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Existence of Canonical Metrics on Non-Kähler Geometry

Simons Foundation Lecture

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Calabi-Yau manifolds

- In this talk I want to discuss the role of Calabi-Yau manifolds in geometry and physics, and discuss the geometry of Calabi-Yau manifolds outside of the Kähler regime.

- For this talk, a Calabi-Yau manifold (or complex manifold that supports non-vanishing holomorphic n-form) will be a complex manifold X with *trivial canonical bundle*. For some parts of this talk the notion of a Calabi-Yau manifold can be weakened to $c_1(X) = 0$, but for simplicity, we'll take the most restrictive definition.

- A very important class of Calabi-Yau manifolds are the *Kähler* Calabi-Yau manifolds. These are the Calabi-Yau manifolds X admitting a hermitian metric whose associated $(1,1)$ form ω satisfies $d\omega = 0$. Such metrics are called Kähler metrics (and the associated cohomology class $[\omega] \in H^2(X, \mathbb{R})$ is called the Kähler class).

Calabi Conjecture

- In 1976 I showed that, for a compact, Kähler Calabi-Yau manifold (X, ω) there is a *unique* Kähler metric g' on X, such that the associated Kähler form ω' has $[\omega'] = [\omega]$ and

$$\mathrm{Ric}(g') = 0$$

- This was achieved by solving the *complex Monge-Ampère* equation

$$(\omega')^n = e^F \omega^n$$

where F is any smooth function.

- These Ricci-flat Kähler metrics are now called *Calabi-Yau* metrics. An important point is that they have holonomy contained in $SU(n)$.

Calabi-Yau manifolds

- In the early 1980s it was realized that Kähler Calabi-Yau three-folds gave fundamental models in string theory, which is an approach to unifying all the fundamental forces in physics (including gravity) into a single Theory of Everything.

- The connection between Calabi-Yau manifolds and theoretical physics has been very fruitful over the past 50 years. In this talk I would like to discuss, among other things, some ideas from physics that I hope will be helpful in improving our understanding of Calabi-Yau manifolds beyond the Käher setting.

Non-Kähler Calabi-Yau manifolds

- A first step to understanding Non-Kähler Calabi-Yau manifolds is to understand to what extent they can be "uniformized". Can they be given some canonical geometry? Is there an analog of Ricci-flat Kähler metrics on non-Kähler Calabi-Yau manifolds?

- A natural generalization of a Kähler metric, which exists on all compact non-Kähler complex manifolds is a Gauduchon metric. A metric g is Gauduchon if the associated $(1,1)$ form α satisfies

$$\partial\overline{\partial}(\alpha^{n-1}) = 0$$

- There is a stronger notion called a *balanced metric*, introduced by Michelson. A metric g is balanced if the associated $(1,1)$ form α satisfies $d\alpha^{n-1} = 0$. Such metrics *do not* always exist.

Non-Kähler Calabi-Yau manifolds

- Michelson gave a sharp characterization of when balanced metrics exist, and Alessandrini-Bassanelli showed that the existence of a balanced metric is invariant under birational transformations.

- So balanced metrics should be useful for birational geometry.

- It is natural to ask if a version of the Calabi conjecture holds for Gauduchon or balanced metrics. Namely, does a non-Kähler Calabi-Yau manifold admit a Gauduchon, or balanced Chern Ricci-flat metric? It was conjectured by Gauduchon that there should always be a Chern Ricci-flat Gauduchon metric.

- Székelyhidi-Tosatti-Weinkove proved that the answer to this question is "Yes".

Non-Kähler Calabi-Yau manifolds

- Székelyhidi-Tosatti-Weinkove showed that every compact Calabi-Yau manifold admits a Gauduchon metric whose Chern-Ricci curvature is zero.

- However, it is not clear how canonical this metric is. For example, the moduli space is not expected to be finite dimensional even modulo automorphisms.

- A natural way is to require the Ricci flat metric to be balanced. We can also impose some further constraint, like minimizing some energy (eg. Yang-Mills energy) on the metric.

Non-Kähler Calabi-Yau manifolds

- Another approach to extending the notion of Calabi-Yau metric beyond the Kähler regime is to look to string theory for motivation.

- Non-Kähler Calabi-Yau manifolds play an important role in string theory, since they can be used to construct theories with *fluxes*. Such theories are important for taking into account non-perturbative effects. The flux will turn out to be determined by the 3-form field $H = \sqrt{-1}(\partial - \bar{\partial})\omega$, which is trivial in the Kähler setting.

- The equations of motion for supersymmetric compactifications of heterotic string theory were obtained independently by Hull and Strominger. I would like to discuss these equations in some detail.

The Hull-Strominger system.

- In the original proposal for compactification of superstring, Candelas, Horowitz, Strominger, and Witten constructed the metric product of a maximal symmetric four-dimensional spacetime M with a six-dimensional Calabi-Yau vacuum X as the ten-dimensional spacetime.

- They identified the Yang-Mills connection with the $SU(3)$ connection of the Calabi-Yau metric and set the dilaton to be a constant.

- Adapting my suggestion of using Donaldson-Uhlenbeck-Yau's theorem on constructing Hermitian-Yang-Mills connections over stable bundles, Witten and later Horava-Witten proposed to use higher rank bundles for strong coupled heterotic string theory so that the gauge groups can be $SU(4)$ or $SU(5)$.

