A brief introduction to quantum groups

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May 5, 2020
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The goal of this talk is to review some of the main ideas and examples of quantum groups and briefly describe some of the applications.
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Quantum physics:
A is deformed to a non-commutative (but still associative) algebra A_\hbar with quantization parameter \hbar, e.g., the Heisenberg uncertainty relation [p, x] = -i\hbar.
What if $X = G$ is a group?

A group is equipped with an associative product, which has a unit and all elements are invertible:

$m : G \times G \to G,\ m(x, y) = xy, (xy)z = (xy)z,\ \exists e : \forall g \in G : eg = ge = g, \forall g \exists g^{-1} : gg^{-1} = g^{-1}g = e.$

Thus the algebra $A = O(G)$ of functions on a (say, finite) group has a natural structure of a coalgebra. Namely, decomposing $O(G \times G)$ as $O(G) \otimes O(G)$, one gets comultiplication, or coproduct on $A$ from the multiplication $m$ in $G$:

$\Delta : A \to A \otimes A : (\Delta f)(x, y) = f(xy) = f(1)(x) \otimes f(2)(y).$

Here we used Sweedler's notation for the coproduct $\Delta f = f(1) \otimes f(2)$ where summation is implied, i.e., $f(1) \otimes f(2)$ is really $\sum_i f_i(1) \otimes f_i(2).$
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The algebra $A$ also has a natural counit and antipode obtained from the unit and inversion in $G$: 

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The following properties could be easily checked:

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- \(\mu \circ (S \otimes \text{id}) \circ \Delta(x) = \mu \circ (\text{id} \otimes S) \circ \Delta(x) = \varepsilon(x),\)
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Definition

A quantum group or Hopf algebra is a unital associative algebra $A$ (not necessary commutative) which is equipped with $\Delta, \varepsilon, S$ and has the properties listed above.
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Proposition 1. If $A$ is a finite dimensional Hopf algebra then $A^*$ is also, with the operations of $A^*$ being dual to the operations of $A$.

2. $\varepsilon$ is an algebra homomorphism.

3. $S$ is an algebra and coalgebra antihomomorphism, i.e., $S(xy) = S(y)S(x)$ and $\Delta(S(x)) = (S \otimes S)(\Delta^{\text{op}}(x))$.

4. $\varepsilon$ and $S$ are uniquely determined by $\Delta$.

5. If $A$ is commutative or cocommutative (i.e., $\Delta = \Delta^{\text{op}}$) then $S^2 = \text{id}$ (even without the assumption that $S$ is invertible).
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### Examples of Hopf algebras

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1. \( A = \mathcal{O}(G) \), \( G \) is finite. Then \( A \) is commutative.

2. \( A = \mathcal{O}(G) \) (the algebra of regular functions), \( G \) is an affine algebraic group, \( A \) is commutative.

3. \( A = \mathbb{C}G \) - the group algebra,
   \( \Delta(g) = g \otimes g \), \( S(g) = g^{-1} \), \( \varepsilon(g) = 1 \), \( A \) is cocommutative: \( \Delta = \Delta^{\text{op}} \).

4. \( g \) a Lie algebra, \( A = U(g) \) (the universal enveloping algebra),
   \( \Delta(x) = x \otimes 1 + 1 \otimes x \), \( S(x) = -x \), \( \varepsilon(x) = 0 \) for \( x \in g \), \( A \) is cocommutative.
Let \( q \in \mathbb{C}, q \neq 0, \pm 1 \). The quantum group \( U_q(\mathfrak{sl}_2) \) is generated by \( e, f, K^{\pm 1} \) with relations

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This is a deformation of $U(\mathfrak{sl}_2)$ because if one sets $K = q^h$ and sends $q \to 1$, one recovers the $\mathfrak{sl}_2$ relations.

This example shows that $S^2 \neq \text{id}$ in general: we have $S^2(x) = KxK^{-1}$. 
Assume that $q$ is not a root of unity.
Representations of $U_q(\mathfrak{sl}_2)$

Assume that $q$ is not a root of unity. Then the representation theory of $U_q(\mathfrak{sl}_2)$ is very similar to the representation theory of $\mathfrak{sl}_2$.

