Hodge structures and the topology of algebraic varieties

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Plan of the talk

- Analysis and differential geometry $\leadsto$ Hodge and Lefschetz decompositions.
- $+$ Algebra $\leadsto$ Consequences on topology.
- The importance of polarizations (signs and Hodge-Riemann relations).
- **Missing**. Variations of Hodge structures.
Kähler and projective complex manifolds

Complex manifold = manifold equipped with an atlas
\[ U_i \cong V_i \subset \mathbb{C}^n, \] with holomorphic change of coordinates maps.

- The tangent space at each point is endowed with a structure de \( \mathbb{C} \)-vector space, hence an operator \( I, I^2 = -Id \), of **almost complex structure** acting on \( T_{X,\mathbb{R}} \). Newlander-Nirenberg integrability condition.

- Notion of **Hermitian metric** on \( X \). In local holomorphic coordinates,
  \[ h = \sum_{ij} h_{ij} dz_i \otimes d\bar{z}_j, \] with imaginary part \( \omega = \frac{1}{i} \sum_{ij} \omega_{ij} dz_i \wedge d\bar{z}_j, \)
  \( \omega_{ij} = \text{Im} h_{ij} \). This is a 2-form “of type (1, 1)”.

**Definition.** The Hermitian metric is Kähler if \( d\omega = 0 \). \( [\omega] = \text{Kähler class} \).

- On \( \mathbb{CP}^N \): Fubini-Study Kähler metric. The Kähler class equals \( c_1(\mathcal{H}^*) \), where \( \mathcal{H} \) is the Hopf line bundle, hence is integral. Idem for \( X \subset \mathbb{CP}^N \) complex submanifold.

- **Kodaira embedding theorem.** A compact Kähler manifold is projective iff it admits a Kähler form with integral cohomology class.
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Complex manifold = manifold equipped with an atlas $U_i \cong V_i \subset \mathbb{C}^n$, with holomorphic change of coordinates maps.

- The tangent space at each point is endowed with a structure of almost complex $\mathbb{C}$-vector space, hence an operator $I$, $I^2 = -Id$, of **almost complex structure** acting on $T_{X,\mathbb{R}}$. Newlander-Nirenberg integrability condition.

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The Frölicher spectral sequence

• $X$ complex manifold, $z_1, \ldots, z_n =$ local holomorphic coordinates. Holomorphic vector bundle $\Omega_X$ generated over $\mathcal{O}_X$ by $dz_i$. Transition matrices given by holomorphic Jacobian matrices.

• Holomorphic de Rham complex $\Omega^k_X := \bigwedge^k \Omega_X$, with exterior differential $d$.

**Thm.** (Holomorphic Poincaré lemma). *The complex*

$$
\mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_X \rightarrow 0
$$

*is exact in degrees $> 0$. This is a resolution of the constant sheaf $\mathbb{C}$.*

**Corollary.** $H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X)$.

• Filtration “bête” $F^p \Omega^\bullet_X := \Omega^{\bullet \geq p}_X \hookrightarrow$ Frölicher spectral sequence. $E_1^{p,q} \Rightarrow H^{p+q}(X, \mathbb{C})$.

• $E_1^{p,q} = H^q(X, \Omega^p_X), \ d_1 = d$.

• On the abutment: “Hodge” filtration

$F^p H^k(X, \mathbb{C}) := \text{Im}(\mathbb{H}^k(X, \Omega^{\bullet \geq p}_X) \rightarrow \mathbb{H}^k(X, \Omega^\bullet_X)), \ E_\infty^{p,q} = Gr^p_F H^k(X, \mathbb{C})$. 
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- \( E_1^{p,q} = H^q(X, \Omega_X^p) \), \( d_1 = d \).
- On the abutment: “Hodge” filtration \( F^p H^k(X, \mathbb{C}) := \text{Im} (H^k(X, \Omega_X^{\geq p}) \to H^k(X, \Omega_X^\bullet)), E_\infty^{p,q} = Gr_F^p H^k(X, \mathbb{C}). \)
Quasiprojective manifolds and logarithmic de Rham complexes

• $j : U \hookrightarrow X$, $U = X \setminus Y$, with $Y \subset X$ closed analytic.

• (Hironaka) By successive blow-ups of $X$ along smooth centers supported over $Y$, one can assume that $Y$ is a normal crossing divisor: i.e. $Y$ is locally defined by a single holomorphic equation of the form $f = z_1 \ldots z_k$ in adequate holomorphic coordinates.