The Hull-Strominger system.

- At around the same time, Hull and Strominger independently analyzed the heterotic super-string background with spacetime supersymmetry and non-zero torsion by allowing a scalar warp factor for the spacetime metric.

- They considered a ten-dimensional spacetime that is a warped product of a maximally symmetric four-dimensional spacetime M and an internal space X; the metric on $M \times X$ takes the form

$$g_0 = e^{2D(y)} \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{ij}(y) \end{pmatrix}, \quad x \in M, \; y \in X;$$

- There is also an auxiliary bundle over X equipped with a Hermitian-Yang-Mills connection:

$$F \wedge \omega^2 = 0, \quad F^{2,0} = F^{0,2} = 0. \tag{1}$$

  Here ω is the Hermitian form $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ defined on the internal space X.

The Hull-Strominger system.

- In this system, the physically relevant quantities are

$$H = -\sqrt{-1}(\bar{\partial} - \partial)\omega, \quad \phi = -\frac{1}{2}\log \|\Omega\| + \phi_0, \quad g^0_{ij} = e^{2\phi_0} \|\Omega\|^{-1} g_{ij}.$$

for a constant ϕ_0.

- In order for the ansatz to provide a supersymmetric configuration, one introduces a Majorana-Weyl spinor ϵ so that

$$\begin{aligned}\delta\psi_M =& \nabla_M \epsilon - \frac{1}{8} H_{MNP}\gamma^{NP}\epsilon = 0,\\ \delta\lambda =& \gamma^M \partial_M \phi\epsilon - \frac{1}{12} H_{MNP}\gamma^{MNP}\epsilon = 0,\\ \delta\chi =& \gamma^{MN} F_{MN}\epsilon = 0,\end{aligned}$$

where ψ_M is the gravitino, λ is the dilatino, χ is the gluino, ϕ is the dilaton and H is the Kalb-Ramond field strength obeying

$$dH = \frac{\alpha'}{2}(\mathrm{tr} F \wedge F - \mathrm{tr} R \wedge R), \quad \alpha' > 0$$

The Hull-Strominger system.

- In order to achieve spacetime super-symmetry, the internal six manifold X must be a complex manifold with a non-vanishing holomorphic three-form Ω; and the anomaly cancellation demands that the Hermitian form ω obey

$$\sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F). \tag{2}$$

- Super-symmetry requires

$$d^*\omega = \sqrt{-1}(\bar{\partial} - \partial) \log \| \Omega \|_\omega .$$

- The last equation is equivalent to

$$d(\| \Omega \|_\omega \, \omega^2) = 0 \tag{3}$$

which was observed by J. Li and myself.

The Hull-Strominger system.

- Equations (1), (2), and (3) compose a system, which is now called the Hull-Strominger system in the literature:

$$
\begin{aligned}
&d(\| \Omega \|_\omega \, \omega^2) = 0; \\
&F_h^{2,0} = F_h^{0,2} = 0; \quad F_h \wedge \omega^2 = 0; \\
&\sqrt{-1}\partial\bar{\partial}\,\omega = \frac{\alpha'}{4}\Big(\mathrm{tr}\big(R_\omega \wedge R_\omega\big) - \mathrm{tr}\big(F_h \wedge F_h\big)\Big).
\end{aligned}
$$

- This system gives a solution of a superstring theory with flux that allows a non-trivial dilaton field and a Yang-Mills field. (It turns out $D(y) = \phi$ and is the dilaton field.)

- Here ω is the Hermitian form and R is the curvature tensor of the Hermitian metric ω; H is the Hermitian metric and F is its curvature of a vector bundle E; tr is the trace of the endomorphism bundle of either E or TX.

Hull-Strominger system

$$d(\| \Omega \|_\omega \omega^2) = 0 \tag{4}$$

$$F_h^{2,0} = F_h^{0,2} = 0 \quad F_h \wedge \omega^2 = 0 \tag{5}$$

$$\sqrt{-1}\partial\bar{\partial}\,\omega = \frac{\alpha'}{4}\Big(\mathrm{tr}\big(R_\omega \wedge R_\omega\big) - \mathrm{tr}\big(F_h \wedge F_h\big)\Big)\,. \tag{6}$$

- Equation (4) is a restricted holonomy equation, which can be viewed as analogous to the Complex Monge-Ampère equation appearing in the Calabi conjecture.

- Equation (5) is the Hermitian-Yang-Mills equation studied by Donaldson, and Uhlenbeck-Yau in the Kähler case and Li-Yau in the non-Kähler case.

- Equation (6) is the *anomaly cancellation* equation (α' is a constant). It is a highly nonlinear equation linking (4) and (5).

Hull-Strominger system

- The Hull-Strominger system is a highly non-linear system of PDEs, and we know very little about its solvability on a general compact Calabi-Yau manifold.

- The solvability of the Hull-Strominger system on a general non-Kähler Calabi-Yau 3-fold is largely open. I conjecture that solvability is equivalent to stability of the gauge bundle.

- More precisely, consider complex threefold (X, Ω) with a balanced class $\tau = [\|\Omega\|_{\omega_0} \omega_0^2] \in H^{2,2}_{BC}(X)$. Let $E \to X$ be a holomorphic bundle which is degree zero and stable with respect to τ. Assume the cohomological condition $ch_2(E) = ch_2(X) \in H^{2,2}_{BC}(X)$. Then the Hull-Strominger system should be solvable for a pair of metrics (ω, h) on (X, E).

