq& C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \\ &\|\nabla p\|_{L^\infty}, |\|(\nabla_N p)^{-1}\||_{L^\infty}, |\|\nabla_N p\||_{L^\infty})\sum_{i=1}^3 E_i,\end{aligned} \tag{4.65}$$

$$
\begin{aligned}
&\|\nabla^3\mathcal{T}\|^2+|||\nabla^3\mathcal{T}|||^2+\|\nabla^2 D_t\mathrm{div}u\|^2+|||\nabla D_t\mathrm{div}u|||^2+|||\nabla^3 p|||^2\\
&\quad\le C(K,K_1,\mathrm{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla p\|_{L^\infty},\\
&\quad|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty},|||\nabla_N\mathcal{T}|||_{L^\infty})\sum_{i=1}^3 E_i,
\end{aligned}\tag{4.66}
$$

$$
\begin{aligned}
&|||\nabla^2\mathrm{div}u|||^2+|||\Pi\nabla^3\mathrm{div}u|||^2+\|\nabla^3\mathrm{div}u\|^2\\
&\quad\le C(K,K_1,\mathrm{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla p\|_{L^\infty},\\
&\quad|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty},|||\nabla_N\mathcal{T}|||_{L^\infty},|||\nabla_N\mathrm{div}u|||_{L^\infty})\sum_{i=1}^3 E_i,
\end{aligned}\tag{4.67}
$$

$$
\begin{aligned}
&|||\nabla^2 D_t p|||^2+|||\Pi\nabla^3 D_t p|||^2+\|\nabla^3 D_t p\|^2\\
&\quad\le C(K,K_1,\mathrm{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla p\|_{L^\infty},\|D_t p\|_{L^\infty},\\
&\quad|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty},|||\nabla_N\mathcal{T}|||_{L^\infty},|||\nabla_N D_t p|||_{L^\infty})\sum_{i=1}^3 E_i.
\end{aligned}\tag{4.68}
$$

Proof. The bound for $\|\nabla^3 u\|$ in (4.62) follows from (2.31) and (4.53). The bound for $|||\nabla^2 u|||$ in (4.62) follows from (2.33), (4.51) and the bound just obtained for $\|\nabla^3 u\|$. The bound for $\|\nabla D_t p\|$ in (4.63) follows from (3.33), (4.59), and Lemmas 4.7 and 4.8. The bound for $\|D_t p\|$ in (4.63) follows from (2.38) and the bound just obtained for $\|\nabla D_t p\|$. The bound for $\|\nabla^2 D_t p\|$ in (4.63) follows from (2.39), (3.33), and the bound just obtained for $\|\nabla D_t p\|$. The bound for $|||\nabla D_t p|||$ in (4.63) follows from (2.33), and the bounds just obtained for $\|\nabla D_t p\|$ and $\|\nabla^2 D_t p\|$. (4.64) follows from (2.41), (4.61) and Lemmas 4.7 and 4.8. (4.65) follows from (2.30) and Lemma 4.8. The bounds for $\|\nabla^3\mathcal{T}\|$ and $|||\nabla^3\mathcal{T}|||$ in (4.66) follow from (2.40), (2.22), (3.1b), (4.65) and Lemmas 4.7 and 4.8. It follows from (3.43), (2.14), Hölder's inequality and Young's inequality that

$$
\begin{aligned}
\|D_t^2\mathrm{div}u-\mathcal{T}\Delta D_t\mathrm{div}u\|^2&\le C(K_1,\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla p\|_{L^\infty})\\
&\quad\times\left(\sum_{i=0}^1\|\nabla^i D_t\mathrm{div}u\|^2+\sum_{i=1}^3\left(\|\nabla^i u\|^2+\|\nabla^i\mathcal{T}\|^2+\|\nabla^i p\|^2\right)\right).
\end{aligned}\tag{4.69}
$$

The bound for $\|\nabla^2 D_t\mathrm{div}u\|$ in (4.66) follows from (2.39), (4.69), Lemmas 4.7 and 4.8, and the bounds just obtained for $\|\nabla^3 u\|$, $\|\nabla^3\mathcal{T}\|$ and $\|\nabla^3 p\|$. The bound for $|||\nabla D_t\mathrm{div}u|||$ in (4.66) follows from (2.33) and the bound just obtained for $\|\nabla^2 D_t\mathrm{div}u\|$. The bound for $|||\nabla^3 p|||$ in (4.66) follows from (2.40), (4.61), Lemmas 4.7 and 4.8, and the bounds just obtained for $\|\nabla^2 D_t\mathrm{div}u\|$, $\|\nabla^3 u\|$, $\|\nabla^3\mathcal{T}\|$ and $\|\nabla^3 p\|$. It follows from (3.42), (2.14), Hölder's inequality and Young's inequality that for $r=1,2,3$,

$$
\|\mathcal{T}\nabla^r\Delta\mathrm{div}u\|^2\le C\|\nabla^r D_t\mathrm{div}u\|^2+C(K_1,\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})\sum_{i=1}^{r+2}\left(\|\nabla^i u\|^2+\|\nabla^i\mathcal{T}\|^2\right).\tag{4.70}
$$

The bound for $|||\nabla^2\mathrm{div}u|||$ in (4.67) follows from (2.40), (2.21), (4.70), Lemmas 4.7 and 4.8, and the bounds just obtained for $\|\nabla^3 u\|$ and $\|\nabla^3\mathcal{T}\|$. The bound for $|||\Pi\nabla^3\mathrm{div}u|||$ in (4.67) follows from (2.22), (4.65), (4.53), and the bound just obtained for $|||\nabla^2\mathrm{div}u|||$. The bound for $\|\nabla^3\mathrm{div}u\|$ in (4.67) follows from (2.41), (4.70), Lemmas 4.7 and 4.8, and the bounds just obtained for $|||\Pi\nabla^3\mathrm{div}u|||$, $\|\nabla^3 u\|$ and $\|\nabla^3\mathcal{T}\|$. It follows from (3.34), (2.14), Hölder's inequality and Young's inequality that for $r=1,2,3$,

$$\|\mathcal{T}\nabla^r\Delta D_tp\|^2 \le C\|\nabla^r D_t^2\mathrm{div}u\|^2 + C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty},$$
$$\|D_tp\|_{L^\infty})\left(\sum_{i=1}^{r+2}\left(\|\nabla^i u\|^2 + \|\nabla^i\mathcal{T}\|^2 + \|\nabla^i p\|^2\right) + \sum_{i=0}^{r+1}\|\nabla^i D_tp\|^2\right). \tag{4.71}$$

With (4.71), we can obtain (4.68) by use of a similar way to the derivation of (4.67). □