• Define $\Omega^1_X(\log Y)$ as the holomorphic vector bundle generated over $\mathcal{O}_X$ by $\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n$.

• $\Omega^k_X(\log Y) = \bigwedge^k \Omega^1_X(\log Y)$, $d : \Omega^k_X(\log Y) \to \Omega^{k+1}_X(\log Y)$. 

• Their sections (= forms with logarithmic growth) are the forms with pole order 1 along $Y$, whose differential also has pole order 1 along $Y$.

**Thm.** The inclusion of the subcomplex $\Omega^\bullet_X(\log Y) \subset j_*\Omega^\bullet_U$ is a quasiisomorphism.

**Corollary.** $H^k(U, \mathbb{C}) = H^k(X, \Omega^\bullet_X(\log Y))$ and Frölicher spectral sequence.

• also $H^k(U, \mathbb{C}) = H^k(U, \Omega^\bullet_U)$ hence two Hodge filtrations on $H^k(U, \mathbb{C})$. 
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The Hodge decomposition theorem

- $X =$ compact oriented Riemannian manifold. $\sim L^2$-metric on forms.
  \[(\alpha, \beta)_{L^2} = \int_X \alpha \wedge * \beta.\]
- Formal adjoint $d^* = \pm * d*$. Laplacian $\Delta_d = d \circ d^* + d^* \circ d$.

**Harmonic forms.** $\Delta_d \alpha = 0$. $X$ compact and $\alpha$ harmonic $\Rightarrow \alpha$ is closed.

**Thm. (Hodge)** *Each de Rham cohomology class contains a unique harmonic representative.*

- Forms of type $(p, q)$ on $X =$cplx mfld: $\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} dz_I \wedge d\bar{z}_J$.
  Any $k$-form writes uniquely as a sum $\sum_{p+q=k} \alpha^{p,q}$.

**Thm. (Hodge)** $X$ Kähler $\Rightarrow \Delta_d \alpha^{p,q}$ is of type $(p, q)$.

**Corollary.** $\alpha$ harmonic, $\alpha = \sum_{p,q} \alpha^{p,q} \Rightarrow$ each $\alpha^{p,q}$ is harmonic.

**Thm. (Hodge)** Let $H^{p,q}(X) := \{\text{classes of closed forms of type } (p, q)\}$. Then $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$ and $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.

- **Hodge symmetry.** $H^{p,q}(X) = H^{q,p}(X)$.

**Cor.** The Frölicher spectral sequence of $X$ degenerates at $E_1$ ($E_1 = E_\infty$).

- Consequences in deformation theory. For example BTT theorem.
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- Consequences in deformation theory. For example BTT theorem.
The category of Hodge structures

**Definition.** (Hodge structure) A Hodge structure of weight $k = \text{ lattice } L$ + decomposition $L_\mathbb{C} = \bigoplus_{p+q=k} L^{p,q}$, with $L^{p,q} = L^{q,p}$.

- Hodge decomposition $\Rightarrow$ Hodge filtration: $F^p L_\mathbb{C} := \bigoplus_{r \geq p} L^{r,k-r}$.

Conversely $L^{p,q} = F^p L_\mathbb{C} \cap \overline{F^q L_\mathbb{C}}$, $p + q = k$.

Condition on $F^\bullet$: $L_\mathbb{C} = F^p L_\mathbb{C} \oplus F^{k-p+1} L_\mathbb{C}$.

- **Variants.** (a) Rational coefficients.
(b) **Effective** Hodge structure: $L^{p,q} = 0$ if $p < 0$ or $q < 0$.

**Definition.** $(L, F^p L_\mathbb{C}), (L', F^p L'_\mathbb{C})$ Hodge structures of weights $k, k+2r$. A morphism of Hodge structures between them is $\phi : L \to L'$, s.t. $\phi_\mathbb{C}(L^{p,q}) \subset L'^{p+r,q+r}$.

**Example** $T = \mathbb{C}^n / \Gamma \simm \Gamma_\mathbb{C} \to \mathbb{C}^n$ with kernel $\Gamma^{1,0} \subset \Gamma_\mathbb{C}$ is an equivalence of categories (Complex tori) $\leftrightarrow$ (effective weight 1 Hodge structures).

- Complex tori up to isogeny $\leftrightarrow$ Weight 1 rational Hodge structures.