Proposition

Finite dimensional representations of $U_q(\mathfrak{sl}_2)$ are semisimple. So it remains to classify the irreducible f.d. representations $V$.

We say that $V$ is of type I if the eigenvalues of $K$ on $V$ are integer powers of $q$.

E.g., the character $\chi: U_q(\mathfrak{sl}_2) \to \mathbb{C}$ given by $\chi(e) = \chi(f) = 0$, $\chi(K) = -1$ is not of type I.

However, if $V$ is not of type I then it has the form $V = V + \otimes \chi$ where $V +$ is of type I.

Thus it suffices to classify irreducibles of type I.

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[e f_j v] &= \begin{cases} \[j] q \[n-j+1] f_{j-1} v & \text{if } j > 0, \\
0 & \text{if } j = 0. \end{cases}
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For a group and a Lie algebra the representation categories $\text{Rep } G$ and $\text{Rep } g$ are endowed with tensor products:

\[ \pi_V \otimes W : (g) = \pi_V(g) \otimes \pi_W(g) \quad \forall g \in G. \]

The formula for the tensor product of representations of a Hopf algebra $A$ is a straightforward generalization:

\[ \pi_V \otimes W : (x) = (\pi_V \otimes \pi_W)(\Delta(x)) = \pi_V(x(1)) \otimes \pi_W(x(2)) \quad \forall x \in A. \]

So one can regard the category $C = \text{Rep } A$ of representations of $A$ as a category equipped with a tensor product bifunctor $\otimes : C \times C \to C$:

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More precisely, a **much better notion** is obtained if, according to the **general yoga of category theory**, we don’t just say simply that \((X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)\), but make this isomorphism a part of the data and impose coherence conditions on this data.
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More precisely, a much better notion is obtained if, according to the general yoga of category theory, we don’t just say simply that $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$, but make this isomorphism a part of the data and impose coherence conditions on this data. This leads to the notion of a monoidal category.
Monoidal categories

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In a similar way, comodules over $A$ (i.e., spaces $V$ with a linear map $\rho : V \rightarrow A \otimes V$ defining an action of the algebra $A^*$ on $V$) form a monoidal category $\text{Comod}\, A$. 
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Duality in monoidal categories

Let us now discuss duality for representations of Hopf algebras, which generalizes duality for group and Lie algebra representations.
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\begin{align*}
\pi_{X^*}(a) &= \pi_X(S(a)) \\
\pi_{{}^*X}(a) &= \pi_X(S^{-1}(a))
\end{align*}
\]

For the pair \( X^*, {}^*X \) there is the evaluation morphism \( X^* \otimes X \to 1 \) (the usual pairing). For finite dimensional representations there is also the coevaluation morphism \( 1 \to X \otimes X^* \). And a pair of functorial isomorphisms \( ({}^*X)^* = X, \quad {}^*(X^*) = X \).
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(the usual pairing). For finite dimensional representations there is also the coevaluation morphism

\[1 \rightarrow X \otimes X^*.\]

and a pair of functorial isomorphisms

\[(\ast X)^* = X, \quad *(X^*) = X.\]
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Duality in monoidal categories, ctd.

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ev : Y \otimes X \to 1, \quad \coev : 1 \to X \otimes Y,$$

such that the following morphisms are the identities:

$$X \coev \otimes 1 \Rightarrow (X \otimes Y) \otimes X \alpha_{XYX} \Rightarrow X \otimes (Y \otimes X),$$

$$Y \otimes \coev \Rightarrow Y \otimes (X \otimes Y) \alpha^{-1}_{1YX} \Rightarrow (Y \otimes X) \otimes Y \ev \otimes 1 \Rightarrow Y.$$

If $Y$ is a left dual to $X$ then we have a functorial isomorphism

$$\text{Hom}(Z, Y) \cong \text{Hom}(Z \otimes X, 1).$$

By the Yoneda lemma, this implies that the left dual, if exists, is unique up to a unique isomorphism.
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An object $X$ is **rigid** if it has both the left and the right dual.