Lemma 4.10 *We have*

$$\sum_{i=0}^{2}\|\nabla^i D_t^2p\|^2 + |||\nabla D_t^2p|||^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty},$$
$$\|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_tp\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty}, |||\nabla_N p|||_{L^\infty})\sum_{i=1}^{4}E_i, \tag{4.72}$$

$$\|\nabla^4 p\|^2 + |||\overline{\nabla}^2\theta|||^2 + \|\nabla^4\mathcal{T}\|^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty},$$
$$\|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N\mathcal{T}|||_{L^\infty})\sum_{i=1}^{4}E_i, \tag{4.73}$$

$$\|\nabla^4 u\|^2 + |||\nabla^3 u|||^2 + |||\nabla^4\mathcal{T}|||^2 + |||\nabla^3\mathrm{div}u|||^2 + |||\Pi\nabla^4\mathrm{div}u|||^2 + \|\nabla^4\mathrm{div}u\|^2$$
$$\le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty},$$
$$|||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N\mathcal{T}|||_{L^\infty}, |||\nabla_N\mathrm{div}u|||_{L^\infty})\sum_{i=1}^{4}E_i, \tag{4.74}$$

$$|||\nabla^2 D_t\mathrm{div}u|||^2 + |||\Pi\nabla^3 D_t\mathrm{div}u|||^2 + \|\nabla^3 D_t\mathrm{div}u\|^2 + |||\nabla^4 p|||^2$$
$$\le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty},$$
$$|||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N\mathcal{T}|||_{L^\infty}, |||\nabla_N\mathrm{div}u|||_{L^\infty}, |||\nabla_N D_t\mathrm{div}u|||_{L^\infty})\sum_{i=1}^{4}E_i, \tag{4.75}$$

$$\|\nabla^2 D_t^2\mathrm{div}u\|^2 + |||\nabla D_t^2\mathrm{div}u|||^2 + |||\nabla^3 D_tp|||^2 + |||\Pi\nabla^4 D_tp|||^2 + \|\nabla^4 D_tp\|^2 \le C(K,$$
$$K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_tp\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty},$$
$$|||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N\mathcal{T}|||_{L^\infty}, |||\nabla_N\mathrm{div}u|||_{L^\infty}, |||\nabla_N D_tp|||_{L^\infty})\sum_{i=1}^{4}E_i, \tag{4.76}$$

$$|||\nabla^2 D_t^2p|||^2 + |||\Pi\nabla^3 D_t^2p|||^2 + \|\nabla^3 D_t^2p\|^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty},$$
$$\|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_tp\|_{L^\infty}, \|D_t^2p\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty},$$
$$|||\nabla_N p|||_{L^\infty}, |||\nabla_N\mathcal{T}|||_{L^\infty}, |||\nabla_N\mathrm{div}u|||_{L^\infty}, |||\nabla_N D_tp|||_{L^\infty}, |||\nabla_N D_t^2p|||_{L^\infty})\sum_{i=1}^{4}E_i, \tag{4.77}$$

$$|||\nabla^2 D_t^2\mathrm{div}u|||^2 + |||\Pi\nabla^3 D_t^2\mathrm{div}u|||^2 + \|\nabla^3 D_t^2\mathrm{div}u\|^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty},$$
$$\|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_tp\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty},$$
$$|||\nabla_N p|||_{L^\infty}, |||\nabla_N\mathcal{T}|||_{L^\infty}, |||\nabla_N\mathrm{div}u|||_{L^\infty}, |||\nabla_N D_t\mathrm{div}u|||_{L^\infty},$$
$$|||\nabla_N D_tp|||_{L^\infty}, |||\nabla_N D_t^2\mathrm{div}u|||_{L^\infty}, \sum_{i=1}^{4}E_i)\sum_{i=1}^{4}E_i. \tag{4.78}$$

Proof. It follows from (3.35), (2.14), Hölder's inequality and Young's inequality that

$$
\begin{aligned}
&\|\mathcal{T}\Delta D_t^2 p + (\nabla\mathcal{T})\cdot\nabla D_t^2 p\|^2\\
\le& C\|D_t^3\mathrm{div}u\|^2 + C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty})\\
&\times\left(\|\nabla D_t\mathrm{div}u\|^2 + \|\nabla^2\mathcal{T}\|^2 + \sum_{i=0}^{2}\|\nabla^i D_t p\|^2 + \sum_{i=1}^{3}\left(\|\nabla^i u\|^2 + \|\nabla^i p\|^2\right)\right).
\end{aligned}
\tag{4.79}
$$

With (4.79), we can obtain (4.72) by use of a similar way to the derivation of (4.63). In a similar way to deriving (4.62) and (4.64), we can obtain, respectively, the bounds for $\|\nabla^4 u\|$ and $|||\nabla^3 u|||$ in (4.74), and the bound for $\|\nabla^4 p\|$ in (4.73). The bound for $|||\overline{\nabla}^2\theta|||$ in (4.73) follows from (2.20), (2.28) and Lemmas 4.8 and 4.9. The bound for $\|\nabla^4\mathcal{T}\|$ in (4.73) follows from (2.41), (2.19), the bound just obtained for $|||\overline{\nabla}^2\theta|||$, and Lemmas 4.7-4.9. The bound for $|||\nabla^4\mathcal{T}|||$ in (4.74) follows from (2.40), (2.19), the bound just obtained for $|||\overline{\nabla}^2\theta|||$, and Lemmas 4.7-4.9. The bound for $|||\nabla^3\mathrm{div}u|||$ in (4.74) follows from (2.40), (4.70), Lemmas 4.7-4.9, and the bounds just obtained for $\|\nabla^4 u\|$ and $\|\nabla^4\mathcal{T}\|$. The bound for $|||\Pi\nabla^4\mathrm{div}u|||$ in (4.74) follows from (2.19), (4.65), (4.53), (4.67), and the bounds just obtained for $|||\overline{\nabla}^2\theta|||$ and $|||\nabla^3\mathrm{div}u|||$. The bound for $\|\nabla^4\mathrm{div}u\|$ in (4.74) follows from (2.41), (4.70), Lemmas 4.7-4.9, and the bounds just obtained for $|||\Pi\nabla^4\mathrm{div}u|||$, $\|\nabla^4 u\|$ and $\|\nabla^4\mathcal{T}\|$. It follows from (3.43), (2.14), Hölder's inequality and Young's inequality that for $r = 1, 2$,

$$
\begin{aligned}
&\|\mathcal{T}\nabla^r\Delta D_t\mathrm{div}u\|^2 \le C\|\nabla^r D_t^2\mathrm{div}u\|^2 + C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty},\\
&\quad\|D_t\mathrm{div}u\|_{L^\infty})\left(\sum_{i=1}^{r+3}\left(\|\nabla^i u\|^2 + \|\nabla^i\mathcal{T}\|^2 + \|\nabla^i p\|^2\right) + \sum_{i=0}^{r+1}\|\nabla^i D_t\mathrm{div}u\|^2\right).
\end{aligned}
\tag{4.80}
$$

With (4.80), we can obtain the bounds for $|||\nabla^2 D_t\mathrm{div}u|||$, $|||\Pi\nabla^3 D_t\mathrm{div}u|||^2$, and $\|\nabla^3 D_t\mathrm{div}u\|$ in (4.75) by use of a similar way to the derivation of (4.67). The bound for $|||\nabla^4 p|||$ in (4.75) follows from (2.40), (4.61), Lemmas 4.7-4.9, and the bounds just obtained for $\|\nabla^3 D_t\mathrm{div}u\|$, $\|\nabla^4 u\|$, $\|\nabla^4\mathcal{T}\|$ and $\|\nabla^4 p\|$. It follows from (3.45), (2.14), Hölder's inequality and Young's inequality that

$$
\begin{aligned}
&\|D_t^3\mathrm{div}u - \mathcal{T}\Delta D_t^2\mathrm{div}u\|^2 \le C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty},\\
&\quad\|D_t\mathrm{div}u\|_{L^\infty})\left(\sum_{i=0}^{1}\|\nabla^i D_t^2\mathrm{div}u\|^2 + \sum_{i=1}^{4}\left(\|\nabla^i u\|^2 + \|\nabla^i\mathcal{T}\|^2 + \|\nabla^i p\|^2\right)\right.\\
&\quad\left.+\sum_{i=0}^{2}\|\nabla^i D_t\mathrm{div}u\|^2 + \sum_{i=0}^{3}\|\nabla^i D_t p\|^2\right).
\end{aligned}
\tag{4.81}
$$