**Fact.** The category of rational Hodge structures is not semi-simple. There are morphisms of complex tori $T \to T'$ which do not split up to isogeny.
The category of Hodge structures

**Definition.** (Hodge structure) A Hodge structure of weight \( k \) is a lattice \( L \) in a decomposition \( L_\mathbb{C} = \bigoplus_{p+q=k} L^{p,q} \), with \( L^{p,q} = L^{q,p} \).

- Hodge decomposition \( \leadsto \) Hodge filtration: \( F^p L_\mathbb{C} := \bigoplus_{r \geq p} L^{r,k-r} \).

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Hodge structures from geometry; functoriality

**Thm.** *X compact Kähler. The cohomology* $H^k(X, \mathbb{Z})/\text{Tors}$ *carries an effective Hodge structure of weight* $k$.

- $\phi : X \to Y$ holomorphic map, with $X$, $Y$ compact Kähler.
  $\phi^* : H^k(Y, \mathbb{Z})_{tf} \to H^k(X, \mathbb{Z})_{tf}$ is a morphism of Hodge structures.

**Prop.** *The Gysin morphism* $\phi_* : H^k(X, \mathbb{Z})_{tf} \to H^{k-2d}(Y, \mathbb{Z})_{tf}$, 
$d = \dim X - \dim Y$, *is a morphism of Hodge structures.*

- **Explanation.**
  (a) Via Poincaré duality, $\phi_*$ is the transpose of $\phi^*$.
  (b) weight $k$: Hodge structure on $L \leftrightarrow$ weight $-k$: Hodge structure on $L^*$: $L^*\to^{p,-q}$ is defined as the orthogonal of $\bigoplus_{(r,s)\neq(p,q)} L^{r,s}$.
  (c) the Hodge structure on $H^{2n-k}(X, \mathbb{Z})_{tf}$ is dual to the Hodge structure on $H^k(X, \mathbb{Z})_{tf}$ up to a shift of bidegree (for "type reasons":
  $\int_X \alpha^{p,q} \wedge \beta^{p',q'} = 0$ for $(p', q') \neq (n-p, n-q))$. \textbf{qed}

**Construction.** Hodge structure on $L$, resp. $M$ of weights $k$, resp. $k'$ $\rightsquigarrow$ Weight $k + k'$ Hodge structure on $L \otimes M$:

$$(L_{\mathbb{C}} \otimes M_{\mathbb{C}})^{p,q} = \bigoplus_{r+r' = p, s+s' = q} L^{r,s} \otimes L'^{r',s'}.$$
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Functoriality and Hodge classes

**Definition.** (Hodge classes) A Hodge class on a weight $2k$ Hodge structure $L$ is an element of $L \cap L^{k,k}$.

**Example.** Hodge classes on $L^* \otimes M$, $L$ of weight $k$, $M$ of weight $k + 2r$, are the morphisms of Hodge structures $L \to M$.

**Corollary.** Hodge classes on a product $X \times Y$ of compact Kähler manifolds identify with the morphisms of Hodge structures $H^*(X,\mathbb{Z})_{tf} \to H^{*+2r}(Y,\mathbb{Z})_{tf}$.

**Example.** $Z \subset X$ closed analytic subset of codimension $k$ has a class $[Z] \in H^{2k}(X,\mathbb{Z})$. If $X$ is compact Kähler, this is a Hodge class.

**Conjecture.** (Hodge conjecture) $X$ smooth complex projective. Rational Hodge classes on $X$ are algebraic, i.e. generated by cycles classes.

**Example.** Künneeth components of the diagonal. $\delta_k \sim Id_{H^k(X,\mathbb{Z})}$.

- Known in degree 2 (Lefschetz (1, 1)-thm) and $2n - 2$ by hard Lefschetz.
- Wrong in the compact Kähler setting, even in a weaker form replacing cycle classes by Chern classes of coherent sheaves (Voisin).
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Mixed Hodge structures

**Definition.** A rational mixed Hodge structure = a $\mathbb{Q}$-vector space $L$ with an increasing (weight) filtration $W_i L$ and a decreasing (Hodge) filtration $F^p L_{\mathbb{C}}$, such that: the induced filtration on $\text{Gr}^W_i L$ defines a Hodge structure of weight $i$.