A category $C$ is **rigid** if all its objects are rigid.

Thus, if $A$ is a Hopf algebra then the category $\text{Rep}_f A$ of finite dimensional representations of $A$ is a rigid monoidal category.
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Examples of rigid monoidal categories

Example

Let $G$ be a finite group, $A = \mathcal{O}(G)$, then $\text{Rep}_f A$ is spanned by 1-dimensional representations parametrized by $g \in G$. 

The tensor product is defined by $g \otimes h = gh$, and $g^* = g^{-1}$. Thus $\text{Rep}_f A$ is a rigid monoidal category. We denote it by $\text{Vec}(G)$ ($G$-graded vector spaces).

This category makes sense for any group $G$ (not necessarily finite).

Example

The previous example has the following twisted version. Let $\alpha_{g, h, k}: (g \otimes h) \otimes k \to g \otimes (h \otimes k) = ghk$, i.e. $\alpha$ satisfies the pentagon identity $\Leftrightarrow \alpha$ is a 3-cocycle of the group $G$. Then $\alpha$ satisfies the pentagon identity $\Leftrightarrow \alpha$ is a 3-cocycle of the group $G$. If so then this equips $C = \text{Vec}(G)$ with another structure of a rigid monoidal category (with the same tensor product functor but different associativity isomorphism). We will denote this category $\text{Vec}(G, \alpha)$. 

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**Definition**

A functor \( F : C \to D \) is a **monoidal functor** if \( F(1_C) \cong 1_D \) and \( F \) is equipped with a functorial (in \( X, Y \)) isomorphism \( J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y) \).
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which makes the diagram

\[
\begin{array}{ccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha_{\mathcal{D}}} & F(X) \otimes (F(Y) \otimes F(Z)) \\
\downarrow J_{X,Y} \otimes id_Z & & \downarrow id_X \otimes J_{Y,Z} \\
F(X \otimes Y) \otimes F(Z) & \downarrow J_{X \otimes Y,Z} & F(X) \otimes F(Y \otimes Z) \\
\downarrow & & \downarrow J_{X,Y \otimes Z} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{\mathcal{C}})} & F(X \otimes (Y \otimes Z))
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The notion of monoidal equivalence is useful because monoidal categories that are monoidally equivalent are “the same for all practical purposes”.

Example

If \( \alpha, \beta \) are two 3-cocycles on \( G \) then the identity functor \( F = \text{Id} : \text{Vec}(G, \alpha) \to \text{Vec}(G, \beta), g \mapsto g \) is monoidal with \( J_g, h \in C^* : g \otimes h \to g \otimes h \iff dJ = \alpha/\beta \), where \( d \) is the differential in the standard complex of \( G \) with coefficients in \( C^* \).

In particular, \( F \) admits a monoidal structure if and only if the cohomology classes of \( \alpha \) and \( \beta \) are the same.

This shows that \( \text{Vec}(G, \alpha) \) is equivalent to \( \text{Vec}(G, \beta) \) iff \( \alpha \) is trivial in \( H_3(G, C^*) \).
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The universal R-matrix of $U_q(\mathfrak{sl}_2)$

If $A$ is a Hopf algebra and $X, Y \in \text{Rep} H$ then $X \otimes Y \not\cong Y \otimes X$ in general, as $\Delta \neq \Delta^\text{op}$ (e.g. $g \otimes h \not\cong h \otimes g$ for $g, h \in \text{Vec}(G)$).
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For $A = U_q(\mathfrak{sl}_2)$ where $q^n \neq 1$ define the universal $R$-matrix

$$R = q^{\frac{h \otimes h}{2}} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{(q - q^{-1})^k}{[k]_q!} e^k \otimes f^k,$$

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**Theorem (Drinfeld)**

The operator $c = P \circ R$ defines an isomorphism of representations $c : X \otimes Y \to Y \otimes X$. In other words, we have $R \Delta(a) = \Delta^\text{op}(a) R$ on $X \otimes Y$ for $a \in U_q(\mathfrak{sl}_2)$.  