The bound for $\|\nabla^2 D_t^2\mathrm{div}u\|$ in (4.76) follows from (2.39), (4.81), Lemmas 4.7-4.9, and the bounds just obtained for $\|\nabla^4 u\|$, $\|\nabla^4 p\|$ and $\|\nabla^4\mathcal{T}\|$. The bound for $|||\nabla D_t^2\mathrm{div}u|||$ in (4.76) follows from (2.33), and the bound just obtained for $\|\nabla^2 D_t^2\mathrm{div}u\|$. In a similar way to deriving the bounds for $|||\nabla^3\mathrm{div}u|||$, $|||\Pi\nabla^4\mathrm{div}u|||$ and $\|\nabla^4\mathrm{div}u\|$ in (4.74), we can obtain the bounds for $|||\nabla^3 D_t p|||$, $|||\Pi\nabla^4 D_t p|||$ and $\|\nabla^4 D_t p\|$ in (4.76) by use of (4.71) and the bound just obtained for $\|\nabla^2 D_t^2\mathrm{div}u\|$. It follows from (3.36), (2.14), Hölder's inequality and Young's inequality that for $r = 1, 2$,

$$
\begin{aligned}
&\|\mathcal{T}\nabla^r\Delta D_t^2 p\|^2 \le C\|\nabla^r D_t^3\mathrm{div}u\|^2 + C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty},\\
&\quad\|D_t\mathrm{div}u\|_{L^\infty}, \|D_t p\|_{L^\infty}, \|D_t^2 p\|_{L^\infty})\left(\sum_{i=1}^{r+2}\left(\|\nabla^i u\|^2 + \|\nabla^i\mathcal{T}\|^2 + \|\nabla^i p\|^2\right)\right.\\
&\quad\left.+\sum_{i=0}^{r+2}\|\nabla^i D_t p\|^2 + \sum_{i=0}^{r+1}\left(\|\nabla^i D_t^2 p\|^2 + \|\nabla^i D_t\mathrm{div}u\|^2\right) + \|\nabla^{r+3}u\|^2 + \|\nabla^{r+3}p\|^2\right).
\end{aligned}
\tag{4.82}
$$

With (4.82), we can obtain (4.77) in a similar way to deriving (4.67). It follows from (3.50), (2.14), Hölder's inequality and Young's inequality that for $r = 1, 2$,

$$\begin{aligned}
&\|\mathcal{T}\nabla^r \Delta D_t^2 \mathrm{div} u\|^2 \le C\|\nabla^r D_t^3 \mathrm{div} u\|^2 + C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla^2 u\|_{L^\infty}, \\
&\|\nabla^2 \mathcal{T}\|_{L^\infty}, \|\nabla^2 p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, \|D_t^2 \mathrm{div} u\|_{L^\infty}, \|\nabla D_t p\|_{L^\infty}) \left(\sum_{i=0}^{r+1} \|\nabla^i D_t^2 \mathrm{div} u\|^2 \right. \\
&\left. + \sum_{i=0}^{r+2} \|\nabla^i D_t \mathrm{div} u\|^2 + \sum_{i=1}^{r+2} \|\nabla^i D_t p\|^2 + \sum_{i=1}^{r+3} \left(\|\nabla^i u\|^2 + \|\nabla^i \mathcal{T}\|^2 + \|\nabla^i p\|^2 \right) \right). \qquad (4.83)
\end{aligned}$$

Due to (2.18), we have

$$\begin{aligned}
&\|\nabla^2 u\|_{L^\infty}^2 + \|\nabla^2 \mathcal{T}\|_{L^\infty}^2 + \|\nabla^2 p\|_{L^\infty}^2 + \|D_t^2 \mathrm{div} u\|_{L^\infty}^2 + \|\nabla D_t p\|_{L^\infty}^2 \\
\le& C(K_1) \sum_{i=0}^{2} \left(\|\nabla^{2+i} u\|^2 + \|\nabla^{2+i} \mathcal{T}\|^2 + \|\nabla^{2+i} p\|^2 + \|\nabla^i D_t^2 \mathrm{div} u\|^2 + \|\nabla^{i+1} D_t p\|^2 \right) \\
\le& C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, \\
&|||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}) \sum_{i=1}^{4} E_i. \qquad (4.84)
\end{aligned}$$

With (4.83) and (4.84), we can obtain (4.78) in a similar way to deriving (4.67). □

Lemma 4.11 *We have*

$$\begin{aligned}
&\|\nabla^5 u\|^2 + |||\nabla^4 u|||^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \\
&\quad \|\nabla p\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}) \sum_{i=1}^{5} E_i,
\end{aligned} \qquad (4.85)$$

$$\begin{aligned}
&\|\nabla^5 p\|^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, \\
&\quad |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, |||\nabla_N D_t \mathrm{div} u|||_{L^\infty}) \sum_{i=1}^{5} E_i,
\end{aligned} \qquad (4.86)$$

$$\begin{aligned}
&\sum_{i=0}^{2} \|\nabla^i D_t^3 p\|^2 + |||\nabla D_t^3 p|||^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \\
&\quad \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, \|D_t^2 p\|_{L^\infty}, \|D_t^2 \mathrm{div} u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, \\
&\quad |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}) \sum_{i=1}^{5} E_i,
\end{aligned} \qquad (4.87)$$

$$\begin{aligned}
&|||\overline{\nabla}^3 \theta|||^2 + \|\nabla^5 \mathcal{T}\|^2 + |||\nabla^4 \mathrm{div} u|||^2 + |||\Pi \nabla^5 \mathrm{div} u|||^2 + \|\nabla^5 \mathrm{div} u\|^2 \le C(K, K_1, \mathrm{Vol}\Omega, \\
&\quad \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, \\
&\quad |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, |||\nabla_N D_t \mathrm{div} u|||_{L^\infty}, \sum_{i=1}^{4} E_i) \sum_{i=1}^{5} E_i,
\end{aligned} \qquad (4.88)$$

$$
\begin{aligned}
&|||\nabla^3 D_t \mathrm{div} u|||^2 + |||\Pi\nabla^4 D_t \mathrm{div} u|||^2 + \|\nabla^4 D_t \mathrm{div} u\|^2 + |||\nabla^5 p|||^2 \\
&\quad \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \\
&\quad \|D_t \mathrm{div} u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, \\
&\quad |||\nabla_N D_t \mathrm{div} u|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, \sum_{i=1}^4 E_i) \sum_{i=1}^5 E_i,
\end{aligned} \tag{4.89}
$$

$$
\begin{aligned}
&\|\nabla^2 D_t^3 \mathrm{div} u\|^2 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \\
&\quad \|D_t p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, \\
&\quad |||\nabla_N D_t p|||_{L^\infty}, \sum_{i=1}^4 E_i) \sum_{i=1}^5 E_i,
\end{aligned} \tag{4.90}
$$

