FRAMING DUALITY

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Abstract. A symmetric quiver with g nodes is described by a symmetric adjacency matrix of size g. The same data defines a "framing" of a certain genus-g Legendrian surface in the five-sphere, and the invariants of the quiver conjecturally relate to the open Gromov-Witten (GW) invariants of a non-exact Lagrangian filling of the surface. (Physically, both data count the same BPS states but from different perspectives.) Further, cluster theory can be exploited to conjecturally obtain all open GW invariants of Lagrangian fillings of a wider class of Legendrian surfaces described by cubic planar graphs.

In this talk, I will describe these observations, which build on prior work of others and are explored in joint works with David Treumann and Linhui Shen.

1. KS (à la HM): COHA of Quiver

Symmetric Quiver $Q = Q^A$ with g nodes $\iff$ symmetric $g \times g$ adjacency matrix A

Dimension vector $d \in \mathbb{Z}^g_{\geq 0}$

Rep space $M_d^{Q^A} = \oplus_{i,j} \mathbb{C}^{A_{ij} d_i d_j} / G_d$, where $G_d = \prod_i GL(d_i, \mathbb{C})$. Note: numerator contractible.

$\mathcal{H} = \oplus_{d,m} \mathcal{H}_{d,m} = \oplus_d H^*_G(M_d^{Q^A})$, a $\mathbb{Z}^g \times \mathbb{Z}$-graded vector space, $m = 2(\text{cohom. deg.}) + d^T(1-A)d$

(Algebraic structure from cup product and inclusion of reps into direct sum. We won't use.)

Poincaré polynomial (DT series) in $x = (x_1, ..., x_g)$, $q^{\frac{1}{2}}$:

$$\Psi^A(x, q^{\frac{1}{2}}) = \sum_{(d,m) \in \mathbb{Z}^g_{\geq 0} \times \mathbb{Z}} (-1)^m \dim \mathcal{H}_{d,m} x^d (q^{\frac{1}{2}})^m$$

Example. $g = 1, A = 1, d \geq 0$. So $H^*_{G_d} M_d^{Q^A} \cong H^*(\mathbb{C}^{d^2}/GL(d,\mathbb{C}) \hookrightarrow \mathbb{Z}[z_1, ..., z_d]$, $z_i \in H^2(B\mathbb{C}^\times_i)$. Map $\hookrightarrow$ is pullback under ("B" of) map $(\mathbb{C}^\times)^d \hookrightarrow GL(d, \mathbb{C})$, with image symmetric ($W$-invt) polys. Young tableaux/partitions give basis. Note for $A = 1$, $m = 2n$ is twice cohomological degree. Then

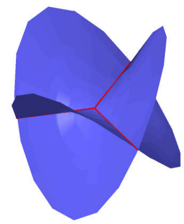
$$\Psi(x, q^{\frac{1}{2}}) = \sum_{d,n} \#\{\text{partitions of } n \text{ into } d \text{ parts}\}\ x^d q^n = \prod_k \frac{1}{1 - q^k x}$$

Note: put $q = e^h$. Then $\lim_{\hbar \to 0} \hbar \log \Psi = \mathrm{Li}_2(x)$, i.e. Ψ is (exponentiated) quantum dilogarithm, Φ.

2. Open GW (à la AV, AKV, OV)

Γ cubic planar graph. So $\#V = 2g+2, \#E = 3g+3, \#F = g+3$ for some $g \geq 0$.

Front in $S^2 \times \mathbb{R}$ for Legendrian Λ_Γ in $T^*S^2 \times \mathbb{R} \cong T^{\infty,-}(S^2 \times \mathbb{R}) \subset T^\infty \mathbb{R}^3 \cong \mathbb{C}^3$ branched over Γ.



$\Lambda_\Gamma \to S^2$ branched over vertices $\Rightarrow \Lambda_\Gamma$ has genus g. (Recall construction: $p_i = \frac{\partial z}{\partial q^i}$.)

$\mathscr{M}_\Gamma$ moduli space of Lag branes $= ObFuk_1(T^*\mathbb{R}^3, \Lambda_\Gamma) \cong Sh_1(\mathbb{R}^3, \Lambda_\Gamma)$

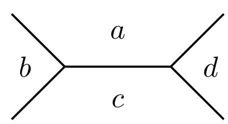
$\mathscr{M}_\Gamma = \{$map colorings of Γ, colors in $\mathbb{P}^1\}/PGL_2$.

"Period map" $\mathscr{M}_\Gamma \to \mathscr{P}_\Gamma = H^1(\Lambda_\Gamma, \mathbb{C}^*)$. Note $\mathscr{P}_\Gamma \cong (\mathbb{C}^\times)^{2g}$, but not canonically (framing).
Map mimics monodromy of rank-1 local system along boundary Λ_Γ of Lagrangian object.

