Matrix inequalities, NC convexity, cp maps and realizations

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Outline

- Linear Matrix inequalities and Spectrahedra.
- Spectrahedral inclusions, cp maps and SDP. The matrix cube.
- Nonlinear Spectrahedral inclusion – a weighted SoS reps.
- Tracially convex sets and quantum channels.
- Convex NC rational functions and realizations.

Linear Pencil, Linear Matrix Inequality (LMI)

- For selfadjoint matrices $A_1 \dots, A_{\mathrm{g}} \in \mathbb{S}_d$, let
$$\Lambda_A(x) = A_1 x_1 + \cdots + A_{\mathrm{g}} x_{\mathrm{g}}.$$
The expression
$$L_A(x) = I - \left[A_1 x_1 + \cdots + A_{\mathrm{g}} x_{\mathrm{g}}\right] = I - \Lambda_A(x)$$
is a (monic) linear pencil of size d.
- $L_A(x) \succeq 0$ is a linear matrix inequality (LMI)
- Its (scalar) solution set
$$\begin{aligned} \mathcal{D}_A(1) &= \left\{x \in \mathbb{R}^{\mathrm{g}} \mid L_A(x) \succeq 0\right\} \\ &= \left\{x \in \mathbb{R}^{\mathrm{g}} \mid I - A_1 x_1 - \cdots - A_{\mathrm{g}} x_{\mathrm{g}} \succeq 0\right\}, \end{aligned}$$
is the feasibility set of an SDP. It is known as an LMI domain or a spectrahedron.

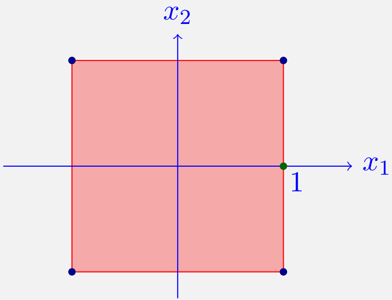
ℹ $\mathcal{D}_A(1) \subseteq \mathbb{R}^{\mathrm{g}}$ is a basic closed convex semialgebraic set.

LMIs: A few examples

Polyhedra

$$L_{\mathfrak{C}^2}(x_1, x_2) = I_4 + \begin{bmatrix} -1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} x_2$$

is a linear matrix pencil. $\mathcal{D}_{\mathfrak{C}^2}(1)$ is the square $[-1, 1]^2 \subseteq \mathbb{R}^2$:



Every (convex) polyhedron is a spectrahedron.

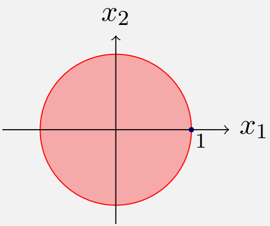
LMIs: A few examples

Balls

$$L_A(x_1,x_2) \;=\; I_3 + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix}$$

$$L_B(x_1,x_2) \;=\; I_2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1+x_1 & x_2 \\ x_2 & 1-x_1 \end{bmatrix}.$$

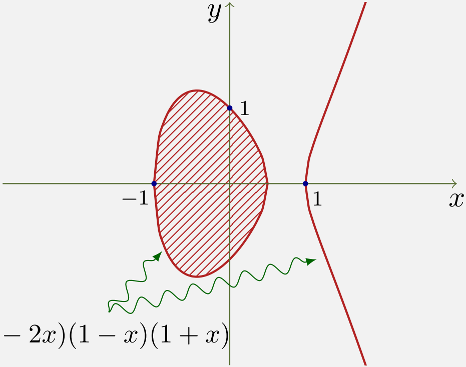
The spectrahedra $\mathcal{D}_A(1)$ and $\mathcal{D}_B(1)$ coincide:



LMIs: A few examples

The interior of an elliptic curve

$$L(x,y) = I + x \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{\sqrt{3}}\, y \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$y^2 = (1-2x)(1-x)(1+x)$$

Thank Igor

LMIs

Their importance: a small sample

- real algebraic geometry: positive polynomials, convexity
- operator algebra and operator theory: complete positivity, matrix convex sets;
- automata theory in theoretical computer science;
- quantum information theory;
- free analysis: nc rational functions
- control theory and optimization: many optimization problems in control theory, system identification and signal processing can be formulated using LMIs; optimizing a linear objective function over a spectrahedron is a semidefinite program (SDP) and can be solved efficiently;

Inclusion of Spectrahedra

Is $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$?

Ex: Determine ϱ such that $\varrho\,\mathfrak{C}^{\mathrm{g}}(1) = [-\rho, \rho]^{\mathrm{g}} \subseteq \mathcal{D}_B(1)$;

Check whether every $x \in [-\varrho, \varrho]^{\mathrm{g}}$ satisfies

$$I - \left[B_1 x_1 + \cdots + B_{\mathrm{g}} x_{\mathrm{g}}\right] \succeq 0. \qquad (\star)$$

Alternately, find the biggest ϱ for which every $x \in [-\varrho, \varrho]^{\mathrm{g}}$ satisfies $(\star)$.