$$
\begin{aligned}
&|||\nabla^4 D_t p|||^2 + |||\Pi\nabla^5 D_t p|||^2 + \|\nabla^5 D_t p\|^2 \\
&\quad \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \\
&\quad \|D_t \mathrm{div} u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, \\
&\quad |||\nabla_N D_t \mathrm{div} u|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, |||\nabla_N D_t^2 \mathrm{div} u|||_{L^\infty}, \sum_{i=1}^4 E_i) \sum_{i=1}^5 E_i,
\end{aligned} \tag{4.91}
$$

$$
\begin{aligned}
&|||\nabla^3 D_t^2 p|||^2 + |||\Pi\nabla^4 D_t^2 p|||^2 + \|\nabla^4 D_t^2 p\|^2 \\
&\quad \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \\
&\quad \|D_t^2 p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, \\
&\quad |||\nabla_N D_t \mathrm{div} u|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, |||\nabla_N D_t^2 p|||_{L^\infty}) \sum_{i=1}^5 E_i,
\end{aligned} \tag{4.92}
$$

$$
\begin{aligned}
&|||\nabla^3 D_t^2 \mathrm{div} u|||^2 + |||\Pi\nabla^4 D_t^2 \mathrm{div} u|||^2 + \|\nabla^4 D_t^2 \mathrm{div} u\|^2 \\
&\quad \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|D_t p\|_{L^\infty}, \\
&\quad \|D_t \mathrm{div} u\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty}, |||\nabla_N \mathrm{div} u|||_{L^\infty}, \\
&\quad |||\nabla_N D_t \mathrm{div} u|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, |||\nabla_N D_t^2 \mathrm{div} u|||_{L^\infty}, \sum_{i=1}^4 E_i) \sum_{i=1}^5 E_i.
\end{aligned} \tag{4.93}
$$

Proof. In a similar way to deriving (4.62) and (4.64), we can obtain, respectively, the bounds for $\|\nabla^5 u\|$ and $|||\nabla^4 u|||$ in (4.85), and the bound for $\|\nabla^5 p\|$ in (4.86). It follows from (3.37), (2.14), Hölder's inequality and Young's inequality that

$$
\begin{aligned}
&\|\mathcal{T}\Delta D_t^3 p + (\nabla \mathcal{T}) \cdot \nabla D_t^3 p\|^2 \le C\|D_t^4 \mathrm{div} u\|^2 + C(K_1, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \\
&\quad \|D_t p\|_{L^\infty}, \|D_t \mathrm{div} u\|_{L^\infty}, \|D_t^2 p\|_{L^\infty}, \|D_t^2 \mathrm{div} u\|_{L^\infty})(\|\nabla D_t^2 \mathrm{div} u\|^2 + \sum_{i=0}^2 (\|\nabla^i D_t \mathrm{div} u\|^2 \\
&\quad + \|\nabla^i D_t^2 p\|^2) + \sum_{i=0}^3 \|\nabla^i D_t p\|^2 + \sum_{i=1}^3 \|\nabla^i \mathcal{T}\|^2 + \sum_{i=1}^4 (\|\nabla^i u\|^2 + \|\nabla^i p\|^2)).
\end{aligned} \tag{4.94}
$$

With (4.94), we can obtain (4.87) by use of a similar way to the derivation of (4.63). The bound for $|||\overline{\nabla}^3 \theta|||$ in (4.88) follows from (2.25), (2.28) and Lemmas 4.8-4.10. The bound for $|||\nabla^4 \mathrm{div} u|||$ in (4.88) follows from (2.40), (4.70), Lemmas 4.7-4.10, and the bounds just obtained for $\|\nabla^5 u\|$ and

$\|\nabla^5\mathcal{T}\|$. The bound for $|||\Pi\nabla^5\text{div}u|||$ in (4.88) follows from (2.24), Lemmas 4.8-4.10, and the bounds just obtained for $|||\overline{\nabla}^3\theta|||$ and $|||\nabla^4\text{div}u|||$. The bound for $\|\nabla^5\text{div}u\|$ in (4.88) follows from (2.41), (4.70), Lemmas 4.7-4.10, and the bounds just obtained for $|||\Pi\nabla^5\text{div}u|||$, $\|\nabla^5 u\|$ and $\|\nabla^5\mathcal{T}\|$. In a similar way to deriving the bounds for $|||\nabla^3\text{div}u|||$, $|||\Pi\nabla^4\text{div}u|||$ and $\|\nabla^4\text{div}u\|$ in (4.74), we can obtain the bounds for $|||\nabla^3 D_t\text{div}u|||$, $|||\Pi\nabla^4 D_t\text{div}u|||$ and $\|\nabla^4 D_t\text{div}u\|$ in (4.89) by use of (4.80). The bound for $|||\nabla^5 p|||$ in (4.89) follows from (2.40), (4.61), Lemmas 4.7-4.10, and the bounds just obtained for $\|\nabla^4 D_t\text{div}u\|$, $\|\nabla^5 u\|$, $\|\nabla^5\mathcal{T}\|$ and $\|\nabla^5 p\|$. It follows from (4.27), (2.18), and Lemmas 4.7-4.10 that

$$\begin{aligned}&\|D_t^4\text{div}u-\mathcal{T}\Delta D_t^3\text{div}u\|^2\le C(K,K_1,\text{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\\&\quad\|\nabla p\|_{L^\infty},\|D_t p\|_{L^\infty},\|D_t\text{div}u\|_{L^\infty},|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty},|||\nabla_N\mathcal{T}|||_{L^\infty},\\&\quad|||\nabla_N\text{div}u|||_{L^\infty},|||\nabla_N D_t p|||_{L^\infty},\sum_{i=1}^4 E_i)\sum_{i=1}^4 E_i,\end{aligned}\tag{4.95}$$

which, together with (2.39) and (4.47), gives (4.90). In a similar way to deriving the bounds for $|||\nabla^4\text{div}u|||$, $|||\Pi\nabla^5\text{div}u|||$ and $\|\nabla^5\text{div}u\|$ in (4.88), we can obtain (4.91) by use of (4.71). In a similar way to deriving the bounds for $|||\nabla^3\text{div}u|||$, $|||\Pi\nabla^4\text{div}u|||$ and $\|\nabla^4\text{div}u\|$ in (4.74), we can obtain, respectively, (4.92) and (4.93) by use of (4.82) and (4.83). □

4.3 Proof of Proposition 4.1

After having seen in Section 4.2 what a bound for the energy implies, we can now prove (4.6).

