A brief introduction to quantum groups

Pavel Etingof

MIT

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The goal of this talk is to review some of the main ideas and examples of quantum groups and briefly describe some of the applications.

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Hopf algebras, ctd.

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Here we used Sweedler's notation for the coproduct $\Delta f = f_{(1)} \otimes f_{(2)}$ where summation is implied, i.e., $f_{(1)} \otimes f_{(2)}$ is really $\sum_i f^i_{(1)} \otimes f^i_{(2)}$.

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Definition

A quantum group or Hopf algebra is a unital associative algebra A (not necessary commutative) which is equipped with Δ, ε, S and has the properties listed above.

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5. If A is commutative or cocommutative (i.e., $\Delta = \Delta^{\mathrm{op}}$) then $S^2 = \mathrm{id}$ (even without the assumption that S is invertible).

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Examples of Hopf algebras

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2. $A = \mathcal{O}(G)$ (the algebra of regular functions), G is an affine algebraic group, A is commutative.
3. $A = \mathbb{C}G$ - the group algebra,
   $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$, $\varepsilon(g) = 1$, A is cocommutative:
   $\Delta = \Delta^{\text{op}}$.
4. $\mathfrak{g}$ a Lie algebra, $A = U(\mathfrak{g})$ (the universal enveloping algebra),
   $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, $\varepsilon(x) = 0$ for $x \in \mathfrak{g}$, A is cocommutative.

Quantum SL_2

Let $q \in \mathbb{C}$, $q \neq 0, \pm 1$. The quantum group $U_q(\mathfrak{sl}_2)$ is generated by $e, f, K^{\pm 1}$ with relations

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This is a deformation of $U(\mathfrak{sl}_2)$ because if one sets $K = q^h$ and sends $q \to 1$, one recovers the $\mathfrak{sl}_2$ relations.

This example shows that $S^2 \neq \mathrm{id}$ in general: we have $S^2(x) = KxK^{-1}$.

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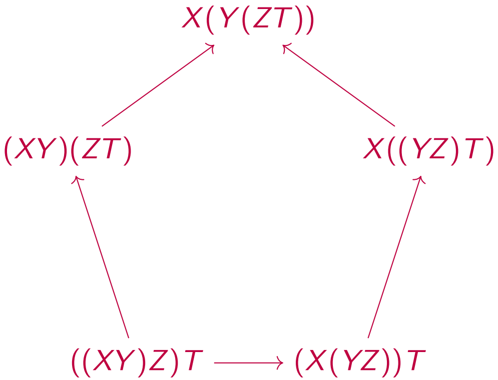

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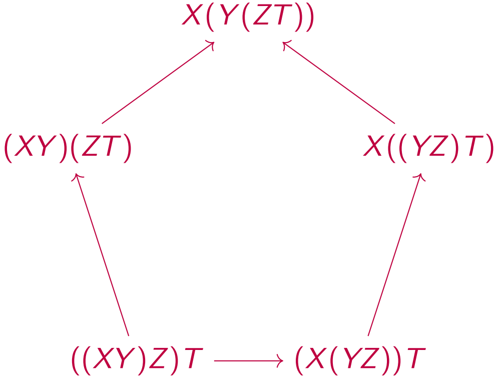

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Duality in monoidal categories

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If Y is a left dual to X then we have a functorial isomorphism $\mathrm{Hom}(Z, Y) \cong \mathrm{Hom}(Z \otimes X, \mathbf{1})$. By the Yoneda lemma, this implies that the left dual, if exists, is unique up to a unique isomorphism.

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Thus, if A is a Hopf algebra then the category $\mathrm{Rep}_f A$ of finite dimensional representations of A is a rigid monoidal category.

Examples of rigid monoidal categories

Example

Let G be a finite group, $A = \mathcal{O}(G)$, then $\mathrm{Rep}_f\, A$ is spanned by 1-dimensional representations parametrized by $g \in G$.

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A functor $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor if $F(\mathbf{1}_{\mathcal{C}}) \cong \mathbf{1}_{\mathcal{D}}$ and F is equipped with a functorial (in X, Y) isomorphism
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If A is a Hopf algebra and $X, Y \in \mathrm{Rep}\, H$ then $X \otimes Y \not\cong Y \otimes X$ in general, as $\Delta \neq \Delta^{\mathrm{op}}$ (e.g. $g \otimes h \not\cong h \otimes g$ for $g, h \in \mathrm{Vec}(G)$).