- In the following slides, I will describe the current state of knowledge, and some work in progress.

Calabi-Yau solutions.

- The Hull-Strominger system, given by

$$
\begin{aligned}
& d(\| \Omega \|_\omega \, \omega^2) = 0; \\
& F_h^{2,0} = F_h^{0,2} = 0; \quad F_h \wedge \omega^2 = 0; \\
& \sqrt{-1}\partial\bar{\partial}\,\omega = \frac{\alpha'}{4}\Big(\mathrm{tr}\big(R_\omega \wedge R_\omega\big) - \mathrm{tr}\big(F_h \wedge F_h\big)\Big),
\end{aligned}
$$

  generalizes the Calabi-Yau condition for a metric ω_{CY} to a system for a pair of metrics (ω, h).

- Indeed, if we take $E = T^{1,0}X$ to be the gauge bundle and set $\omega = h = \omega_{\mathrm{CY}}$ to be a Kähler Ricci-flat metric, then

$$
d\omega_{\mathrm{CY}} = 0, \quad \|\Omega\|_{\omega_{\mathrm{CY}}} = \mathrm{const}, \quad R_{\omega_{\mathrm{CY}}} \wedge \omega_{\mathrm{CY}}^2 = 0
$$

  which implies that $\omega = h = \omega_{\mathrm{CY}}$ solves the system.

- In superstring theory, these are the solutions of Candelas-Horowitz-Strominger-Witten.

Li-Yau's solution.

- Assume that Y is a Kähler Calabi-Yau threefold and ω is a Calabi-Yau metric. Take $V = \mathbb{C}_Y^{\oplus r} \oplus TY$ and $H = H_1 \oplus \omega$, where H_1 is a standard constant metric on $\mathbb{C}_Y^{\oplus r}$. Then (Y, ω, V, H) is a solution which is called a reducible solution.

- For any small deformations $\bar{\partial}_s$ of the holomorphic structure of $\mathbb{C}_Y^{\oplus r} \oplus TY$, J. Li and I derived a sufficient condition for the Hull-Strominger system being solvable for $(Y, \bar{\partial}_s)$: it is that the Kodaira-Spencer class of the family $\bar{\partial}_s$ at $s = 0$ satisfies a certain non-degeneracy condition.

- We showed that this condition holds for

1. $X \subset \mathbb{P}^4$: a smooth quintic threefold;
2. $X \subset \mathbb{P}^3 \times \mathbb{P}^3$: cut out by three homogeneous polynomials of bi-degree $(3, 0)$, $(0, 3)$ and $(1, 1)$.

thereby constructing the first examples of regular irreducible solution to the Hull-Strominger system with gauge group $U(4)$ and $U(5)$:

Fu-Yau's solution.

- Fu and I constructed solutions of the Hull-Strominger system on a class of non-Kähler Calabi-Yau threefolds. These are T^2-bundles over $K3$-surfaces constructed by Calabi-Eckmann's method.

- On these manifolds, there exist natural metrics:

$$\omega_u = e^u \omega_{K3} + i\theta \wedge \bar{\theta},$$

  which satisfy the first equation of the Hull-Strominger system. Here u is any function of $K3$ surface, θ is the connection 1-form on the T^2-bundle. Such ansatz were first considered by Dasgupta-Rajesh-Sethi, Becker-Becker-Dasgupts-Green, and Goldstein-Prokushkin.

Fu-Yau's solution.

- Under this ansatz Fu and I reduced the anomaly cancellation equation of the Hull-Strominger system to the following Monge-Ampere equation:

$$\triangle(e^u - \frac{\alpha'}{2} f e^{-u}) + 4\alpha' \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0,$$

- Here f and μ are functions on $K3$ surface satisfying $f \geq 0$ and $\int_S \mu \omega_{K3}^2 = 0$.

- This equation is more complicated than the equation in the Calabi conjecture. For example, the estimate of volume form gives extra complication. We obtained some crucial a priori estimates up to third order derivatives and then used the continuity method to solve the equation.

Fu-Yau's solution.

Fu and I obtained the following existence theorem on the Hull-Strominger system:

Theory 1. (Fu-Y.) *Let S be a $K3$ surface with a Calabi-Yau metric ω_S. Let ω_1 and ω_2 be anti-self-dual $(1,1)$-forms on S such that $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^2(S,\mathbb{Z})$. Let X be a T^2-bundle over S constructed (twisted) by ω_1 and ω_2. Let E be a stable bundle over S with the gauge group $SU(r)$. Suppose ω_1, ω_2 and $c_2(E)$ satisfy the topological constraint*

$$\alpha'(24 - c_2(E)) = -\left(Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right)\right).$$

Then there exists a smooth function u on $K3$ surface and a Hermitian-Yang-Mills metric H on E such that $(\pi^*E, \pi^*H, X, \omega_u)$ is a solution to the Hull-Strominger system.

Fu-Yau's solution.

- From Mukai's theory on the stable vector bundle over $K3$ surface, we know that a sufficient condition for the existence of a stable bundle E with $(r, c_1(E) = 0, c_2(E))$ on $K3$ surface is given by the inequality

$$2rc_2(E) - 2r^2 \geq -2.$$

  Thus, we can determine all $(\omega_1, \omega_2, c_2(E))$ which satisfy topological constraint in the theorem.