It follows from (4.31), and Lemmas 4.7 and 4.8 that

$$\frac{d}{dt}E_0\le C(\text{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty})(E_1+E_2).\tag{4.96}$$

It follows from (4.16), (4.32), and Lemmas 4.7 and 4.8 that

$$\frac{d}{dt}E_1\le C(K,\text{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty})(E_1+E_2).\tag{4.97}$$

It follows from (4.17), (4.41), and Lemmas 4.7 and 4.8 that

$$\begin{aligned}&\frac{d}{dt}E_2\le C(K,K_1,\text{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla p\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\|\nabla\text{div}u\|_{L^\infty},\\&\|\nabla^2\mathcal{T}\|_{L^\infty},|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty},|||\nabla_N D_t p|||_{L^\infty},|||\nabla u|||_{L^\infty},|||\nabla^2 p|||_{L^\infty})(E_1+E_2).\end{aligned}\tag{4.98}$$

It follows from (4.18), (4.41), and Lemmas 4.7-4.9 that

$$\begin{aligned}&\frac{d}{dt}E_3\le C(K,K_1,\text{Vol}\Omega,\|\mathcal{T}^{-1}\|_{L^\infty},\|\mathcal{T}\|_{L^\infty},\|\nabla u\|_{L^\infty},\|\nabla p\|_{L^\infty},\|\nabla\mathcal{T}\|_{L^\infty},\\&\|\nabla\text{div}u\|_{L^\infty},\|\nabla^2\mathcal{T}\|_{L^\infty},\|D_t\text{div}u\|_{L^\infty},\|D_t p\|_{L^\infty},|||(\nabla_N p)^{-1}|||_{L^\infty},|||\nabla_N p|||_{L^\infty},\\&|||\nabla_N\mathcal{T}|||_{L^\infty},|||\nabla_N\text{div}u|||_{L^\infty},|||\nabla_N D_t p|||_{L^\infty},|||\nabla u|||_{L^\infty},|||\nabla^2 p|||_{L^\infty})\sum_{i=1}^3 E_i.\end{aligned}\tag{4.99}$$

It follows from (2.18) that

$$\|\nabla D_t p\|_{L^\infty}\le C(K_1)\sum_{i=1}^3\|\nabla^i D_t p\|,$$

which, together with (4.19), (4.41), and Lemmas 4.7-4.10, implies that

$$\begin{aligned}&\frac{d}{dt}E_4 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla \mathrm{div} u\|_{L^\infty},\\ &\|\nabla^2\mathcal{T}\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty}, \|D_t p\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty},\\ &|||\nabla_N \mathrm{div}u|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, |||\nabla_N D_t \mathrm{div}u|||_{L^\infty}, |||\nabla u|||_{L^\infty}, |||\nabla^2 p|||_{L^\infty}, \sum_{i=1}^{3} E_i) \sum_{i=1}^{4} E_i.\end{aligned} \tag{4.100}$$

It follows from (2.18) that

$$\begin{aligned}&\|D_t^2 p\|_{L^\infty} + \|D_t^2 \mathrm{div}u\|_{L^\infty} + \|\nabla D_t p\|_{L^\infty} + \|\nabla^2 D_t p\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty} + \|\nabla^2 p\|_{L^\infty}\\ &\quad+\|\nabla^2 \mathrm{div}u\|_{L^\infty} \le C(K_1)\sum_{j=0}^{2}(\|\nabla^j D_t^2 p\| + \|\nabla^j D_t^2 \mathrm{div}u\| + \|\nabla^{1+j} D_t p\|\\ &\qquad+\|\nabla^{2+j} D_t p\| + \|\nabla^{2+j} u\| + \|\nabla^{2+j} p\| + \|\nabla^{2+j}\mathrm{div}u\|),\end{aligned}$$

which, together with (4.20), (4.41), and Lemmas 4.7-4.11, implies that

$$\begin{aligned}&\frac{d}{dt}E_5 \le C(K, K_1, \mathrm{Vol}\Omega, \|\mathcal{T}^{-1}\|_{L^\infty}, \|\mathcal{T}\|_{L^\infty}, \|\nabla u\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla \mathcal{T}\|_{L^\infty}, \|\nabla \mathrm{div} u\|_{L^\infty},\\ &\|\nabla^2\mathcal{T}\|_{L^\infty}, \|D_t\mathrm{div}u\|_{L^\infty}, \|D_t p\|_{L^\infty}, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, |||\nabla_N \mathcal{T}|||_{L^\infty},\\ &|||\nabla_N \mathrm{div}u|||_{L^\infty}, |||\nabla_N D_t p|||_{L^\infty}, |||\nabla_N D_t \mathrm{div}u|||_{L^\infty}, |||\nabla_N D_t^2 p|||_{L^\infty}, |||\nabla_N D_t^2 \mathrm{div}u|||_{L^\infty},\\ &|||\nabla u|||_{L^\infty}, |||\nabla^2 p|||_{L^\infty}, \sum_{i=1}^{4} E_i) \sum_{i=1}^{5} E_i.\end{aligned} \tag{4.101}$$

It is produced by substituting (4.2), (4.3), (4.5) into (4.96)-(4.101) that there are continuous functions C_r $(0 \le r \le n+2)$ such that

$$\begin{aligned}&\frac{d}{dt}E_0(t) \le C_0\left(\overline{V}, M, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right)(E_1(t) + E_2(t)),\\ &\frac{d}{dt}E_1(t) \le C_1\left(\overline{V}, K, M, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right)(E_1(t) + E_2(t)),\\ &\frac{d}{dt}E_2(t) \le C_2\left(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}\right)(E_1(t) + E_2(t)),\\ &\frac{d}{dt}E_r(t) \le C_r\left(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, \sum_{s=1}^{r-1} E_s(t)\right)\sum_{s=1}^{r} E_s(t), \quad 3 \le r \le n+2.\end{aligned}$$

This concludes the proof of Proposition 4.1.

4.4 Proof of Proposition 4.2

Let us now show how Proposition 4.2 follows.

Lemma 4.12 *Let $n = 2, 3$. Then there are continuous functions $T_{n1} > 0$ such that*

$$\begin{aligned}&\sum_{s=0}^{r} E_s(t) \le 2\sum_{s=0}^{r} E_s(0), \quad 2 \le r \le n+2, \qquad 2^{-1}\mathrm{Vol}\mathscr{D}_0 \le \mathrm{Vol}\Omega(t) \le 2\mathrm{Vol}\mathscr{D}_0,\\ &|||\theta(t,\cdot)|||_{L^\infty} + \iota_0^{-1}(t) \le 18K_0, \qquad -\nabla_N p(t,y) \ge 2^{-1}\epsilon_0 \ \text{ for } \ y \in \partial\Omega,\\ &\|\nabla p(t,\cdot)\|_{L^\infty} + \|\nabla u(t,\cdot)\|_{L^\infty} + \|\nabla \mathcal{T}(t,\cdot)\|_{L^\infty} \le 2M_0,\end{aligned} \tag{4.102}$$

for $t \le T_{n1}(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0))$.

Proof. The cases $n = 2$ and $n = 3$ can be shown in the same manner, so we present here only the proof of the case of $n = 2$. Let $n = 2$ in the rest of this proof.

