Bartnik minimizing initial data sets

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joint with Lan-Hsuan Huang

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Initial data sets

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- We say that (U^n, g, k) sits inside a Lorentzian spacetime $(\mathbf{N}^{n+1}, \mathbf{g})$ if (U, g) isometrically embeds into $(\mathbf{N}, \mathbf{g})$ with k as its second fundamental form.

Einstein constraints

- If (U, g, k) sits inside $(\mathbf{N}, \mathbf{g})$ and G is the Einstein tensor of $\mathbf{g}$, then

$$G_{00} = \frac{1}{2}\left[R_g + (\operatorname{tr}_g k)^2 - |k|_g^2\right]$$
$$G_{0i} = (\operatorname{div}_g k)_i - \nabla_i(\operatorname{tr}_g k)$$

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$$G_{00} = \frac{1}{2}\left[R_g + (\operatorname{tr}_g k)^2 - |k|_g^2\right] =: \mu$$
$$G_{0i} = (\operatorname{div}_g k)_i - \nabla_i(\operatorname{tr}_g k) =: J_i$$

  where e_0 is orthogonal to U and $e_1, \ldots, e_n$ are tangent to U.
- We call μ and J the energy and current densities of (g, k), and we define the constraint map to be

$$\Phi(g, k) = (\mu, J)$$

Dominant energy condition

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  which we will refer to as the dominant energy condition, or DEC for (g, k).
- When $k \equiv 0$ (the time-symmetric case), $\mu = \frac{1}{2}R_g$, $J = 0$, and the DEC for $(g, 0)$ is just nonnegativity of scalar curvature.

Asymptotic flatness

- For $n \geq 3$, an initial data set (M^n, g, k), possibly with boundary, is asymptotically flat if there exists $q > \frac{n-2}{2}$, $\alpha \in (0,1)$, a compact $K \subset M$, and a diffeomorphism $M \smallsetminus K \cong \mathbb{R}^n \smallsetminus B$ such that

$$
\begin{aligned}
g_{ij} - \delta_{ij} &= O_{2,\alpha}(|x|^{-q}) \\
k_{ij} &= O_{1,\alpha}(|x|^{-q-1}) \\
(\mu, J) &\in L^1(M).
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Spacetime positive mass theorem (Schoen-Yau, Witten, EHLS)

For $n < 8$ or M^n spin, if (M, g, k) is an asymptotically flat initial data set (without boundary) satisfying the DEC, then $E \geq |P|$. We define the ADM mass to be $m := \sqrt{E^2 - |P|^2}$.

Bartnik mass

- Let (Ω_0, g_0, k_0) be a compact initial data set with nonempty smooth boundary satisfying the DEC. The concept of "quasi-local mass" is an attempt to understand how much Ω_0 contributes to the ADM mass of an asymptotically flat extension of Ω_0.

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- Bartnik mass $m_B(\Omega_0, g_0, k_0)$ is the infimum of the ADM masses of all admissible asymptotically flat extensions (M, g, k) that satisfy the DEC.
- Admissibility imposes a matching condition at $\partial\Omega_0 \cong \partial M$ and also a "no horizon" condition.

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Bartnik's stationary conjecture

If (M, g, k) *is a Bartnik minimizer, then its interior sits inside a vacuum (i.e.* $G \equiv 0$*) spacetime* $(\mathbf{N}, \mathbf{g})$ *that admits a global timelike Killing vector field, i.e.* $\mathbf{g}$ *is vacuum stationary.*

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- Fact: g sits inside a vacuum static spacetime iff $R_g = 0$ (i.e. "g is vacuum") and the adjoint linearization $DR|_g^*$ admits a nontrivial kernel.

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- Moncrief showed that (g,k) sits inside a vacuum spacetime admitting a global Killing field iff $\mu = |J|_g = 0$ (i.e. "(g,k) is vacuum") and the adjoint linearization $D\Phi|^*_{(g,k)}$ admits a nontrivial kernel.