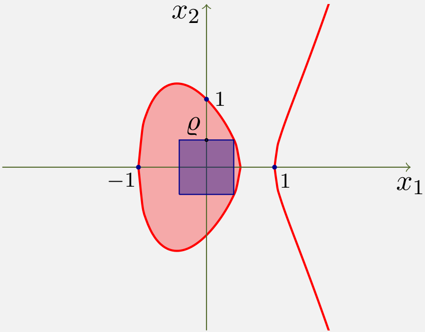
Inclusion of Spectrahedra

Including a cube

Determine ϱ such that $\varrho\,\mathfrak{C}^{\mathfrak{g}}(1) = [-\rho, \rho]^{\mathfrak{g}} \subseteq \mathcal{D}_B(1)$; Check whether every $x \in [-\varrho, \varrho]^{\mathfrak{g}}$ satisfies

$$I - \left[B_1 x_1 + \cdots + B_{\mathfrak{g}} x_{\mathfrak{g}}\right] \succeq 0. \qquad (\star)$$

Alternately, find the biggest ϱ for which every $x \in [-\varrho, \varrho]^{\mathfrak{g}}$ satisfies $(\star)$.



Spectrahedral Inclusion

... and positive maps

Given tuples $A \in M_d(\mathbb{C})^{\mathrm{g}}$ and $B \in M_e(\mathbb{C})^{\mathrm{g}}$, let

$$\mathcal{A} = \operatorname{span}\{I_d, A_1, \ldots, A_{\mathrm{g}}\},$$
$$\mathcal{B} = \operatorname{span}\{I_e, B_1, \ldots, B_{\mathrm{g}}\}.$$

$\mathcal{A}$ and $\mathcal{B}$ are examples of *unital operator systems*.

Assume $\mathcal{D}_A$ is bounded. The inclusion $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$ is equivalent to positivity of the unital linear map

$$\tau : \mathcal{A} \longmapsto \mathcal{B},$$
$$A_j \longmapsto B_j.$$

Positive maps can be evil, without redeeming structure.

Completely Positive Maps

Let $\mathcal{S} \subseteq M_d$ and $\mathcal{T} \subseteq M_e$ be linear subspaces of matrices containing I and invariant under adjoint (operator systems). A linear map $\varphi : \mathcal{S} \longmapsto \mathcal{T}$ is completely positive if, for each $n \in \mathbb{N}$, the linear map (ampliation of φ)

$$\varphi_n = \varphi \otimes I_n : \mathcal{S} \otimes M_n \to \mathcal{T} \otimes M_n$$

determined by

$$\varphi_n(S \otimes X) = \varphi(S) \otimes X$$

Completely Positive Maps

Let $\mathcal{S} \subseteq M_d$ and $\mathcal{T} \subseteq M_e$ be linear subspaces of matrices containing I and invariant under adjoint (operator systems). A linear map $\varphi : \mathcal{S} \longmapsto \mathcal{T}$ is completely positive if, for each $n \in \mathbb{N}$, the linear map (ampliation of φ)

$$\varphi_n \left(\begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} \right) := \begin{bmatrix} \varphi(S_{11}) & \cdots & \varphi(S_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(S_{n1}) & \cdots & \varphi(S_{nn}) \end{bmatrix} \in M_n(M_d),$$

where $S_{j,k} \in \mathcal{S}$. is positive.

- Transpose: $M_2 \to M_2$ is positive but not cp.

Completely Positive Maps ...

... have structure

Given linear subspaces $\mathcal{S} \subseteq M_d$ and $\mathcal{T} \subseteq M_e$ of matrices containing I and invariant under transpose (operator systems).

`Theorem.` [Arveson, Choi, Kraus, Stinespring]

- If $\varphi : \mathcal{S} \to \mathcal{T}$ is completely positive, then φ extends to a cp map $\varphi : M_d \to M_e$.
- $\varphi : M_d \to M_e$ is ucp if and only if there is an isometry $V : \mathbb{C}^d \to \mathbb{C}^e \otimes \mathbb{C}^d$ such that

$$\varphi(X) = V^* \left(I_{de} \otimes X\right) V = \sum^{de} V_j^* X V_j.$$

- if and only if the Choi matrix is PsD,

$$0 \preceq C_\varphi = \left(\varphi(E_{j,k})\right)_{j,k} = \begin{bmatrix} \varphi(E_{1,1}) & \dots & \varphi(E_{1,d}) \\ \vdots & \ddots & \vdots \\ \varphi(E_{d,1}) & \dots & \varphi(E_{d,d}) \end{bmatrix} \in M_d(M_e).$$

Intermezzo

Theorem. [Arveson] [Klep, M., Šivic, Zalar] The probability $p_{n,m}$ that a positive map $\Phi : \mathbb{S}_n \to \mathbb{S}_m$ is cp satisfies

$$(0,1) \ni p_{n,m} \approx \min(m,n)^{-1/2} \quad \text{as } m,n \to \infty.$$

Proof uses Real Algebraic Geometry, Convexity and Harmonic Analysis.