**Thm.** (Deligne) The cohomology of quasiprojective complex varieties, or analytic-Zariski open in compact Kähler manifolds, or relative (co)homology of such pairs, carries functorial mixed Hodge structures.

- **Smooth case:** $U = X \setminus Y \hookrightarrow X$, $Y$=normal crossing divisor. Use $H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X(\log Y))$. Filtration $F$ on $\Omega^\bullet_X(\log Y)$ is the usual one ("bête"). Filtration $W$ on $\Omega^\bullet_X(\log Y)$: up to a shift, this is given by $W_i \Omega^\bullet_X(\log Y) = \Omega^i_X(\log Y) \wedge \Omega^{\bullet-i}_X$.

- A posteriori, the induced $W$- filtration is defined on rational cohomology and related to the Leray filtration of $j$.

- **The s.s. for $F$ degenerates at $E_1$, the s.s. for $W$ degenerates at $E_2$.**

- In this case, the smallest weight part of $H^k(U, \mathbb{Q})$ is $\text{Im} (j^* : H^k(X, \mathbb{Q}) \to H^k(U, \mathbb{Q}))$ (weight $k$).
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Formal properties and application to the coniveau

- Morphisms of MHS: \( \phi : L \to L', \phi(W_iL) \subset W_iL', \phi_{\mathbb{C}}(F^pL_{\mathbb{C}}) \subset F^pL'_{\mathbb{C}}. \)

**Thm.** (Deligne) **Morphisms of mixed Hodge structures are strict for both filtrations** (i.e.: \( F^pL'_{\mathbb{C}} \cap \text{Im} \phi_{\mathbb{C}} = \phi_{\mathbb{C}}(F^pL_{\mathbb{C}}), W_iL' \cap \text{Im} \phi = \phi(W_iL). \))

**Sketch of proof.** Follows from an algebra lemma: There exists a functorial decomposition \( L_{\mathbb{C}} = \bigoplus_{p,q} L^{p,q} \) for mixed Hodge structures \((L, W, F)\), with \( F^pL_{\mathbb{C}} = \bigoplus_{r \geq p} L^{r,q}, W_i L_{\mathbb{C}} = \bigoplus_{p+q \leq i} L^{p,q}. \)

Let \( \alpha \in W_iL' \cap \text{Im} \phi. \) Write \( \alpha = \phi(\beta), \beta = \sum_{p,q} \beta^{p,q}. \) Then \( \phi(\beta^{p,q}) = 0 \) for \( p + q > i \) so \( \alpha = \phi(\beta') \) with \( \beta' = \sum_{p+q \leq i} \beta^{p,q} \in W_iL_{\mathbb{C}}. \) qed

**Definition.** A class \( \alpha \in H^k(X, \mathbb{Q}) \) is of coniveau \( \geq c \) if \( \alpha|_{X \setminus Y} = 0 \) with \( Y \) closed analytic of codim \( \geq c. \)

If \( X \) is smooth compact, \( j : Y \hookrightarrow X, \) equivalent condition: \( \alpha = j_*\beta \) in \( H_{2n-k}(X, \mathbb{Q}) \) for some \( \beta \in H_{2n-k}(Y, \mathbb{Q}). \)

**Strictness \Rightarrow** If \( X \) is smooth projective, \( j : Y \hookrightarrow X \) with desingularization \( \tilde{j} : \tilde{Y} \to X, \) then \( \text{Im} j_* = \text{Im} \tilde{j}_* \subset H_{2n-k}(X, \mathbb{Q}). \)

**Corollary.** (Deligne) **The set of cohomology classes of coniveau \( \geq c \) is a Hodge substructure of** \( H^k(X, \mathbb{Q}), \) **of Hodge coniveau \( \geq c. \)**
Morphisms of MHS: \( \phi : L \to L', \phi(W_iL) \subset W_iL', \phi_C(F^pL_C) \subset F^pL'_C. \)

**Thm. (Deligne)** Morphisms of mixed Hodge structures are strict for both filtrations (i.e.: \( F^pL'_C \cap \text{Im } \phi_C = \phi_C(F^pL_C), W_iL' \cap \text{Im } \phi = \phi(W_iL) \)).