It follows from (4.14) that

$$\left|\frac{d}{dt}\mathrm{Vol}\Omega(t)\right| = \left|\frac{d}{dt}\int_\Omega d\mu_g\right| = \left|\int_\Omega \mathrm{div}u d\mu_g\right| \leq M\mathrm{Vol}\Omega(t),$$

which implies $\mathrm{Vol}\Omega(0)\exp\{-Mt\} \leq \mathrm{Vol}\Omega(t) \leq \mathrm{Vol}\Omega(0)\exp\{Mt\}$. Thus we have, due to the fact $\mathrm{Vol}\Omega(0) = \mathrm{Vol}\mathscr{D}_0$, that for $t \leq M^{-1}\ln 2$,

$$2^{-1}\mathrm{Vol}\mathscr{D}_0 \leq \mathrm{Vol}\Omega(t) \leq 2\mathrm{Vol}\mathscr{D}_0. \tag{4.103}$$

It follows from (4.39), (2.9), (4.2c), (4.2d) and (4.2e) that

$$|D_t(-\nabla_N p)^{-1}| \leq |(-\nabla_N p)^{-1}|^2(|h_{NN}\nabla_N p| + |\nabla_N D_t p|) \leq \epsilon_b^{-2}(M^2 + L) \quad \text{on} \quad \partial\Omega,$$

which, together with $\partial_{\mathcal{N}} p = \nabla_N p$ and (1.8a), implies that for $t \leq \epsilon_b^2(M^2 + L)^{-1}\underline{\epsilon_0}^{-1}$,

$$-\nabla_N p(t, y) \geq 2^{-1}\underline{\epsilon_0} \quad \text{for} \quad y \in \partial\Omega. \tag{4.104}$$

Let $\epsilon_1 \in (0, 1/2]$ be a fixed constant (for example, $\epsilon_1 = 1/4$), $\iota_1(0)$ the largest number such that

$$\begin{aligned}&|\mathcal{N}(x(0, y_1)) - \mathcal{N}(x(0, y_2))| \leq \epsilon_1/2,\\ &\qquad \text{whenever} \quad |x(0, y_1) - x(0, y_2)| \leq 2\iota_1(0), \quad y_1, y_2 \in \partial\Omega.\end{aligned} \tag{4.105}$$

Thus we have from Lemma 2.7 and (1.8a) that

$$\iota_1(0) \geq 2^{-1}K_0^{-1}\epsilon_1. \tag{4.106}$$

Due to $\partial_t x(t, y) = v(t, x(t, y))$ in $\bar{\Omega}$, $|D_t\mathcal{N}| \leq 2|\nabla u| \leq 2M$ on $\partial\Omega$, and

$$\|v(t, x(t, \cdot))\|_{L^\infty(\bar{\Omega})} \leq 2\|v(0, x(0, \cdot))\|_{L^\infty(\bar{\Omega})} \quad \text{for} \quad t \leq (\overline{\mathcal{T}}M)^{-1}\|v(0, x(0, \cdot))\|_{L^\infty(\bar{\Omega})}, \tag{4.107}$$

we have

$$|x(t, y) - x(0, y)| \leq 2^{-1}\iota_1(0) \quad \text{for} \quad y \in \bar{\Omega} \quad \text{and} \quad t \leq T_1, \tag{4.108}$$

$$|\mathcal{N}(x(t, y)) - \mathcal{N}(x(0, y))| \leq 4^{-1}\epsilon_1 \quad \text{for} \quad y \in \partial\Omega \quad \text{and} \quad t \leq 8^{-1}M^{-1}\epsilon_1, \tag{4.109}$$

where $T_1 = \min\{(\overline{\mathcal{T}}M)^{-1}\|v(0, x(0, \cdot))\|_{L^\infty(\bar{\Omega})},\ 4^{-1}\|v(0, x(0, \cdot))\|^{-1}_{L^\infty(\bar{\Omega})}\iota_1(0)\}$. Indeed, the bound $|D_t v| = |\mathcal{T}\partial p| = |\mathcal{T}\nabla p| \leq \overline{\mathcal{T}}M$ in $\bar{\Omega}$, which follows from (1.1a), (4.5) and (4.2e), has been used to derive (4.107). It follows from (4.105), (4.108) and (4.109) that for $t \leq \min\{T_1,\ 8^{-1}M^{-1}\epsilon_1\}$,

$$\begin{aligned}&|\mathcal{N}(x(t, y_1)) - \mathcal{N}(x(t, y_2))| \leq \epsilon_1,\\ &\qquad \text{whenever} \quad |x(t, y_1) - x(t, y_2)| \leq \iota_1(0), \quad y_1, y_2 \in \partial\Omega.\end{aligned} \tag{4.110}$$

This, together with (4.106), implies that for $t \leq \min\{T_1,\ 8^{-1}M^{-1}\epsilon_1\}$,

$$\iota_1(t) \geq \iota_1(0) \geq 2^{-1}K_0^{-1}\epsilon_1. \tag{4.111}$$

It follows from Proposition 4.1 that there are continuous functions $T_r > 0$ such that

$$\sum_{s=0}^r E_s(t) \leq 2\sum_{s=0}^r E_s(0), \quad 2 \leq r \leq 4, \tag{4.112}$$

for $t \le T_r(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_1(0), \cdots, E_{r-1}(0))$. This, together with (2.18), (2.16), Lemmas 4.8-4.10, (4.2), (4.3) and (4.5), gives that

$$\begin{aligned}\|\nabla D_t p(t,\cdot)\|_{L^\infty}^2 \le & C(K_1)\sum_{i=1}^3 \|\nabla^i D_t p(t,\cdot)\|^2 \\ \le & C(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}})\sum_{s=0}^3 E_s(0), \quad t \le T_3,\end{aligned} \tag{4.113}$$

$$\begin{aligned}\|\nabla^2 p(t,\cdot)\|_{L^\infty}^2 + \|\|\nabla^2 u(t,\cdot)\|\|_{L^\infty}^2 \le & C(K_1)\left(\sum_{i=2}^4 \|\nabla^i p(t,\cdot)\|^2 + \sum_{i=2}^3 \|\|\nabla^i u(t,\cdot)\|\|^2\right) \\ \le & C(\overline{V}, K, \epsilon_b^{-1}, M, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}})\sum_{s=0}^4 E_s(0), \quad t \le T_4.\end{aligned} \tag{4.114}$$

Notice that

$$|D_t \nabla p| = |\nabla D_t p| \le \|\nabla D_t p\|_{L^\infty} \text{ in } \Omega, \quad |D_t \nabla p| = |\nabla D_t p| \le L \text{ on } \partial\Omega,$$

$$|D_t \nabla u| \le \mathcal{T}|\nabla^2 p| + |\nabla \mathcal{T}||\nabla u| + |\nabla u|^2 \le \overline{\mathcal{T}}\|\nabla^2 p\|_{L^\infty} + 2M^2 \text{ in } \Omega, \tag{4.115}$$

$$|D_t \nabla \mathcal{T}| \le \mathcal{T}|\nabla \mathrm{div} u| + |\nabla \mathcal{T}||\nabla u| \le \overline{\mathcal{T}}M + M^2 \text{ in } \Omega, \tag{4.116}$$

$$|D_t \theta| \le |\nabla^2 u| + C|\theta||\nabla u| \le \|\|\nabla^2 u\|\|_{L^\infty} + CKM \text{ on } \partial\Omega.$$

Here (4.115) (or respectively, (4.116)) follows from (3.2) (or respectively, (3.10)), (4.5) and (4.2e). Then we have from (1.8a), (1.8c), the fact that $|\partial p| = |\nabla p|$, $|\partial v| = |\nabla u|$ and $|\partial \mathcal{T}| = |\nabla \mathcal{T}|$, (4.113) and (4.114) that there is a continuous function $T_5 > 0$ such that

$$\|\nabla p(t,\cdot)\|_{L^\infty} + \|\nabla u(t,\cdot)\|_{L^\infty} + \|\nabla \mathcal{T}(t,\cdot)\|_{L^\infty} + \|\|\nabla p(t,\cdot)\|\|_{L^\infty} \le 2M_0, \tag{4.117}$$

$$\|\|\theta(t,\cdot)\|\|_{L^\infty} \le 2K_0, \tag{4.118}$$

for $t \le T_5(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_0(0), \cdots, E_4(0), M_0, K_0)$. Moreover, we can derive from Lemma 2.7, (4.118) and (4.111) that for $t \le \min\{T_1,\ 8^{-1}M^{-1}\epsilon_1,\ T_5\}$,

$$\iota_0^{-1}(t) \le \max\{2\iota_1^{-1}(t),\ 2K_0\} \le 4\epsilon_1^{-1}K_0. \tag{4.119}$$