Φ is associated to the bihomogeneous biquadratic polynomial

$$q_\Phi(x,y) = y^t \Phi(xx^t) y.$$

- Φ is positive iff q_Φ is nonnegative on $\mathbb{R}^{n+m}$;
- Φ is cp iff q_Φ is a sum of squares (SoS) in $\mathbb{R}[x,y]$.

Greg Blekherman developed machinery to investigate the gap between SoS and positive polynomials.

Relax. Be Free

For $A_1, \ldots, A_g \in \mathbb{S}_d$

$$\Lambda_A(x) = \sum A_j x_j, \quad L_A(x) = I_d - \Lambda_A(x).$$

For $X = (X_1, \ldots, X_{\mathrm{g}}) \in \mathbb{S}_n^{\mathrm{g}}$

- $\Lambda_A(X) = \sum_j A_j \otimes X_j$ and

$$L_A(X) = I_d \otimes I_n - \Lambda_A(X).$$

- For each dimension $n \in \mathbb{N}$,

$$\mathcal{D}_A(n) := \left\{ X \in \mathbb{S}_n^{\mathrm{g}} \mid L_A(X) \succeq 0 \right\}$$

  are natural relaxations of $\mathcal{D}_A(1)$.

- Completely relaxed: free spectrahedron is

$$\mathcal{D}_A := (\mathcal{D}_A(n))_{n=1}^{\infty}.$$

Free Spectrahedra are Matrix Convex

Free spectrahedra are matrix convex.

- $\mathcal{D}_A$ is closed wrt direct sums: If $X \in \mathcal{D}_A(n)$ and $Y \in \mathcal{D}_A(m)$, then

$$X \oplus Y = (X_1 \oplus Y_1, \ldots, X_{\mathrm{g}} \oplus Y_{\mathrm{g}}) \in \mathcal{D}_A(n+m),$$

$$X_j \otimes Y_j = \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}$$

- $\mathcal{D}_A$ is closed under isometric similarity: If $X \in \mathcal{D}_A(n)$ and $V : \mathbb{C}^n \to \mathbb{C}^m$ is an isometry, then

$$V^* X V = (V^* X_1 V, \ldots V^* X_{\mathrm{g}} V) \in \mathcal{D}_A(m),$$

Since $(I_d \otimes V^*)\, L_A(X)(I_d \otimes V) = L_A(V^* X V)$.

Inclusion of Spectrahedra and Complete Positivity

Geometric Intepretation of Interpolation

$A \in \mathbb{S}_d^{\mathrm{g}}$ and $B \in \mathbb{S}_e^{\mathrm{g}}$. Suppose $\mathcal{D}_A(1)$ is bounded. The unital linear map

$$\tau : \operatorname{span}\{I, A_1, \ldots, A_{\mathrm{g}}\} \longmapsto \operatorname{span}\{I, B_1, \ldots, B_{\mathrm{g}}\}$$
$$A_j \longmapsto B_j$$
$$\varphi_n \left(I \otimes X_0 + \sum A_j \otimes X_j \right) = I \otimes X_0 + \sum B_j \otimes X_j .$$

- is positive if and only if $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$;
- is completely positive if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$. Ahh. Relax.
- $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$ can be NP hard.
- The free relaxation $\mathcal{D}_A \subseteq \mathcal{D}_B$ is an SDP and implies $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$.

Free inclusion as an SDP ...

... return of the Choi matrix

$$A \in \mathbb{S}_d^{\mathrm{g}}, \quad B \in \mathbb{S}_e^{\mathrm{g}}$$

$$\varphi\left(I \otimes X_0 + \sum A_j \otimes X_j\right) = I \otimes X_0 + \sum B_j \otimes X_j.$$

The map φ is cp if and only if there is a solution to the SDP:

- $C \in M_d(M_e)$ (Choi Matrix).
- $C \succeq 0$; (positivity)
- $\sum C_{jj} = I$ (unital);
- $\sum_{s,t} (A_j)_{s,t} C_{s,t} = B_j$ $(\varphi(A_j) = B_j)$.

if and only if $\varphi(X) = \sum V_j^* X V_j$, for some $\sum V_j^* V_j = I$.

Free inclusion as a relaxation ...

. . . ... just how relaxed is it?

$A \in \mathbb{S}_d^{\mathrm{g}}$ and $B \in \mathbb{S}_e^{\mathrm{g}}$.

- $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$ can be NP hard.
- The free relaxation $\mathcal{D}_A \subseteq \mathcal{D}_B$ is an SDP of size de and implies $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$.
- What is the smallest $\rho = \rho(A, e)$, depending only upon the size e of B, such that

$$\rho\, \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies \mathcal{D}_A \subseteq \mathcal{D}_B?$$

We call ρ the A-inclusion scale.

☞ Dilation theory provides an answer.

ℹ Recall: X dilates to/is a compression of T if

$$T = \begin{pmatrix} X & * \\ * & * \end{pmatrix}, \quad X = V^* T V.$$

Dilation theory . . .

. . . and the error of the free relaxation

If $X_j = V^* T_j V$ and $T \in \mathcal{D}_A$, then $X \in \mathcal{D}_A$ by matrix convexity.