$H_1(\Lambda_\Gamma) \cong coker(\delta : \mathbb{Z}^F \to \mathbb{Z}^E)$, $\partial f = \sum_{e \in \partial f} e$. So edge e gives function $x_e : \mathscr{P}_\Gamma \to \mathbb{C}^\times$.

Poisson str. on $\mathbb{Z}^E$: $e \cdot e' = \pm 1$ if $e \cap e'$. Poisson torus $\mathrm{Hom}(\mathbb{Z}^E, \mathbb{C}^\times)$. Symp. leaf: $\forall f$, $\prod_{e \in \partial f} x_e = 1$.

Then map $\mathscr{M}_\Gamma \to \mathscr{P}_\Gamma$ given by: x_e is the cross ratio $x_e = -\frac{a-b}{b-c} \cdot \frac{c-d}{d-a}$ where local data is



Theorem [TZ]: Γ simple $\Rightarrow$ no exact fillings. Proof: $\#\mathscr{M}_\Gamma/\mathbb{F}_q$ too small for torus of dim $\geq g$.

So geometric objects non-exact Lagrangrian fillings $L_\Gamma \Rightarrow$ obstructed. Try counting disks à la AV:

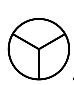
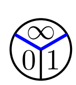
Choose coords and lift to $\widetilde{\mathscr{M}_\Gamma} \subset T^*\mathbb{C}^g$. Coords $\mathbb{C}^g_u \subset \mathbb{C}^{2g}$ "phase," transverse space $\mathbb{C}^g_v$ "framing."
Invariantly/geometrically, phase is map $H_1(\Lambda_\Gamma) \to H_1(L_\Gamma)$ and framing f is an isotropic splitting.

Now try to write $\widetilde{\mathscr{M}_\Gamma}$ as $Graph(dW^f_\Gamma)$ and hope/conjecture (à la OV), for any f:

$$W^f_\Gamma(x) = \sum_{d \in \mathbb{Z}^g_{\geq 0}} K^f_d x^d = \sum_d a^f_d \,\mathrm{Li}_2(x^d), \quad x = \pm e^u \qquad \text{with} \quad a^f_d \in \mathbb{Z}$$

Here K^f_d are open GW invariants defined by framing f (cf. Katz-Liu, Solomon-Tukachinsky w.i.p.).

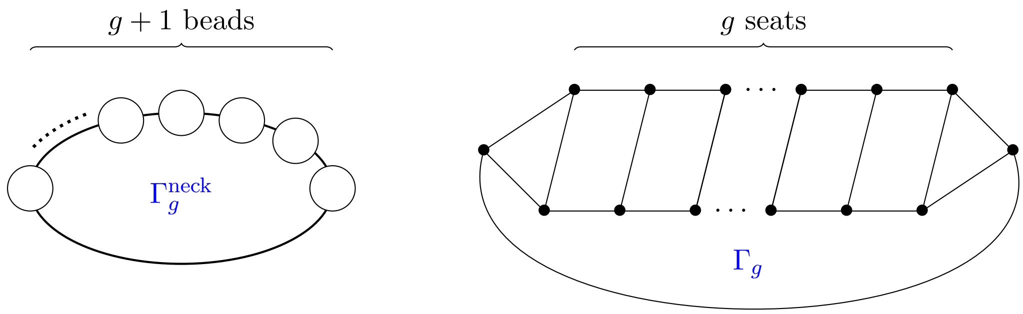
In [TZ] we did this for some examples. Here's one (also done in [Luo-Zhao], but not generalized):

Example. $\Gamma_1 =$ , $\mathscr{M}_\Gamma =$ p.o.p. H : 

z . $/ = -\frac{\infty - 0}{0-1} \cdot \frac{1-z}{z-\infty} = z-1$, $\backslash = -\frac{0-1}{1-\infty} \cdot \frac{\infty - z}{z-0} = -\frac{1}{z}$.
Framing p coords: $/ = -xy^{p-1} = z-1$. $\backslash = -\frac{1}{y} = -\frac{1}{z}$. So $y + xy^{p-1} = 1$. $f = p = 1$: $x + y = 1$.
$y = e^{-v}, x = e^u \Rightarrow v = -\log(1-e^u) = \partial_u W^{f=1}_{\Gamma_1} \Rightarrow W^1_{\Gamma_1} = \mathrm{Li}_2(e^u) = \mathrm{Li}_2(x)$. N.B.: No MS!