Clearly, there is a continuous function $T_6 > 0$ such that (4.103), (4.104), (4.112), (4.117), (4.118) and (4.119) hold for $t \le T_6(\overline{V}, K, \epsilon_b^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_0(0), \cdots, E_4(0), \underline{\epsilon_0}, M_0, K_0, \mathrm{Vol}\mathscr{D}_0)$, since $\|v(0, x(0,\cdot))\|_{L^\infty(\bar{\Omega})}^2 \le C(\mathrm{Vol}\mathscr{D}_0, \overline{\mathcal{T}})\sum_{s=0}^2 E_s(0)$ which follows from $|v| = |u|$, (2.18), (2.16), Lemmas 4.7 and 4.8, and (4.5). So, (4.102) holds for $t \le T_{21}$ for some continuous function

$$T_{21}(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, L, M, \widetilde{M}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, E_0(0), \cdots, E_4(0), M_0) > 0,$$

by choosing $\overline{V} = 4\mathrm{Vol}\mathscr{D}_0$, $\epsilon_b = 4^{-1}\underline{\epsilon_0}$, and $K = 2(2 + 4\epsilon_1^{-1})K_0 = 36K_0$ with ϵ_1 being 4^{-1} in T_6. □

Lemma 4.13 *Let $n = 2, 3$. Then there are continuous functions $T_{n2} > 0$ such that*

$$\begin{aligned}&\sum_{i=1}^{n-1}\left(\|\|\nabla D_t^i p(t,\cdot)\|\|_{L^\infty} + \|\|\nabla D_t^i \mathrm{div} u(t,\cdot)\|\|_{L^\infty}\right) + \|\|\nabla^2 p(t,\cdot)\|\|_{L^\infty} \\ &+ \|\nabla \mathrm{div} u(t,\cdot)\|_{L^\infty} + \|D_t p(t,\cdot)\|_{L^\infty} + \|D_t \mathrm{div} u(t,\cdot)\|_{L^\infty} + \|\nabla^2 \mathcal{T}(t,\cdot)\|_{L^\infty} \\ &\le C\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0)\right)\end{aligned} \tag{4.120}$$

for $t \le T_{n2}(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_{n+2}(0))$.

Moreover, (4.102) also holds for $t \le T_{n2}$.

Proof. The proof consists of two cases of $n = 2$ and $n = 3$.

Case 1. Let n=2. It follows from (2.42) and (4.56) that

$$\begin{aligned}|||\nabla\mathcal{T}|||_{L^\infty} \le& \|\nabla\Delta\mathcal{T}\| + C\left(K, K_1, |||\theta|||, \mathrm{Vol}\Omega\right)\|\Delta\mathcal{T}\| \\ \le& \|\nabla\Delta\mathcal{T}\| + C\left(K, K_1, |||(\nabla_N p)^{-1}|||_{L^\infty}, |||\nabla_N p|||_{L^\infty}, E_2, \mathrm{Vol}\Omega\right)\|\Delta\mathcal{T}\|,\end{aligned}$$

which, together with (3.1b), Lemma 4.7, (4.3), (4.117) and (4.102), gives that for $t \le T_{21}$,

$$|||\nabla\mathcal{T}(t,\cdot)|||_{L^\infty} \le C_1\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, M_0, E_0(0), \cdots, E_2(0)\right). \tag{4.121}$$

It follows from (2.18) and (2.16) that

$$\|D_t p\|_{L^\infty} \le C(K_1)\sum_{i=0}^{2}\|\nabla^i D_t p\| \quad\text{and}\quad |||\nabla u|||_{L^\infty} \le C(K_1)\sum_{i=1}^{2}|||\nabla^i u|||,$$

which, together with Lemmas 4.8 and 4.9, (4.3), (4.117) and (4.102), gives that for $t \le T_{21}$,

$$\|D_t p(t,\cdot)\|_{L^\infty} + |||\nabla u(t,\cdot)|||_{L^\infty} \le C_2\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_3(0)\right). \tag{4.122}$$

Similarly, we have that for $t \le T_{21}$,

$$\begin{aligned}&|||\nabla\mathrm{div}u(t,\cdot)|||_{L^\infty} + |||\nabla D_t p(t,\cdot)|||_{L^\infty} + \|\nabla\mathrm{div}u(t,\cdot)\|_{L^\infty} + \|D_t\mathrm{div}u(t,\cdot)\|_{L^\infty} \\ &\quad + |||\nabla^2 p(t,\cdot)|||_{L^\infty} \le C_3\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_3(0)\right),\end{aligned} \tag{4.123}$$

$$\begin{aligned}&\|\nabla^2\mathcal{T}(t,\cdot)\|_{L^\infty} + |||\nabla D_t\mathrm{div}u(t,\cdot)|||_{L^\infty} \\ &\quad \le C_4\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0)\right).\end{aligned} \tag{4.124}$$

Indeed, the bounds for $|||\nabla\mathrm{div}u|||_{L^\infty}$ and $|||\nabla D_t p|||_{L^\infty}$ in (4.123) follows from (2.42), (4.56), (4.70), (4.71), Lemmas 4.7-4.9, the bounds just obtained for $|||\nabla\mathcal{T}|||_{L^\infty}$ and $\|D_t p\|_{L^\infty}$, (4.3), (4.117) and (4.102). The bounds for $\|\nabla\mathrm{div}u\|_{L^\infty}$, $\|D_t\mathrm{div}u\|_{L^\infty}$ and $|||\nabla^2 p|||_{L^\infty}$ in (4.123) follows from (2.18), (2.16), Lemmas 4.7-4.9, the bounds just obtained for $|||\nabla\mathcal{T}|||_{L^\infty}$ and $|||\nabla\mathrm{div}u|||_{L^\infty}$, (4.3), (4.117) and (4.102). The bound for $\|\nabla^2\mathcal{T}\|_{L^\infty}$ in (4.124) follows from (2.18), Lemmas 4.7-4.10, (4.3), (4.117) and (4.102). The bound for $|||\nabla D_t\mathrm{div}u|||_{L^\infty}$ in (4.124) follows from (2.42), (4.56), (4.80), Lemmas 4.7-4.10, the bounds just obtained for $\|D_t\mathrm{div}u\|_{L^\infty}$, $|||\nabla\mathcal{T}|||_{L^\infty}$ and $|||\nabla\mathrm{div}u|||_{L^\infty}$, (4.3), (4.117) and (4.102).

So, it follows from (4.121)-(4.124) that (4.120) holds for $t \le T_{22}$ for some continuous function $T_{22}(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0)) > 0$, by choosing $L = 2\sum_{i=3}^{4} C_i$, $M = 4M_0 + 2\sum_{i=1}^{3} C_i$, and $\widetilde{M} = 2\sum_{i=2}^{4} C_i$ in the continuous function T_{21} given by Lemma 4.12. Clearly, (4.102) holds for $t \le T_{22}$.