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- Zhongshan An recently showed that this is true if one assumes that (g, k) is a vacuum Bartnik minimizer.
- Corvino recently showed that a Bartnik minimizer admits a nontrivial kernel of $D\overline{\Phi}_{(g,k)}\big|^*_{(g,k)}$ where $\overline{\Phi}_{(g,k)}$ is the modified constraint operator introduced by Corvino and Huang.

Modified constraint operator

- Definition:
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- Corvino-Huang: Injectivity of $D\overline{\Phi}_{(g,k)}\Big|^*_{(g,k)}$ allows one to find compactly supported deformations of (nearby) initial data such that the quantity σ increases.
- We call this property improvability of the dominant energy scalar.

Main result

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2. *The interior of (M, g, k) sits inside a null dust spacetime $(\mathbf{N}, \mathbf{g})$ which satisfies the DEC and admits a global Killing vector field $\mathbf{Y}$. The vector field $\mathbf{Y}$ is null wherever $\mathbf{g}$ is not vacuum.*

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- $\mathbf{Y}$ arises from a lapse-shift (f, X) solving $D\overline{\Phi}_{(g,k)}\big|_{(g,k)}^{*}(f, X) = 0$.

Main ideas behind proof, part 1

- As mentioned, Corvino-Huang says that non-improvability implies the existence of a nontrivial kernel of $D\overline{\Phi}_{(g,k)}\big|^*_{(g,k)}$.

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If the dominant energy scalar is NOT improvable in (M, g, k), then there exists (f, X) satisfying the system

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- This is proved by introducing an infinite-dimensional family of deformations of the modified constraint operator.
- Meanwhile, improvability of the dominant energy scalar in (M, g, k) allows one to reduce the mass while maintaining DEC and leaving a neighborhood of ∂M unchanged. Thus Bartnik minimizers do NOT have the improvability property.

Main ideas behind proof, part 2

- If f is nonvanishing, this (M, g, k) sits inside its Killing development $(\mathbb{R} \times M, \mathbf{g})$, where

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- For *non-vacuum* initial data, it has been unclear what geometric or physical significance there is to a nontrivial kernel of either $D\overline{\Phi}_{(g,k)}\big|^*_{(g,k)}$ or $D\Phi|^*_{(g,k)}$.
- We observed that a nontrivial solution to $(\star)$ must have $\mu - |J|_g$ constant, and it also yields a Killing development which is a null perfect fluid. In the asymptotically flat case, this $\mu - |J|_g$ must be zero, and we obtain null dust.

Examples of solutions to $(\star)$

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- The most interesting examples have $J \neq 0$. Such examples can be constructed as spacelike hypersurfaces in pp-waves:

$$\mathbf{g} = 2dudz + Sdz^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2$$

where S is independent of u. This is null dust with Killing vector $\frac{\partial}{\partial u}$. It satisfies DEC iff S is superharmonic with respect to the x variables.

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For $n > 8$, S can be chosen so that the $u = 0$ slice of the pp-wave is an asymptotically flat initial data set satisfying DEC and $E = |P| > 0$.

Consequences

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- If you take a large open ball B in one of these asymptotically flat pp-wave initial data sets (M, g, k), which has mass equal to zero, one can see that $(M \setminus B, g, k)$ is a Bartnik minimizer for $(\overline{B}, g, k)$ which is not vacuum. This shows that Bartnik's strict positivity and stationary conjectures both fail in high dimension.

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5. Thank you for your attention. What are your questions?

More about the proof

- For an arbitrary function φ, we construct an infinite-dimensional family of φ-modified constraint operators:

$$\overline{\Phi}^{\varphi}_{(g,k)}(\gamma,\tau) := \overline{\Phi}_{(g,k)}(\gamma,\tau) + \left(\varphi|J|^2_{\gamma}, \varphi|J|_g \gamma \cdot J\right)$$

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- Idea: We use the freedom to vary φ to show that if there is a nontrivial kernel for *every* choice of φ, then $(\star)$ must hold.
- The advantage that we get from introducing φ is that we can analyze the linear algebraic dependence of the equation $D\overline{\Phi}^{\varphi}_{(g,k)}\Big|^*_{(g,k)}(f,X) = 0$ (and its derivatives) on φ at a point.