Matricial singularities of noncommutative polynomials

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Outline

(1) Matricial “zero sets” and Nullstellensätze

(2) Free loci of nc polynomials

(3) Singulärstellensatz for monic pencils

(4) Factorization in free algebra

(5) Realizations, irreducibility

(6) General Singulärstellensatz

(7) Open questions

Nc polynomials and linear matrix pencils

Let $\mathbb{k}$ be a field of characteristic 0 and $x = (x_1, \dots, x_d)$ freely noncommuting variables. Elements of $\mathbb{k}{<}x{>}^{\delta\times\delta}$ are **(matrix-valued) nc polynomials**. If

$$f = \sum_{w\in{<}x{>}} F_w w \in \mathrm{M}_\delta(\mathbb{k}) \otimes \mathbb{k}{<}x{>} = \mathbb{k}{<}x{>}^{\delta\times\delta}$$

and $X \in \mathrm{M}_n(\mathbb{k})^d$, then

$$f(X) = \sum_{w\in{<}x{>}} F_w \otimes w(X) \in \mathrm{M}_\delta(\mathbb{k}) \otimes \mathrm{M}_n(\mathbb{k}) = \mathrm{M}_{\delta n}(\mathbb{k}),$$

where $\otimes$ is the Kronecker product.

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Affine polynomials are called **linear matrix pencils:**

$$L = A_0 + A_1 x_1 + \cdots + A_d x_d, \qquad A_j \in \mathrm{M}_\delta(\mathbb{k}).$$

If $A_0 = I$, then L is **monic**.

Free “zero sets” of an nc polynomial

Let $f, g \in \Bbbk{<}x{>}$.

1. $Z(f) = \bigcup_n \{X \in \mathrm{M}_n(\Bbbk)^d : f(X) = 0\}$
   nc zero set; Amitsur's Nullstellensatz **for fixed** n:

$$Z(f) \cap \mathrm{M}_n(\Bbbk)^d \subseteq Z(g) \cap \mathrm{M}_n(\Bbbk)^d \implies g^r \in (f) + \mathrm{PI}_n$$

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Free (singularity) locus

For $f \in \mathbb{k}\langle x\rangle^{\delta\times\delta}$ we define its **free locus** as

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- $\mathscr{Z}_n(f)$ is a (possibly degenerate) determinantal hypersurface in $\mathrm{M}_n(\Bbbk)^d$, invariant under simultaneous conjugation: $X \in \mathscr{Z}_n(f) \implies PXP^{-1} \in \mathscr{Z}_n(f)$ for $P \in \mathrm{GL}_n(\Bbbk)$

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(I) $\mathscr{Z}(f) \subseteq \mathscr{Z}(g)$? **(II) Components of $\mathscr{Z}(f)$?**

Why do we care about free loci?

- $\{X\colon \det f(X) = 0\}$ is the Zariski closure of the boundary of the free semialgebraic set $\{X\colon f(X) \text{ psd}\}$
- Convexity (free spectrahedra)
- free loci are the complements of domains of nc rational functions
- factorization over $\mathbb{k}{<}x{>}$
- smooth points on free loci have implications for bianalytic maps between free spectrahedra
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Singulärstellensatz for monic pencils

For a monic pencil $L_A = I + \sum_j A_j x_j$ let $\mathcal{A}$ be the matrix subalgebra generated by $A_1, \ldots, A_d$. Let $\operatorname{rad} \mathcal{A}$ be its Jacobson radical (largest nilpotent ideal).

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Theorem (Klep–V)

$\mathscr{Z}(L_A) \subseteq \mathscr{Z}(L_B)$ iff the map $B_j \mapsto A_j$ induces a (surj.) $\Bbbk$-algebra homomorphism $\mathcal{B}/\operatorname{rad}\mathcal{B} \to \mathcal{A}/\operatorname{rad}\mathcal{A}$.

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Algebraization trick:

$$\det(I + A_1 \otimes X_1 + A_2 \otimes X_2 + A_1A_2 \otimes X_3)$$
$$= \det\left(I + A_1 \otimes \begin{pmatrix} 0 & 0 \\ X_3 & X_1 \end{pmatrix} + A_2 \otimes \begin{pmatrix} 0 & -I \\ 0 & X_2 \end{pmatrix}\right).$$

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There are probabilistic algorithms for working with $\mathcal{A}/\operatorname{rad}\mathcal{A}$ (finding radical is the most technical bit, Cohen–Ivanyos–Wales)

Factorization in free algebra (P. M. Cohn)

$f \in \mathbb{k}\langle x\rangle^{\delta\times\delta}$ is **full** if it cannot be factored as $f = f_1 f_2$ where $f_1 \in \mathbb{k}\langle x\rangle^{\delta\times\varepsilon}$, $f_2 \in \mathbb{k}\langle x\rangle^{\varepsilon\times\delta}$ and $\varepsilon < \delta$.