Case 2. Let $n = 3$. In this case, we can use the arguments similar to the way which we dealt with case 1 to obtain that for $t \le T_{31}$,

$$\begin{aligned}&\|D_t p(t,\cdot)\|_{L^\infty} \le C_5\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_3(0)\right), \\ &|||\nabla\mathcal{T}(t,\cdot)|||_{L^\infty} + \|D_t\mathrm{div}u(t,\cdot)\|_{L^\infty} \le C_6\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_3(0)\right), \\ &\|\nabla^2\mathcal{T}(t,\cdot)\|_{L^\infty} \le C_7\left(\mathrm{Vol}\mathscr{D}_0, K_0, \underline{\epsilon_0}^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0)\right).\end{aligned} \tag{4.125}$$

It follows from (2.41), (2.22) and (4.4) that for any $\delta \in (0, 1]$,

$$\begin{aligned}\|\nabla^3\mathrm{div}u\| + |||\nabla^2\mathrm{div}u||| \le& \delta|||\Pi\nabla^3\mathrm{div}u||| + C(\delta^{-1}, K, \mathrm{Vol}\Omega)\sum_{s=0}^{1}\|\nabla^s\Delta\mathrm{div}u\| \\ \le& \delta\left(|||\nabla_N\mathrm{div}u|||_{L^\infty}|||\overline{\nabla}\theta||| + 3K|||\nabla^2\mathrm{div}u||| + 2K^2|||\nabla\mathrm{div}u|||\right) \\ &+ C(\delta^{-1}, K, \mathrm{Vol}\Omega)\sum_{s=0}^{1}\|\nabla^s\Delta\mathrm{div}u\|,\end{aligned}$$

which implies, by choosing $\delta = \min\{(6K)^{-1}, 1\}$, that

$$\|\nabla^3 \mathrm{div} u\| + 2^{-1}|\|\nabla^2 \mathrm{div} u\|| \leq |\|\nabla_N \mathrm{div} u\||_{L^\infty} |\|\overline{\nabla}\theta\|| + C(K)|\|\nabla \mathrm{div} u\|| + C(K, \mathrm{Vol}\Omega) \sum_{s=0}^{1} \|\nabla^s \Delta \mathrm{div} u\|.$$

This, together with (2.31), gives

$$\|\nabla^4 u\|^2 \leq C\|\nabla^3 \mathrm{div} u\|^2 + C(\|\mathcal{T}\|_{L^\infty} + 1)E_4 \leq C|\|\nabla_N \mathrm{div} u\||^2_{L^\infty} |\|\overline{\nabla}\theta\||^2 + L_1, \tag{4.126}$$

where

$$L_1 = C(K)|\|\nabla \mathrm{div} u\||^2 + C(K, \mathrm{Vol}\Omega) \sum_{s=0}^{1} \|\nabla^s \Delta \mathrm{div} u\|^2 + C(\|\mathcal{T}\|_{L^\infty} + 1)E_4.$$

It follows from (2.42), (4.70) and (4.126) that for any $\delta \in (0, 1]$,

$$\begin{aligned}
|\|\nabla \mathrm{div} u\||^2_{L^\infty} \leq & \delta\|\nabla^2 \Delta \mathrm{div} u\|^2 + C(\delta^{-1}, K, K_1, |\|\overline{\nabla}\theta\||, \mathrm{Vol}\Omega) \sum_{s=0}^{1} \|\nabla^s \Delta \mathrm{div} u\|^2 \\
\leq & \delta\|\mathcal{T}^{-1}\|^2_{L^\infty} C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty})\|\nabla^4 u\|^2 + L_2 \\
\leq & \delta\|\mathcal{T}^{-1}\|^2_{L^\infty} C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty})|\|\nabla_N \mathrm{div} u\||^2_{L^\infty} |\|\overline{\nabla}\theta\||^2 + L_3,
\end{aligned} \tag{4.127}$$

where

$$\begin{aligned}
L_2 = & C(\delta^{-1}, K, K_1, |\|\overline{\nabla}\theta\||, \mathrm{Vol}\Omega) \sum_{s=0}^{1} \|\nabla^s \Delta \mathrm{div} u\|^2 + \delta\|\mathcal{T}^{-1}\|^2_{L^\infty} C\|\nabla^2 D_t \mathrm{div} u\|^2 \\
& + \delta\|\mathcal{T}^{-1}\|^2_{L^\infty} C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}) \left(\sum_{i=1}^{3} \|\nabla^i u\|^2 + \sum_{i=1}^{4} \|\nabla^i \mathcal{T}\|^2 \right),
\end{aligned}$$

and

$$L_3 = \delta\|\mathcal{T}^{-1}\|^2_{L^\infty} C(K_1, \|\nabla u\|_{L^\infty}, \|\nabla\mathcal{T}\|_{L^\infty}) L_1 + L_2.$$

By choosing suitable small δ in (4.127), and using Lemmas 4.7-4.10, the bound just obtained for $|\|\nabla\mathcal{T}\||_{L^\infty}$ in (4.125), (4.3) and (4.102), we have that for $t \leq T_{31}$,

$$|\|\nabla \mathrm{div} u(t, \cdot)\||_{L^\infty} \leq C_8 \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0)\right). \tag{4.128}$$

In a similar way to deriving (4.128), we have, using (4.71), (4.80), (4.82) and (4.83), that for $t \leq T_{31}$,

$$|\|\nabla D_t p(t, \cdot)\||_{L^\infty} \leq C_9 \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0)\right), \tag{4.129}$$

$$\begin{aligned}
& |\|\nabla D_t \mathrm{div} u(t, \cdot)\||_{L^\infty} + |\|\nabla D_t^2 p(t, \cdot)\||_{L^\infty} + |\|\nabla D_t^2 \mathrm{div} u(t, \cdot)\||_{L^\infty} \\
& \quad \leq C_{10} \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_5(0)\right).
\end{aligned} \tag{4.130}$$

With these bounds, we can obtain, in the same manner as the case of $n = 2$, that for $t \leq T_{31}$,

$$\|\nabla \mathrm{div} u(t, \cdot)\|_{L^\infty} + |\|\nabla u(t, \cdot)\||_{L^\infty} \leq C_{11} \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_4(0)\right), \tag{4.131}$$

$$|\|\nabla^2 p(t, \cdot)\||_{L^\infty} \leq C_{12} \left(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_5(0)\right). \tag{4.132}$$

It is produced from (4.125), (4.127)-(4.132) that (4.120) holds for $t \leq T_{32}$ for some continuous function $T_{32}(\mathrm{Vol}\mathscr{D}_0, K_0, \epsilon_0^{-1}, \underline{\mathcal{T}}^{-1}, \overline{\mathcal{T}}, M_0, E_0(0), \cdots, E_5(0)) > 0$, by choosing $L = 2(C_9 + C_{10} + C_{12})$,

$M = 4M_0 + 2(C_6 + C_8 + C_{11})$, and $\widetilde{M} = 2\sum_{i=5}^{7} C_i$ in the continuous function T_{31} given by Lemma 4.12. Clearly, (4.102) holds for $t \leq T_{32}$. □

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