**Sketch of proof.** Follows from an algebra lemma: There exists a functorial decomposition \( L_C = \bigoplus_{p,q} L^{p,q} \) for mixed Hodge structures \( (L, W, F') \), with \( F^pL_C = \bigoplus_{r \geq p, q} L^{r,q}, W_iL_C = \bigoplus_{p+q \leq i} L^{p,q}. \)

Let \( \alpha \in W_iL' \cap \text{Im } \phi. \) Write \( \alpha = \phi(\beta), \beta = \sum \beta_{p,q}. \) Then \( \phi(\beta_{p,q}) = 0 \) for \( p + q > i \) so \( \alpha = \phi(\beta') \) with \( \beta' = \sum_{p+q \leq i} \beta_{p,q} \in W_iL_C. \) qed

**Definition.** A class \( \alpha \in H^k(X, \mathbb{Q}) \) is of coniveau \( \geq c \) if \( \alpha|_{X \setminus Y} = 0 \) with \( Y \) closed analytic of codim \( \geq c. \)

If \( X \) is smooth compact, \( j : Y \hookrightarrow X, \) equivalent condition: \( \alpha = j_*\beta \) in \( H_{2n-k}(X, \mathbb{Q}) \) for some \( \beta \in H_{2n-k}(Y, \mathbb{Q}). \)

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**Thm.** (Hard Lefschetz, proved by Hodge) *Let* \( X \) *be compact Kähler of dimension* \( n \), \( \omega \) *a Kähler form on* \( X \). *Then* \( \forall k \leq n \), 
\[
\bigcup [\omega]^{n-k} := L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})
\] is an isomorphism.

**• Projective case**: One can take \([\omega]\) rational. Then the Lefschetz isomorphism is an isomorphism of Hodge structures.

**Coro.** (Lefschetz decomp.) \( H^k(X, \mathbb{R}) = \bigoplus_{k-2r \geq 0} L^r H^{k-2r}(X, \mathbb{R})_{\text{prim}} \), where \( H^{k-2r}(X, \mathbb{R})_{\text{prim}} := \text{Ker} L^{n-k+2r+1} \subset H^{k-2r}(X, \mathbb{R}) \).

**• Lefschetz intersection pairing on* \( H^k \): \( \langle \alpha, \beta \rangle_{\text{Lef}} = \int_X L^{n-k} \alpha \cup \beta \).
\( h_{\text{Lef}}(\alpha, \beta) := i^k(\alpha, \overline{\beta})_{\text{Lef}} \).

**easy**: The Lefschetz decomposition is orthogonal for \( (\ , \ )_{\text{Lef}} \), and the Hodge decomposition is orthogonal for \( h_{\text{Lef}} \). (HR1).

**Thm.** 2nd H-R bilinear relations: \((-1)^{p+r} h_{\text{Lef} \mid L^r H^{p-r,q-r}(X, \mathbb{R})_{\text{prim}}} \) is positive definite Hermitian (up to a global sign depending on \( k \)). (HR2).

**Corollary.** Let \([\omega]\) be rational. On \( L^r H^{k-2r}(X, \mathbb{Q})_{\text{prim}} \), multiply \((\ , \ )_{\text{Lef}}\) by \((-1)^r\): one gets a polarized Hodge structure on \( H^k(X, \mathbb{Q}) \).
**Polarizations**

**Thm.** (Hard Lefschetz, proved by Hodge) Let $X$ be compact Kähler of dimension $n$, $\omega$ a Kähler form on $X$. Then $\forall k \leq n, \quad \cup [\omega]^{n-k} := L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$ is an isomorphism.

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Polarizations, ctd

**Thm.** Let $H =$ rational polarized Hodge structure, $H' \subset H$ a Hodge substructure, then $H = H' \oplus H''$ for some Hodge substructure $H'' \subset H$. *(The category of polarized Hodge structures is semisimple).*

**Proof.** Choose a polarization $(\cdot, \cdot)$ on $H$. First prove that $(\cdot, \cdot)|_{H'}$ is nondegenerate using HR2, then define $H'' = H' \perp$. $H''$ is a Hodge substructure by HR1. qed

- Polarizations on the cohomology of smooth projective varieties are almost motivic, but one needs the Lefschetz decomposition and the change of signs. To make them **motivic**, one needs:

  **Lefschetz standard conjecture.** $X$ projective. There exists a codimension $k$ closed algebraic subset $Z_{\text{Lef}} \subset X \times X$ such that $[Z_{\text{Lef}}]^* : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ is the inverse $(L^{n-k})^{-1}$ of the Lefschetz isomorphism.