- Additionally, Fu and I generalized the Hull-Strominger system to the higher dimensional case and derived the following equation:

$$\sqrt{-1}\partial\bar{\partial}u \wedge (\omega^{n-1} - \rho \wedge \omega^{n-2}) - n\partial\bar{\partial}u \wedge \partial\bar{\partial}u \wedge \omega^{n-2} + \mu\frac{\omega^n}{n!} = 0.$$

  where ρ is a real $(1,1)$-form on X, μ is a smooth function on X satisfying the integrable condition $\int_X \mu\frac{\omega^n}{n!} = 0$. This equation is now called the Fu-Yau equation in the literature. Further analytic techniques in studying the Fu-Yau equation have been developed by Phong-Picard-Zhang in recent years.

Fu-Tseng-Yau's solution.

- We can replace $K3$ surface by a non-compact ALE space.

- The simplest one is the Eguchi-Hanson space: blow up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin of the $\mathbb{Z}_2$ action $\sigma(z_1, z_2) = (-z_1, -z_2)$. Alternatively, it is $T^*\mathbb{P}^1$.

- There is a Ricci-flat metric on it

$$\omega_{EH} = \frac{i}{2}(k(r^2)\partial\bar{\partial}r^2 + k'(r^2)\partial r^2 \wedge \bar{\partial}r^2),$$

  where $k = \sqrt{1 + \frac{a^4}{r^4}}$, r^2 is the radius on $\mathbb{C}^2$, and a is the size of the blow-up $\mathbb{P}^1$. On Eguchi-Hanson space, there is an anti-self dual $(1,1)$-form. We can use this form to twist the torus and as $U(1)$ gauge fields.

Fu-Tseng-Yau's solution.

- We only need to satisfy the anomaly equation. Due to dependence only on the radial coordinate for all quantities on $\mathbb{C}^2/\mathbb{Z}_2$, the anomaly equation can be reduced to an ODE:

$$2(1+s)^{\frac{1}{2}}v^2v' + 4\alpha s v'^2 - \frac{3\alpha}{(1+s)^2}v^2 + \frac{2\alpha^2|n|^2}{(1+s)^{\frac{3}{2}}}v' + \frac{4\alpha^2|n|^2}{(1+s)^{\frac{5}{2}}}v = 0,$$

where $v(s) = e^{u(r^2)}$ and $s = \frac{r^4}{a^4}$.

- For $\alpha = \frac{\alpha'}{a^2}$ sufficiently small, we can get a convergent solution:

$$e^u = \sum_{k=0}^{\infty} \frac{a_k}{(1+\frac{r^4}{a^4})^{\frac{k}{2}}} = 1 - \alpha \frac{1}{(1+\frac{r^4}{a^4})^{\frac{3}{2}}} + \alpha^2 \frac{|n|^2}{(1+\frac{r^4}{a^4})^2} + \alpha^3 \frac{(|n|^2+9/7)}{(1+\frac{r^4}{a^4})^{\frac{7}{2}}} + \cdots .$$

Fei-Picard-Huang's solution.

- A class of compact non-Kähler Calabi-Yau 3-folds was constructed by Calabi using vector cross product on $\mathbb{R}^7$ and triply periodic minimal surfaces in $\mathbb{R}^3$. Calabi's construction was later generalized by A. Gray.

- These Calabi-Gray manifolds are total spaces of certain holomorphic fibrations, whose fibers are hyperkähler 4-manifolds (T^4, K3 surfaces), and whose bases are triply periodic minimal surfaces in $\mathbb{R}^3$.

- Compared to the Calabi-Eckmann case, the new feature in the Calabi-Gray construction is that the complex structures on the fibers are varying from point to point. In fact, my student T. Fei identified the Calabi-Gray construction with pullbacks of the twistor family of hyperkähler manifolds using holomorphic maps from a Riemann surface to $\mathbb{P}^1$ coming from vanishing theta characteristics. This observation allows him to generalize the Calabi-Gray construction. Fei also showed that there is a natural balanced metric on any generalized Calabi-Gray manifold.

Fei-Picard-Huang's solution.

- Moreover, Fei-Huang-Picard showed that the full Hull-Strominger system can be solved on a large class of generalized Calabi-Gray manifolds:

- First, they write down an ansatz solving the conformally balanced equation based on the Fu-Yau method.

- Then, by carefully choosing the gauge bundle, they showed that the Hull-Strominger system can be reduced to a linear elliptic equation on a Riemann surface coupled with an algebraic equation, which can be solved explicitly in many cases.

Fei-Picard-Huang's solution.

- In particular, F-P-Z obtain solutions to the Hull-Strominger system on generalized Calabi-Gray manifolds whose base (a Riemann surface) can be of any genus greater or equal than 3. Therefore they produce solutions on non-Kähler Calabi-Yau 3-folds with unbounded topological types and Hodge numbers. Another nice feature of their construction is that the solution metrics can be written down explicitly.

- There is also a local version of the Fei-Huang-Picard construction. A Landau-Ginzburg model for these local solutions is developed by Chen-Pantev-Sharpe.