The largest $\gamma = \gamma(A, n)$ such that for each $X \in \mathcal{D}_A(n)$ the tuple γX dilates to a commuting tuple of self-adjoint operators with joint spectrum in $\mathcal{D}_A(1)$ is called the A-commutability index.

- $T_j = \begin{pmatrix} \gamma X_j & * \\ * & * \end{pmatrix}$;
- $T_j T_k = T_k T_k$; and $T_j^* = T_j$,
- $\sigma(T) \subset \mathcal{D}_A(1)$.

Dilation theory . . .

. . . and the error of the free relaxation

`Theorem.` [Commutability index = Inclusion scale] (Bounded $\mathcal{D}_A(1)$)

The commutability index for A equals its inclusion scale,

$$\rho(A, e) = \gamma(A, e).$$

That is $\gamma(A, e)$ is the largest constant such that

$$\gamma(A, e)\ \mathcal{D}_A \subseteq \mathcal{D}_B$$

for each $B \in \mathbb{S}_e^{\mathrm{g}}$ satisfying $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$.

- If $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$ and if $T \in \mathcal{D}_A(e)$ is a commuting tuple, then $T \in \mathcal{D}_B(e)$.

Matrix Cube Problem

Nemirovskii, ICM 2006

Given a tuple $B \in \mathbb{S}_\nu^{\mathrm{g}}$ does

$$[-1,1]^{\mathrm{g}} \subseteq \mathcal{D}_B(1) \ ?$$

`Matrix Cube Relaxation.` Does the (free) matrix cube

$$\mathfrak{C}^{\mathrm{g}} = \{(X_1, \ldots, X_{\mathrm{g}}) : \|X_j\| \leq 1\}$$

include into $\mathcal{D}_B$?

Determining the universal inclusion scale

$$\rho(\nu) = \sup\{\rho(\mathfrak{C}^{\mathrm{g}}, \nu) : \mathrm{g}\}$$

is the matrix cube problem of Ben-Tal and Nemirovskii.

Matrix Cube Problem

Nemirovskii, ICM 2006

Determining the universal inclusion scale

$$\rho(\nu) = \sup\{\rho(\mathfrak{C}^{\mathrm{g}}, \nu) : \mathrm{g}\}$$

is the matrix cube problem of Ben-Tal and Nemirovskii.

- problems of robust control, e.g. Lyapunov stability analysis for uncertain dynamical systems;
- various combinatorial problems can be reduced to maximizing a positive definite quadratic form over the unit cube;
- the case $\nu = 2$ contains the Nesterov $\frac{\pi}{2}$-Theorem about optimizing a positive definite quadratic form over the unit cube. Also the symmetric (little) Grothendieck inequality.

Matrix Cube Problem

Nemirovskii, ICM 2006

Determining the universal inclusion scale

$$\rho(\nu) = \sup\{\rho(\mathfrak{C}^{\mathrm{g}}, \nu) : \mathrm{g}\}$$

is the matrix cube problem of Ben-Tal and Nemirovskii.

`Free relaxation.` Given A such that $\mathcal{D}_A(1) = [-1,1]^{\mathrm{g}}$, solve the SDP $\mathcal{D}_A \subseteq \mathcal{D}_B$.

- the most conservative choice of A is $\mathfrak{C}^{\mathrm{g}}$, the matrix cube. Any other choice of A satisfies $\mathcal{D}_A \subset \mathcal{D}_{\mathfrak{C}^{\mathrm{g}}}$.
- The size of the tuple $\mathfrak{C}^{\mathrm{g}}$ is $2\mathrm{g}$ and hence the SDP has size $2g\nu$.
- Other choices of A that potentially produce better inclusion scales have larger size and thus lead to larger SDPs. There is no optimal choice.

Spectrahedral Inclusions

The theorem of Ben-Tal and Nemirovski

`Matrix Cube Problem.` [Ben-Tal and Nemirovski]. If $B \in \mathbb{S}_n(\mathbb{C})^{\mathfrak{g}}$, the ranks of the B_j are at most ν and $\mathcal{D}_{\mathfrak{C}\mathfrak{g}}(1) \subseteq \mathcal{D}_B(1)$, then $\mathcal{D}_{\mathfrak{C}\mathfrak{g}} \subseteq \vartheta(\nu)\mathcal{D}_B$, where

$$\frac{1}{\vartheta(\nu)} = \min\{\int_{S^{\nu-1}} |\xi^* T\xi| \, d\xi : T \in \mathbb{S}_\nu, \quad \mathrm{tr}\,|T| = \nu\} \geq \frac{2}{\pi\sqrt{\nu}}.$$

`Theorem.` [HKMS] The estimate $\vartheta(\nu)$ is sharp and for ν even

$$\vartheta(\nu) = \frac{\sqrt{\pi}\,\Gamma(1+\frac{\nu}{4})}{\Gamma(\frac{1}{2}+\frac{\nu}{4})} \approx \frac{\pi\sqrt{\nu}}{2}.$$

How good is the relaxation?

The matrix cube meets dilation theory

(1) The bound $\vartheta(\nu)$ is a corollary of a Dilation Theoretic result.