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**Theorem (Drinfeld)**

The operator $c = P \circ R$ defines an isomorphism of representations $c : X \otimes Y \rightarrow Y \otimes X$. In other words, we have $R \Delta(a) = \Delta^{\text{op}}(a)R$ on $X \otimes Y$ for $a \in U_q(\mathfrak{sl}_2)$.
A prototypical example of a monoidal category where $X \otimes Y \cong Y \otimes X$ is the Drinfeld center of a monoidal category $C$. 

**Definition**

The Drinfeld center $Z(C)$ of $C$ is the category of pairs $(Y, \phi)$ where $Y \in C$ and $\phi: Y \otimes ? \to ? \otimes Y$ is a functorial isomorphism given by $\phi_X: Y \otimes X \cong - \to X \otimes Y \forall X \in C$, satisfying the following commutative diagram:

\[
\begin{array}{ccc}
Y \otimes (X_1 \otimes X_2) & \xrightarrow{\phi_{X_1} \otimes \id} & (X_1 \otimes X_2) \otimes Y \\
\downarrow{\alpha^{X_1}} & & \downarrow{\phi_{X_1} \otimes \id} \\
X_1 \otimes (Y \otimes X_2) & \xrightarrow{\id \otimes \phi_{X_2}} & X_1 \otimes (Y \otimes X_2)
\end{array}
\]

Morphisms of such pairs are morphisms in $C$ which preserve $\phi$. 

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The Drinfeld center

A prototypical example of a monoidal category where $X \otimes Y \cong Y \otimes X$ is the Drinfeld center of a monoidal category $C$.

**Definition**

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Y \otimes (X_1 \otimes X_2) & \xrightarrow{\varphi_{X_1} \otimes X_2} & (X_1 \otimes X_2) \otimes Y \\
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(Y \otimes X_1) \otimes X_2 & & X_1 \otimes (X_2 \otimes Y) \\
\downarrow \varphi_{X_1} \otimes \text{id} & & \uparrow \text{id} \otimes \varphi_{X_2} \\
(X_1 \otimes Y) \otimes X_2 & \xrightarrow{\alpha_{X_1YX_2}} & X_1 \otimes (Y \otimes X_2)
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The Drinfeld center, ctd.

The Drinfeld center $\mathcal{Z}(C)$ has a natural monoidal structure defined by $(Y, \varphi) \otimes (Z, \psi) = (Y \otimes Z, \eta)$,
The Drinfeld center, ctd.

The Drinfeld center $\mathcal{Z}(C)$ has a natural **monoidal structure** defined by $(Y, \varphi) \otimes (Z, \psi) = (Y \otimes Z, \eta)$, where (suppressing $\alpha$)

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Note also that we have a monoidal **forgetful functor** $\mathcal{Z}(C) \to C$, $(Y, \varphi) \mapsto Y$. 

Recall that $B_n = \pi_1(C_n \setminus \text{diagonals}) = \langle s_1, \ldots, s_{n-1} \mid s_i s_j = s_j s_i \text{ if } |i - j| \geq 2, s_i s_i+1 s_i = s_i+1 s_i s_i+1 \rangle$.
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Proposition

There is an action of $B_n$ on $V \otimes^n$ defined by

$$\rho: B_n \to \text{Aut}(V \otimes^n), \quad \rho(s_i) = c_{i, i+1}.$$ 

Recall that $B_n = \pi_1(C_n \setminus \text{diagonals}) = \langle s_1, \ldots, s_{n-1} | s_i s_j = s_j s_i \text{ if } |i - j| \geq 2, s_i s_i + 1 s_i = s_i s_i + 1 s_i s_i \rangle$. 

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The Drinfeld center, ctd.

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Proof.

This follows from the hexagon relations for $X, Y, Z \in \mathcal{Z}(\mathcal{C})$:

$$X \otimes Y \otimes Z \rightarrow Y \otimes X \otimes Z$$

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The Drinfeld center, ctd.