The Anomaly flow

- A geometric flow approach to the system was proposed by Phong-Picard-Zhang. The flow of the pair $(\omega(t), h(t))$ is determined by

$$h^{-1}\partial_t h = -\sqrt{-1}\Lambda_\omega F_h;$$
$$\partial_t(\| \Omega \|_\omega \, \omega^2) = \sqrt{-1}\partial\bar{\partial}\,\omega - \frac{\alpha'}{4}\Big(\mathrm{tr}\Big(R_\omega \wedge R_\omega\Big) - \mathrm{tr}\Big(F_h \wedge F_h\Big)\Big)\,,$$

  with initial metric ω_0 satisfying $d(\|\Omega\|_{\omega_0}\omega_0^2) = 0$.

- For fixed ω, the flow of metrics h on the gauge bundle is the Donaldson heat flow.

- The flow preserves the balanced condition $d(\|\Omega\|_{\omega(t)}\omega(t)^2) = 0$.

- Fixed points solve the Hull-Strominger system.

The Anomaly flow

- The flow provides a path in the space of balanced metrics inside an initial class $[\|\Omega\|_{\omega(0)}\,\omega(0)^2] \in H^{2,2}(X)$.

- Short-time existence is known for initial metrics with $|\alpha' R_\omega| < 1$.

- The analysis of this parabolic approach was recently developed in works by Fei, Phong, Picard, and Zhang. The flow can be used to recover several of the special ansatz solutions discussed in the previous slides.

- A special case ($\alpha' = 0$) of the flow is $\partial_t(\| \, \Omega \, \|_\omega \, \omega^2) = \sqrt{-1}\partial\bar{\partial}\,\omega$ with initial conformally balanced metric $d(\|\Omega\|_{\omega(0)}\omega(0)^2) = 0$. Stationary points in this case are Kähler Ricci-flat. Thus the flow deforms Calabi-Yau geometry with torsion towards torsion free Calabi-Yau geometry.

Connecting Kähler and non-Kähler Calabi-Yau manifolds

- Having discussed its appearance in string compactifications with non-trivial flux, we will next discuss motivation for non-Kähler geometry in topology and algebraic geometry.

- A fundamental open question is to understand the possible topological types of Kähler (or projective) Calabi-Yau threefolds. To date, over 30,000 distinct Hodge diamonds have been constructed.

- To understand this landscape, it is conjectured that Calabi-Yau threefolds with different topologies can be connected by an operation called a geometric transition.

- In the mathematics literature, this proposal was developed by Clemens, Friedman and Reid (called "Reid's Fantasy"). In the physics literature, an analogous proposal was made by Candelas-Green-Hubsch.

Connecting Kähler and non-Kähler Calabi-Yau manifolds

- However, under these geometric transitions it is possible to pass from Kähler to non-Kähler Calabi-Yau manifolds. This suggests that understanding the non-Kähler geometry emerging from these geometric transitions may shed light on the various topologies of Kähler threefolds.

- I will explain below how the Hull-Strominger system is a natural PDE which seems to respect these transitions. I hope that this will help us to understand Reid's Fantasy.

Conifold transitions

- An example of a geometric transition is the conifold transition

$$Y \to X_0 \rightsquigarrow X_t.$$

- This is defined as follows. Let Y be a smooth Kähler Calabi-Yau threefold that contains a collection of mutually disjoint smooth curves E_i isomorphic to $\mathbb{P}^1$ and having normal bundles isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. By contracting these E_i, we obtain a singular Calabi-Yau threefold X_0 with l ordinary double points $p_1, \ldots, p_l$:

$$\psi : Y \setminus \cup_{i=1}^{l} E_i \cong X_0 \setminus \{p_1, \cdots, p_l\}.$$

Conifold transitions

We have the following facts:

- (Friedman) There is an infinitesimal smoothing of X_0 if and only if the fundamental classes $[E_i]$ in $H^2(Y;\Omega_Y^2)$ satisfy a relation

$$\sum_i n_i[E_i] = 0$$

  where $n_i \neq 0$ for each i.

- (Kawamata, Tian) The infinitesimal smoothing can always be realized by a real smoothing, i.e., X_0 can be smoothed to a family of smooth complex manifolds X_t.

- (Friedman) 1. $\pi_1(X_t) = \pi_1(Y)$;
  2. The canonical bundle of X_t is also trivial;
  3. The $\partial\bar{\partial}$-lemma holds on X_t for general t (2017).

Motivating non-Kähler geometry

- Let $Y \to X_0 \rightsquigarrow X_t$ be a conifold transition as just described. The complex manifolds X_t have trivial canonical bundle. But in general, X_t are non-Kähler.

- This is because if the curves $[E_i]$ generate $H_2(Y, \mathbb{R})$, then contracting them can produce a manifold X_t with zero second homology $H_2(X_t, \mathbb{Z}) = 0$.

- For example, $\#_k(S^3 \times S^3)$ for any $k \geq 2$ can be given a complex structure in this way (Friedman: $k \geq 103$ and Lü-Tian: $2 \leq k \leq 102$).

Motivating non-Kähler geometry

- Reid proposes that all Kähler Calabi-Yau threefolds fit into a single universal moduli space in which families of smooth Calabi-Yau's are connected by conifold transitions and their inverses, even though they are of different homotopy types.