(2) Worst case analysis (inspired by dilation theory) shows the bound is sharp;

(3) New results for the beta distribution identify $\vartheta(\nu)$ analytically.

How good is the relaxation?

Proof ingredients. Dilation theory

$$\frac{1}{\vartheta(\nu)} = \min_{\substack{T \in \mathbb{S}_\nu \\ \operatorname{trace}|T| = \nu}} \int_{S^{\nu-1}} |\xi^* T \xi| \, d\xi \approx \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{d}}.$$

Theorem. [Simultaneous Dilation] Let $d \in \mathbb{N}$. There is a Hilbert space $\mathcal{K}$, a family $\mathscr{C}_d$ of commuting self-adjoint contractions on $\mathcal{K}$, and an isometry $V : \mathbb{R}^d \to \mathcal{H}$ such that for each symmetric $d \times d$ contraction matrix X there exists a $T \in \mathscr{C}_d$ such that

$$X = \vartheta(d)\, V^* T V.$$

Moreover, $\vartheta(d)$ is the smallest such constant.

Simultaneous Dilation Theorem

Let $d \in \mathbb{N}$. There is a Hilbert space $\mathcal{K}$, a family $\mathscr{C}_d$ of commuting self-adjoint contractions on $\mathcal{K}$, and an isometry $V : \mathbb{R}^d \to \mathcal{K}$ such that for each symmetric $d \times d$ contraction matrix X there exists a $T \in \mathscr{C}_d$ such that $X = \vartheta(d)\ V^*TV$. Moreover, $\vartheta(d) \approx \frac{\sqrt{d\pi}}{2}$ is the smallest such constant.

- $\mathcal{O}(d) \subseteq M_d(\mathbb{R})$ is the orthogonal group with its Haar measure dU;
- Let $\mathcal{K} = \mathbb{R}^d \otimes L^2(dU) = L^2_{\mathbb{R}^d}(dU)$ square integrable functions $f : \mathcal{O}(d) \to \mathbb{R}^d$;
- Define $V : \mathbb{R}^d \to \mathcal{K}$ by $Vx(U) = x$ ($Vx =$ constant function x);
  $V^*f = \int_{\mathcal{O}(d)} f(U)\, dU$;
- $\mathscr{D}_d$ is the set of contractive $d \times d$ diagonal matrices;
- For $D : \mathcal{O}(d) \to \mathscr{D}_d$, define the twisted multiplication operator $M_D : \mathcal{K} \to \mathcal{K}$,
$$M_D f(U) = UD(U)U^*\, f(U);$$
- $\|M_D\| \leq \|D\|_\infty \leq 1$;
- If $E : \mathcal{O}(d) \to \mathscr{D}_d$, then $M_D M_E = M_E M_D$ (D and E pointwise commute);
- $\mathscr{C}_d = \{M_D \mid D : \mathcal{O}(d) \to \mathscr{D}_d\}$ is a collection of commuting self-adjoint contractions on $\mathcal{K}$.

Linear Mutual Domination ...

... and the Linear Gleichstellensatz

- The LMI L_A dominates L_B if $L_A(X) \succeq 0$ implies $L_B(X) \succeq 0$.
- Equivalently $\mathcal{D}_A \subseteq \mathcal{D}_B$.
- Mutual domination is then $\mathcal{D}_A = \mathcal{D}_B$.

`Linear Gleichstellensatz.` If A and B are minimal and $\mathcal{D}_A = \mathcal{D}_B$, then $A = U^*BU$ for some unitary U.

- One proof uses the existence of the Silov ideal (boundary reps for operator systems).

Free analytic polynomials

Warning: Switching to not so selfadjoint matrices - $M(\mathbb{C})$.

- $x = (x_1, \ldots, x_g)$ freely noncommuting indeterminants;
- $\langle x \rangle$ the set of words in x,

$$\alpha = x_{i_1} x_{i_2} \cdots x_{i_m}.$$

- $\mathbb{C}\langle x \rangle$, the free polynomials:

$$p(x) = \sum_{\alpha \in \langle x \rangle}^{\text{finite}} p_\alpha \, \alpha, \quad p_\alpha \in \mathbb{C};$$

- Matrix-valued free polynomials, defined in the obvious way(s),

$$p(x) = \sum_{\alpha \in \langle x \rangle}^{\text{finite}} p_\alpha \otimes \alpha \in M_\mu(\mathbb{C}\langle x \rangle), \quad p_\alpha \in M_\mu(\mathbb{C}).$$

Free polynomials ...

... and their evaluations

- $M_n(\mathbb{C})^g$, the set of g-tuples $X = (X_1, \dots, X_g)$ of $n \times n$ matrices;
- $M(\mathbb{C})^g = (M_n(\mathbb{C})^g)_n$;
- given $\langle x \rangle \ni \alpha = x_{i_1} x_{i_2} \cdots x_{i_m}$, and $X \in M(\mathbb{C})^g$,
$$X^\alpha = X_{i_1} X_{i_2} \cdots X_{i_m};$$
- $p = \sum p_\alpha \otimes \alpha \in M_\mu(\mathbb{C}\langle x \rangle)$ is evaluated at $X \in M_n(\mathbb{C})^g$ by
$$p(X) = \sum p_\alpha \otimes X^\alpha \in M_\mu(\mathbb{C}) \otimes M_n(\mathbb{C}).$$

Free polynomials ...