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They are call “hexagon relations” because they would have been hexagons had we not suppressed \( \alpha \).

This motivates the definition of a braided monoidal category.
Braided monoidal categories

**Definition**

A **braided monoidal category** is a monoidal category endowed with a functorial isomorphism \( c : \otimes \rightarrow \otimes^{\text{op}} \), \( c_{X,Y} : X \otimes Y \rightarrow Y \otimes X \) which satisfies the hexagon relations.

Thus we obtain

**Theorem**

The Drinfeld center \( Z(C) \) of a monoidal category \( C \) is a braided monoidal category. Moreover, every braided monoidal category \( C \) is a braided subcategory of its Drinfeld center using the inclusion \( \iota : C \rightarrow Z(C) \) given by \( X \mapsto (X, c_X) \). In this sense the Drinfeld center is a prototypical example of a braided category.

**Theorem**

The category of finite dimensional (type I) representations of \( \mathcal{U}_q(\mathfrak{sl}_2) \) is a braided monoidal category, with \( c = P \circ R \).
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The category of finite dimensional (type I) representations of \( U_q(\mathfrak{sl}_2) \) is a braided monoidal category, with \( c = P \circ R \).
Note that as a consequence $\mathcal{R}$ satisfies the **Quantum Yang-Baxter equation**

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}$$

(this follows from the relation $s_1 s_2 s_1 = s_2 s_1 s_2$).

Remark. A braided category is called symmetric if $c_{XY}c_{YX} = id_X \otimes Y$ for all $X, Y$. For example, the categories $\text{Rep}\, G$ and $\text{Rep}\, g$ are symmetric. However, $\text{Z}(\mathbb{C})$ is usually not symmetric, and $\text{Rep}\, U_q(\mathfrak{sl}_2)$ isn't either. We'll explain that the theorem on $U_q(\mathfrak{sl}_2)$, and in fact the construction of $\mathcal{R}$, are consequences of the theorem about the Drinfeld center. To this end, let us compute the Drinfeld center of the category $\text{Rep}\, A$ of representations of a Hopf algebra $A$. 162
Braided monoidal categories ctd.

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Braided monoidal categories ctd.

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We’ll explain that the theorem on $U_q(sl_2)$, and in fact the construction of $\mathcal{R}$, are consequences of the theorem about the Drinfeld center. To this end, let us compute the Drinfeld center of the category $\text{Rep } A$ of representations of a Hopf algebra $A$. 
Let \( Y \in \mathcal{Z} (\text{Rep } A) \).
Let $Y \in Z(\text{Rep } A)$. Then the map
\[
\varphi_A : \text{Ind}_C^A Y = Y \otimes A \to A \otimes Y
\]
defines a comodule structure
\[
\tau = \varphi_A|_Y : Y \to A \otimes Y.
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\[ \tau(ay) = a_1 y_1 S(a_2) \otimes y_2 a_3. \]
Yetter-Drinfeld modules

Let $Y \in \mathcal{Z}(\text{Rep} \ A)$. Then the map

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**Definition**

A **Yetter-Drinfeld module over $A$** is an $A$-module $Y$ which is also an $A$-comodule with $\tau : Y \to A \otimes Y$ satisfying the above compatibility condition.
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Proposition

One has $\mathcal{Z}(\text{Rep} \ A) \cong \text{YD}(A)$. 
Let $Y \in \mathcal{Z}(\text{Rep} A)$. Then the map

$$\varphi_A : \text{Ind}^A_C Y = Y \otimes A \to A \otimes Y$$

defines a comodule structure $\tau = \varphi_A|_Y : Y \to A \otimes Y$. The compatibility condition between this comodule structure and the $A$-module structure on $Y$ is

$$\tau(ay) = a(1)y(1)S(a(2)) \otimes y(2)a(3).$$

where $y \in Y$, $\tau(y) = y(1) \otimes y(2)$ and

$$a \in A, \ (1 \otimes \Delta) \circ \Delta(a) = a(1) \otimes a(2) \otimes a(3).$$

**Definition**

A Yetter-Drinfeld module over $A$ is an $A$-module $Y$ which is also an $A$-comodule with $\tau : Y \to A \otimes Y$ satisfying the above compatibility condition. The category of such modules is denoted $\text{YD}(A)$.