- In fact, P. S. Green and Hübsch found that for a very large number (perhaps all) of distinct Calabi-Yau threefolds, the relevant moduli spaces join together to form a connected "web" by including certain singular limit points. Then, Candelas, Green, and Hübsch studied the geometry of these moduli spaces, especially near the interfacing regions which correspond to conifolds. They showed that the Weil-Petersson metric coincides with the point field limit of the Zamolodchikov metric and all the distances in the web are finite.

Motivating non-Kähler geometry

- From the point of view of differential geometry and moduli space, we would like to understand conifold transitions $Y \dashrightarrow X_0 \rightsquigarrow X_t$ with metrics.

$$(Y, \omega_{\mathrm{CY}}) \rightsquigarrow ?$$

- If the threefold X_t emerging from a conifold transition is Kähler, then according to my solution to the Calabi conjecture, there exists a unique Ricci-flat Kähler metric in each Kähler class of the threefold. Such metrics, known as Calabi-Yau metrics, are the bed rocks of geometric studies of Calabi-Yau threefolds.

- However, conifold transitions may leave the realm of Kähler geometry. To study this process, we need an analog of Calabi-Yau metrics in the non-Kähler setting.

- We believe that Hull-Strominger system provides a natural candidate. Let me explain why.

Conifold transitions with metrics

- Let $(Y, \omega_{\mathrm{CY}})$ be a Kähler threefold with Kähler Ricci-flat metric.

- Let $Y \to X_0 \rightsquigarrow X_t$ be a conifold transition.

- We would like to understand this transition at the level of Riemannian metrics. Since X_t may not be Kähler, we expect the Calabi-Yau metric to break into a pair of Hermitian metrics

$$\omega_{\mathrm{CY}} \rightsquigarrow (\omega_t, h_t)$$

on the tangent bundle of the smoothing X_t solving the Hull-Strominger system.

$$\begin{aligned}
&d(\| \Omega_t \|_{\omega_t} \omega_t^2) = 0; \\
&F_{h_t}^{2,0} = F_{h_t}^{0,2} = 0; \quad F_{h_t} \wedge \omega_t^2 = 0; \\
&\sqrt{-1}\partial\bar{\partial}\,\omega_t = \frac{\alpha'}{4}\Big(\mathrm{tr}\Big(R_{\omega_t} \wedge R_{\omega_t}\Big) - \mathrm{tr}\Big(F_{h_t} \wedge F_{h_t}\Big)\Big).
\end{aligned}$$

- Note: if $\omega_t = h_t$, then the system implies ω_t is Kähler Ricci-flat.

Fu-Li-Yau metrics

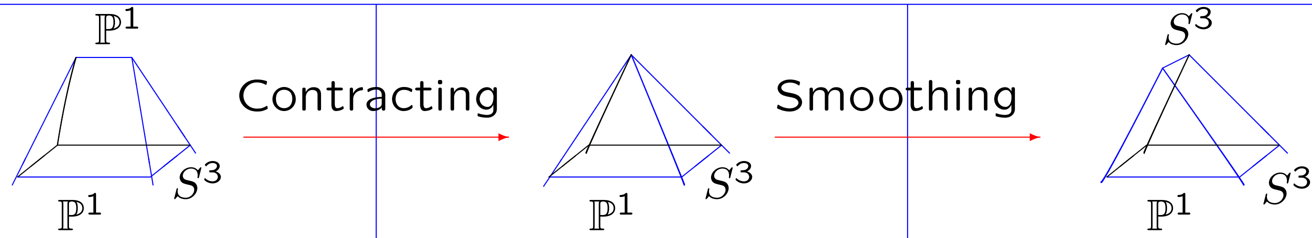
The first equation of the system, $d(\|\Omega\|_\omega \omega^2) = 0$ has been solved by Fu-Li-Yau.

Theory 2. (Fu-Li-Y.) *Let Y be a smooth Kähler Calabi-Yau threefold and let $Y \to X_0$ be a contraction of mutually disjoint $(-1,-1)$-curves. Suppose X_0 can be smoothed to a family of smooth complex manifolds X_t. Then for sufficiently small t, X_t admits a smooth balanced metric.*

Our construction provides a balanced metric on a large class of threefolds. In particular, we have

Corollary 1. (Fu-Li-Y.) *There exists a balanced metric on $\#_k(S^3 \times S^3)$ for any $k \geq 2$.*

Local Models



	Small resolution	Local neighborhoods of the node	Deformation
Description	$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$	$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0$	$w_1^2 + w_2^2 + w_3^2 + w_4^2 = t$
Radii	$r^2 = (1+\|z\|^2)(\|u\|^2 + \|v\|^2)$	$r^2 = \|w_1\|^2 + \|w_2\|^2 + \|w_3\|^2 + \|w_4\|^2$	
Neighborhoods $(0 < c < 1)$	$U(c) = \{(z,u,v) \| r^2 < c\}$	$V_t(c) = \{\sum w_i^2 = t \mid r^2 < c\}$	
Candelas-de la Ossa's metrics	$\omega_{co} = \cdots$	$\omega_{co,0} = i\partial\bar{\partial} f_0(r^2)$	$\omega_{co,t} = i\partial\bar{\partial} f_t(r^2)$

where

$$f_0(r^2) = \frac{3}{2} r^{\frac{4}{3}},$$

$$f_t(r^2) = 2^{-\frac{1}{3}} |t|^{\frac{2}{3}} \int_0^{\cosh^{-1}(\frac{r^2}{|t|})} (\sinh 2\tau - 2\tau)^{\frac{1}{3}} d\tau.$$

Sketch of the proof.