... invariance properties

We view $p \in \mathbb{C}\langle x \rangle$ as a function (sequence of functions $(p[n])$), $p : M(\mathbb{C})^{\mathrm{g}} \to M(\mathbb{C})$ ($p[n] : M_n(\mathbb{C})^{\mathrm{g}} \to M_n(\mathbb{C})$).

- p respects direct sums; For $X \in \mathcal{D}_A(n)$ and $Y \in \mathcal{D}_A(m)$,

$$p(X \oplus Y) = p(X) \oplus p(Y).$$

- p respects unitary similarity: For $X \in \mathcal{D}_A(n)$ and $U \in M_n$ unitary,

$$p(U^* X U) = U^* p(X) U.$$

Free analytic functions

... informally

A free domain $\mathcal{S} \subset M(\mathbb{C})^{\mathrm{g}}$ is a sequence $S = (S_n)$ such that

- $\mathcal{S}_n \subset M_n(\mathbb{C})^{\mathrm{g}}$.
- $\mathcal{S}$ is closed with respect to direct sums: If $X \in S_n$ and $Y \in S_m$, then
$$X \oplus Y \in \mathcal{S}_{n+m}$$
- $\mathcal{S}$ is closed under unitary similarity: if $X \in \mathcal{S}_n$ and $U \in M_n(\mathbb{C})$ is unitary, then
$$U^* X U \in \mathcal{S}_n.$$

A free (analytic) function $f : \mathcal{S} \to M(\mathbb{C})$ is a sequence $f_n : \mathcal{S}_n \to M_n(\mathbb{C})$ satisfying

- $f(X \oplus Y) = f(X) \oplus f(Y)$.
- $f(U^* X U) = U^* f(X) U$.

Free Rational Functions

- A free rational function (regular at 0) has a realization

$$q = b^*(I - \Lambda_E(x))^{-1}c, \quad E \in M_e(\mathbb{C})^h, \quad b, c \in \mathbb{C}^h.$$

- Up to poles, it defines a free analytic function

$$q : M(\mathbb{C})^{\mathrm{g}} \to M(\mathbb{C}).$$

- There is a minimal choice.
- Its domain is the set where $I - \Lambda_E(X)$ is invertible. (Victor and Jurij)

Beyond Linear

For $A \in M_d(\mathbb{C})^{\mathrm{g}}$,

$$L_A(x) = I - \Lambda_A(x) - \Lambda_A(x)^*.$$

and

$$\mathcal{D}_A = \{X : L_A(X) \succeq 0\}.$$

$\mathcal{D}_A \subseteq \mathcal{D}_B$ is equivalent to the inclusion

$$\mathrm{id} : \mathcal{D}_A \to \mathcal{D}_B.$$

A nonlinear version of this inclusion is

$$f : \mathcal{D}_A \to \mathcal{D}_B, \quad f(0) = 0, \quad f'(0) = I.$$

The analytic hereditary convex postivstellensatz

A shaggy dog story

- $f : \mathcal{D}_A \to \mathcal{D}_B$ iff $L_A(X) \succ 0 \implies L_B \circ f(X) \succ 0.$
- Equivalently,

$$L_B \circ f = I - \Lambda_A(f(x)) - \Lambda_A(f(x))^* \text{ is positive on } \mathcal{D}_A.$$

`Theorem.` [Special case of the AHCP] ($\mathcal{D}_A$ bounded). If $f : \mathcal{D}_A \to \mathcal{D}_B$ and f is defined and bounded on some free pseudoconvex set containing $\mathcal{D}_A$, then there exists a Hilbert space $\mathcal{H}$ and W a formal power series with coefficients $W_\alpha : \mathbb{C}^e \to \mathcal{H} \otimes \mathbb{C}^d$ such that

$$L_B(f(x)) = W(x)^* L_{I_{\mathcal{H}} \otimes A}(X) W(x).$$

Nonlinear Mutual Domination

The equality $\mathcal{D}_A = \mathcal{D}_B$ says that the identity map $\mathrm{id} : \mathcal{D}_A \to \mathcal{D}_B$ is bianalytic (at each level).

When does there exist a binalytic free map $f : \mathcal{D}_A \to \mathcal{D}_B$ s.t.

- $f(0) = 0$; and
- $f'(0) = I$?

ℹ An analog of rigidity in several complex variables.

Nonlinear Mutual Domination

Theorem [AHKMV] Suppose $A \in M_d(\mathbb{C})^{\mathrm{g}}$ and $B \in M_e(\mathbb{C})^{\mathrm{g}}$.

If $f : \mathcal{D}_A \to \mathcal{D}_B$ is free bianalytic mapping such that

- $f(0) = 0$; and
- $f'(0) = I$,
- plus some natural irreducibility/minimality hypotheses on A and B,

then

(i) $d = e$;

(ii) $B = U^* CAU$, where C and U are unitary;

(iii) $(C - I)A$ spans an algebra;

(iv) f is a free rational mapping.