3. Framing Duality Conjecture

Let Γ_g^{neck} be the necklace (Chekanov) with $g+1$ beads; let Γ_g be the canoe (Clifford) with g seats:



Γ_g is the g-fold blow-up of the Θ graph at the bow, equal to the tetrahedron when $g = 1$.

Blow-up formula: $\mathscr{M}_{\widehat{\Gamma}} = \mathscr{M}_\Gamma \times H$, so $\mathscr{M}_{\Gamma_g} \cong H^g$.

Also: framings transfer across mutations, since edge lattices isomorphic.

Γ_g is the g-fold mutation of necklace graph with g beads $\Rightarrow$ framing symm. $g \times g$ matrix A.

$\Rightarrow$ 0-framing coordinates u_i, v_i. In framing A have u_i, $v_i + A_{i,j}u_j$. OGW problem defined.

Conjecture (Framing Duality): Generating function of disk invariants is

$$W^A_{\Gamma_g}(x) = \lim_{\hbar \to 0} \hbar \log \Psi^{Q^A}(x; q^{\frac{1}{2}}), \qquad \text{i.e. } \Psi \sim e^{W/\hbar}.$$

OV integrality follows from KS admissibility. All-genus, too (w.i.p.).

LHS/RHS arise from different physical viewpoints ([CCV,CEHRV,DGoGu,DGaGo,AV,AENV]):

- BPS Quiver QM (closest to KS)
- String/M-theory “compactified” on $\mathbb{R}^4 \times L \Rightarrow$ BPS states via hol disks or membranes
- Effective 3d SQFT on $S^3 \leftrightarrow$ CS Theory on L (hence dilog) $\Rightarrow$ wavefunction for quantum T^2

Question: $\exists$ a quiver/DT interpretation of W^A_Γ for $\Gamma \neq \Gamma_g$? Non-symmetric quivers?

4. Computations via Cluster Theory

We can transfer more than just a framing across mutations!

Cubic planar graphs are dual to triangulations.

[FG]: these label cluster charts of $\mathscr{P} = \{\text{decorated } PGL_2 \text{ loc. systs on } (S^2, p_1, \cdots, p_{g+3})\}/PGL_2$.

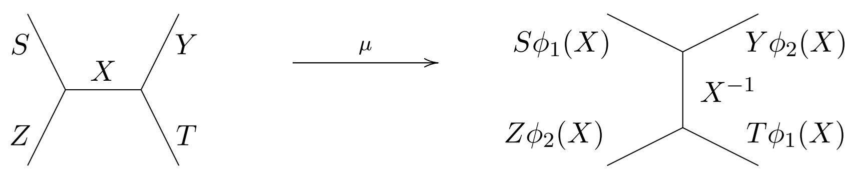
[DGG] (later [TZ],[STZ] in this context): There is a global Lagrangian $\mathscr{M} \subset \mathscr{P}$ s.t. $\mathscr{M} \cap \mathscr{P}_\Gamma = \mathscr{M}_\Gamma$.

Each $\mathcal{O}(\mathscr{P}_\Gamma)$ has quantization. Now $\prod_{e \in \partial f} q x_e = 1$ for all f and $x_v x_w = q^{\omega(v,w)} x_w x_v$.

Ideal of functions vanishing on $\mathscr{M}_\Gamma$ also has quantization, also compatible with gluing.

So $\mathscr{P}$ is quantized and $\mathscr{M}$ quantized to sheaf of modules (like global, exponentiated D-module)

Ideal of $\mathscr{M}_\Gamma$ is $\langle P_f \rangle$: for face with ccw edges e_i, $P_f = 1 + q x_{e_1} + q^2 x_{e_1} x_{e_2} + ... + q^{n-1} x_{e_1} \cdots x_{e_{n-1}}$.



Here $\phi_1(X) = 1 + qX$ and $\phi_2(X) = (1 + qX^{-1})^{-1}$. Note the P_f are preserved! (Warning: signs.)

Lesson: *One quantum Lagrangian determines everything.*

Sheaf of $\mathcal{D}$-modules is cyclic in each patch $\mathscr{P}_\Gamma$ and we can compute the generator Ψ_Γ:

Mutate μ_e: $\Gamma \rightsquigarrow \Gamma'$. Edge lattice isomorphism $\Rightarrow$ phase/framing transfers. Then (Reineke, Keller):

$$\Psi_{\Gamma'} = \Phi(x_e)\Psi_\Gamma, \qquad \text{where } \Phi \text{ is q-dilogarithm}$$

But $\Psi_{\Gamma_g^{\text{neck}}} \equiv 1$ in canonical framing. So all Ψ_Γ determined, thus all conjectural OGW invts W_Γ^f.