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$(x_1x_2+1)(x_3x_2x_1+x_3+x_1) = (x_1x_2x_3+x_1+x_3)(x_2x_1+1)$

Stable association

$f \in \Bbbk\langle x\rangle^{\delta\times\delta}$ and $g \in \Bbbk\langle x\rangle^{\varepsilon\times\varepsilon}$ are **stably associated** if there are $P, Q \in \mathrm{GL}_{\delta+\varepsilon}(\Bbbk\langle x\rangle)$ such that

$$g \oplus I_\delta = P(f \oplus I_\varepsilon)Q.$$

E.g.

$$\begin{pmatrix} 1+x_1x_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & 1+x_1x_2 \\ -1 & -x_2 \end{pmatrix}\begin{pmatrix} 1+x_2x_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 & -1 \\ 1+x_1x_2 & x_1 \end{pmatrix}$$

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Cohn: a complete factorization of an nc polynomial is unique up to stable association of irreducible factors.
Stable association preserves irreducibility.

Note: f, g stably associated $\implies \mathscr{Z}(f) = \mathscr{Z}(g)$.

Linearizations and realizations

Higman, Cohn: every $f \in \mathbb{k}\langle x\rangle^{\delta\times\delta}$ is stably associated to a linear matrix pencil L.

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In the monic case, we get such L efficiently via realizations of nc rational functions. Every $r \in \Bbbk(\!\langle x\rangle\!)^{\delta\times\delta}$ defined at 0 admits an **Fornasini–Marchesini realization**

$$r = r(0) + c^{\mathrm{t}} L^{-1} b$$

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Let $f \in \Bbbk{<}x{>}^{\delta\times\delta}$ with $\det f(0) \neq 0$. If L is the monic pencil from a minimal realization for f^{-1}, then L is stably associated to f.

Indecomposable pencils

An irreducible $f \in \Bbbk\!<\!x\!>^{\delta\times\delta}$ is stably associated to an **indecomposable** $L = A_0 + A_1x_1 + \cdots + A_dx_d \in \Bbbk\!<\!x\!>^{\varepsilon\times\varepsilon}$: there are no $P, Q \in \mathrm{GL}_\varepsilon(\Bbbk)$ such that

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If $\Bbbk$ is alg. closed, this is equivalent (Burnside) to $A_1, \dots, A_d$ generating $\mathrm{M}_\varepsilon(\Bbbk)$ as a $\Bbbk$-algebra. This is why monic is better!

Invariant theory: three group actions to consider

1. $\mathsf{GL}_n(\Bbbk) \curvearrowright \mathsf{M}_n(\Bbbk)^d$ by simultaneous conjugation.
(Procesi) Invariants: polynomials in $\mathrm{tr}(X_{j_1} \cdots X_{j_\ell})$ (trace polys).
Generators: of degree $\leq n^2$. Relations (trace ids): of degree $\geq n+1$. Closed orbits: semisimple representations.
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3. $\mathrm{GL}_\varepsilon(\Bbbk) \times \mathrm{GL}_\varepsilon(\Bbbk) \curvearrowright \mathrm{M}_\varepsilon(\Bbbk)^{d+1}$ by simultaneous left-right multiplication; on coefficients of general pencils.
(King; Schofield–van den Bergh) indecomposable pencils have closed orbits and trivial stabilizers.
If L, M and $\mathscr{Z}(L) = \mathscr{Z}(M)$, are L, M in the same orbit?

Irreducibility - monic case

From now on let $\mathbb{k}$ be **algebraically closed**.

Theorem (Helton–Klep–V)

Let L be an indecomposable pencil. Then $\mathscr{Z}_n(L)$ is an irreducible hypersurface for all large enough n.

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Main steps when L is monic:

1. Invariant theory and algebraization trick: for large enough n, if $\mathscr{Z}_n(I+\sum_j A_j x_j + A_1A_2x_{d+1})$ is irreducible, then $\mathscr{Z}_{2n}(I+\sum_j A_j x_j)$ is irreducible.

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Let L be an indecomposable pencil. Then $\mathscr{Z}_n(L)$ is an irreducible hypersurface for all large enough n.

Main steps when L is monic:

1. Invariant theory and algebraization trick: for large enough n, if $\mathscr{Z}_n(I+\sum_j A_jx_j+A_1A_2x_{d+1})$ is irreducible, then $\mathscr{Z}_{2n}(I+\sum_j A_jx_j)$ is irreducible.

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2. Coefficients of L generate the full matrix algebra.

3. Determinant is an irreducible polynomial.

Irreducibility - general case

What to do if L (of size ε) is not monic?

Let $X \in \mathrm{M}_n(\Bbbk)^d$ be such that $L(X)$ is invertible. Let

$$\mathfrak{x} = (\mathfrak{x}_{j\imath\jmath} \colon 1 \leq j \leq d, 1 \leq \imath, \jmath \leq n)$$

be dn^2 freely noncommuting variables. Define the **point-centered ampliation** of L at X,

$$L^X = L\left(X_1 + (\mathfrak{x}_{1\imath\jmath})_{\imath,\jmath}, \dots, X_d + (\mathfrak{x}_{d\imath\jmath})_{\imath,\jmath}\right) \in \Bbbk<\mathfrak{x}>^{\varepsilon n \times \varepsilon n}.$$

Note that $\det L(X) = \det L^X(0)$, and $L(X)^{-1} L^X$ is monic.