Convexotonic maps

- Let $R_j = (C - I)A_j$ so that $R = (C - I)A$.
- Since R spans an algebra,

$$R_j R_k = \sum_{s=1}^{\mathrm{g}} (\Xi_j)_{k,x} R_s$$

- It follows that

$$\Xi_j \Xi_k = \sum_{s=1}^{\mathrm{g}} (\Xi_j)_{k,s} \Xi_s.$$

-

$$f(x) = x\,(I - \Lambda_\Xi(x))^{-1} = \left(x_1, \dots, x_{\mathrm{g}}\right) \left(I - \sum \Xi_j x_j\right)^{-1}.$$

Matrix Convex Hulls ...

... and images of ucp maps

Unwarning - return to selfadjoint matrices - $\mathbb{S}$.

- The matrix convex hull of a free set $\mathcal{S}$ is

$$\operatorname{matco}\mathcal{S} = \{V^*XV : X \in \mathcal{S}, \quad V^*V = I\}.$$

- Given $A \in \mathbb{S}_d^{\mathrm{g}}$ and $B \in \mathbb{S}_d^{\mathrm{g}}$, there is a ucp map $\varphi(A_j) = B_j$ if and only if $B \in \operatorname{matco}\{A\}$ – ucp interpolation.

$$\begin{aligned}\operatorname{matco}\{A\} =& \{\varphi(A) : \varphi \text{ is a ucp map}\} \\ =& \{\sum V_j^* A V_j : \sum V_j^* V_j = I\}.\end{aligned}$$

Matrix Convex Sets ...

... and the Effros Winkler Theorem

Recall free spectrahedra $\mathcal{D}_A$ are matrix convex.

`Effros-Winkler Separation Theorem.` [Special Case] If $\mathcal{K} \subseteq \mathbb{S}^{\mathrm{g}}$ is closed, matrix convex and contains 0 and if $Y \in \mathbb{S}_\ell^{\mathrm{g}} \setminus \mathcal{K}(\ell)$, then there is a $A \in \mathbb{S}_\ell^{\mathrm{g}}$ such that $Y \notin \mathcal{D}_A \supset \mathcal{K}$.

- Thus $\mathcal{K}$ is an intersection of free spectrahedra.

Intermezzo

convex free basic semialgebraic sets

Suppose q is a free (matrix-valued symmetric) rational function. The set

$$\mathfrak{P}_{\mathfrak{q}} = \{X \in \mathbb{S}^{\mathrm{g}} : q(X) \succeq 0\}^{\circ} \quad \text{(component of } 0\text{)}$$

is a free set. It is convex at each level if and only if it is matrix convex.

`Theorem.` [Helton, M] $\mathfrak{P}_{\mathfrak{q}}$ is convex iff it is a free spectrahedron.

Tracially convex sets

Replacing ucp maps with quantum channels trace non-increasing cp maps leads to the notion of tracial convexity.

A subset $\mathcal{S} \subseteq \mathbb{S}^g$ is tracially convex provided:

- It is closed wrt convex direct sums;

$$\lambda_k \geq 0, \sum \lambda_k \leq 1, \, X_k \in \mathcal{S} \implies \oplus \lambda_k X_k \in \mathcal{S}.$$

- S is closed under co-contractive similarity: If $X \in \mathcal{S}$ and $\sum V_j^* V_j \preceq I$, then

$$\sum V_j X V_j^* \in \mathcal{S}.$$

- The tracial hull of a point is

$$\begin{aligned} \operatorname{traceco}\{Z\} &= \{\sum V_j Z V_j^* : \sum V_j^* V_j \preceq I\} \\ &= \{X : X = \varphi(Z) \text{ for a quantum channel } \varphi\}. \end{aligned}$$

Matrix vs. Tracial convexity ...

... go toe to toe

$$\begin{aligned}
\text{matco}\{Z\} &= \{\sum V_j^* Z V_j : \sum V_j^* V_j = I\} \\
\text{matco}\{Z, 0\} &= \{\sum V_j^* Z V_j : \sum V_j^* V_j \preceq I\} \\
\text{traceco}\{Z\} &= \{\sum V_j Z V_j^* ; \sum V_j^* V_j \preceq I\}
\end{aligned}$$

Tracially convex sets

The tracial EW Theorem

A tuple $B \in \mathbb{S}_n^{\mathrm{g}}$ determines a *tracial spectrahedron*,

$$\mathcal{H}_B = \{X : \exists T \succeq 0,\ \operatorname{tr}(T) \leq 1,\ \ T \otimes I - \sum B_j \otimes X_j \succeq 0\}$$
$$\mathcal{D}_B = \{X : \quad I \otimes I - \sum B_j \otimes X_j \succeq 0\}.$$

$\mathcal{H}_B$ is tracially convex and closed.

`Theorem.` [HKM] (Tracial Effrow-Winkler Separation Theorem).