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For $A = U_q(\mathfrak{sl}_2)$ where $q^n \neq 1$ define the universal R-matrix

$$\mathcal{R} = q^{\frac{h \otimes h}{2}} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{(q - q^{-1})^k}{[k]_q!} e^k \otimes f^k,$$

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Definition

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is the category of pairs (Y, φ) where $Y \in \mathcal{C}$ and $\varphi: Y \otimes ? \to ? \otimes Y$ is a functorial isomorphism given by $\varphi_X: Y \otimes X \xrightarrow{\sim} X \otimes Y \; \forall X \in \mathcal{C}$, satisfying the following commutative diagram:

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Morphisms of such pairs are morphisms in $\mathcal{C}$ which preserve φ.

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$$\eta_X=(\phi_X\otimes \mathrm{id}_Z)\circ(\mathrm{id}_Y\otimes\psi_X): Y\otimes Z\otimes X\to Y\otimes X\otimes Z\to X\otimes Y\otimes Z.$$

Note also that we have a monoidal forgetful functor $Z(\mathcal{C})\to\mathcal{C}$, $(Y,\varphi)\mapsto Y$. Moreover, if $Y,Z\in\mathcal{Z}(\mathcal{C})$ then there are two ways $Y\otimes Z\xrightarrow{c_{YZ},c_{ZY}^{-1}} Z\otimes Y$ to identify $Y\otimes Z$ and $Z\otimes Y$, $c_{YZ}=\varphi_Z$, $c_{ZY}=\psi_Y$. This gives an action of the braid group B_n on $V^{\otimes n}$ for $V\in\mathcal{Z}(\mathcal{C})$. Recall that $B_n=\pi_1(\mathbb{C}^n\backslash\text{diagonals}/S_n)=$

$$\langle s_1,\dots,s_{n-1}\mid s_is_j=s_js_i \text{ if } |i-j|\geq 2,\ s_is_{i+1}s_i=s_{i+1}s_is_{i+1}\rangle$$

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$$\rho:\ B_n\to\mathrm{Aut}(V^{\otimes n}),\quad \rho(s_i)=c_{i,i+1}=(c_{V,V})_{i,i+1}$$

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Proof.

This follows from the hexagon relations for $X, Y, Z \in \mathcal{Z}(\mathcal{C})$:

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This motivates the definition of a braided monoidal category.

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The Drinfeld center $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category.

Moreover, every braided monoidal category $\mathcal{C}$ is a braided subcategory of its Drinfeld center using the inclusion $\iota : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ given by $X \mapsto (X, c_{X?})$. In this sense the Drinfeld center is a prototypical example of a braided category.

Theorem

The category of finite dimensional (type I) representations of $U_q(\mathfrak{sl}_2)$ is a braided monoidal category, with $c = P \circ \mathcal{R}$.

Braided monoidal categories ctd.

Note that as a consequence $\mathcal{R}$ satisfies the Quantum Yang-Baxter equation

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}$$

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A braided category is called symmetric if $c_{XY} c_{YX} = \mathrm{id}_{X \otimes Y}$ for all X, Y.

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We'll explain that the theorem on $U_q(\mathfrak{sl}_2)$, and in fact the construction of $\mathcal{R}$, are consequences of the theorem about the Drinfeld center. To this end, let us compute the Drinfeld center of the category $\mathrm{Rep}\, A$ of representations of a Hopf algebra A.

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Moreover, the commutation relation between A *and* A^* *is* *uniquely determined* *by this equation.*

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i.e., it is just the truncation of the formula for $U_q(\mathfrak{sl}_2)$ for general q. In fact, the punchline is that $U_q(\mathfrak{sl}_2)$ for general q can also be constructed by an infinite dimensional version of the quantum double construction, which naturally produces both the commutation relation between e and f and the R-matrix!

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These commutation relations, as well as the R-matrix (which is now much more complicated) are produced automatically by the double construction. In fact, this works more generally, for any symmetrizable Kac-Moody algebra.

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For V_i for $i > 1$ one gets the colored Jones polynomial, and for other Lie algebras – more complex invariants of knots called the Reshetikhin-Turaev invariants.

Thank you!