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Proposition

If L is indecomposable iff $L(X)^{-1}L^X$ is indecomposable.

Gleichstellensatz

Let $\mathbb{k}$ be algebraically closed of characteristic 0.

Theorem (Helton–Klep–V)

Let L, M be indecomposable pencils. Then $\mathscr{Z}(L) = \mathscr{Z}(M)$ if and only if L, M are of the same size ε and there are $P, Q \in \mathrm{GL}_\delta(\mathbb{k})$ such that $M = PLQ$.

Gleichstellensatz; Singulärstellensatz

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Theorem (Helton–Klep–V)

Let f, g be full nc polynomials. Then $\mathscr{Z}(f) \subseteq \mathscr{Z}(g)$ if and only if every irreducible factor of f is stably associated to a factor of g.

Real points

Let $x^* = (x_1^*, \ldots, x_d^*)$ be formal adjoints of x. For $f \in \mathbb{C}\langle x, x^* \rangle^{\delta \times \delta}$ let

$$\mathscr{Z}^{\mathrm{re}}(f) = \bigcup_n \mathscr{Z}_n^{\mathrm{re}}(f), \quad \mathscr{Z}_n^{\mathrm{re}}(f) = \{X \in \mathrm{M}_n(\mathbb{C})^d : \det f(X, X^*) = 0\}$$

be its **real free locus**. (complex pts (X, Y) vs real points (X, X^*))

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be its **real free locus**. (complex pts (X, Y) vs real points (X, X^*))

If $f \in \mathbb{C}\langle x, x^* \rangle^{\delta\times\delta}$ is **hermitian** $(f^* = f)$, then it describes a **free semialgebraic set**

$$\{X \colon f(X, X^*) \text{ is PSD}\}.$$

Then $\mathscr{Z}^{\mathrm{re}}(f)$ is more or less the real Zariski closure of the boundary of this set. convexity, Positivstellensätze, etc

Real Singulärstellensätze

$\mathbb{C}\langle x \rangle^{\delta\times\delta}$ inside $\mathbb{C}\langle x, x^* \rangle^{\delta\times\delta}$ are called **analytic** polynomials.

A hermitian f is **unsignatured** if there are some $X, Y \in \mathrm{M}_n(\mathbb{C})^d$ such that $f(X), f(Y)$ are invertible with different signatures.

E.g. hermitian monic pencils

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Theorem (Helton–Klep–V)

Let f, g be full nc polynomials in x, x^*.

(i) *If f is analytic and irreducible, then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(g)$ iff f or f^* is stably associated to a factor of g.*

(ii) *If f is hermitian, unsignatured and irreducible, then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(g)$ iff f is stably associated to a factor of g.*

Smooth points

For $f \in \mathbb{k}\langle x \rangle^{\delta\times\delta}$ let

$$\mathscr{Z}_n^{\mathbb{1}}(f) = \{X \in \mathrm{M}_n(\mathbb{k})^d \colon \dim\ker f(X) = 1\},$$
$$\mathscr{Z}_n^{\mathbb{1}\mathbb{a}}(f) = \{X \in \mathrm{M}_n(\mathbb{k})^d \colon \dim\ker f(X)^2 = 1\}.$$

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Proposition

If f is irreducible, then for large enough n,
$\emptyset \neq \mathscr{Z}_n^{\mathbb{1}}(f) \subseteq \{\text{smooth points of } \mathscr{Z}_n(f)\} \subseteq \mathscr{Z}_n^{\mathbb{1}\mathrm{a}}(f).$

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Corollary

Let L be an irreducible hermitian pencil in x, x^*. If n is large enough, then $X_0 \in \{X \in \mathrm{M}_n(\mathbb{C})^d : L(X, X^*) \succeq 0\}$ is a smooth point on the boundary iff $\dim\ker L(X_0, X_0^*) = 1$.

Kippenhahn's conjecture '51: does $n = 1$ work? **No** (Laffey '83)

Open questions

- Polynomial bounds for testing $\mathscr{Z}(L_A) \subseteq \mathscr{Z}(L_B)$ on n?
  If L is indecomposable, for which n is $\mathscr{Z}_n(L_A)$ irreducible?
  (currently, sufficient n is exponential in size of A, B)

- Intrinsic definition of a free locus $\mathscr{Z}$:
  family of hypersurfaces $\mathscr{Z}_n \subset \mathrm{M}_n(\Bbbk)^d$, $\mathrm{GL}_n(\mathbb{C})$-invariant,
  $\deg \mathscr{Z}_n$ grows linearly with n, $X \in \mathscr{Z} \implies X \oplus Y \in \mathscr{Z}$?

- $\mathscr{Z}_n^k(f) = \{X \in \mathrm{M}_n(\Bbbk)^d : \dim \ker f(X) \geq k\}$. Find reasonable
  $n = n(k, \deg f)$ such that $\mathscr{Z}_n^k(f) \neq \emptyset$.

- $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(g)$ for general f, g; possibly with some SOS
  techniques? Test case $\mathscr{Z}^{\mathrm{re}}(f) = \emptyset$.