If $\mathcal{S}$ is closed and tracially convex, then

$$\mathcal{S} = \bigcap \{\mathcal{H}_B : \mathcal{H}_B \supset \mathcal{S}\}.$$

Quantum Channel Interplation ...

... as an SDP

There is a quantum channel φ such that $\varphi(A_j) = B_j$
if and only if
there is a solution to the SDP:

- $C \in M_d(M_e)$ (Choi Matrix).
- $C \succeq 0$; (positivity)
- $\sum_{s,t}(A_j)_{s,t}C_{s,t} = B_j$ ($\varphi(A_j) = B_j$).
- $(\mathrm{tr}(C_{p,q}))_{p,q} \preceq I$ (trace nonincreasing).

A symmetric NC rational function has a descriptor realization,

$$r(x) = b^* \left[J - \Lambda_E(x)\right]^{-1} b,$$

where, for some $d \in \mathbb{N}^+$,

- $E = (E_1, \dots, E_{\mathsf{g}}) \in \mathbb{S}_d^{\mathsf{g}}$; $b \in \mathbb{C}^d$;
- $\Lambda_E(x) = \sum E_j x_j$;
- $J = J^* = J^{-1}$ (signature matrix).

It is evaluated at $X \in \mathbb{S}_n^{\mathsf{g}}$ as

$$r(X) = (b^* \otimes I_n) \left[J \otimes I_n - \Lambda_E(X)\right]^{-1} (b \otimes I_n) \in \mathbb{S}_n.$$

Convex Rational Functions

Convexity near 0

$$r(x) = b^*[J - \Lambda_E(x)]^{-1} b, \quad \text{descriptor realization}$$

The rational function r is convex on a free set $S \subseteq \mathbb{S}^{\mathrm{g}}$ if

$$r\left(\frac{X+Y}{2}\right) \preceq \frac{1}{2}\left(r(X) + r(Y)\right), \quad X, Y \in S(n)$$

Convex Rational Functions

Convexity near 0

$$r(x) = b^*[J - \Lambda_E(x)]^{-1}b,$$

$$r\left(\frac{X+Y}{2}\right) \preceq \frac{1}{2}\left(r(X) + r(Y)\right).$$

Theorem. [mashup] Assume the realization is minimal.

- r is convex in a nghd of 0 if and only if $P_{\mathrm{range}\,E} J P_{\mathrm{range}\,E} \succeq 0$;
- r is convex on the set

$$\operatorname{dom} r^+ = \{X : P_{\mathrm{range}\,E}\,[J - \Lambda_E(x)]^{-1}\,P_{\mathrm{range}\,E} \succeq 0\}.$$

Convex Rational Functions

Butterfly/Kraus realizations

$r(x) = b^*[J - \Lambda_E(x)]^{-1}b,$ descriptor realization

Theorem. [HMV] r is convex in a nghd of 0

iff

it admits a minimal butterfly realization

$$r(x) = a + \ell(x) + b(x)^* \; [I - \Lambda_A(x)]^{-1} \; b(x),$$

- In this case, r is convex on the set $I - \Lambda_A(X) \succ 0$.
- If r is a polynomial, then $r = a + \ell(x) + b(x)^* b(x)$ (quadratic);
- $f(x) = a + cx + \int_{-1}^{1} \frac{x^2}{1-tx}\, d\mu(t)$.

Partial Convexity

... a reality check

Let $(x, z) = (x_1, \ldots, x_{\mathtt{g}}, z_1, \ldots, z_{\mathtt{h}})$ be a tuple of freely noncommuting variables.

- Let $r(x, z)$ be a rational function, with descriptor realization,

$$r(x, z) = b^* \left[J - \Lambda_T(x) - \Lambda_S(z)\right]^{-1} b;$$

- r is convex in x (on a free set $S \subseteq \mathbb{S}^{\mathtt{g}} \times \mathbb{S}^{\mathtt{h}}$) if

$$r\left(\frac{X+Y}{2}, Z\right) \preceq \frac{1}{2}\left(r(X, Z) + r(Y, Z)\right);$$

- Let $R(x, z) = P_{\mathrm{range}\, T} \left[J - \Lambda_T(x) - \Lambda_S(z)\right]^{-1} P_{\mathrm{range}\, T}$.
- Let $\mathrm{dom}\, r^+ = \{(X, Y) : R(X, Y) \succeq 0\}$

Partial Convexity

… a reality check

- Let $r(x, z)$ be a rational function, with descriptor realization,

$$r(x, z) = b^* \left[J - \Lambda_T(x) - \Lambda_S(z)\right]^{-1} b;$$

- Let $R(x, z) = P_{\mathrm{range}\,T} \left[J - \Lambda_T(x) - \Lambda_S(z)\right]^{-1} P_{\mathrm{range}\,T}$.
- Let $\operatorname{dom} r^+ = \{(X, Y) : R(X, Y) \succeq 0\}$

`Theorem.` [JKMMP]

(1) $\operatorname{dom} r^+$ is the largest set on which r is convex in x.

- If $\operatorname{dom} r^+ \neq \emptyset$, then r has a root butterfly realization.