**Proposition**

One has $\mathcal{Z}(\text{Rep} A) \cong \text{YD}(A)$. 
If $A$ is finite dimensional, an $A$-comodule is the same as an $A^*$-module,
If $A$ is finite dimensional, an $A$-comodule is the same as an $A^\ast$-module, so the category $YD(A)$ can be realized as the category of modules over some algebra generated by $A$ and $A^\ast$ with some commutation relation between them.
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$$ba = (a_{(1)}, b_{(1)})(a_{(3)}, b_{(3)})a_{(2)}S^{-1}(b_{(2)}),$$

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**Theorem (Drinfeld)**
The braiding of $\text{Rep} D(A) = \text{YD}(A)$ is given by the formula $c = P \circ R$, where $R \in A \otimes A^\ast \subset D(A) \otimes D(A)$ is given by the universal $R$-matrix $R = \sum a_i \otimes a_i^\ast$, where $a_i$ is basis of $A$ and $a_i^\ast$ the dual basis of $A^\ast$.

In particular, $R$ satisfies the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in D(A)^3$.

Moreover, the commutation relation between $A$ and $A^\ast$ is uniquely determined by this equation.
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Let $A = \mathbb{C}G$ where $G$ is a finite group.
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Note that $D_0$ is nothing but the quotient of $U_q(sl_2)$ by the relations $E^\ell = F^\ell = K^\ell - 1 = 0$. 
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$$R = q^{\frac{h \otimes h}{2}} \sum_{k=0}^{\ell-1} q^{k(k-1)/2} \frac{(q - q^{-1})^k}{[k]_q!} e^k \otimes f^k,$$

where $[k]_q = \frac{q^k - 1}{q - 1}$.
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i.e., it is just the truncation of the formula for $U_q(\mathfrak{sl}_2)$ for general $q$. In fact, the punchline is that $U_q(\mathfrak{sl}_2)$ for general $q$ can also be constructed by an infinite dimensional version of the quantum double construction, which naturally produces both the commutation relation between $e$ and $f$ and the $R$-matrix!
Quantum groups attached to any simple Lie algebra can be constructed similarly.

Given a Cartan matrix \((a_{ij})\) with symmetrizing numbers \(d_i \in \mathbb{Z}^+\) (i.e., \(d_i a_{ij}\) is symmetric), we start with the Hopf algebra \(U_{q}(b)\) (quantum Borel) generated by \(e_i, K_{\pm 1}^i\) with relations

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K^i e_j = q^{d_i a_{ij}} e_j K^i, \\
[K^i, K^j] = 0
\]

with coproduct defined by 
\[
\Delta(K^i) = K^i \otimes K^i, \\
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and also quantum Serre relations, which are the most constraining relations you can impose preserving the Hopf algebra structure without killing any of the \(e_i\).

Then the quantum group \(U_{q}(g)\) is the quantum double of \(U_{q}(b)\) (understood appropriately) modded out by the “redundant copy of the maximal torus” created by the quantum double construction.

This algebra has additional generators \(f_i\) also satisfying quantum Serre relations and 
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Higher rank quantum groups

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Theorem

The trace of $b \cdot \mathcal{K}$ in $V \otimes^n$ (called the quantum trace of $b$) is the Jones polynomial of $\mathcal{K}$ (up to normalization). For $V_i$ for $i > 1$ one gets the colored Jones polynomial, and for other Lie algebras – more complex invariants of knots called the Reshetikhin-Turaev invariants.
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Theorem

The trace of $b \cdot K$ in $V \otimes^n$ (called the quantum trace of $b$) is the Jones polynomial of $\mathcal{K}$ (up to normalization).

For $V_i$ for $i > 1$ one gets the colored Jones polynomial, and for other Lie algebras – more complex invariants of knots called the Reshetikhin-Turaev invariants.
Thank you!