We need to construct a d-closed positive $(2,2)$-form on X_t for sufficiently small t.

First Step. Constructing a Balanced Metric on the Conifold X_0.

In this step we use the direct method. Let ω be a Kähler metric on the threefold Y. Denote by $\omega_{co,0}$ the Candelas-de la Ossa metric on a neighbourhood of the node. Near a $(-1,-1)$-curve E_i we glue $\omega^2_{co,0}$ onto ω^2 directly to construct a d-closed positive $(2,2)$-form Ω_0 on $X_{0,sm} = X_0 \setminus \{p_i\}$. This construction is very delicate.

Second Step: Constructing Balanced Metrics on the smoothing.

We glue the Candelas-de la Ossa metric $\omega_{co,t}^2$ on the deformation to Ω_0 to get a real d-closed 4-form Φ_t. Φ_t can be decomposed as

$$\Phi_t = \Phi_t^{3,1} + \Phi_t^{2,2} + \Phi_t^{1,3}.$$

For t sufficiently small, $\Phi_t^{2,2}$ is positive definite. Let ω_t be the hermitian form on X_t such that $(\omega_t)^2 = \Phi_t^{2,2}$. We use ω_t as our background metric on X_t.

We shall modify the form $\Phi_t^{2,2}$ to make it both closed and positive definite. Since Φ_t is d-closed on X_t,

$$\bar{\partial}_t \Phi_t^{2,2} = -\partial_t \Phi_t^{1,3}.$$

On the other hand, we can prove that $H_{\bar{\partial}}^{1,3}(X_t, \mathbb{C}) = 0$. Therefore there is a $(1,2)$-form ν_t on X_t such that $i\bar{\partial}_t \nu_t = -\Phi_t^{1,3}$. So $i\partial_t \bar{\partial}_t \nu_t = -\partial_t \Phi_t^{1,3}$. We let μ_t be a $(1,2)$-form on X_t such that

$$i\partial_t \bar{\partial}_t \mu_t = -\partial_t \Phi_t^{1,3} = \bar{\partial}_t \Phi_t^{2,2} \quad \text{and} \quad \mu_t \perp_{\omega_t} \ker \partial_t \bar{\partial}_t.$$

We then define

$$\Omega_t = \Phi_t^{2,2} + \theta_t + \bar{\theta}_t, \quad \theta_t = i\partial_t \mu_t.$$

Clearly Ω_t is d-closed and is positive for sufficiently small t if

$$\lim_{t\to 0} \| \theta_t \|_{C^0}^2 = 0.$$

In fact, we can prove

Proposition 1. (Fu-Li-Y.) *If* $\kappa > -\frac{4}{3}$, *then*

$$\lim_{t\to 0} (t^\kappa \| \theta_t \|_{C^0}^2) = 0.$$

This fact is the key point in M.-T. Chuan's paper on existence of Hermitian-Yang-Mills metrics under conifold transitions.

The Hull-Strominger system through conifold transitions.

- By Fu-Li-Yau's result, the first equation in the system

$$d(\| \Omega_t \|_{\omega_t} \omega_t^2) = 0$$

can be solved for a Hermitian metric ω_t. We now consider the Hermitian-Yang-Mills equation with respect to ω_t.

- M.-T. Chuan has constructed Hermitian-Yang-Mills metrics h_t solving

$$F_{h_t} \wedge \omega_t^2 = 0$$

on holomorphic bundles $E_t \to X_t$ which are deformations of a bundle $E \to Y$ which is stable with respect to $(Y, \omega_{\mathrm{CY}})$ and **holomorphically trivial** in a neighborhood of the contracted $(-1,-1)$ curves.

The Hull-Strominger system through conifold transitions.

- The most natural setting to consider conifold transitions is when the holomorphic bundle E in the Hull-Strominger system is the tangent bundle $T^{1,0}X$. However, this is not holomorphically trivial near the $(-1,-1)$ curves, so Chuan's theorem does not apply.

- In ongoing joint work with T. Collins and S. Picard, we are constructing a Hermitian metric h_t on $T^{1,0}X_t$ such that the pair (g_t, h_t) of metrics on $T^{1,0}X_t$ satisfy

$$d(\| \Omega_t \|_{\omega_t} \omega_t^2) = 0, \qquad F_{h_t} \wedge \omega_t^2 = 0, \tag{7}$$

and such that both metrics (g_t, h_t) are perturbations of the Kähler Ricci-flat Candelas-de la Ossa metrics $g_{co,t}$ in a neighborhood U of the deformation of a node, e.g. for $C > 0, \delta > 0$ then

$$\|g_{co,t} - g_t\|_{L^\infty(U \cap X_t, g_{co,t})} \leq C|t|^\delta$$
$$\|g_{co,t} - h_t\|_{L^\infty(U \cap X_t, g_{co,t})} \leq C|t|^\delta.$$

The Hull-Strominger system through conifold transitions.

- To complete the program of solving the Hull-Strominger system through conifold transitions, it remains to understand the anomaly cancellation equation. We expect to construct a solution of the full Hull-Strominger system by perturbing around the solution (g_t, h_t) of (7).