- $([Z_{\text{Lef}}] \in H^{2k}(X \times X, \mathbb{Q}) = \text{cohomology class of } Z_{\text{Lef}}$.)

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Hodge structures on cohomology algebras and applications to topology

- A cohomology algebra = graded, graded commutative, algebra of finite dimension over \( \mathbb{Q} \), with \( A^{2n} = \mathbb{Q} \) and Poincaré duality.

**Definition.** A Hodge structure on a cohomology algebra \( A^\ast \), = Hodge structure of weight \( k \) on \( A^k \), such that \( A^k \otimes A^l \rightarrow A^{k+l} \) is a morphism of Hodge structures.

**Example.** \( H^\ast(X, \mathbb{Q}) \) for \( X \) compact Kähler.

**Thm.** (Voisin) There exist compact Kähler manifolds (\( \dim \geq 4 \)) whose cohomology algebra is not isomorphic to \( H^\ast(X, \mathbb{Q}) \) for \( X \) complex projective.

**Idea of proof.** (1) Construct an \( X \) such that the structure of its cohomology algebra \( \Rightarrow \) the Hodge structure on \( H^1(X, \mathbb{Q}) \) (or \( H^2(X, \mathbb{Q}) \) for simply connected examples) has endomorphisms.

(2) Certain endomorphisms on weight 1 (or weight 2) HS prevent the existence of a polarization.

**Case of \( \dim 2 \) (Kodaira), \( \dim 3 \) (Lin):** Any compact Kähler \( X \) has small deformations which are projective.
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**Thm.** (Blanchard, Deligne) *If* $f : X \to Y$ *is smooth projective, the Leray spectral sequence of* $f$ *with* $\mathbb{Q}$-*coefficients degenerates at* $E_2$.

**Proof.** Relative Lefschetz operator $L = c_1(\mathcal{L}) \cup$ acts on the whole spectral sequence, and induces Lefschetz decomposition

$$R^k f_* \mathbb{Q} = \bigoplus_r L^r (R^{k-2r} f_* \mathbb{Q})_{\text{prim}}.$$  

Suffices to prove $d_2 \alpha = 0$ for $\alpha \in H^p(Y, R^q f_* \mathbb{Q}_{\text{prim}})$. But $L^{n-q+1} \alpha = 0 \Rightarrow L^{n-q+1} d_2 \alpha = 0$. But $d_2 \alpha \in H^{p+2}(Y, R^{q-1} f_* \mathbb{Q})$ and $L^{n-q+1} : R^{q-1} f_* \mathbb{Q} \cong R^{2n-q+1} f_* \mathbb{Q}$. **qed**

- **Monodromy.** Local system $R^k f_* \mathbb{Q} \rightsquigarrow$ monodromy representation $\rho : \pi_1(Y, 0) \to \text{Aut } H^k(X_0, \mathbb{Q})$. Thus $H^k(X_0, \mathbb{Q})^\rho = H^0(Y, R^k f_* \mathbb{Q}) = \text{Im } (H^k(X, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}))$ by degeneracy at $E_2$.

**Thm** (Deligne) $X \subset \overline{X}$ smooth projective, $f : X \to Y$ as above with $Y$ quasi-projective. Then $H^k(X_0, \mathbb{Q})^\rho = \text{Im } (H^k(\overline{X}, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}))$. This is a Hodge substructure of $H^k(X_0, \mathbb{Q})$.

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The Hodge bundles

- Algebraic de Rham complex $\Omega^\bullet_{X/\mathbb{C}}$, relative version $\Omega^\bullet_{X/Y}$ for $f : X \to Y$ algebraic, smooth morphism.

**Thm.** (Serre-Grothendieck) *X* smooth quasiprojective over $\mathbb{C}$. Then $\mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{C}}) \cong H^k_B(X, \mathbb{C})$.

- So, for $X$ projective, the Hodge filtration and Frölicher s.s. are algebraic.

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- Katz-Oda construction: relative holomorphic de Rham complex $\Omega^\bullet_{X/Y}$. $R^k f_* \Omega^\bullet_{X/Y} \cong \mathcal{H}^k := H^k \otimes \mathcal{O}_Y$. Hodge filtration $F^p\mathcal{H}^k = R^k f_* \Omega^\bullet_{X/Y}^{\geq p}$ with fiber $F^p H^k(X_t)$.

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**Thm.** (Katz-Oda) The Gauss-Manin connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_Y$ is the connecting map.

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