- The key point is that both (g_t, h_t) are approximately equal to the Kähler Ricci-flat metric $g_{co,t}$ near the nodes and converge back to the Calabi-Yau metric on the central fiber. Thus, we can solve the anomaly cancellation equation using perturbation techniques.

The Hull-Strominger system through conifold transitions.

Geometrically, we expect that solutions of the Hull-Strominger system on the pair $(X, T^{1,0}X)$ can provide a "canonical metric" which is continuous, in an appropriate sense, through conifold transitions. Note that, when X is Kähler, such a solution of the Hull-Strominger system is just the Calabi-Yau metric, as I discussed before. Thus, if the picture is correct, it would show that the Hull-Strominger system is the correct generalization of the Calabi-Yau equation to non-Kähler manifolds.

Moduli space

- An important future direction is to understand the moduli space of solutions to the Hull-Strominger system.

- There is promising preliminary work by Svanes-de la Ossa in the physics literature, and Garcia-Fernandez, Rubio, Tipler in the mathematics literature.

- Some of the difficulties here include understanding the regime where the Hull-Strominger system is elliptic, and proving uniqueness theorems modulo automorphisms in certain topological classes determined by a solution to the system (e.g. balanced classes, string classes).

Generalized Transitions Another direction I am considering with T. Collins and S. Picard is constructing solutions of the Strominger system through "generalized" conifold transitions. Namely, if one contracts a chain an $(-1, -1)$ curves, then we can obtain singular Calabi-Yau manifolds with singularities modeled on

$$\{x^2 + y^2 + z^p + w^p = 0\} \subset \mathbb{C}^4.$$

These affine varieties admit Calabi-Yau metrics due to Collins-Székelyhidi, generalizing the Candelas- de la Ossa metric on the conifold (which is the $p = 2$ case). They also admit natural smoothings. We are currently considering the generalization of my work with Fu and Li to construct balanced metrics through these generalized transitions.

Symplectic conifold transitions: Smith-Thomas-Yau.

- There is a symplectic version of Clemens-Friedman's conifold transition for complex manifolds. In Clemens-Friedman, the contraction of a rational curve, $\mathbb{CP}^1$ (and the inverse operation of resolution) is naturally a complex operation. The smoothing of a conifold singularity by S^3 on the other hand is naturally a symplectic operation. The condition that Friedman gave is necessary to ensure that the smoothed out Calabi-Yau has a global complex structure.

- But instead of the complex structure, we can ask to preserve the symplectic structure throughout the conifold transition. This would be the symplectic *mirror* of the Clemens-Friedman's conifold transition. In this case, we would collapse disjoint Lagrangian three-spheres, and replace them by symplectic two-spheres. Such a symplectic transition was proposed by Smith, Thomas, and myself in 2002.

Locally, there is a natural symplectic form in resolving the singularity by a two-sphere. But there may be obstructions to patching the local symplectic forms to get a global one. In Smith-Thomas-Yau, we wrote down the condition (analogous to Friedman's complex condition) that ensures a global symplectic structure. This symplectic structure however may not be compatible with a genuine complex structure.

The resulting real six-manifold would be symplectic and have $c_1 = 0$. But its third Betti number may be zero. Hence, such a manifold would generally be non-Kähler, and is called a *sympletic Calabi-Yau*.

Smith-Thomas-Yau used such symplectic conifold transitions to construct many real six-dimensional symplectic Calabi-Yaus. We showed that if such a conifold transition collapsed all disjoint three-spheres, then the resulting space is a manifold that is diffeomorphic to connected sums of $\mathbb{CP}^3$. This mirrors the complex case, where after the collapsed of all disjoint rational curves gives a connected sums of $S^3 \times S^3$

It is known that six-dimensional symplectic Calabi-Yaus come in all different topological types. For example, Fine-Panov (2013) constructed an infinite collection of six-dimensional symplectic Calabi-Yaus for any given finitely-presented fundamental group.

SYZ mirror of non-Kähler Calabi-Yaus.

Symplectic Calabi-Yaus can also be considered from the perspective of Strominger-Yau-Zaslow (SYZ) mirror symmetry.

Recall that the SYZ mirror of a Kähler Calabi-Yau threefolds with a T^3-fibration is found by applying three T-dualities on the torus fibration. This viewpoint was clarified in much detail in the work of Gross-Siebert. Extending the SYZ idea, we can consider applying T-dualities on complex non-Kähler Calabi-Yau spaces that are torus fibration. In doing so, we can find interesting mirrors that are symplectic Calabi-Yau manifolds.

In this non-Kähler SYZ mirror symmetry context, there are also equations from physics that let us study the geometry of the mirror pairs. The equations come from supersymmetry conditions of Type IIA and IIB string theory. The relevant Type IIB equations are similar to the Hull-Strominger system though the Hermitian-Yang-Mills bundles are now replaced by D5-brane and orientifold 5-brane source charges.

In particular, for six-dimensional manifolds and in the semi-flat limit, Lau-Tseng-Yau (2015) related the symplectic, non-Kähler, supersymmetry conditions of type IIA strings to those of complex, non-Kähler, supersymmetry conditions of type IIB strings by means of SYZ and Fourier-Mukai transform. In dimensions eight and higher, we also found a mirror pair system of equations - a symplectic system and a complex system for non-Kähler Calabi-Yaus - that are dual to each other under SYZ mirror symmetry.

Thank you for joining us!

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